

# FUZZY SWITCHING SYSTEMS: MINIMIZING DISCONTINUITIES AND RIPPLE MAGNITUDE AND ENERGY

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## Abstract

This paper presents an efficient and effective method to determine optimal switching instants of fuzzy switching systems such that both the ripple magnitude and energy of the fuzzy switching systems are minimized. The method is based on optimal switching control techniques, where an optimal enhancing control method is used. This method has several advantages over the traditional methods. Firstly, it does not require the process of linearization. Secondly, it guarantees to achieve optimality. For illustration, a practical example of an optimal pulse width modulated fuzzy control of a switched-capacitor DC/DC power converter is presented.

## Key Words

Optimal switching, enhancing control method, fuzzy switching systems, ripple magnitude and energy

## 1. Introduction

There are many fuzzy systems storing and processing fuzzy association memory (FAM) rules in parallel and the defuzzification of these systems is to integrate these rules to the output. There are many defuzzification strategies, such as the max criterion method, the mean of maximum method, the center of area method, etc. For the max criterion method, the output is switched to the fuzzy sub-system with the one having the highest fuzzy membership value. And it was found that this kind of defuzzification strategy will produce a very good result for edge detection [1].

Since the fuzzy systems are usually dynamic systems, ripples will usually occur at the output. Moreover, this kind of defuzzification strategy may also cause discontinuities at the output because the value of the output before the switching time instants may be different from that after the switching time instants. These ripples and discontinuities are sometimes harmful for some systems, such as fuzzy DC/DC power converter systems.

To work on this problem, the sizes of ripples and the discontinuities should be as small as possible. There are commonly two types of criteria of measuring the ripple sizes, one is to use the  $H_2$  criterion and the another is to use the  $H_\infty$  criterion because the use of the  $H_2$  criterion can minimize the average error energy, while the use of the  $H_\infty$  criterion can minimize the worst case error. However, it is worth to note that it is insufficient to formulate the problem using only either one of the  $H_2$  criterion or  $H_\infty$  criterion. This is because the use of the  $H_2$  criterion without the use of the  $H_\infty$  criterion may turn out to have excessively large ripples at particular time instants even if the total ripple energy is small. On the other hand, the use of the  $H_\infty$  criterion without the use of the  $H_2$  criterion may turn out to produce excessively large bias values. Hence, we formulate the problem as an optimal control problem in which the optimal switching time instants are to be determined such that both of the ripple magnitude and energy, as well as the continuities are minimized. In the traditional optimal control problem, the optimal control signal is to be determined such that a smooth cost function is minimized subject to some smooth constraints. However, this optimal control problem is different from the traditional optimal control problem. In this control problem, we determine the exact switching time instants when the system switches from one sub-system to another. As different sub-systems have different transient responses, the optimal switching time instants based on minimizing the output ripple is not necessary equally distributed over the period.

The rest of the paper is organized as follows. The formulation is given in Section II. In Section III, a solution method is developed. In Section IV, a practical example of an optimal pulse width modulated fuzzy control of a switched-capacitor DC/DC power converter is presented for illustration. Finally, a conclusion is summarized in Section V.

## 2. Formulation

Consider a fuzzy switching system in which the dynamics is subject to switching from one state to another at various instants. Without loss of generality, we assume that there are totally  $N-1$  switching instants during a cycle from  $t_0=0$  to  $t_N=T$ . Those switching times, denoted by  $t_i$ , for  $i=1,2,\dots,N-1$ , divides the time period  $[0,T]$  into  $N$  time intervals  $[t_{i-1},t_i)$ , for  $i=1,2,\dots,N$ . In general, these time intervals are not uniformly distributed and are going to be determined.

During the  $i^{\text{th}}$  intervals  $[t_{i-1},t_i)$ , the state space model of the fuzzy switching system is assumed to be described by a linear system  $S_i=(\mathbf{A}_i,\mathbf{B}_i,\mathbf{C}_i,\mathbf{D}_i)$  as follows:

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t), \quad (1a)$$

$$y(t)=\mathbf{C}_i\mathbf{x}(t)+\mathbf{D}_i\mathbf{u}(t), \quad (1b)$$

where  $\mathbf{A}_i \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{B}_i \in \mathfrak{R}^{n \times p}$ ,  $\mathbf{C}_i \in \mathfrak{R}^{1 \times n}$  and  $\mathbf{D}_i \in \mathfrak{R}^{1 \times p}$ , in which  $\mathfrak{R}^{k \times q}$  represents the set of  $k \times q$  real matrices.  $\mathbf{x}(t)$  and  $y(t)$  represent, respectively, the state vector and the output of the fuzzy switching system, and  $\mathbf{u}(t)$  represents the input of fuzzy switching system.

At the switching time  $t_i$ , state jump  $\Delta\mathbf{x}(t_i)$  is expected. That is:

$$\mathbf{x}(t_i)=\boldsymbol{\varphi}_i(\mathbf{x}(t_i-0))=\mathbf{x}(t_i-0)+\Delta\mathbf{x}(t_i-0), \text{ for } i=1,2,\dots,N-1, \quad (2)$$

where  $\boldsymbol{\varphi}_i$  is supposed to be continuously differentiable.

The fuzzy switching system can be described by:

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t), \text{ for } t_{i-1}<t<t_i \text{ and for } i=1,2,\dots,N, \quad (3a)$$

$$y(t)=\mathbf{C}_i\mathbf{x}(t)+\mathbf{D}_i\mathbf{u}(t), \text{ for } t_{i-1}<t<t_i \text{ and for } i=1,2,\dots,N, \quad (3b)$$

$$\mathbf{x}(0)=\mathbf{x}_0, \quad (3c)$$

and

$$\mathbf{x}(t_i)=\boldsymbol{\varphi}_i(\mathbf{x}(t_i-0))=\mathbf{x}(t_i-0)+\Delta\mathbf{x}(t_i-0), \text{ for } i=1,2,\dots,N-1. \quad (3d)$$

Let  $\boldsymbol{\xi}=[t_1,t_2,\dots,t_{N-1}]^T \in \mathfrak{R}^{N-1}$ , and

$$\mathfrak{S}=\{\boldsymbol{\xi} \in \mathfrak{R}^{N-1} : t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{N-1} \leq t_N\}, \quad (4)$$

where  $\mathfrak{R}^{N-1}$  denotes the set of  $(N-1)$ -dimensional real vectors.

For a fuzzy switching system, the discontinuities at the output should be as small as possible, that means,  $\max_{1 \leq i \leq N-1} |y(t_i)-y(t_i-0)|$  should be as small as possible, where  $|\cdot|$  denotes the absolute value. The ripple magnitude and energy of the fuzzy switching system should also be as small as possible. Thus, both  $\max_{1 \leq i \leq N} |y(t_i)-y(t_{i-1})|$  and  $\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (y(t)-y(t_{i-1}))^2 dt$  should be as small as possible. Hence, the control objective is to determine the optimal switching time instants  $\boldsymbol{\xi} \in \mathfrak{S}$  such that both the ripple magnitude and energy, as well as the discontinuities at the output are minimized. The problem can be expressed as follows:

### Problem (P)

Given the dynamic system (3a)-(3d), find a set of switching instants such that

$$J(\boldsymbol{\xi})=\max_{1 \leq i \leq N-1} |y(t_i)-y(t_i-0)|+\max_{1 \leq i \leq N} |y(t_i)-y(t_{i-1})|+\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (y(t)-y(t_{i-1}))^2 dt \quad (5)$$

is minimized.

### 3. Solution Method

For simplicity, we assume that the input signal is a constant vector, that is  $\mathbf{u}(t) = \mathbf{u}$  for  $t \geq 0$ . To work on this problem, we first apply the model transformation technique [2] to convert the problem with non-differentiable cost function into a problem with smooth cost function subject to some constraints. Then we apply the enhancing control method to convert the problem with variable switching time instants into a problem with uniform switching time instants [3]. The discontinuities of the state variables are handled by computing the state variables segment by segment [4]. The value of the initial condition of the current segment can be computed by the value of the state variables at the end of the previous segment.

#### 3.1 Model Transformation

Problem (P) is a min-max type problem, which involves a non-differentiable cost function. By using the model transformation, Problem (P) can be converted into the following constrained optimization problem:

##### Problem (Q)

$$\begin{aligned} \min_{\xi \in \mathfrak{R}, \mathbf{v} \in \mathfrak{R}^{2+}} \quad & \hat{J}(\xi, \mathbf{v}) = v_1 + v_2 + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (y(t) - y(t_{i-1}))^2 dt, & (6a) \\ \text{subject to} \quad & y(t_i) - y(t_{i-1}) \leq v_1, & i = 1, 2, \dots, N, & (6b) \\ & y(t_{i-1}) - y(t_i) \leq v_1, & i = 1, 2, \dots, N, & (6c) \\ & y(t_i) - y(t_i - 0) \leq v_2, & i = 1, 2, \dots, N - 1, & (6d) \\ & y(t_i - 0) - y(t_i) \leq v_2, & i = 1, 2, \dots, N - 1, & (6e) \\ & \frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t), & \text{for } t_{i-1} < t < t_i \text{ and for } i = 1, 2, \dots, N, & (6f) \\ & y(t) = \mathbf{C}_i \mathbf{x}(t) + \mathbf{D}_i \mathbf{u}(t), & \text{for } t_{i-1} < t < t_i \text{ and for } i = 1, 2, \dots, N, & (6g) \\ & \mathbf{x}(0) = \mathbf{x}_0, & & (6h) \end{aligned}$$

and

$$\mathbf{x}(t_i) = \boldsymbol{\Phi}_i(\mathbf{x}(t_i - 0)) = \mathbf{x}(t_i - 0) + \Delta \mathbf{x}(t_i - 0), \text{ for } i = 1, 2, \dots, N - 1, \quad (6i)$$

in which  $\mathbf{v} \equiv [v_1, v_2]^T$  and  $\mathfrak{R}^{2+}$  denotes the set of two-dimensional real vectors with each elements are positive. For the details, please refer to Section 6.8.1 of [2].

#### 3.2 Enhancing Control Transform

Since the switching time intervals are generally not of the same length, we apply the control enhancing method to rescale the time in such a way that the switching time instants occur at consecutive integer time instants. The method is summarized below. For details, please refer to [3].

Let  $\mu(\cdot | \boldsymbol{\tau}): [0, N] \rightarrow \mathfrak{R}$  denote a function of the form:

$$\mu(s | \boldsymbol{\tau}) = \sum_{i=1}^N \tau_i \chi_{[i-1, i)}(s), \quad (7)$$

where  $\tau_i = t_i - t_{i-1}$ ,  $\boldsymbol{\tau} = [\tau_1, \tau_2, \dots, \tau_N]^T$  and  $\chi_{[i-1, i)}(s) = \begin{cases} 1 & s \in [i-1, i) \text{ for } i = 1, 2, \dots, N \\ 0 & \text{otherwise} \end{cases}$

Let  $\Lambda = \left\{ \mu(\cdot | \boldsymbol{\tau}): \tau_i \geq 0, \sum_{i=1}^N \tau_i = T \right\}$  and define  $U: \boldsymbol{\tau} \mapsto \mu(\cdot | \boldsymbol{\tau})$  and  $t(\cdot | \mu): [0, N] \rightarrow [0, T]$  as follows:

$$U(\boldsymbol{\tau}) = \mu, \quad (8a)$$

and

$$t(s | \mu) = \int_0^s \mu(\theta | \boldsymbol{\tau}) d\theta. \quad (8b)$$

Let  $\mathbf{z}(s | \mu) = \mathbf{x}(t(s | \mu) | \boldsymbol{\tau})$  and  $w(s | \mu) = y(t(s | \mu) | \boldsymbol{\tau})$ . Then we have:

$$\frac{d}{ds} \mathbf{z}(s | \mu) = \mu(s | \boldsymbol{\tau}) (\mathbf{A}_i \mathbf{z}(t(s | \mu)) + \mathbf{B}_i \mathbf{u}), \text{ for } i-1 < s < i \text{ and } i = 1, 2, \dots, N - 1, \quad (9a)$$

$$\frac{d}{ds} t(s | \mu) = \mu(s | \boldsymbol{\tau}), \quad (9b)$$

$$\mathbf{z}(t(0 | \mu)) = \mathbf{x}_0, \quad (9c)$$

$$\mathbf{z}(t(i|\mu)) = \boldsymbol{\varphi}_i(\mathbf{z}(t(i-0|\mu))) = \mathbf{z}(t(i-0|\mu)) + \Delta \mathbf{z}(t(i-0|\mu)), \quad i = 1, 2, \dots, N-1, \quad (9d)$$

$$t(i|\mu) = t_i, \quad i = 0, 1, \dots, N, \quad (9e)$$

and

$$w(t(s|\mu)) = \mathbf{C}_i \mathbf{z}(t(s|\mu)) + \mathbf{D}_i \mathbf{u}, \quad i = 1, 2, \dots, N-1. \quad (9f)$$

Define the following problem:

### Problem (R)

$$\min_{\mu \in \Lambda, \mathbf{v} \in \mathbb{R}^{2^2}} \tilde{J}(\mu, \mathbf{v}) = v_1 + v_2 + \int_0^N \mu(s|\boldsymbol{\tau}) (\mathbf{C}_i(\mathbf{z}(t(s|\mu)) - \mathbf{z}(t(i-1|\mu))))^2 ds, \quad (10a)$$

$$\text{subject to } \mathbf{C}_i(\mathbf{z}(t(i|\mu)) - \mathbf{z}(t(i-1|\mu))) \leq v_1, \quad i = 1, 2, \dots, N, \quad (10b)$$

$$\mathbf{C}_i(\mathbf{z}(t(i-1|\mu)) - \mathbf{z}(t(i|\mu))) \leq v_1, \quad i = 1, 2, \dots, N, \quad (10c)$$

$$\mathbf{C}_{i+1} \boldsymbol{\varphi}_i(\mathbf{z}(t(i-0|\mu))) - \mathbf{C}_i \mathbf{z}(t(i-0|\mu)) + (\mathbf{D}_{i+1} - \mathbf{D}_i) \mathbf{u} \leq v_2, \quad i = 1, 2, \dots, N-1, \quad (10d)$$

$$\mathbf{C}_i \mathbf{z}(t(i-0|\mu)) - \mathbf{C}_{i+1} \boldsymbol{\varphi}_i(\mathbf{z}(t(i-0|\mu))) + (\mathbf{D}_i - \mathbf{D}_{i+1}) \mathbf{u} \leq v_2, \quad i = 1, 2, \dots, N-1, \quad (10e)$$

$$\frac{d}{ds} \mathbf{z}(s|\mu) = \mu(s|\boldsymbol{\tau}) (\mathbf{A}_i \mathbf{z}(s|\mu) + \mathbf{B}_i \mathbf{u}), \quad i-1 < s < i, \text{ and } i = 1, 2, \dots, N-1, \quad (10f)$$

$$\frac{d}{ds} t(s|\mu) = \mu(s|\boldsymbol{\tau}), \quad (10g)$$

$$\mathbf{z}(t(0|\mu)) = \mathbf{x}_0, \quad (10h)$$

$$\mathbf{z}(t(i|\mu)) = \boldsymbol{\varphi}_i(\mathbf{z}(t(i-0|\mu))) = \mathbf{z}(t(i-0|\mu)) + \Delta \mathbf{z}(t(i-0|\mu)), \quad i = 1, 2, \dots, N-1, \quad (10i)$$

$$t(i|\mu) = t_i, \quad i = 0, 1, \dots, N, \quad (10j)$$

and

$$w(t(s|\mu)) = \mathbf{C}_i \mathbf{z}(t(s|\mu)) + \mathbf{D}_i \mathbf{u}, \quad i = 1, 2, \dots, N-1. \quad (10k)$$

### Theorem 1

Problem (Q) and Problem (R) are equivalent in the sense that  $(\xi^*, \mathbf{v}^*)$  is a solution of Problem (Q) if and only if  $(\mu^*, \mathbf{v}^*)$  is a solution of Problem (R), where  $U(\xi^*) = \mu^*$ . Furthermore,  $\hat{J}(\xi^*, \mathbf{v}^*) = \tilde{J}(\mu^*, \mathbf{v}^*)$ .

### Proof

The proof is omitted in this paper. ■

### 3.3 State Jump Problem

To find  $\mathbf{z}(t(i|\mu))$  for  $i = 1, 2, \dots, N-1$ , we first compute the value of  $\mathbf{z}(t(1-0|\mu))$  using the initial condition  $\mathbf{z}(t(0|\mu))$ . Then  $\mathbf{z}(t(1|\mu))$  can be obtained by using the equation  $\mathbf{z}(t(1|\mu)) = \boldsymbol{\varphi}_1(\mathbf{z}(t(1-0|\mu)))$ . Similarly,  $\mathbf{z}(t(2-0|\mu))$  can be computed using  $\mathbf{z}(t(1|\mu))$ , and  $\mathbf{z}(t(2|\mu))$  can be obtained by  $\mathbf{z}(t(2|\mu)) = \boldsymbol{\varphi}_2(\mathbf{z}(t(2-0|\mu)))$ . The procedures are repeated until all the values of  $\mathbf{z}(t(i|\mu))$  for  $i = 1, 2, \dots, N-1$  are obtained. The problem becomes a standard optimal parameter selection problem and can be solved easily using MISER3.

### 4. An Illustrative Example

In this paper, an optimal pulse width modulated fuzzy control of a switched-capacitor DC/DC power converter in [5] is employed for illustration. The circuit can be modeled as one which switches between two linear systems, so  $N = 2$ . Figure 1 shows the corresponding switching topologies. The state space models for these two linear systems are:

$$\mathbf{A}_1 = \frac{1}{\alpha_1} \begin{bmatrix} -\frac{\alpha_1}{C_1(R_{C1} + 2R_{on})} & 0 & 0 \\ 0 & -\frac{R_L + R_{C0}}{C_2} & -\frac{R_L}{C_2} \\ 0 & -\frac{R_L}{C_0} & -\frac{R_L + R_{C2} + 2R_{on}}{C_0} \end{bmatrix}, \quad (11a)$$

$$\mathbf{A}_2 = \frac{1}{\alpha_2} \begin{bmatrix} -\frac{\alpha_2}{C_2(R_{C2} + 2R_{on})} & 0 & 0 \\ 0 & -\frac{R_L + R_{C0}}{C_1} & -\frac{R_L}{C_1} \\ 0 & -\frac{R_L}{C_0} & -\frac{R_L + R_{C1} + 2R_{on}}{C_0} \end{bmatrix}, \quad (11b)$$

$$\mathbf{B}_1 = \frac{1}{\alpha_1} \begin{bmatrix} \alpha_1 & R_L + R_{C0} & R_L \\ C_1(R_{C1} + 2R_{on}) & C_2 & C_0 \end{bmatrix}^T, \quad (11c)$$

$$\mathbf{B}_2 = \frac{1}{\alpha_2} \begin{bmatrix} \alpha_2 & R_L + R_{C0} & R_L \\ C_2(R_{C2} + 2R_{on}) & C_1 & C_0 \end{bmatrix}^T, \quad (11d)$$

$$\mathbf{C}_1 = \frac{1}{\alpha_1} [0 \quad -R_L R_{C0} \quad R_L(R_{C2} + 2R_{on})], \quad (11e)$$

$$\mathbf{C}_2 = \frac{1}{\alpha_2} [0 \quad -R_L R_{C0} \quad R_L(R_{C1} + 2R_{on})], \quad (11f)$$

$$D_1 = \frac{R_{C0}R_L}{\alpha}, \quad (11g)$$

and

$$D_2 = \frac{R_{C0}R_L}{\alpha}, \quad (11h)$$

where

$$\alpha_1 = R_{C0}R_L + (R_{C2} + 2R_{on})R_L + R_{C0}(R_{C2} + 2R_{on}), \quad (12a)$$

$$\alpha_2 = R_{C0}R_L + (R_{C1} + 2R_{on})R_L + R_{C0}(R_{C1} + 2R_{on}), \quad (12b)$$

the values of the components and the input voltage are shown in the circuit schematic (Figure 1). We set the switching frequency as 1MHz, that is  $T = t_N = 10^{-6}$  s, and we assume that the circuit is initially at rest, that is  $\mathbf{x}(0) = \mathbf{0}$ . Since the discontinuities of the state variables are smaller than the ripple magnitude, we neglect the effect of the discontinuities in the cost function, that is,  $y(t_i) = y(t_i - 0)$ . By applying the above method, we found that  $t_1 = 4 \times 10^{-7}$  s and the steady state ripple is  $6 \times 10^{-2}$  V as shown in Figure 2.

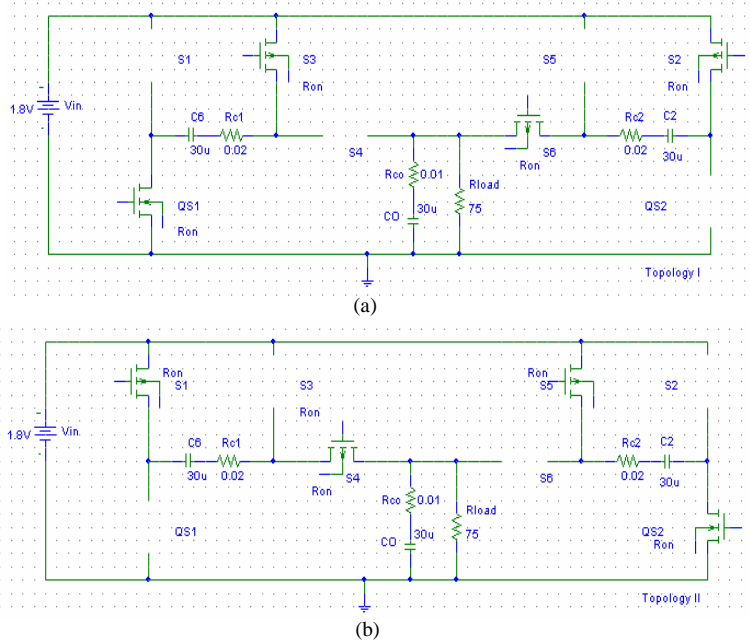


Figure 1. Switching topologies of the switched-capacitor DC/DC power converter in [5]. (a) Topology 1. (b) Topology 2.

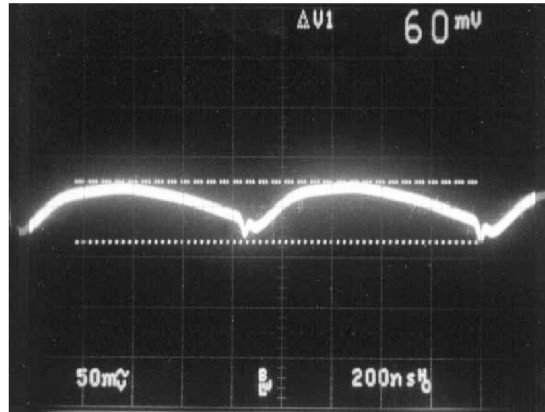


Figure 2. The steady state output voltage of the switched-capacitor DC/DC power converter in [5].

## 5. Conclusion

In this paper, we apply the optimal parameter selection method to determine the optimal switching time instants for fuzzy switching systems so that both the ripple magnitude and energy, as well as the discontinuities at the output of the systems are minimized. The problem is solved via a model transformation and optimal enhancing control method. The optimization process is carried out via MISER and a practical example of an optimal pulse width modulated fuzzy control of a switched-capacitor DC/DC power converter is presented for illustration.

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## References

- [1] B.W.K. Ling and P.K.S. Tam, Edge detection via fuzzy switch, *SPIE's International Technical Group Newsletter*, 12(2), 2002, 2.
- [2] K.L. Teo, C.J. Goh and K.H. Wong, *A unified computational approach to optimal control problems* (55, Pitman Monographs and Surveys in Pure and Applied Mathematics: Longman Scientific and Technical, England, 1991).
- [3] K.L. Teo, L.S. Jennings, H.W. J. Lee and V. Rehbock, The control parameterization enhancing transform for constrained optimal control problems, *Journal of Australian Mathematical Society, Series B*, 40, 1999, 314-335.
- [4] Y. Liu, K.L. Teo, L.S. Jennings and S. Wang, On a class of optimal control problems with state jumps, *Journal of Optimization Theory and Applications*, 98(1), 1998, 65-82.
- [5] H. Chung and Y.K. Mok, Development of a switched-capacitor DC/DC boost converter with continuous input current waveform, *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, 46(6), 1999, 756-759.