On the Role of Inhibition Processes in Modeling Control Strategies for Composting Plants

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Abstract. We introduce a mathematical model for the composting process in biocells where several chemical phenomena, like the aerobic biodegradation, the hydrolysis of insoluble substrate and the biomass decay, occur. We investigate the best aeration strategies in presence of inhibition processes due to high concentrations of oxygen. Optimal strategies are obtained as result of a suitable optimal control problem. The dynamics exhibits an enhanced level of the oxygen concentration that guarantees the aerobic feature of the biodegradation process. Then, a nonlinear bioeconomic term is included in the objective functional to take into account of the external operational cost. The role of the economic cost in the control policy is analyzed and discussed.

Keywords: Waste - composting - bioreactor - inhibition - bioeconomic cost - optimal control

1 Introduction

Waste management is an important challenge, especially for local authorities [32], since the traditional stockage is not an efficient technique at long term [16]. In fact, in a containment vessel the risk of soil and aquifers contamination is high due to the formation of leachate. In addition, new stockage sites are often required in order to face the increasing amount of waste [4].

In order to overcome such critical issues a different approach has been proposed by conceiving the vessel as a bioreactor. From this perspective, the containment site can be seen as a biologically active environment where the stocked matter can be degraded by suitable natural processes [26], possibly improved by means of external operation like mechanical aeration [10, 12].

A significant fraction of solid waste is given by the organic matter, i.e. a mixture of green, food, agricultural waste, biodegradable matter. Such portion can be used to produce both an agricultural fertilizer and biogas (similar to methane but with low calorific value). As a consequence, a consistent reduction of the total amount and an increasing capacity of landfills follow [29].
The degradation process is a natural and spontaneous phenomenon but it can be improved by external operations, like aeration or leachate recirculation [10, 12, 18, 3, 33]. The degradation takes place by means of a bacterial population, that uses the waste as a nutrient, and it can occur both in absence or in presence of oxygen, respectively in anaerobic and aerobic processes [8, 9]. Anaerobic digestion is usually involved in the biogas production and in the leachate treatment [13, 17] while the aerobic one is more useful in composting techniques, like aerated static pile, in-vessel or windrow composting [16].
The benefits of both approaches can be combined in the so-called integrated systems where the organic matter is conceived as a source of energy for biogas production, and as raw material for the high quality composting [21]. The waste is pressed and mashed and the liquid part is separated by the solid one. The first is treated by an anaerobic process in a biodigester by producing burnable biogas. The second is used mainly to produce a high quality compost. The product of this process is not still usable as fertilizer and it has to undergo an additional maturation phase out of the bioreactor [16].
In this paper, we focus our attention to the aerobic phase involved in biocell composting; this means that aerobic biodegradation occurs in a closed system where additional oxygen can be injected by external operation.
Inspired by the investigation in [19], a new mathematical model has been proposed in [23] to describe the digestion process in biocell . Such description tries to ensure the aerobic feature of the digestion phenomenon.
As well described in [30], an aerobic bacterial population requires a sufficient level of oxygen for its survival; if the oxygen concentration in the system atmosphere goes under a given threshold part of the process can become anaerobic. The optimal control proposed in [23] shows an aeration strategy to maintain the oxygen concentration level close to an optimal operational value, identified as the value corresponding to the fastest degradation of the organic matter.
In this paper we are interested in including other distinctive features.
In particular, first we would like to model and discuss possible inhibition effects due to high oxygen concentration values. As indicated in [30], a high oxygen fraction in the cell atmosphere can overdry the organic matter, so that a necessary level of moisture cannot be maintained in the entire evolution and the aerobic degradation is not guaranteed. Such effects can be introduced in the model by a suitable modification of the biodegradation term in the evolution equations governing the biological system. More precisely, the dependence of the degradation rate on the oxygen variable has to be modified in order to have small contributions for high oxygen concentration.
A second characteristic feature to be taken into account is the economic cost related to the artificial aeration. Such operation corresponds to an economic

contribution in term of technology, electricity or working hours and its cost has a crucial role in the decision policy, since any intervention requires to be cost effective [1].
The paper is organized as follows: after a presentation of the mathematical description of the aerobic process in Section 2 and of the optimal control problem in Section 3, we will introduce the inhibition effect due to excessive aeration in Section 4 and present a comparison with the results obtained in [23] in Section 5. In Section 6 we will introduce the bioeconomic cost due to artificial aeration and discuss the control strategy for varying weigth of the cost in decision policy in Section 7.

2 The aerobic biodegradation model

We indicate by S the soluble substrate, i.e. the waste fraction ready to be degraded, by I the insoluble one, that is not yet available for the digestion process and has to be decomposed, by X the biomass, by L the liquid part and by M the inert mass, i.e the pre-compost.
The time evolution of these variables is driven by three different chemical phenomena; more precisely, (i) the aerobic biodegradation of the soluble substrate in presence of the oxygen Ω, where the biomass concentration increases and water and inert mass are produced, (ii) the hydrolysis, where the insoluble substrate is decomposed giving the soluble one, (iii) the biomass decay, that converts part of the biomass in pre-compost and insoluble substrate.
Such phenomena are mathematically described by the following system of nonlinear ordinary differential equations

$$\begin{aligned}
\dot{S} &= -\frac{1}{Y_S}\,\tilde{g}\,(S,\Omega)\;X \;+\; K_h I \\
\dot{I} &= -K_h I \;+\; \frac{1}{Y_I}\,bX \\
\dot{X} &= \tilde{g}\,(S,\Omega)\;X \;-\; bX \\
\dot{L} &= \frac{1}{Y_L}\,\tilde{g}\,(S,\Omega)\;X \\
\dot{M} &= -\left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\tilde{g}\,(S,\Omega)\;X+\left(1-\frac{1}{Y_I}\right)bX
\end{aligned}$$

where the upper dot denotes the derivative with respect to the time variable $\tilde{t}$, $\tilde{g}$ is the biomass growth function describing the soluble substrate degradation, K_h and b are respectively the hydrolitic and biomass decay constant and Y_S, Y_I and Y_L are the yield coefficients. All the parameters are given positive constants.
The time variation of the oxygen is given by

$$\dot{\Omega} \;=\; -\frac{1}{Y_\Omega}\,\tilde{g}\,(S,\Omega)\;X\,.$$

We can notice that the inert mass variation is obtained by imposing the conservation of the total (solid+liquid) mass

$$\dot{M} = -\left(\dot{S} + \dot{I} + \dot{X} + \dot{L}\right) .$$

Let us denote by $\tilde{\mathcal{M}}$ the total mass, $\tilde{\mathcal{M}} = S + I + X + L + M$. It follows that $\tilde{\mathcal{M}}(t) = \tilde{\mathcal{M}}(0) = S(0) + I(0) + X(0) + L(0) + M(0) =: \tilde{\mathcal{M}}_0$ for all $t \geq 0$.
The system can be rewritten by introducing the non dimensional variables

$$t = \mu\tilde{t}\,,\; s = \frac{S}{\tilde{\mathcal{M}}_0}\,,\; i = \frac{I}{\tilde{\mathcal{M}}_0}\,,\; x = \frac{X}{\tilde{\mathcal{M}}_0}\,,\; \ell = \frac{L}{\tilde{\mathcal{M}}_0}\,,\; m = \frac{M}{\tilde{\mathcal{M}}_0}\,,\; \omega = \frac{\Omega}{\Omega_0}\,,$$

where Ω_0 is the optimal operational value of the oxygen concentration for the biodigestion process. As indicated in [38] the aerobic process occurs in presence of a suitable oxygen level. The aerobic biomass can survive in presence of as little as 5% oxygen concentration in the system atmosphere but if the oxygen level goes under 10% part of the biodegradation can become anaerobic. Moreover an oxygen level around 10% guarantees a fast degradation of the organic matter. It follows that such level can be considered an optimal operational value for the oxygen concentration.
The non-dimensional differential equations have the following form

$$\begin{aligned}
\dot{s} &= -\frac{1}{Y_S} g\,(s,\omega)\; x \,+\, c_h i \\
\dot{i} &= -\,c_h i \,+\, \frac{1}{Y_I}\beta x \\
\dot{x} &= g\,(s,\omega)\, x \,-\, \beta x \\
\dot{\ell} &= \frac{1}{Y_L} g\,(s,\omega)\, x \\
\dot{m} &= -\left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right) g\,(s,\omega)\; x \,+\, \left(1 - \frac{1}{Y_I}\right)\beta x
\end{aligned} \tag{1}$$

and

$$\dot{\omega} = -\gamma\, g\,(s,\omega)\, x. \tag{2}$$

where now the upper dots denote the derivative with respect to the nondimensional variable t, g is the non-dimensional version of the bacterial growth function $\tilde{g}$, c_h and β are respectively the non-dimensional hydrolitic and decay coefficients, $\gamma = T_0/\left(Y_\Omega \Omega_0\right)$,

$$0 \leq s\,,\, i\,,\, x\,,\, \ell\,,\, m \leq 1\,,$$

$\omega > 0$ and its optimal operational value is equal to 1.
From equation (2) immediately follows that ω decreases and the oxygen will be, at least partially, consumed. In order to ensure the survival of the composting process for a long time, it is necessary to inject additional oxygen in the system by an external operation, like mechanical aeration, since a sufficient level of oxygen can not be guaranteed.

3 Optimal aeration control problem

The aeration operation can be modeled by introducing the control function $u = u(t)$ that describes the addition of oxygen in the cell atmosphere. The model (1)-(2) is modified by substituting equation (2) by

$$\dot{\omega} \,=\, -\gamma\, g\,(s, \omega)\, x \,+\, u\,(t)\ , \tag{3}$$

where $0 \leq u(t) \leq u_{\max}$ $\forall t \geq 0$. The positive value $u_{\max}$ is an upper bound for the control variable u corresponding to the maximal value of oxygen that can be introduced in the biological system.
We assume that the optimal time profile is in the admissible control set

$$\mathcal{U} \,=\, \{u \text{ Lebesgue measurable on } (0, t_f)\,|\, 0 \leq u(t) \leq u_{\max},\ \forall t \geq 0\}$$

and the objective functional is given by

$$J\,(u) \,=\, -m(t_f) + (\omega\,(t_f) - 1)^2 + \int_0^{t_f} (\omega\,(t) - 1)^2\, dt \tag{4}$$

where $[0, t_f]$ is the time range, $m(t_f)$ is the inert mass final concentration and the other terms express the deviation of the oxygen level from the operational optimal one at the final time t_f (first term) and in the time range $(0, t_f)$ (integral part).
The aim is to determine the state $(s^\star, i^\star, x^\star, \ell^\star, m^\star, \omega^\star)$ associated to an admissible control $u^\star \in \mathcal{U}$ satisfying (1)-(3) and minimizing the objective functional (4), i.e.

$$J\,(u^\star) \,=\, \min_{u \in \mathcal{U}} J\,(u)\ . \tag{5}$$

To determine a solution of the optimal control problem we consider the Pontryagin's principle [11, 28] that converts the problem (5) into the problem of minimizing the following Hamiltonian function

$$\begin{aligned}\mathcal{H}\,(\mathbf{X}, u, \boldsymbol{\Lambda}, t) \,=\, & (\omega\,(t) - 1)^2 \\ & - g\,(s, \omega)\, x \left[\frac{\lambda_s}{Y_S} - \lambda_x - \frac{\lambda_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\lambda_m + \gamma\lambda_\omega\right] \\ & + c_h i\,(\lambda_s - \lambda_i) + \beta x \left[\frac{\lambda_i}{Y_I} - \lambda_x + \left(1 - \frac{1}{Y_I}\right)\lambda_m\right] + u\,(t)\,\lambda_\omega\end{aligned}$$

where $\mathbf{X} = (s, i, x, \ell, m, \omega)$ and $\boldsymbol{\Lambda} = (\lambda_s, \lambda_i, \lambda_x, \lambda_\ell, \lambda_m, \lambda_\omega)$ are, respectively, the sets of state and adjoint variables.
The adjoint variables solve a system of six ordinary differential equations which can be described in vectorial form by

$$\dot{\boldsymbol{\Lambda}} \,=\, A\boldsymbol{\Lambda} + \mathbf{b}$$

with final condition

$$\boldsymbol{\Lambda}\,(t_f) \,=\, (0, 0, 0, 0, -1, 2\,(\omega\,(t_f) - 1))^T$$

where $\mathbf{b} = (0,0,0,0,0,2\,(1-\omega\,(t)))^T$,

$$A = \begin{bmatrix} \frac{\alpha_s}{Y_S} & 0 & -\alpha_s & -\frac{\alpha_s}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_s & \gamma\alpha_s \\ -c_h & c_h & 0 & 0 & 0 & 0 \\ \frac{\alpha_x}{Y_s} & -\frac{\beta}{Y_I} & -\alpha_x+\beta & -\frac{\alpha_x}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_x - \left(1-\frac{1}{Y_I}\right)\beta & \gamma\alpha_x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha_\omega}{Y_S} & 0 & -\alpha_\omega & -\frac{\alpha_\omega}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_\omega & \gamma\alpha_\omega \end{bmatrix}$$

and

$$\alpha_s = \frac{\partial g}{\partial s}\,(s,\omega)\,x\,,\ \alpha_x = g\,(s,\omega)\ ,\ \alpha_\omega = \frac{\partial g}{\partial \omega}\,(s,\omega)\,x\,.$$

We can give a characterization of the control when the Hamiltonian is linear in the control variable u. For such reason we recall some definitions.
The optimal control $u^\star$ is called a singular control on $[\underline{t},\bar{t}]$ if

$$\frac{\partial \mathcal{H}}{\partial u}\,(\mathbf{X}^\star, u^\star, \mathbf{\Lambda}, t)\ =\ 0\,,$$

for every $t \in [\underline{t},\bar{t}]$ and the corresponding solution $(\mathbf{X}^\star, u^\star)$ is called singular arc.
If u^* is a singular control, the problem order is the smallest number $q^\star$ such that the $2q$-th derivative

$$\frac{d^{2q}}{dt^{2q}}\frac{\partial \mathcal{H}}{\partial u}\,(\mathbf{X}^\star, u^\star, \mathbf{\Lambda}, t)\ ,$$

explicitly contains the control variable u (if no derivative satisfies this condition then $q=\infty$).
In our case, the switching function is given as

$$\sigma(t)\ =\ \frac{\partial \mathcal{H}}{\partial u}\ =\ \lambda_\omega$$

and we can give the following characterization to the control function

$$u\,(t)\ =\ \begin{pmatrix} 0 \\ \text{singular} \\ u_{\max} \end{pmatrix} \qquad \text{if} \qquad \lambda_\omega \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0\,. \tag{6}$$

It is also easy to show that

$$\begin{aligned}\frac{d^2}{dt^2}\frac{\partial \mathcal{H}}{\partial u} &= -2u\,(t) + 2\gamma g\,(s,\omega)\,x \\ &\quad + \dot{\alpha}_\omega\,(\mathbf{X})\left[\frac{\lambda_s}{Y_S} - \lambda_x - \frac{\lambda_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\lambda_m + \gamma\lambda_\omega\right] \\ &\quad + \alpha_\omega\,(\mathbf{X})\left[\frac{\dot{\lambda}_s}{Y_S} - \dot{\lambda}_x - \frac{\dot{\lambda}_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\dot{\lambda}_m + \gamma\dot{\lambda}_\omega\right]\end{aligned}$$

and hence our problem is of order 1.
Possible singular arcs may exist since they cannot be ruled out by the generalized Legendre-Clebsch condition for linear problem of order 1

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial \mathcal{H}}{\partial u}\,(\mathbf{X}^\star, u^\star, \mathbf{\Lambda}, t) \;\geq\; 0 \tag{7}$$

that will be checked numerically in Section 5.

4 Modeling oxygen uptake

Let us discuss some possible choices of the bacterial growth function $g = g\,(s,\omega) = g_1\,(s)\,g_2\,(\omega)$.
The dependence on the soluble substrate variable s has to be modeled to catch two main features: first, there is not any biomass growth in absence of substrate, since it is the nutrient for the bacterial population; second, a limiting value can be reached corresponding to a maximal rate. Mathematically, this means that

$$g_1\,(0) \;=\; 0\;,\; g_1\,(s) \;\leq\; g_{\max}\;\forall s \geq 0\,.$$

These conditions are satisfied by a Monod function [24]

$$g_1\,(s) \;=\; \frac{s}{s + c_s} \tag{8}$$

where c_s is the half-saturation constant.
We can notice that the maximal value is reached for high substrate concentrations ($s \to 1$).
As concerns the dependence on the oxygen variable, we remind that we are interested in modeling a degradation process in presence of a suitable level of oxygen. In particular, we have stressed that a low concentration of oxygen in the cell atmosphere can not guarantee the aerobic process. This feature has to be taken into account in the formulation of the aerobic biomass growth function.
A possible formulation is given again by a Monod-type function

$$g_2\,(\omega) \;=\; \frac{\omega}{\omega + c_\omega} \tag{9}$$

where c_ω is the half-saturation constant.
As said above, the Monod function reaches the maximal value for high concentration of the oxygen variable when $\omega \to \infty$. This means that the substrate consumption occurs more rapidly if the oxygen level is sufficiently high.
This formulation does not take into account of the optimal operational value $\omega_0 = 1$ around which a fast degradation occurs (according to [30] an optimal value of oxygen concentration ranges around 5-15%). Moreover, it is known that high values of oxygen can overdry the organic matter and the aerobic process can be compromised, since it requires a minimal value of moisture. Including this feature requires a different formulation of the oxygen dependence in g_2. In this case we look for a function g_2 such that

$$g_2(0) = 0 \ , \ g_2(\omega = 1) = g_{\max} \ , \ g_2(\omega) \leq \delta \text{ if } \omega \in \mathbb{R} \setminus (0.5, 1.5) \tag{10}$$

where δ is a small given quantity and $(0.5, 1.5)$ corresponds to the optimal operational range for oxygen concentration.
Such features can be reproduced by a Haldane-like function [14], usually proposed to model substrate inhibition [5, 35, 37]. We set

$$g_2(\omega) = \frac{\omega}{c_0 + c_1\omega + c_2\omega^2} \tag{11}$$

where c_0, c_1 and c_2 are given constants. We choose as a reference value $\delta = 0.25$ and we obtain $c_0 = c_2 = 18$ and $c_1 = -35$.
We will compare the effects of inhibition due to a high level of oxygen by comparing the dynamics in presence of Monod and Haldane growth functions (illustrated in Figure 1)

$$g^M(s,\omega) = \frac{s}{c_s + s}\,\frac{\omega}{c_\omega + \omega} \ , \ g^H(s,\omega) = \frac{s}{c_s + s}\,\frac{\omega}{c_0 + c_1\omega + c_2\omega^2} \tag{12}$$

where $c_s = 1.36e{-5}$, $c_\omega = 0.74$ (as indicated in [15]), $c_0 = c_2 = 18$ and $c_1 = -35$.

5 Numerical investigation of the optimal control problem

5.1 Optimal time-profiles

The optimal control solutions are computed by using a gradient method described in [2].
The choice of the numerical values of parameters and initial data is inspired by the paper [15]. In particular, the values for the physical quantities and the corresponding nondimensional ones are given in Table 1.
In order to give a value to the optimal operational one Ω_0 composting guidelines [30] suggest that the best percentage of oxygen in the cell atmosphere for the evolution of the aerobic digestion is around 10%.
By reminding that the concentration of wet air is $1220\, g/m^3$ and that oxygen percentage is approximately 21%, the oxygen concentration is around $256.2\, g/m^3$

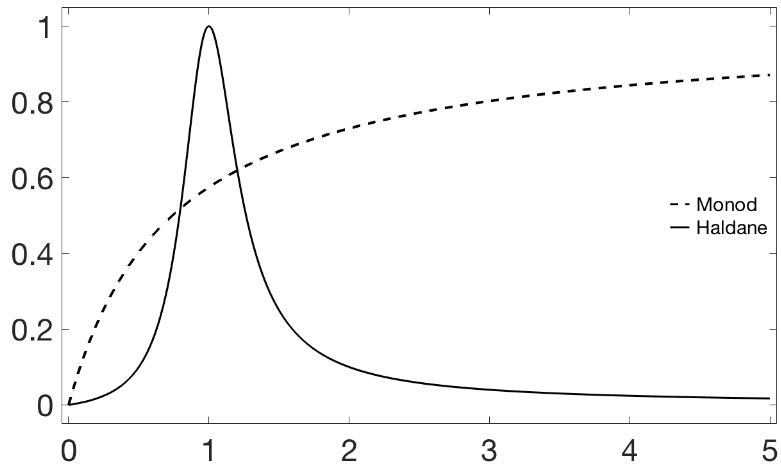

Fig. 1. Trend of the growth function g_2 versus the oxygen concentration ω in Monod (dashed line) and Haldane (solid line) formulation.

Parameter	value
K_h	4.9e-7 s^-1
b	3.8e-5 s^-1
Y_S	0.53
Y_I	1.02
Y_L	1.34
Y_Ω	1.12
S_0	1420 mol/m^3
I_0	4750 mol/m^3
X_0	1.5 mol/m^3
L_0	12297 mol/m^3
M_0	450 mol/m^3
T_0	18918,5 mol/m^3

Parameter	value
c_s	1.36e-5
c_ω	0.74
c_0, c_2	18
c_1	-35
c_h	2.45e-3
β	1.9e-1
γ	3.7e3
s_0	0.07506
i_0	0.25108
x_0	0.00008
ℓ_0	0.64999
m_0	0.02379

Table 1. Values of the physical parameters and the relative nondimensional ones.

corresponding to $8\,mol/m^3$. By proportion we can set $\Omega_0 = 3.81\,mol/m^3$.
As illustrated in Figure 2 the Haldane-like dependence on the oxygen variable ω shows a slower degradation of the organic matter compared to the Monod case; in fact, the growth of bacteria, the production of pre-compost and the reduction of soluble substrate reach a lower value at the final time. We can also notice an improvement in terms of oxygen level along the entire process; the oxygen concentration is in the optimal operation range for every time and the aerobic phenomenon is guaranteed.
As concerns the control variable u, we can observe in Figure 3 a similar strategy

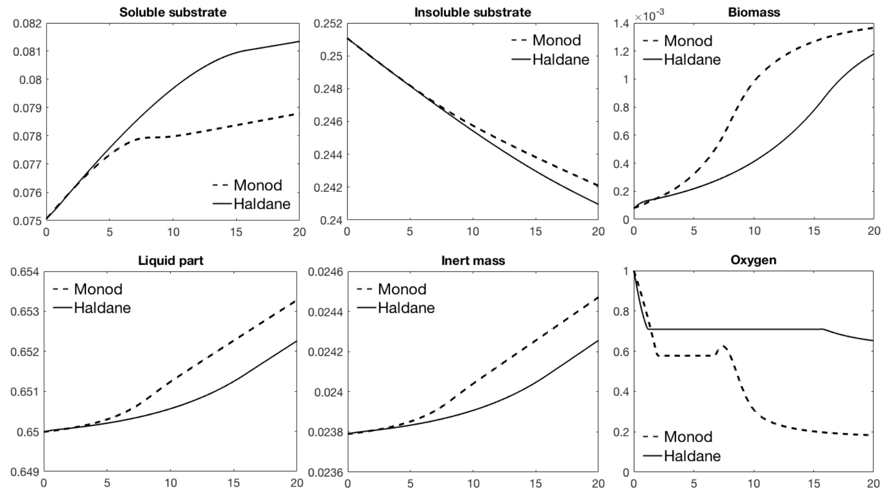

Fig. 2. State variable profiles versus time for Monod-like (dashed line) and Haldane-like (solid line) formulation of the bacterial growth function. The nondimensional final time is set equal to 20 (corresponding to approximately 28 hours).

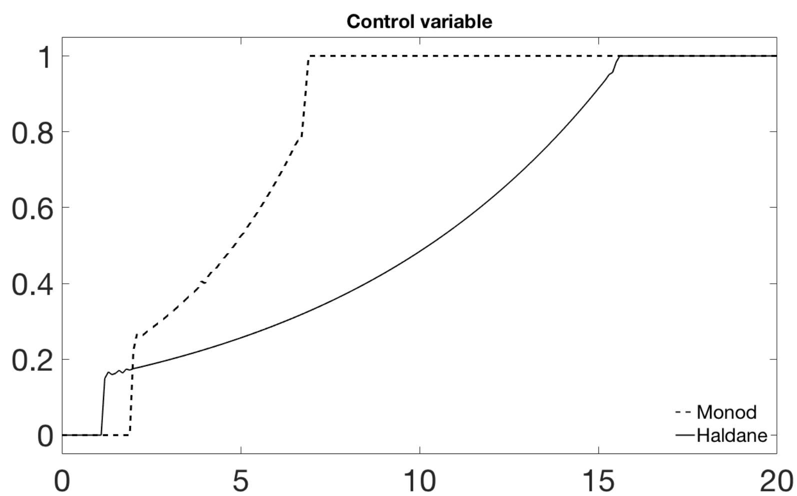

Fig. 3. Control time profile for Monod (dashed line) and Haldane (solid line) dynamics. The control upper bound $u_{\max}$ is set equal to 1.

in aeration for both formulations. Initially, the system does not require any external operation and an oxygen injection gradually occurs only later till to reach the maximal value $u_{\max} = 1$. The main differences between the two approaches concern the first time for injection and its velocity; in fact, when inhibition effects

	Monod	Haldane	Index value
$\mathbf{I}_1 = m(t_f) - m(0)$	yes	no	+78kg (+2.863%)
	no	yes	+54kg (+1.960%)
$\mathbf{I}_2 = (s+i)(t_f) - (s+i)(0)$	yes	no	-605kg (-1.612%)
	no	yes	-442kg (-1.178%)
$\mathbf{I}_3 = J(u)$	yes	no	8.651
	no	yes	1.820
$\mathbf{I}_4 = \max_{0 \le t \le t_f} \|\omega(t) - 1\|$	yes	no	0.817
	no	yes	0.348
$\mathbf{I}_5 = \int_0^{t_f} u(t)dt$	yes	no	15.502
	no	yes	10.925

Table 2. Values of the performance indices for Monod and Haldane growth rates.

are taken into account, the injection starts earlier but continues more slowly and reaches the maximal value later.
As last remark, we point out that the plot (not reported here) of the left hand side term in (7) is positive where the control does not assume the extremal values 0 and $u_{\max}$ and consequently a singular arc may exist.

5.2 Performance indices

In order to evaluate the plant performance, we introduce some quantitative indices as follows: (i) the gain of precompost with respect to the initial configuration, $\mathbf{I}_1 = m(t_f) - m(0)$; (ii) the consumption of substrate with respect to the initial configuration, $\mathbf{I}_2 = (s+i)(t_f) - (s+i)(0)$; (iii) the value of the objective functional, $\mathbf{I}_3 = J(u)$; (iv) the maximal distance of the oxygen variable from the optimal operational value, $\mathbf{I}_4 = \max_{0 \le t \le t_f} |\omega(t) - 1|$; (v) the required global effort for aeration, $\mathbf{I}_5 = \int_0^{t_f} u(t)dt$.
In the realistic case of a biocell capacity of 115 tonnes [27], the performance improvements can be computed in terms of kilos of produced precompost and reduced substrate in a time range of about 28 hours. As shown in Table 2, the presence of inhibition effects due to high concentrations of oxygen implies a lower amount of produced precompost at the final time and a higher level of (soluble+insoluble) substrate compared to the Monod case; we also notice again that the oxygen level along the process is higher with respect to the case of Monod–like dynamics and stays in the optimal range around $\omega = 1$. It is interesting to observe also that the objective functional and the required global effort assume lower values, as well as the relative cost of the operation.

6 The bioeconomic model

Up to now, we have considered the effects of the aeration and discussed the best strategy to improve the biocell performance. The objective functional only

depends on the state variables by including the deviation of the oxygen level from its optimal operational value (along the entire process and at the final time) and the final amount of the inert material.
In this section we want to describe the economic cost that an external operation like aeration requires in terms of technology, energy and working hours. This cost should be taken into account to give a more realistic description of the problem, since any control strategy must be cost-effective to be adapted.
For such reason, we introduce a function $\varphi = \varphi(u)$ that describes the economic cost of the aeration. We expect a nonlinear trend of the cost that can be modeled by a quadratic function

$$\varphi(u) = k_1 u + k_2 u^2 \tag{13}$$

where k_1 and k_2 are some weight constants.
The cost description by means of a quadratic term has been widely proposed in control problems [7, 22, 25, 31]. As indicated in [34] the square terms are often introduced to amplify (respectively de-emphasize) the effects of large (respectively small) variations of the involved variables from a desired value. In our case, the desired value 0 corresponds to no intervention. An alternative interpretation consists in considering the control as the effort to be expended to achieve the target [36]. A physical significance is finally suggested in [6], and cited afterwards in [20], where the control function is supposed to be proportional to the voltage or electric current used in aeration and then the square of the control is proportional to the electrical power and its integral is proportional to the energy expended in the observation time interval.
We are interested in minimizing the cost in the time range $(0, t_f)$, in addition to the minimization of the deviation of the oxygen variable from its optimal operational value 1 and the maximization of the final precompost amount. Therefore, the objective functional becomes

$$\begin{aligned} J(u) = & -m(t_f) + (\omega(t_f) - 1)^2 \\ & + \int_0^{t_f} (\omega(t) - 1)^2 \, dt + \int_0^{t_f} k_1 u(t) + k_2 u(t)^2 dt \end{aligned} \tag{14}$$

and the Hamiltonian function has to be modified as follows

$$\begin{aligned} \mathcal{H}(\mathbf{X}, u, \mathbf{\Lambda}, t) = & (\omega(t) - 1)^2 + k_1 u(t) + k_2 u(t)^2 \\ & - g(s, \omega) x \left[\frac{\lambda_s}{Y_S} - \lambda_x - \frac{\lambda_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right) \lambda_m + \gamma \lambda_\omega \right] \\ & + c_h i (\lambda_s - \lambda_i) + \beta x \left[\frac{\lambda_i}{Y_I} - \lambda_x + \left(1 - \frac{1}{Y_I}\right) \lambda_m \right] + u(t) \lambda_\omega \,. \end{aligned}$$

The optimality condition

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \,, \tag{15}$$

gives a characterization of the control variable in terms of the state and adjoint variables since the Hamiltonian function is nonlinear in the variable u. In

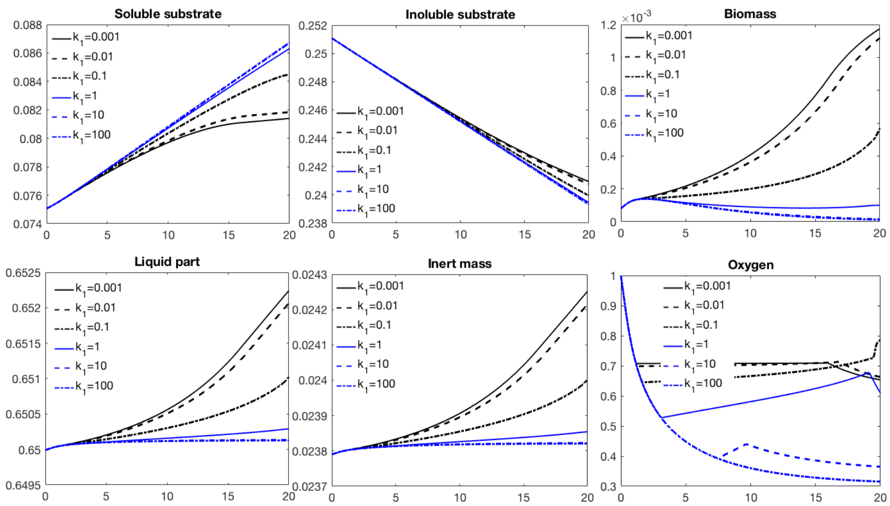

Fig. 4. State variable profiles versus time for Monod-like (dashed line) and Haldane-like (solid line) formulation of the bacterial growth function. The nondimensional final time is set equal to 20 (corresponding to approximately 28 hours).

particular, the characterization below follows

$$k_1 + 2k_2 u + \lambda_\omega = 0 \iff u = -\frac{\lambda_\omega + k_1}{2k_2}, \tag{16}$$

when a singular arc exists. We can notice that such characterization does not depend on the growth function explicitly.

7 Numerical investigations of the bioeconomic problem

We are interested in discussing the role of the parameter k_1 in the control strategy and hence we focus our attention to the case of weak nonlinearity ($k_2 = 10^{-8}$). As said above, we analyze how sensitive the control strategy is with respect to the varying parameter k_1; in particular we consider $k_1 = 10^n$, $n = -3, -2, \ldots, 2, 3$. The aerobic process is described by means of a Haldane growth function in the dependence on the oxygen concentration.

When the bioeconomic cost has a low weight in the objective functional, we can observe in Figure 4 that the control strategy mainly depends on the control of the oxygen level and on the maximization of the final product. In this case the performance is very good by exhibiting a low amount of the soluble and insoluble substrate and a high level of product. Moreover the oxygen concentration assumes values close to the optimal operational one and stays in the range that guarantees the survival of the aerobic process. When the cost has a crucial role, unlike the previous case, the performance shows that the global substrate is less

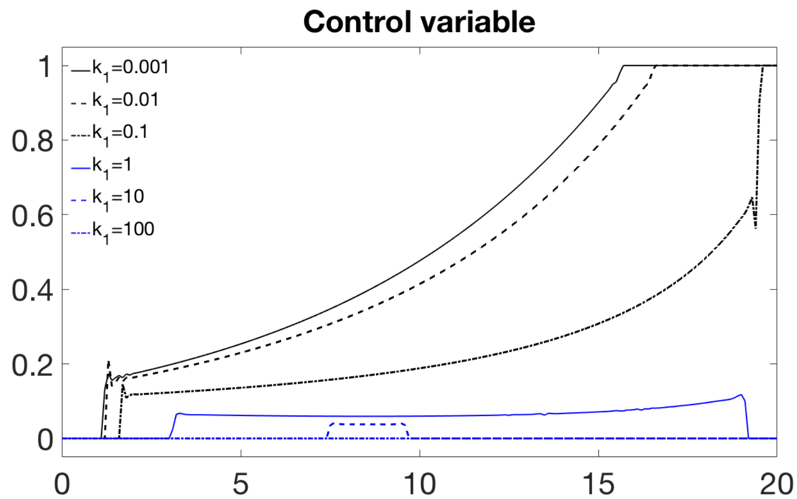

Fig. 5. Control time profile for Monod (dashed line) and Haldane (solid line) dynamics. The control upper bound $u_{\max}$ is set equal to 1.

degraded and the oxygen concentration tends to settle on lower levels and out of the survival range for the aerobic process.

As concerns the control variable, it is interesting to notice in Figure 5 that a high weight of the bioeconomic cost in the objective functional implies that no external operation (or a negligible one) is operated. Any external action is not cost-effective and it would give a high value of the objective functional. On the contrary, when the bioeconomic cost has not a strong role in the decision policy, additional oxygen is injected in order to maintain its concentration at satisfactory levels.

We present some indices to give a quantification of the plant performance; in particular we introduce the following ones: (i) $\mathbf{J}_1 = m(t_f)$ which gives the amount of the produced pre-compost at the final time t_f; (ii) $\mathbf{J}_2 = \int_0^{t_f} (\omega(t) - 1)^2 \, dt$ which computes the deviation of the oxygen concentration from the optimal operational value along the process; (iii) the value of the objective functional which is denoted by $\mathbf{J}_3$.

We indicate by $\mathbf{J}_i^n$ the value of $\mathbf{J}_i$, $i = 1, 2, 3$, corresponding to $k_1 = 10^n$ and consider the quantity $\delta_i^n = \left(\mathbf{J}_i^n - \mathbf{J}_i^{-8}\right)/\mathbf{J}_i^{-8}$, $i = 1, 2, 3$, $n = -7, -6, \ldots, 2, 3$, which gives the relative difference between the indices.

We can observe that Figure 6 and Table 3 individuate three different ranges for the parameter k_1: more precisely $\mathcal{I}_1 = (-\infty, \tilde{k}_1)$, $\mathcal{I}_2 = [\tilde{k}_1, \hat{k}_1]$ and $\mathcal{I}_3 = (\hat{k}_1, +\infty)$ where $\tilde{k}_1 \approx 10^{-2}$ and $\hat{k}_1 \approx 10$.

In $\mathcal{I}_1$ the bioeconomic contribution in the objective functional has a very limited role and the choice of the control variable mainly has to maximize the final product and minimize the deviation of the oxygen concentration from its optimal

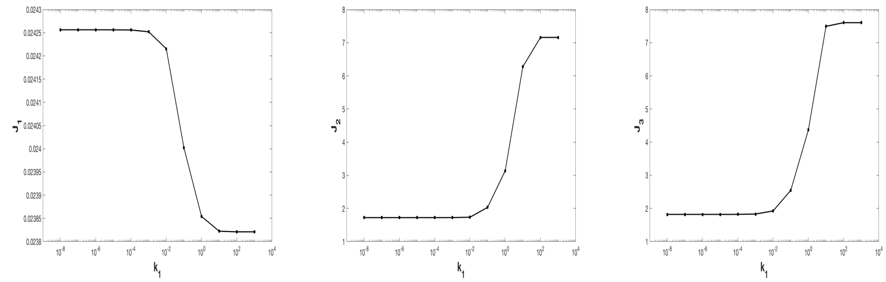

Fig. 6. The figure shows the trend of the performance indices $\mathbf{J}_1$ (left), $\mathbf{J}_2$ (middle) and $\mathbf{J}_3$ (right) versus the parameter k_1.

	$\mathbf{J}_1^n$	δ_1^n	$\mathbf{J}_2^n$	δ_2^n	$\mathbf{J}_3^n$	δ_3^n
$n = -7$	0.02426	negligible	1.72293	negligible	1.81950	negligible
$n = -6$	0.02426	negligible	1.72293	negligible	1.81950	negligible
$n = -5$	0.02426	negligible	1.72294	negligible	1.81961	+0.006%
$n = -4$	0.02426	-0.002%	1.72300	+0.004%	1.82059	+0.060%
$n = -3$	0.02425	-0.017%	1.72363	+0.040%	1.83037	+0.598%
$n = -2$	0.02422	-0.169%	1.73459	+0.676%	1.92387	+5.736%
$n = -1$	0.02400	-1.048%	2.03183	+17.928%	2.54502	+39.875%
$n = 0$	0.02385	-1.658%	3.13705	+82.076%	4.36778	+140.05%
$n = 1$	0.02382	-1.789%	6.27330	+264.11%	7.49204	+311.76%
$n = 2$	0.02382	-1.795%	7.15799	+315.45%	7.60291	+317.86%
$n = 3$	0.02382	-1.795%	7.15799	+315.45%	7.60291	+317.86%

Table 3. Values of the performance indices and their difference with respect to the refence case ($k_1 = 10^{-8}$).

operational value. In this range, if one considers two values of the parameter, $-\infty < k_1^m << k_1^M < \tilde{k}_1$, significant differences in the parameter value do not influence substantially the performance, i.e. $\mathbf{J}_1(k_1^m) \approx \mathbf{J}_1(k_1^M)$, $\mathbf{J}_2(k_1^m) \approx \mathbf{J}_2(k_1^M)$ and $\mathbf{J}_3(k_1^m) \approx \mathbf{J}_3(k_1^M)$.

Analogously in $\mathcal{I}_3$, where the control strategy is chosen mainly to minimize the bioeconomic cost, different values of the parameter do not imply significant changes in the indices, i.e. $\mathbf{J}_1(k_1^m) \approx \mathbf{J}_1(k_1^M)$, $\mathbf{J}_2(k_1^m) \approx \mathbf{J}_2(k_1^M)$ and $\mathbf{J}_3(k_1^m) \approx \mathbf{J}_3(k_1^M)$ where $\hat{k}_1 < k_1^m << k_1^M < +\infty$.

Therefore the most interesting case is when k_1 ranges into the interval $\mathcal{I}_2$. In fact, this produces remarkable differences in the composting plant evolution corresponding to different choices of the parameter (see Figure 6 and Table 3). In particular, the system results less performing for increasing values of k_1 in $\mathcal{I}_2$ by showing a significant loss of pre-compost at the final time for higher values of the parameter k_1.

In Table 4 such loss is quantified in kilos by starting from a realistic case of composting in biocell; each biocell has a capacity of 110 tonnes and we consider

	$m_n(t_f) - m_n(0)$ (in Kg)	$m_n(t_f) - m_8(t_f)$ (in Kg)
$n = -8$	+193	–
$n = -7$	+193	negligible
$n = -6$	+193	negligible
$n = -5$	+193	negligible
$n = -4$	+193	negligible
$n = -3$	+192	-1
$n = -2$	+188	-5
$n = -1$	+164	-29
$n = 0$	+147	-46
$n = 1$	+143	-50
$n = 2$	+143	-50
$n = 3$	+143	-50

Table 4. Amount of the precompost for varying parameter k_1 in the realistic case of composting in biocell. The biocell capacity is around 115 tonnes and the observation period is around 28 hours.

an observation period of around 28 hours. In the best case around 193 kilos of precompost are produced in the observation period; such amount reduces to 143 kilos (-26%) in the less performing case.

8 Concluding remarks

The mathematical modeling of aerobic degradation of the organic matter is a topical issue in waste management and it is far to be deeply explored.

We have proposed a mathematical description for composting in biocells to investigate (i) the optimal aeration strategies to ensure a sufficient level of oxygen in the cell atmosphere and therefore the aerobic nature of the digestion process, (ii) possible inhibition effects for over aerated systems.

In particular an optimal control problem has been proposed and analyzed to individuate the best aeration strategy that minimizes the distance of the oxygen concentration level from its optimal operational value and the maximization of the precompost amount at the final time of our observation.

We have compared the results in absence and presence of inhibition and we have observed that more satisfactory levels of oxygen have been reached when the inhibition is included in the model. Moreover, in this case the different control strategy requires a overall aeration effort.

In the second part of the work we have considered the economic cost of the aeration operation in order to give a more realistic description of the problem. We have included the minimization of a nonlinear quadratic cost in the objective functional and we have observed that different aeration strategies can occur depending on the weight of the economic term in the control policy. In particular there is a negligible injection of additional oxygen in the biocell as soon as the aeration has a significant cost while the best performance is obtained when the

cost has not a crucial role in the control strategy.
Other optimal control problems for aerobic processes can be formulated. In particular it could be interesting to find the best strategy to minimize the time to reach a fixed target on the variable states, like the reduction of the global substrate below a giving threshold value.
Let us observe that we focused on the composting phase in the integrated system but it would be important to investigate also the anaerobic process for the biogas production as well as the treatment of the wastewater sludge. This will be the subject of future investigations.

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