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The connection between topological and algebraic entropy

Dikran Dikranjan
dikran.dikranjan@uniud.it
Dipartimento di Matematica e Informatica,
Università di Udine,
Via delle Scienze, 206 - 33100 Udine, Italy

Anna Giordano Bruno
anna.giordanobruno@uniud.it
Dipartimento di Matematica e Informatica,
Università di Udine,
Via delle Scienze, 206 - 33100 Udine, Italy

Abstract

We show that the topological entropy of a continuous endomorphism of a compact abelian group coincides with the algebraic entropy of the dual endomorphism of the (discrete) Pontryagin dual group. As an application a relation is given between the topological Pinsker factor and the algebraic Pinsker subgroup.

Key words: topological entropy, algebraic entropy, Pontryagin duality, compact abelian group, Pinsker factor, Pinsker subgroup
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1 Introduction

Inspired by the notion of measure entropy in Ergodic Theory given by Kolmogorov [17] and Sinai [22], in [1] Adler, Konheim and McAndrew introduced the topological entropy for continuous self-maps of compact spaces as follows.

For a compact space X and for an open cover $\mathcal{U}$ of X, let $N(\mathcal{U})$ be the minimal cardinality of a subcover of $\mathcal{U}$. Since X is compact, $N(\mathcal{U})$ is finite. Let $H(\mathcal{U}) = \log N(\mathcal{U})$ be the *entropy of* $\mathcal{U}$. For any two open covers $\mathcal{U}$ and $\mathcal{V}$ of X, let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Define analogously $\mathcal{U}_1 \vee \ldots \vee \mathcal{U}_n$, for open covers $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of X. Let $\psi : X \to X$ be a continuous self-map and $\mathcal{U}$ an open cover of X. Then $\psi^{-1}(\mathcal{U}) = \{\psi^{-1}(U) : U \in \mathcal{U}\}$. The *topological entropy of* ψ *with respect to* $\mathcal{U}$ is

$$H_{top}(\psi, \mathcal{U}) = \lim_{n\to\infty} \frac{H(\mathcal{U} \vee \psi^{-1}(\mathcal{U}) \vee \ldots \vee \psi^{-n+1}(\mathcal{U}))}{n},$$

and the *topological entropy* of ψ is

$$h_{top}(\psi) = \sup\{H_{top}(\psi, \mathcal{U}) : \mathcal{U} \text{ open cover of } X\}.$$

In Section 2.1 we recall the main properties of the topological entropy in the context of continuous endomorphisms of compact abelian groups.

Later on, using ideas briefly sketched in [1], Weiss developed in [26] the definition of algebraic entropy for endomorphisms of torsion abelian groups (see also [8]). Moreover, Peters modified this definition in [20] for automorphisms of abelian groups, and this approach was extended to all endomorphisms of abelian groups in [5]. We recall now this general definition.

Let G be an abelian group and let $\phi : G \to G$ be an endomorphism. For a non-empty subset F of G and n a positive integer, the *n-th* ϕ*-trajectory* of F is

$$T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F),$$

and the ϕ*-trajectory* of F is

$$T(\phi, F) = \sum_{n\in\mathbb{N}} \phi^n(F).$$

For a non-empty finite subset F of G, the limit

$$H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n} \tag{1.1}$$

exists (see [5]) and $H_{alg}(\phi, F)$ is the *algebraic entropy of ϕ with respect to F*. Let $[G]^{<\omega}$ denote the family of all non-empty finite subset of G. The *algebraic entropy* of ϕ is

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) : F \in [G]^{<\omega}\}.$$

In Section 2.2 we recall the basic properties of the algebraic entropy for endomorphisms of abelian groups. One can immediately see that these properties are dual with respect to the properties of the topological entropy.

Both Weiss and Peters connected the algebraic entropy with the topological entropy using the Pontryagin duality, in the following Bridge Theorems.

Fact 1.1. *Let G be a abelian group and $\phi : G \to G$ an endomorphism.*

(a) [26] *If G is torsion, then $h_{alg}(\phi) = h_{top}(\widehat{\phi})$.*

(b) [20] *If G is countable and ϕ is an automorphism, then $h_{alg}(\phi) = h_{top}(\widehat{\phi})$.*

The main result of this paper is the generalization of the Bridge Theorem to all endomorphisms of all abelian groups:

Bridge Theorem. *Let G be an abelian group and $\phi : G \to G$ an endomorphism. Then*

$$h_{alg}(\phi) = h_{top}(\widehat{\phi}).$$

Let us mention here that, using Pontryagin's duality theorem, one can reformulate the theorem also in its dual form (as in the abstract): if K is a compact abelian group and $\psi : K \to K$ is a continuous endomorphism, then $h_{top}(\psi) = h_{alg}(\widehat{\psi})$.

As a first application of the Bridge Theorem, we deduce from the Uniqueness Theorem for the algebraic entropy (see Theorem 2.13 below) a Uniqueness Theorem for the topological entropy in the category of compact abelian groups and continuous endomorphisms (see Corollary 3.3).

Inspired by the Pinsker subalgebra introduced in Ergodic Theory with respect to the measure entropy, similar Pinsker-like constructions are considered for both the topological entropy and the algebraic entropy, as we describe in the sequel.

A pair (X, ψ) is a *topological flow* if X is a compact Hausdorff space and $\psi : X \to X$ is a homeomorphism. A *factor* $(\pi, (Y, \eta))$ of (X, ψ) is a topological flow (Y, η) together with a continuous surjective map $\pi : X \to Y$ such that $\pi \circ \psi = \eta \circ \pi$. In [2] (see also [16]) it is proved that a topological flow (X, ψ) admits a largest factor $\mathbf{P}_{top}(X, \psi)$ with zero topological entropy, called *topological Pinsker factor*. Moreover, a topological flow (X, ψ) has *completely positive topological entropy* if all its non-trivial factors have positive topological entropy [2], that is $\mathbf{P}_{top}(X, \psi)$ is trivial.

In analogy with the topological case, in [4] an *algebraic flow* is defined as a pair (G, ϕ), where G is an abelian group and $\phi : G \to G$ is an endomorphism. Moreover, the *Pinsker subgroup* of G with respect to ϕ is the greatest ϕ-invariant subgroup $\mathbf{P}_{alg}(G, \phi)$ of G such that $h_{alg}(\phi\restriction_{\mathbf{P}_{alg}(G,\phi)}) = 0$. It is clear that for an algebraic flow (G, ϕ), $h_{alg}(\phi) = 0$ if and only if $G = \mathbf{P}_{alg}(G, \phi)$. In the opposite direction, the following property is considered in [4]; we say that ϕ has *completely positive algebraic entropy* if $h_{alg}(\phi\restriction_H) > 0$ for every non-trivial ϕ-invariant subgroup H of G, that is $\mathbf{P}_{alg}(G, \phi) = 0$.

More generally, for an arbitrary category $\mathfrak{X}$, a *flow* in $\mathfrak{X}$ is a pair (X, ϕ), where X is an object in $\mathfrak{X}$ and $\phi : X \to X$ an endomorphism in $\mathfrak{X}$. The category $\mathrm{Flow}_{\mathfrak{X}}$ of flows of $\mathfrak{X}$ has as objects all flows in $\mathfrak{X}$, and a morphism in $\mathbf{Flow}_{\mathfrak{X}}$ between two flows (X, ϕ) and (Y, ψ) is a morphism $u : X \to Y$ in $\mathfrak{X}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[\ \psi\ ]{} & Y \end{array} \tag{1.2}$$

in $\mathfrak{X}$ commutes. So an entropy function can be seen as a function $h : \mathrm{Flow}_{\mathfrak{X}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, defined on flows (X, ϕ). If the domain X of ϕ is clear from the context, we can denote the flow (X, ϕ) only by ϕ. Let **Comp**, **CompAb**, **AbGrp** be respectively the categories of compact groups, compact abelian groups and abelian groups.

As an application of the Bridge Theorem we have the following relation between the Pinsker factor and the Pinsker subgroup.

Theorem A. *Let K be a compact abelian group and $\psi : K \to K$ a continuous endomorphism. Then*

$$\mathbf{P}_{top}(K, \psi) = (K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp}, \overline{\psi}),$$

where $\overline{\psi} : K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp} \to K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp}$ is the continuous endomorphism induced by ψ.

Every compact group K has its unique Haar measure μ. Halmos [13] noticed that a continuous surjective endomorphism ψ of K is measure preserving with respect to μ. So both the topological and the measure entropy are available in this case and they coincide as proved by Stoyanov [23]. Moreover, ψ is *ergodic* if one of the following two equivalent properties is satisfied:

(a) $\psi^{-1}(B) = B$ implies $\mu(B) = 0$ or $\mu(B) = 1$ for every measurable set B;

(b) for every pair A, B of measurable subsets of K with $\mu(A) > 0$ and $\mu(B) > 0$ there exists $n \in \mathbb{N}$ such that $\mu(\psi^{-n}(A) \cap B) > 0$.

In analogy to this definition, in [4] an endomorphism $\phi : G \to G$ of an abelian group G is called *algebraically ergodic* if ϕ has no non-trivial quasi-periodic points.

As a consequence of the Bridge Theorem, we prove the following relations among ergodic, topological and algebraic properties related to entropy.

Theorem B. *Let K be a compact abelian group and $\psi : K \to K$ a continuous automorphism. Then the following conditions are equivalent:*

(a) ψ is ergodic;

(b) ψ has completely positive topological entropy;

(c) $\widehat{\psi}$ is algebraically ergodic;

(d) $\widehat{\psi}$ has completely positive algebraic entropy.

Moreover, we give the following:

Definition 1.2. *Let K be a compact abelian group and $\psi : K \to K$ a continuous injective endomorphism of K. The* greatest domain of ergodicity *of ψ is the greatest closed ψ-invariant subgroup $\mathcal{E}(K, \psi)$ of K such that the restriction $\psi\restriction_{\mathcal{E}(K,\psi)}$ is ergodic.*

As an application of Theorem B we show that $\mathcal{E}(K, \psi)$ is precisely the annihilator of $\mathbf{P}_{alg}(\widehat{K}, \widehat{\phi})$ in K (see Corollary 4.1).

Both the topological entropy and the algebraic entropy are defined also for continuous endomorphisms of locally compact groups (see [3, 15] and [24] respectively; see also [7] and [6]). So one can consider the possibility of extending the Bridge Theorem to the class of all locally compact abelian groups.

Problem 1.3. *Extend the Bridge Theorem to continuous endomorphisms of locally compact abelian groups.*

Some results in this direction were announced in [21], but the proofs given there contain gaps, as pointed out in [11].

2 Basic properties of both entropy functions

For an abelian group G the Pontryagin dual $\widehat{G}$ is $\mathrm{Hom}(G, \mathbb{T})$ endowed with the compact-open topology [19] (here and in the sequel, when no specific topology is given, the group is assumed to be discrete as far as topology is concerned). The Pontryagin dual of an abelian group is compact. Moreover, for an endomorphism $\phi : G \to G$, its dual endomorphism $\widehat{\phi} : \widehat{G} \to \widehat{G}$ is continuous. For basic properties concerning the Pontryagin duality see [10] and [14]. For a subset A of G, the annihilator of A in $\widehat{G}$ is $A^{\perp} = \{\chi \in \widehat{G} : \chi(A) = 0\}$, while for a subset B of $\widehat{G}$, the annihilator of B in G is $B^{\top} = \{x \in G : \chi(x) = 0 \text{ for every } \chi \in B\}$.

Remark 2.1. Let $n \in \mathbb{N}_+$. If $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ is an endomorphism, then ϕ is $\mathbb{Q}$-linear. Then the action of ϕ on $\mathbb{Q}^n$ is given by an $n \times n$ matrix A_ϕ with rational coefficients. The characteristic polynomial $p_\phi(t)$ of ϕ over $\mathbb{Z}$ is the characteristic polynomial of A_ϕ over $\mathbb{Z}$ (namely, the primitive polynomial $st^n + a_1 t^{n-1} + \ldots + a_n \in \mathbb{Z}[t]$ with $s > 0$ obtained from the monic characteristic polynomial of A_ϕ over $\mathbb{Q}$).

Let now $\psi : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ be a continuous endomorphism. Due to the isomorphism $\widehat{\mathbb{Q}^n} \cong_{top} \widehat{\mathbb{Q}}^n$, we identify ψ with the corresponding continuous endomorphism $\widehat{\mathbb{Q}^n} \to \widehat{\mathbb{Q}^n}$. Then $\phi = \widehat{\psi} : \mathbb{Q}^n \to \mathbb{Q}^n$ is an endomorphism, with its matrix A_ϕ. The action of ψ on $\widehat{\mathbb{Q}}^n$ is given by the transposed matrix ${}^tA_\phi$, that we call A_ψ. Consequently, the characteristic polynomial $p_\psi(t)$ of ψ over $\mathbb{Z}$ is the characteristic polynomial of A_ψ over $\mathbb{Z}$, which coincides with the characteristic polynomial of A_ϕ over $\mathbb{Z}$, that is $p_\phi(t)$.

2.1 Topological entropy

In the following fact we collect the basic properties of the topological entropy.

Fact 2.2. *Let K be a compact group and $\psi : K \to K$ a continuous endomorphism.*

(a) [Invariance under conjugation] *If $\eta : H \to H$ a continuous endomorphism of a compact group H and ψ and η are conjugated (i.e., there exists a topological isomorphism $\xi : K \to H$ such that $\eta = \xi\psi\xi^{-1}$), then $h_{top}(\psi) = h_{top}(\eta)$.*

(b) [Logarithmic law] *For every $k \in \mathbb{N}_+$, $h_{top}(\psi^k) = k \cdot h_{top}(\psi)$. If ψ is an automorphism, then $h_{top}(\psi^k) = |k| h_{top}(\psi)$ for every $k \in \mathbb{Z}$.*

(c) [Continuity for inverse limits] *If K is an inverse limit of an inverse system of quotient groups $\{G/K_i : i \in I\}$, where each K_i is a closed ψ-invariant normal subgroup of G, then $h_{top}(\phi) = \sup_{i\in I} h_{top}(\phi\restriction_{K_i})$.*

(d) [Weak Addition Theorem] *If $K = K_1 \times K_2$ and $\psi = \psi_1 \times \psi_2$ with $\psi_i : K_i \to K_i$ continuous, $i = 1, 2$, then $h_{top}(\psi_1 \times \psi_2) = h_{top}(\psi_1) + h_{top}(\psi_2)$.*

As in the case of measure entropy, the left Bernoulli shift plays a fundamental role for the topological entropy:

Example 2.3. [Bernoulli Axiom] For any compact abelian group K the *left Bernoulli shift* is ${}_K\beta : K^{\mathbb{N}} \to K^{\mathbb{N}}$ defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3 \ldots).$$

It is a well-known fact (see [23]) that $h_{top}({}_K\beta) = \log |K|$, with the usual convention that $\log |K| = \infty$, if $|K|$ is infinite. In particular, $h_{top}({}_{\mathbb{Z}(p)}\beta) = \log p$, for every prime p.

For continuous endomorphisms ψ of compact groups K, the *hyperimage* of ψ is defined by

$$\mathrm{Im}_\infty \psi = \bigcap_{n\in\mathbb{N}} \psi^n(K);$$

so, it is a closed ψ-invariant subgroup of K. The next result shows that, as far as the computation of the topological entropy of continuous endomorphisms of compact groups is concerned, one can restrict to surjective endomorphisms.

Lemma 2.4. [23] *Let K be a compact group and $\psi : K \to K$ a continuous endomorphism. Then $\psi\restriction_{\mathrm{Im}_\infty\psi}$ is surjective and $\mathrm{Im}_\infty\psi$ is the largest closed ψ-invariant subgroup of K with this property. Moreover, $h_{top}(\psi) = h_{top}(\psi\restriction_{\mathrm{Im}_\infty\psi})$.*

The following is the Yuzvinski Formula for the topological entropy proved by Yuzvinski in [28] (see also [18]).

Theorem 2.5 (Yuzvinski Formula)**.** [27] *Let $n \in \mathbb{N}_+$ and $\psi : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ a topological automorphism. Then*

$$h_{top}(\psi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|, \tag{2.1}$$

where λ_i are the roots of the characteristic polynomial $p_\psi(t) \in \mathbb{Z}[t]$ of ψ over $\mathbb{Z}$ and s is the leading coefficient of $p_\psi(t)$.

The next fundamental property of the topological entropy, showing that it is additive, is due to Yuzvinski [28].

Theorem 2.6 (Addition Theorem)**.** [28] *Let K be a compact abelian group, $\psi : K \to K$ a continuous endomorphism, N a closed ψ-invariant subgroup of K and $\overline{\psi} : K/N \to K/N$ the endomorphism induced by ψ. Then $h_{top}(\psi) = h_{top}(\psi\restriction_N) + h_{top}(\overline{\psi})$.*

Moreover, a Uniqueness Theorem holds for the topological entropy in the category of compact groups and continuous endomorphisms.

Theorem 2.7 (Uniqueness Theorem)**.** [23] *The topological entropy* $h_{top} : \mathrm{Flow}_{\mathbf{Comp}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *is the unique function such that:*

(a) h_{top} *is monotone;*

(b) if $(K, \psi) \in \mathrm{Flow}_{\mathbf{Comp}}$*, then* $h_{top}(\psi) = h_{top}(\psi\restriction_{\mathrm{Im}_\infty \psi})$*;*

(c) h_{top} *satisfies the Logarithmic Law;*

(d) if ψ *is an inner automorphism of* $K \in \mathbf{Comp}$*, then* $h_{top}(\psi) = 0$*;*

(e) h_{top} *is continuous on inverse limits;*

(f) h_{top} *satisfies the Addition Theorem;*

(g) h_{top} *satisfies the Bernoulli Axiom;*

(h) h_{top} *satisfies the Yuzvinski Formula.*

In Corollary 3.3 we obtain a counterpart of this Uniqueness Theorem for the full subcategory of compact abelian groups using only five axioms.

2.2 Algebraic entropy

The next fact gives the basic properties of the algebraic entropy.

Fact 2.8. [5] *Let* G *be an abelian group and* $\phi : G \to G$ *an endomorphism.*

(a) [Invariance under conjugation] *If* H *is another abelian group,* $\eta : H \to H$ *an endomorphism and* ϕ *and* η *are conjugated (i.e., there exists an isomorphism* $\xi : G \to H$ *such that* $\eta = \xi\phi\xi^{-1}$*), then* $h_{alg}(\phi) = h_{alg}(\eta)$*.*

(b) [Logarithmic law] *For every* $k \in \mathbb{N}_+$*,* $h_{alg}(\phi^k) = k h_{alg}(\phi)$*. If* ϕ *is an automorphism, then* $h_{alg}(\phi^k) = |k| h_{alg}(\phi)$ *for every* $k \in \mathbb{Z}$*.*

(c) [Continuity for direct limits] *If* G *is a direct limit of* ϕ*-invariant subgroups* $\{G_i : i \in I\}$*, then* $h_{alg}(\phi) = \sup_{i\in I} h_{alg}(\phi\restriction_{G_i})$*.*

(d) [Weak Addition Theorem] *If* $G = G_1 \times G_2$ *and* $\phi = \phi_1 \times \phi_2$ *with* $\phi_i : G_i \to G_i$ *an endomorphism,* $i = 1, 2$*, then* $h_{alg}(\phi_1 \times \phi_2) = h_{alg}(\phi_1) + h_{alg}(\phi_2)$*.*

A fundamental example in the context of algebraic entropy is given by the right Bernoulli shift:

Example 2.9. [Bernoulli axiom] For any abelian group K the *right Bernoulli shift* $\beta_K : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ is defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots).$$

Then $h_{alg}(\beta_K) = \log |K|$, with the usual convention that $\log |K| = \infty$ if $|K|$ is infinite.

The following is a fundamental result on the values of the algebraic entropy for endomorphisms of $\mathbb{Q}^n$. It was recently proved in [12], and a shorter proof in the case of zero algebraic entropy was previously given in [9].

Theorem 2.10 (Algebraic Yuzvinski Formula)**.** *For* $n \in \mathbb{N}_+$ *an automorphism* ϕ *of* $\mathbb{Q}^n$ *is described by a matrix* $A \in GL_n(\mathbb{Q})$*. Then*

$$h_{alg}(\phi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|, \tag{2.2}$$

where λ_i *are the roots of the characteristic polynomial* $p_\phi(t) \in \mathbb{Z}[t]$ *of* ϕ *over* $\mathbb{Z}$ *and* s *is the leading coefficient of* $p_\phi(t)$*.*

Applying the Algebraic Yuzvinski Formula, in [5] the following important property is given, showing that the algebraic entropy is additive.

Theorem 2.11 (Addition Theorem)**.** *Let* G *be an abelian group,* $\phi : G \to G$ *an endomorphism,* H *a* ϕ*-invariant subgroup of* G *and* $\overline{\phi} : G/H \to G/H$ *the endomorphism induced by* ϕ*. Then*

$$h_{alg}(\phi) = h_{alg}(\phi\restriction_H) + h_{alg}(\overline{\phi}).$$

Theorems 2.10 and 2.11 are the main ingredient in the proof of the BridgeTheorem.

Let G be an abelian group and $\phi : G \to G$ an endomorphism; the *hyperkernel* of ϕ is

$$\ker_\infty \phi = \bigcup_{n\in\mathbb{N}_+} \ker \phi^n.$$

The subgroup $\ker_\infty \phi$ is ϕ-invariant and also invariant for inverse images. Hence the induced endomorphism $\overline{\phi} : G/\ker_\infty \phi \to G/\ker_\infty \phi$ is injective, and the next lemma proved in [5] shows that ϕ and $\overline{\phi}$ have the same algebraic entropy.

Lemma 2.12. *Let G be an abelian group and $\phi : G \to G$ an endomorphism. Then $h_{alg}(\phi\restriction_{\ker_\infty \phi}) = 0$ and $h_{alg}(\phi) = h_{alg}(\overline{\phi})$, where $\overline{\phi} : G/\ker_\infty \phi \to G/\ker_\infty \phi$ is the endomorphism induced by ϕ.*

Proof. The equality $h_{alg}(\phi\restriction_{\ker_\infty \phi}) = 0$ follows easily from the definitions since for every finite subset F of $\ker_\infty \phi$ the n-th ϕ-trajectory $T_n(\phi, F)$ becomes invariant (i.e., coincides with $T(\phi, F)$) for sufficiently large $n \in \mathbb{N}_+$. Applying the Addition Theorem 2.11 one obtains immediately the equality $h_{alg}(\phi) = h_{alg}(\overline{\phi})$. □

Roughly speaking, this lemma is a counterpart of Lemma 2.4, that is, it reduces the computation of the algebraic entropy of arbitrary endomorphisms of abelian groups to the injective ones.

Moreover, a Uniqueness Theorem was proved in [5] for the algebraic entropy in the class of all endomorphisms of abelian groups.

Theorem 2.13 (Uniqueness Theorem). [5] *The algebraic entropy $h_{alg} : \mathrm{Flow}_{\mathbf{AbGrp}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the unique function such that:*

(a) h_{alg} is invariant under conjugation;

(b) h_{alg} is continuous on direct limits;

(c) h_{alg} satisfies the Addition Theorem;

(d) h_{alg} satisfies the Bernoulli Axiom;

(e) h_{alg} satisfies the Algebraic Yuzvinski Formula.

The non-trivial proof of this theorem is given in [5]. We use it here only for an alternative proof of the Bridge Theorem, leaving the main proof completely self-contained, leaning only on the Yuzvinski Formulas, the Addition Theorems, the main properties of the entropies and the basic examples of Bernoulli shifts.

3 The Bridge Theorem

As an application of the Pontryagin duality we have the following relation between the left and the right Bernoulli shifts (see [7, Proposition 6.1]).

Example 3.1. If K is a compact abelian group, then ${}_K\beta = \widehat{\beta_{\widehat{K}}}$.

Let G be an abelian group and $\phi : G \to G$ an endomorphism. Let $K = \widehat{G}$ and $\psi = \widehat{\phi}$. Let also H be a ϕ-invariant subgroup of G. By the Pontryagin duality, $N = H^\perp$ is a closed ψ-invariant subgroup of K, and $N^\perp = H$. Moreover, we have the following commutative diagrams:

$$\begin{array}{ccccc} H & \hookrightarrow & G & \twoheadrightarrow & G/H \\ \downarrow{\scriptstyle \phi\restriction_H} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ H & \hookrightarrow & G & \twoheadrightarrow & G/H \end{array} \qquad \begin{array}{ccccc} K/N & \twoheadleftarrow & K & \hookleftarrow & N \\ \uparrow{\scriptstyle \overline{\psi}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi\restriction_N} \\ K/N & \twoheadleftarrow & K & \hookleftarrow & N \end{array} \tag{3.1}$$

The second diagram is obtained from the first one by applying the Pontryagin duality functor. In particular,

$$\begin{aligned} \widehat{K/N} \cong H \text{ and } \widehat{N} \cong G/H; \\ K/N \cong \widehat{H} \text{ and } N \cong \widehat{G/H}. \end{aligned} \tag{3.2}$$

Moreover,

$$\begin{aligned} \widehat{\overline{\psi}} \text{ is conjugated to } \phi\restriction_H \text{ and } \widehat{\psi\restriction_N} \text{ is conjugated to } \overline{\phi}; \\ \overline{\psi} \text{ is conjugated to } \widehat{\phi\restriction_H} \text{ and } \psi\restriction_N \text{ is conjugated to } \widehat{\overline{\phi}}. \end{aligned} \tag{3.3}$$

So, in these notations, Fact 2.8(a) gives the following:

Proposition 3.2. *Let G be an abelian group, $\phi : G \to G$ an endomorphism and H a ϕ-invariant subgroup of G. Then $\psi = \widehat{\phi}$ satisfies:*

(a) $h_{alg}(\phi\restriction_H) = h_{alg}(\widehat{\overline{\psi}})$;

(b) $h_{alg}(\overline{\phi}) = h_{alg}(\widehat{\psi\restriction_N})$.

Let G be an abelian group and $\phi : G \to G$ an endomorphism. For a subset F of G, the trajectory $T(\phi, F)$ needs not be a subgroup of G. So let

$$V(\phi, F) = \langle \phi^n(F) : n \in \mathbb{N} \rangle = \langle T(\phi, F) \rangle .$$

This is the smallest ϕ-invariant subgroup containing $T(\phi, F)$ (and so also F). If $F = \{g\}$ we denote $V(\phi, \{g\})$ simply by $V(\phi, g)$. For $F \in [G]^{<\omega}$, $V(\phi, F) = \sum_{g \in F} V(\phi, g)$. As proved in [5], it is easy to see that

$$G = \varinjlim\{V(\phi, F) : F \in [G]^{<\omega}\}. \tag{3.4}$$

We can now prove the Bridge Theorem stated in the Introduction.

Bridge Theorem. *Let G be an abelian group and $\phi : G \to G$ an endomorphism. Then $h_{alg}(\phi) = h_{top}(\widehat{\phi})$.*

Proof. Let $K = \widehat{G}$ and $\psi = \widehat{\phi}$. We split the proof of the main assertion into several steps by reduction to more and more restricted cases.

(i) It is possible to assume that G is torsion-free (i.e., K is connected). Indeed, consider the torsion part $t(G)$ of G and the endomorphism $\overline{\phi} : G/t(G) \to G/t(G)$ induced by ϕ. Then the connected component $c(K)$ of K is $c(K) = t(G)^\perp$, and so the diagram in (3.1) becomes

$$\begin{array}{ccccc} t(G) & \hookrightarrow & G & \twoheadrightarrow & G/t(G) \\ \downarrow{\scriptstyle \phi\restriction_{t(G)}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ t(G) & \hookrightarrow & G & \twoheadrightarrow & G/t(G) \end{array} \qquad \begin{array}{ccccc} K/c(K) & \twoheadleftarrow & K & \hookleftarrow & c(K) \\ \uparrow{\scriptstyle \overline{\psi}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi\restriction_{c(K)}} \\ K/c(K) & \twoheadleftarrow & K & \hookleftarrow & c(K) \end{array}$$

where $\overline{\psi} : K/c(K) \to K/c(K)$ is the induced endomorphism. By the Addition Theorems 2.11 and 2.6 we have respectively $h_{alg}(\phi) = h_{alg}(\phi\restriction_{t(G)}) + h_{alg}(\overline{\phi})$ and $h_{top}(\psi) = h_{top}(\psi\restriction_{c(K)}) + h_{top}(\overline{\psi})$. Proposition 3.2(a) and Fact 1.1(a) give $h_{alg}(\phi\restriction_{t(G)}) = h_{top}(\overline{\psi})$, while Proposition 3.2(b) yields $h_{alg}(\overline{\phi}) = h_{alg}(\widehat{\psi\restriction_{c(K)}})$. So if we show that $h_{alg}(\widehat{\psi\restriction_{c(K)}}) = h_{top}(\psi\restriction_{c(K)})$, this implies $h_{alg}(\phi) = h_{top}(\psi)$, so the thesis.

(ii) We can assume that G is torsion-free of finite rank. Indeed, by (i) we can suppose that G is torsion-free. If there exists $g \in G$ such that $r(V(\phi, g))$ is infinite, then

$$V(\phi, g) \cong \mathbb{Z}^{(\mathbb{N})} \text{ and } \phi\restriction_{V(\phi,g)} \text{ is conjugated to } \beta_{\mathbb{Z}}. \tag{3.5}$$

So $h_{alg}(\phi\restriction_{V(\phi,g)}) = \infty$ by Fact 2.8(a) and Example 2.9. Moreover, $h_{alg}(\phi) \geq h_{alg}(\phi\restriction_{V(\phi,g)})$ as a consequence of the Addition Theorem 2.11, hence $h_{alg}(\phi) = \infty$.

Let $N = V(\phi, g)^\perp$ and $\overline{\psi} : K/N \to K/N$ the endomorphism induced by ψ. Then N is a closed ψ-invariant subgroup of K such that, in view of (3.2) and (3.3),

$$K/N \cong_{top} \widehat{V(\phi, g)} \text{ and } \overline{\psi} \text{ is conjugated to } \widehat{\phi\restriction_{V(\phi,g)}}$$

Moreover,

$$\widehat{V(\phi, g)} \cong_{top} \mathbb{T}^{\mathbb{N}} \text{ and } \widehat{\phi\restriction_{V(\phi,g)}} \text{ is conjugated to } {}_{\mathbb{T}}\beta.$$

according to (3.5) and Example 3.1. Then $h_{top}(\overline{\psi}) = \infty$ by Fact 2.2(a) and Example 2.3. The Addition Theorem 2.6 gives $h_{top}(\psi) \geq h_{top}(\overline{\psi})$. Hence $h_{top}(\psi) = \infty = h_{alg}(\phi)$.

Assume now that $r(V(\phi, g))$ is finite for every $g \in G$. Then $r(V(\phi, F))$ is finite for every $F \in [G]^{<\omega}$. By (3.4) $G = \varinjlim\{V(\phi, F) : F \in [G]^{<\omega}\}$, so $h_{alg}(\phi) = \sup_{F \in [G]^{<\omega}} h_{alg}(\phi\restriction_{V(\phi,F)})$ by Fact 2.8(c). For every $F \in [G]^{<\omega}$ let $N_F = V(\phi, F)^\perp$; then $V(\phi, F) \cong \widehat{K/N_F}$. So Proposition 3.2(a) gives

$$h_{alg}(\phi\restriction_{V(\phi,F)}) = h_{alg}(\widehat{\overline{\psi}_F}), \tag{3.6}$$

where $\overline{\psi}_F : K/N_F \to K/N_F$ is the endomorphism induced by ψ. Since $G = \varinjlim\{V(\phi, F) : F \in [G]^{<\omega}\}$, by Pontryagin duality $K = \varprojlim\{K/N_F : F \in [G]^{<\omega}\}$. Since $h_{alg}(\widehat{\overline{\psi}_F}) = h_{top}(\overline{\psi}_F)$ for every $F \in [G]^{<\omega}$, Fact 2.8(c), (3.6) and Fact 2.2(c) gives

$$h_{alg}(\phi) = \sup_{F\in[G]^{<\omega}} h_{alg}(\phi\restriction_{V(\phi,F)}) = \sup_{F\in[G]^{<\omega}} h_{alg}(\widehat{\overline{\psi}_F}) = \sup_{F\in[G]^{<\omega}} h_{top}(\overline{\psi}_F) = h_{top}(\psi).$$

This shows that we can consider only torsion-free abelian groups of finite rank.

(iii) It suffices to prove the assertion for G a divisibile torsion-free abelian group of finite rank. Indeed, by (ii) we can assume that G is a torsion-free abelian group of finite rank $n \in \mathbb{N}_+$; by Pontryagin duality this is equivalent to say that K is a connected compact abelian group of dimension n. Assume without loss of generality (by Fact 2.8(a) and Fact 2.2(a)) that $D(G) = \mathbb{Q}^n$, and so that $\widehat{D(G)} = \widehat{\mathbb{Q}}^n$ (as we noted above $\widehat{\mathbb{Q}^n} \cong_{top} \widehat{\mathbb{Q}}^n$), where $D(G)$ is the divisible hull of G and $\widetilde{\phi} : D(G) \to D(G)$ denotes the unique extension of ϕ. Let $\varphi : \mathbb{Q}^n/G \to \mathbb{Q}^n/G$ be the endomorphism induced by $\widetilde{\phi}$, $\eta : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ the dual endomorphism of $\widetilde{\phi}$, and $N = G^\perp$. The diagrams in (3.1) give the following corresponding diagrams:

$$\begin{array}{ccccc} G & \hookrightarrow & \mathbb{Q}^n & \twoheadrightarrow & \mathbb{Q}^n/G \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \varphi} \\ G & \hookrightarrow & \mathbb{Q}^n & \twoheadrightarrow & \mathbb{Q}^n/G \end{array} \qquad \begin{array}{ccccc} K & \twoheadleftarrow & \widehat{\mathbb{Q}}^n & \hookleftarrow & N \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \eta\restriction_N} \\ K & \twoheadleftarrow & \widehat{\mathbb{Q}}^n & \hookleftarrow & N \end{array}$$

We show that $h_{alg}(\phi) = h_{alg}(\widetilde{\phi})$ and $h_{top}(\psi) = h_{top}(\eta)$. By the Addition Theorems 2.11 and 2.6, it suffices to prove that $h_{alg}(\varphi) = 0$ and $h_{top}(\eta\restriction_N) = 0$. By Proposition 3.2(b) $h_{alg}(\varphi) = h_{alg}(\widehat{\eta\restriction_N})$. Moreover, since G is essential in $\mathbb{Q}^n$, so $\mathbb{Q}^n/G$ is torsion. Therefore, $h_{alg}(\widehat{\eta\restriction_N}) = h_{top}(\eta\restriction_N)$ by Fact 1.1(a), and so $h_{alg}(\varphi) = h_{top}(\eta\restriction_N)$. Then it remains to verify that $h_{alg}(\varphi) = 0$.

Let $H = \mathbb{Q}^n/G \cong \bigoplus_p \mathbb{Z}(p^\infty)^{k_p}$ with each $k_p \in \mathbb{N}$, $k_p \leq n$. For every $m \in \mathbb{N}_+$, the fully invariant subgroup $H[m] = \{x \in H : mx = 0\}$ of H is finite, so $h_{alg}(\varphi\restriction_{H[m]}) = 0$. Since $H = \varinjlim H[m]$, Fact 2.8(c) yields $h_{alg}(\varphi) = \sup_{m\in\mathbb{N}_+} h_{alg}(\varphi\restriction_{H[m]}) = 0$.

So $h_{alg}(\phi) = h_{alg}(\widetilde{\phi})$ and $h_{top}(\psi) = h_{top}(\eta)$. Then $h_{alg}(\widetilde{\phi}) = h_{top}(\eta)$ would imply $h_{alg}(\phi) = h_{top}(\psi)$. In other words, it suffices to prove the thesis for endomorphisms $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$.

(iv) We can suppose that ϕ is injective (i.e., ψ surjective). Indeed, consider the corresponding diagrams given by (3.1):

$$\begin{array}{ccccc} \ker_\infty \phi & \hookrightarrow & G & \twoheadrightarrow & G/\ker_\infty \phi \\ \downarrow{\scriptstyle \phi\restriction_{\ker_\infty \phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ \ker_\infty \phi & \hookrightarrow & G & \twoheadrightarrow & G/\ker_\infty \phi \end{array} \qquad \begin{array}{ccccc} K/\mathrm{Im}_\infty\psi & \twoheadleftarrow & K & \hookleftarrow & \mathrm{Im}_\infty\psi \\ \uparrow{\scriptstyle \overline{\psi}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi\restriction_{\mathrm{Im}_\infty\psi}} \\ K/\mathrm{Im}_\infty\psi & \twoheadleftarrow & K & \hookleftarrow & \mathrm{Im}_\infty\psi \end{array}$$

Indeed, $\mathrm{Im}_\infty\psi = (\ker_\infty \phi)^\perp$. By Lemma 2.12 $h_{alg}(\phi) = h_{alg}(\overline{\phi})$, where the induced endomorphism $\overline{\phi} : G/\ker_\infty \phi \to G/\ker_\infty \phi$ is injective. By Lemma 2.4 $h_{top}(\psi) = h_{top}(\psi\restriction_{\mathrm{Im}_\infty\psi})$, where $\psi\restriction_{\mathrm{Im}_\infty\psi}$ is surjective. Proposition 3.2(b) yields $h_{alg}(\overline{\phi}) = h_{alg}(\widehat{\psi\restriction_{\mathrm{Im}_\infty\psi}})$. So if we prove that $h_{lag}(\widehat{\psi\restriction_{\mathrm{Im}_\infty\psi}}) = h_{top}(\psi\restriction_{\mathrm{Im}_\infty\psi})$, this will imply $h_{alg}(\phi) = h_{top}(\psi)$.

(v) By (iii) we can assume that G is a divisible torsion-free abelian group of finite rank $n \in \mathbb{N}$, that is, $G = \mathbb{Q}^n$ for $n = r(G)$. By (iv) we can suppose that $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ is injective; then ϕ is also surjective and so ϕ is an automorphism of $\mathbb{Q}^n$. By Pontryagin duality, ψ is an automorphism of $\widehat{\mathbb{Q}}^n$. Then $h_{alg}(\phi) = h_{top}(\psi)$ by the Yuzvinski Formula 2.5 and the Algebraic Yuzvinski Formula 2.10, since the characteristic polynomials $p_\phi(t)$ and $p_\psi(t)$ of ϕ and ψ over $\mathbb{Z}$ coincide as noted in Remark 2.1. □

The Bridge Theorem can be proved also making use of the Uniqueness Theorem 2.13.

Second proof of the Bridge Theorem. Let $h_G(\phi) = h_{top}(\widehat{\phi})$ for every endomorphism $\phi : G \to G$. By Fact 2.2, Example 2.3, the Yuzvinski Formula 2.5, and in view of the properties of Pontryagin duality, h_G satisfies the conditions (a), (b), (c), (d), (e) of the Uniqueness Theorem 2.13. Hence, $h_G = h_{alg}$. In particular, $h_{alg}(\phi) = h_{top}(\widehat{\phi})$. □

The next corollary is a counterpart of the Uniqueness Theorem 2.7, but cannot be obtained immediately "by restriction" from that theorem. An easy proof, along the lines of the above proof, can be obtained making use of the Uniqueness Theorem 2.13 and the Bridge Theorem. (This time the generic topological entropy function $\psi \mapsto h^*_{top}(\psi)$ defined on continuous endomorphisms ψ of compact abelian groups can be used to define a function $\phi \mapsto h^*(\phi) := h^*_{top}(\widehat{\phi})$ on endomorphisms ϕ of abelian groups that must coincide with the algebraic entropy function $\phi \mapsto h_{alg}(\phi)$ by the Uniqueness Theorem 2.13.)

Corollary 3.3 (Uniqueness Theorem)**.** *The topological entropy* $h_{top} : \mathrm{Flow}_{\mathbf{CompAb}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *is the unique function such that:*

(a) h_{top} *is invariant under conjugation;*

(b) h_{top} *is continuous on inverse limits;*

(c) h_{top} *satisfies the Addition Theorem;*

(d) h_{top} *satisfies the Bernoulli Axiom;*

(e) h_{top} *satisfies the Yuzvinski Formula.*

4 Pinsker factor and Pinsker subgroup

We start this section proving Theorem A stated the Introduction as an application of the Bridge Theorem.

Theorem A. *Let K be a compact abelian group and $\psi : K \to K$ a continuous endomorphism. Then*

$$\mathbf{P}_{top}(K, \psi) = (K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp}, \overline{\psi}),$$

where $\overline{\psi} : K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp} \to K/\mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^{\perp}$ is the continuous endomorphism induced by ψ.

Proof. Let $G = \widehat{K}$, $\phi = \widehat{\psi}$, $\overline{\psi} : K/\mathbf{P}_{alg}(G, \phi)^{\perp} \to K/\mathbf{P}_{alg}(G, \phi)^{\perp}$ the endomorphism induced by ψ and $\overline{\phi} : G/\mathbf{P}_{alg}(G, \phi) \to G/\mathbf{P}_{alg}(G, \phi)$ the endomorphism induced by ϕ. By Pontryagin duality we have the following corresponding diagrams.

$$\begin{array}{ccccc} \mathbf{P}_{alg}(G,\phi) & \hookrightarrow & G & \twoheadrightarrow & G/\mathbf{P}_{alg}(G,\phi) \\ \downarrow{\scriptstyle \phi\restriction_{\mathbf{P}_{alg}(G,\phi)}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ \mathbf{P}_{alg}(G,\phi) & \hookrightarrow & G & \twoheadrightarrow & G/\mathbf{P}_{alg}(G,\phi) \end{array} \qquad \begin{array}{ccccc} K/\mathbf{P}_{alg}(G,\phi)^{\perp} & \twoheadleftarrow & K & \hookleftarrow & \mathbf{P}_{alg}(G,\phi)^{\perp} \\ \uparrow{\scriptstyle \overline{\psi}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi\restriction_{\mathbf{P}_{alg}(G,\phi)^{\perp}}} \\ K/\mathbf{P}_{alg}(G,\phi)^{\perp} & \twoheadleftarrow & K & \hookleftarrow & \mathbf{P}_{alg}(G,\phi)^{\perp} \end{array}$$

By the Bridge Theorem and Proposition 3.2(b), we have $h_{top}(\overline{\psi}) = h_{alg}(\phi\restriction_{\mathbf{P}(G,\phi)}) = 0$. Therefore

$$(K/\mathbf{P}_{alg}(G, \phi)^{\perp}, \overline{\psi})$$

is a factor of (K, ψ) of zero topological entropy. Since $\mathbf{P}(G, \phi)$ is the greatest ϕ-invariant subgroup of G where the restriction of ϕ has zero algebraic entropy, it follows from Proposition 3.2(b) that $(K/\mathbf{P}(G, \phi)^{\perp}, \overline{\psi})$ is the greatest factor of (K, ψ) with zero topological entropy, that is $\mathbf{P}_{top}(K, \psi) = (K/\mathbf{P}_{alg}(G, \phi)^{\perp}, \overline{\psi})$. □

The ergodic transformations are the core of Ergodic Theory (see [25] for the definition and main properties). For our purposes it is enough to recall here the following characterization of ergodicity of a continuous automorphism ψ of a compact abelian group K considered with its Haar measure proved independently by Halmos and Rohlin (see also [25]). Let $G = \widehat{K}$ and $\phi = \widehat{\psi}$; then ψ is ergodic if and only if ϕ has no non-zero periodic point. Since ϕ is an automorphism this is equivalent to say that

$$\psi \text{ is ergodic if and only if } \phi \text{ has no non-zero quasi-periodic point, that is } \phi \text{ is algebraically ergodic.} \tag{4.1}$$

In [4], for an abelian group G and an endomorphism $\phi : G \to G$, the smallest ϕ-invariant subgroup $\mathfrak{Q}(G, \phi)$ of G such that the induced endomorphism $\overline{\phi}$ of $G/\mathfrak{Q}(G, \phi)$ is algebraically ergodic was introduced. A consequence of the Main Theorem of [4] is that

$$\mathfrak{Q}(G, \phi) = \mathbf{P}_{alg}(G, \phi); \tag{4.2}$$

therefore ϕ is algebraically ergodic if and only if ϕ has completely positive algebraic entropy.

Now we are in position to give the proof of Theorem B of the Introduction, that connects ergodicity, topological entropy and algebraic entropy in the above sense.

Theorem B. *Let K be a compact abelian group and $\psi : K \to K$ a continuous automorphism. Then the following conditions are equivalent:*

(a) ψ *is ergodic;*

(b) ψ *has completely positive topological entropy;*

(c) $\widehat{\psi}$ *is algebraically ergodic;*

(d) $\widehat{\psi}$ *has completely positive algebraic entropy.*

Proof. (a)⇔(c) is (4.1), (b)⇔(d) follows directly from Theorem A, (c)⇔(d) follows from (4.2). □

We conclude with the following result:

Corollary 4.1. *Let K be a compact abelian group and $\psi : K \to K$ a continuous injective endomorphism of K. Then $\mathcal{E}(K, \psi) = \mathbf{P}_{alg}(\widehat{K}, \widehat{\psi})^\perp$.*

Proof. Let $G = \widehat{K}$, $\phi = \widehat{\psi} : G \to G$. Since $\mathfrak{Q}(G, \phi) = \mathbf{P}_{alg}(G, \phi)$ by (4.2), one has $N := \mathfrak{Q}(G, \phi)^\perp = \mathbf{P}_{alg}(G, \phi)^\perp$. We verify now that $N = \mathcal{E}(K, \psi)$. By Pontryagin duality ϕ is surjective. It was proved in [4] that the induced endomorphism $\overline{\phi} : G/\mathfrak{Q}(G, \phi) \to G/\mathfrak{Q}(G, \phi)$ is an automorphism. By Pontryagin duality, $G/\mathbf{P}(G, \phi) \cong \widehat{N}$ and $\overline{\phi}$ is conjugated to $\widehat{\psi\restriction_N}$. Since $\mathfrak{Q}(G, \phi)$ is the smallest ϕ-invariant subgroup of G such that $\overline{\phi}$ is algebraically ergodic, by Pontryagin duality and Theorem B N is the greatest closed ψ-invariant subgroup of K such that $\psi\restriction_N$ is ergodic, that is $N = \mathcal{E}(K, \psi)$. □

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