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Weakly metrizable pseudocompact groups

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ABSTRACT. We study various weaker versions of metrizability for pseudocompact abelian groups G: singularity (G possesses a compact metrizable subgroup of the form mG, $m > 0$), almost connectedness (G is metrizable modulo the connected component) and various versions of extremality in the sense of Comfort and co-authors (s-extremal, if G has no proper dense pseudocompact subgroups, r-extremal, if G admits no proper pseudocompact refinement). We introduce also weakly extremal pseudocompact groups (weakening simultaneously s-extremal and r-extremal). It turns out that this "symmetric" version of extremality has nice properties that restore the symmetry, to a certain extent, in the theory of extremal pseudocompact groups obtaining simpler uniform proofs of most of the known results. We characterize doubly extremal pseudocompact groups within the class of s-extremal pseudocompact groups. We give also a criterion for r-extremality for connected pseudocompact groups.

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1. Introduction

The metric spaces and their generalizations are of major interest in General Topology. We consider here several weak versions of metrizability that work particularly well for pseudocompact groups. The class of pseudocompact groups, introduced by Hewitt [25] was proved to be an important class of topological groups in the last forty years.

1.1. **Extremal pseudocompact groups.**

In the framework of pseudocompact groups, extremality is a generalization of metrizability discovered by Comfort and co-authors. The starting point was a theorem of Comfort and Soundararajan [15] who proved that for a compact, metrizable and totally disconnected group there exists no strictly finer pseudocompact group topology. In 1982 Comfort and Robertson [9] extended this result to all compact, metrizable groups and proved also that a compact abelian group is metrizable if and only if it has no proper dense pseudocompact subgroup (see also Corollary 4.14 here for a short simultaneous proof of both theorems). This result motivated the study of the extremal pseudocompact groups, which can be considered as a generalization of the metrizable (compact) ones.

In 1988 Comfort and Robertson [11] defined extremal pseudocompact groups; however, they did not distinguish among different kinds of extremality. This was done explicitly somewhat later (see [1]).

A pseudocompact group G is

- *s-extremal* if G has no proper dense pseudocompact subgroups;
- *r-extremal* if there exists no strictly finer pseudocompact group topology on G;
- *doubly extremal* if it is both s-extremal and r-extremal.

Since the pseudocompact groups of countable pseudocharacter are compact, they cannot contain proper dense subgroups and every continuous bijection between such groups is a homeomorphism; hence it follows that *every metrizable pseudocompact group is doubly extremal.* Note that this result holds for arbitrary pseudocompact groups (not necessarily abelian) and introduces an important class of doubly extremal pseudocompact groups.

Extremality of zero-dimensional pseudocompact abelian groups was considered by Comfort and Robertson [11]. It was shown that a *zero-dimensional pseudocompact abelian group that is either s- or r-extremal is metrizable* [11, Theorem 7.3]. An important step in the proof was the case of abelian elementary p-groups, for which s- and r-extremality coincide and both imply metrizability (cf. [11, Theorem 5.19]). Hence it is worth asking whether there exist non-metrizable pseudocompact groups that are s-, r- or doubly extremal and what is the relation between s- and r-extremality. In the following list we collect questions posed in [11], [6], [4], [23] and [1]:

(A) Is every s-extremal pseudocompact group metrizable?
(B) Is every r-extremal pseudocompact group metrizable?
(C) Is every doubly extremal pseudocompact group metrizable?

(D) Is every s-extremal pseudocompact group also r-extremal?
(E) Is every r-extremal pseudocompact group also s-extremal?

Clearly, (A) implies (C) and (D), while (B) implies (C) and (E). Moreover, if the conjunction of (D) and (E) is true, then (A), (B) and (C) are equivalent. Nevertheless, none of the above questions has a complete answer, even when the attention is restricted to the context of abelian groups, hence it is worth studying extremality in the class of pseudocompact abelian groups.

In many particular cases it has been proved that some forms of extremality are equivalent to metrizability (e.g., for totally disconnected pseudocompact abelian groups, countably compact abelian groups, pseudocompact abelian groups of weight at most $\mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum). As far as the relation between s- and r-extremality is concerned, in some cases these two notions turn out to be equivalent.

1.2. **Further levels of extremality.** An essential tool in the study of pseudocompact groups is the notion of *G_δ-density* (a subgroup $H \leq G$ is G_δ-dense if non-trivially meets every non-empty G_δ-set of G). The first very important theorem on pseudocompact group is due to Comfort and Ross (cf. Theorem 2.1); it shows the relation between a pseudocompact group G, its completion $\widetilde{G}$ and G_δ-density.

In this paper we intend to face the problem of a better description of the relations among the three levels of extremality. To this end we introduce some weaker forms of extremality (in what follows $r_0(G)$ will denote the free rank of G).

Definition 1.1. *A pseudocompact abelian group is*

(a) d-extremal *if G/H is divisible for every G_δ-dense subgroup H of G;*
(b) c-extremal *if $r_0(G/H) < \mathfrak{c}$ for every G_δ-dense subgroup H of G;*
(c) weakly extremal *if it is both d-extremal and c-extremal.*

While there exist plenty of non-metrizable c-extremal (e.g., torsion) or d-extremal (e.g., divisible) pseudocompact abelian groups, we are not aware whether every weakly extremal pseudocompact abelian group is metrizable. This property as well as the one given in the following theorem make the weak extremality the most relevant extremality property:

Theorem A. (Theorem 3.12) *If a pseudocompact abelian group G is either s-extremal or r-extremal, then G is weakly extremal.*

The same conclusion of the above theorem can be deduced from [3, Theorem 4.4]; the proof is given in §3.2 and is based on the dense graph theorem (cf. Theorem 3.8), proved in 1988 by Comfort and Robertson [11].

In particular, Theorem 3.12 implies that if the problem of extremality for a certain class of pseudocompact abelian groups admits a solution in the case of weak extremality, then this problem is solved also for s-extremality, r-extremality and double extremality, which are the main forms of extremality. The notion of weakly extremal group introduces the following relevant question:

(F) Is every weakly extremal pseudocompact group metrizable?

Let us note that a positive answer to this question will lead to a simultaneous positive answer to all questions (A)-(E), since r-extremal or s-extremal implies weakly extremal by Theorem A. On the other hand, one may anticipate that in all cases where a positive answer to one of questions (A)-(E) is given for both s- and r-extremality, it is possible to extend these results to the case of weakly extremal groups. Moreover, the notion of weak extremality introduces a new approach to the study of extremal groups and in most cases allows us to extend to r-extremal groups results proved only for s-extremal ones (cf. Theorem 4.15).

1.3. **Singularity and almost connectedness of pseudocompact groups and their connection to extremality.** The original aim of this paper (having large intersection with [24]), was to be a comprehensive survey on extremal pseudocompact groups. Gradually some original ideas and results appeared. In June 2004, shortly before the submission of this paper, we received a copy of [3], kindly sent by the authors before the publication. Although most of the results were announced earlier [22, 23], the proofs became accessible to us only at that point.

Let us describe in detail the content of the present paper.

Following [10], we denote by $\Lambda(G)$ the family of all closed (normal) G_δ-subgroups of a pseudocompact topological group G and by $m(G)$ the minimum cardinality of a dense pseudocompact subgroup of G. Then every G_δ-set containing 0 contains also a subgroup $N \in \Lambda(G)$.

In §2 we recall some important facts concerning pseudocompact groups G and the family $\Lambda(G)$, that will be essential in the sequel. Most of the proofs are omitted but hints or due references are given in all cases.

In §3 we consider some stability properties of the various classes of extremal groups, with particular attention to the behavior of extremal groups under taking closed (pseudocompact) subgroups and quotients with respect to subgroups $N \in \Lambda(G)$. §3.2 is opened by the dense graph theorem (Theorem 3.8), which is at the basis of the proofs of other results of §§3.2 and 6.1, concerning in particular the relations among various kinds of extremality.

Comfort, Gladdines and van Mill showed in [4, Corollary 4.6] that s-extremal groups have free rank at most $\mathfrak{c}$. The same conclusion was proved for r-extremal groups by Comfort and Galindo in [3, Theorem 5.10 (b)] (announced in [23, Theorem 7.3]). These results are simultaneously extended to c-extremal groups in Theorem 3.6 (although c-extremal pseudocompact abelian groups G with $r_0(G) = \mathfrak{c}$ need not be metrizable, Example 4.4). On the other hand, d-extremal pseudocompact abelian groups need not have bounded free rank (take any large pseudocompact divisible abelian group).

In §4 we introduce the concept of *singular* group generalizing simultaneously metrizability and torsion (cf. Theorem 2.14). If m is a positive integer and G is a group, let $G[m] = \{x \in G : mx = 0\}$.

Definition 1.2. *A topological group G is* singular *if there exists $m \in \mathbb{N}_+$ such that $G[m] \in \Lambda(G)$.*

The concept of singular abelian group implicitly appeared already in [20] in the case of compact groups.

We prove that for singular pseudocompact abelian groups d-extremality is equivalent to metrizability (cf. Theorem 4.6). Note that d-extremality and c-extremality behave in a completely different way w.r.t. singularity: d-extremality complements singularity to metrizability, whereas c-extremality is a *weaker version of singularity* (Proposition 4.7). On the other hand, Theorem 4.13 shows that a c-extremal pseudocompact abelian group G is singular whenever $w(G) \leq \mathfrak{c}$ or G is compact. In particular, Theorems 4.6 and 4.13 imply Theorem 4.15, imposing the restraint $w(G) > \mathfrak{c}$ for non-metrizable weakly extremal pseudocompact abelian groups G. So this theorem simultaneously strengthens the known results of Comfort, Gladdines and van Mill [4, Theorem 4.11] and Comfort and Galindo [3, Theorem 5.10] where the same conclusion is obtained for s-extremal (resp., r-extremal) pseudocompact abelian groups. It is not possible to replace in Theorem 4.15 weak extremality neither by c-extremality nor by d-extremality (just take a non-metrizable pseudocompact abelian group of weight $\leq \mathfrak{c}$ that is either torsion or divisible).

In §5 we define almost connected groups as follows (here $c(G)$ denotes the connected component of G):

Definition 1.3. *A topological group G is* almost connected *if $c(G) \in \Lambda(G)$.*

If G is an almost connected pseudocompact group, then the quotient $G/c(G)$ is compact. Let us recall here, that some authors [5, p.82] call a locally compact group G *almost connected* if only compactness is required for the quotient $G/c(G)$ (whereas, in our setting this quotient is compact metrizable). Throughout this paper almost connected will be understood always in the sense of Definition 1.3. Since the locally compact pseudocompact groups are compact, there is no interference at all except in the case of compact groups.

It follows from the definition that both connected groups and metrizable groups are almost connected. Moreover, totally disconnected pseudocompact abelian groups are almost connected if and only if they are metrizable (cf. Proposition 5.6). As a matter of fact, almost connectedness and singularity form a well balanced pair of generalizations of metrizability for pseudocompact abelian groups: G is singular if mG is metrizable for some $m > 0$ (Lemma 4.1), whereas G is almost connected if $G/\overline{mG}$ is metrizable for every $m > 0$ (Theorem 5.8). Hence G is metrizable whenever it is simultaneously singular and almost connected.

The almost connectedness is stable under taking G_δ-dense subgroups and closed G_δ-subgroups (cf. Theorem 5.8). Moreover, if a topological group G admits a dense almost connected pseudocompact subgroup, then G itself is pseudocompact and almost connected (cf. Lemma 5.2).

For non-metrizable almost connected pseudocompact abelian groups some cardinal invariants are preserved by subgroups that are in $\Lambda(G)$. In particular, the connected component $c(G)$ of a non-metrizable almost connected pseudocompact abelian group satisfies $m(G) = m(c(G)) \leq r_0(G)$ and $w(G) \leq 2^{r_0(G)}$ (cf. Theorem 5.13).

The properties of almost connected groups are used in §5 to generalize some results concerning s-extremality of connected pseudocompact abelian groups proved in [6]. The main tool is given by Theorem 5.14 which shows that a d-extremal pseudocompact abelian group is almost connected. Note that c-extremal pseudocompact abelian groups need not be almost connected (take any torsion, non-metrizable pseudocompact abelian group).

The proof of the following result, due to Comfort and Robertson, is not covered by the proofs given here. For the rest this paper is self-contained.

Theorem 1.1. ([11, Corollary 7.5]) *Let G be a torsion pseudocompact abelian group that is s- or r-extremal. Then G is metrizable.*

This theorem can easily be generalized to d-extremal groups:

Theorem B. (Corollary 5.17) *Let G be a pseudocompact abelian group that is either totally disconnected or singular. Then G is d-extremal if and only if it is metrizable.*

Since torsion pseudocompact abelian groups are singular, this is a generalization of Theorem 1.1. Moreover, since a zero-dimensional topological group is totally disconnected, this corollary generalizes also [11, Theorem 7.3], limited to the case of zero-dimensional groups.

Comfort and van Mill [6, Theorem 7.1] proved that every connected s-extremal group is divisible. An analogous result for r-extremal groups was announced in [1, Corollary 5.11] and proved in [3, Corollary 5.11]. These results are simultaneously extended to d-extremal groups in Corollary 5.22 where it is proved that a d-extremal pseudocompact abelian group is divisible if and only if it is connected. This follows by the more general fact (relying on properties of almost connectedness) that the connected component of an almost connected pseudocompact abelian group is divisible if and only if it is d-extremal (Theorem 5.20).

The following theorem, due to Comfort and Galindo (announced in [23, Theorem 8.2], [1, Theorem 6.1]), establishes some relevant necessary conditions satisfied by a non-metrizable r-extremal (resp. s-extremal) group. It introduces a certain asymmetry between s-extremality and r-extremality for pseudocompact abelian groups.

Theorem 1.2. ([3, Theorem 7.1]) *Let G be a pseudocompact abelian group. Then:*

(a) *if G is s-extremal and non-metrizable, then there exists $p \in \mathbb{P}$ such that $G[p]$ is not G_δ-dense in $\widetilde{G}[p]$;*
(b) *if G is r-extremal, then $G[p]$ is dense in $\widetilde{G}[p]$ for every $p \in \mathbb{P}$.*

As an immediate corollary one can see that for a non-metrizable doubly extremal pseudocompact abelian group G, there exists $p \in \mathbb{P}$ such that $G[p]$ is dense but not G_δ-dense in $\widetilde{G}[p]$, i.e., $G[p]$ is not pseudocompact ([3, Theorem 7.2(ii)]). According to [30]: an abelian group G is *almost torsion-free* if $G[p]$ is finite for every prime p. This generalization of the notion "torsion-free group" immediately gives: *if G is a doubly extremal pseudocompact almost torsion-free abelian group, then G is metrizable.* For torsion-free groups this result was obtained in [3].

The problem in attacking question (D) is the current lack of *sufficient* conditions for r-extremality. Theorem 3.8 and its consequences allow us to obtain a sufficient condition for a s-extremal group G to be doubly extremal (hence, can be considered as a partial solution to (D)).

Theorem C. (Theorem 6.10) *Let G be a s-extremal pseudocompact abelian group. Then G is doubly extremal if and only if $\overline{G[p]} = \widetilde{G}[p]$ for every $p \in \mathbb{P}$.*

This result strengthens Theorem 4.4 (b) of [3] where it is shown that every *divisible* s-extremal pseudocompact abelian group G such that $\overline{G[m]} = \widetilde{G}[m]$ for every $m \in \mathbb{N}$ is r-extremal.

In §6.2 we give a criterion for r-extremality of connected pseudocompact groups making no recourse to "external" issues such as characters or even different topologies on the group: a connected pseudocompact abelian group G is r-extremal if and only if G is weakly extremal and every $N \in \Lambda(G)$ with $G/N \cong \mathbb{T}$ is d-extremal (Theorem 6.11).

According to [4, Theorem 4.8], every s-extremal pseudocompact abelian group has cardinality at most $\mathfrak{c}$. In §7 we generalize this result as follows: an infinite weakly extremal pseudocompact abelian group G has $|G| = \mathfrak{c}$ if and only if $m(G) = |G|$ (cf. Theorem 7.1 and Corollary 7.2).

It turns out that c-extremality is stable with respect to taking G_δ-subgroups:

Theorem D. (Theorem 4.11) *Every closed G_δ-subgroup of a c-extremal pseudocompact abelian group G is c-extremal.*

Nevertheless, such a stability need not be available for *all* extremality properties. Indeed, it was proved by Comfort and Galindo [3, Theorem 6.1] (announced in [23, Lemma 8.1] and [1, Theorem 5.16 (b)]) that a pseudocompact abelian group G is metrizable whenever every $N \in \Lambda(G)$ is s-extremal. This motivates the necessity to define stronger versions of extremality as follows.

Definition 1.4. *A pseudocompact group G is said to be:*

(a) *strongly extremal* if every $N \in \Lambda(G)$ is r-extremal*;*
(b) *strongly d-extremal* if every $N \in \Lambda(G)$ is d-extremal.

The property in item (a) (without a specific term) was introduced and used in [3]. We will prove in Theorem 6.7 that s-extremal pseudocompact abelian groups that are also strongly d-extremal must be metrizable. Moreover, in Theorem 6.8 we will see that if every $N \in \Lambda(G)$ is weakly extremal, then G is strongly extremal (i.e., every c-extremal and strongly d-extremal pseudocompact abelian group is strongly extremal).

The next diagram shows the relations among all forms of weak metrizability that appear in the paper:

$$\begin{array}{ccccccccc}
 & & & & \text{metrizable} & & & & \\
 & & & & \downarrow & & & & \\
 & & \text{strongly extremal} & \longleftarrow & \text{doubly extremal} & & & & \\
 & \swarrow & & \searrow \quad \swarrow & & \searrow & & & \\
\text{str. } d\text{-extr.} & & & r\text{-extremal} & & & s\text{-extremal} & & \\
 & & & & \searrow & \swarrow & & & \\
 & \searrow & & & \text{weakly extremal} & & & \text{singular} & \\
 & & & \swarrow & & \searrow & & \swarrow & \\
 & \text{almost conn.} & \longleftarrow & d\text{-extremal} & & & c\text{-extremal} & &
\end{array}$$

In the compact case only three distinct classes remain: metrizable groups (as weakly extremal compact groups are metrizable by Corollary 4.14), singular groups (as c-extremal compact groups are singular by Theorem 4.13) and almost connected groups (as almost connected compact groups are strongly d-extremal according to Theorem 5.15). The intersection of that last two is the class of metrizable groups. Since every non-metrizable compact connected abelian group is strongly d-extremal, but not singular (hence not c-extremal), strongly d-extremal does not imply c-extremal.

1.4. **Notation and terminology.** The symbols $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ are used for the set of integers, the set of primes, the set of natural numbers and the set of positive integers, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology.

Let G be an abelian group. The subgroup of torsion (p-torsion) elements of G is denoted by $t(G)$ (resp., $t_p(G)$). The group G is said to be *bounded torsion* if there exists $n \in \mathbb{N}_+$ such that $nG = 0$. If m is a positive integer, $\mathbb{Z}(m)$ is the cyclic group of order m. We denote by $r_0(G)$ the free rank of G (if G is free abelian this is simply its rank, otherwise this is the maximum rank of a free subgroup of G). For $n \in \mathbb{N}$ let $\varphi_n : G \to G$ be defined by $\varphi_n(x) = nx$ for every $x \in G$. Then $\ker \varphi_n = G[n]$. If H is a group and $h : G \to H$ is a homomorphism, then we denote by $\Gamma_h := \{(x, h(x)),\ x \in G\}$ the graph of h.

For a topological space X, we denote by $w(X)$ the *weight* of X (i.e., the minimum cardinality of a base for the topology on X). A space X is said to be *zero-dimensional* if X has a base consisting of clopen sets. A space X is *pseudocompact* if every continuous real-valued function on X is bounded, *ω-bounded* if the closure of every countable subset of X is compact.

Throughout this paper all topological groups are Hausdorff and completeness is intended with respect to the two-sided uniformity, so that every topological group has a completion which we denote by $\widetilde{G}$. A group G is *precompact* if $\widetilde{G}$ is compact. With $c(G)$ we indicate the connected component of 0 in G; a group G is *totally disconnected* if $c(G)$ is trivial.

For a topological group G, we will denote by $\mathcal{V}_G(0)$ the filter of 0-neighborhoods in G, by $\chi(G)$ the *character* of G (that is, the minimal cardinality of a basis of $\mathcal{V}_G(0)$) and by $\psi(G)$ the *pseudocharacter* of G (i.e., the minimum size of a family $\mathcal{B}$ of 0-neighborhoods of G such that $\bigcap_{U\in\mathcal{B}} U = \{0\}$).

If M is a subset of a topological group G, then $\langle M\rangle$ is the smallest subgroup of G containing M and $\overline{M}$ is the closure of M. For any abelian group G let $Hom(G,\mathbb{T})$ be the group of all homomorphisms from G to the circle group $\mathbb{T}$. When (G,τ) is an abelian topological group, the set of τ-continuous homomorphisms $\chi : G \to \mathbb{T}$ (*characters*) is a subgroup of $Hom(G,\mathbb{T})$ and is denoted by $\widehat{G}$. For a subset H of G the *annihilator* of H in $\widehat{G}$ is the subgroup $A(H) = \{\chi \in \widehat{G} : \chi(H) = \{0\}\}$ of $\widehat{G}$.

For undefined terms see [19, 26].

2. Background on pseudocompact groups

The following theorem guarantees precompactness of the pseudocompact groups and characterizes the pseudocompact groups among the precompact ones.

Theorem 2.1. ([13, Theorems 1.2 and 4.1])

(a) *Every pseudocompact group is precompact.*

(b) *Let G be a precompact group. Then the following conditions are equivalent:*

(b_1) *G is pseudocompact;*

(b_2) *G is G_δ-dense in $\widetilde{G}$;*

(b_3) *$\widetilde{G}$ is the Stone-Cech compactification of G.*

For the sake of easier reference we isolate the following lemma from Theorem 2.1.

Lemma 2.2. *Let G be a topological group and let H be a dense, pseudocompact subgroup of G. Then G is pseudocompact and H is G_δ-dense in G.*

Lemma 2.2 immediately yields that a dense subgroup H of a pseudocompact group G is pseudocompact if and only if H is G_δ-dense in G.

Corollary 2.3. *Let G be a pseudocompact topological group.*

(a) *If G is metrizable, then G is compact;*

(b) *G is connected if and only if $\widetilde{G}$ is connected;*

(c) *G is zero-dimensional if and only if $\widetilde{G}$ is zero-dimensional.*

Lemma 2.4. ([11, Theorem 3.2]) *Let G be a pseudocompact group such that $\{0\}$ is a G_δ-set. Then G is metrizable and compact.*

A useful pseudocompact criterion via quotients follows.

Theorem 2.5. ([11, Lemma 6.1]) *Let G be a precompact group. Then G is pseudocompact if and only if G/H is compact metric for every $H \in \Lambda(G)$.*

Corollary 2.6. *Let G be a pseudocompact abelian group. Then:*

(a) *N is pseudocompact for every $N \in \Lambda(G)$;*
(b) *if $w(G) > \omega$, then $w(G) = w(N)$ for every $N \in \Lambda(G)$;*
(c) *if $N \in \Lambda(G)$ and H is a closed subgroup of G such that $N \subseteq H$, then $H \in \Lambda(G)$;*
(d) *if $N \in \Lambda(G)$ and $L \in \Lambda(N)$, then $L \in \Lambda(G)$.*

Now we see that G_δ-density in G is completely controlled by the subgroups $N \in \Lambda(G)$.

Lemma 2.7. *Let G be a pseudocompact abelian group and let D be a subgroup of G. Then:*

(a) *D is G_δ-dense in G if and only if $N + D = G$ for every $N \in \Lambda(G)$;*
(b) *if D is G_δ-dense in G and $N \in \Lambda(G)$, then $G/D \cong N/(D \cap N)$;*
(c) *if D is G_δ-dense in G and $N \in \Lambda(G)$, then $D \cap N$ is G_δ-dense in N.*

Proof. (a) follows from the fact that D is G_δ-dense in G if and only if $(x + N) \cap D \neq \varnothing$ for every $x \in G$ and every $N \in \Lambda(G)$.

(b) By (a) $G = N + D$, hence $G/D \cong N/(D \cap N)$.

(c) follows from (a). □

Lemma 2.8. *Let G be a pseudocompact abelian group. Then:*

(a) *if $N \in \Lambda(G)$, then $\overline{N} \in \Lambda(\widetilde{G})$;*
(b) *if G is dense in G_1 and $N \in \Lambda(G)$, then $\overline{N}^{G_1} \in \Lambda(G_1)$.*

Proof. (b) follows from (a) and $\widetilde{G} = \widetilde{G}_1$. □

Corollary 2.9. *Let G be a pseudocompact abelian group. If G is dense in G_1 and M is a closed subgroup of G_1 such that $M \cap G \in \Lambda(G)$, then $M \in \Lambda(G_1)$. In particular, $M \in \Lambda(\widetilde{G})$ for every closed subgroup of $\widetilde{G}$ such that $M \cap G \in \Lambda(G)$.*

Lemma 2.10. *Let G be a precompact abelian group, $N \in \Lambda(G)$ and $n \in \mathbb{N}_+$. Then $\overline{nN}^{\widetilde{G}} = n\overline{N}^{\widetilde{G}}$ and $N \cap n\overline{N}^{\widetilde{G}} = \overline{nN}^{G}$.*

Lemma 2.11. *Let G be a topological group. Let τ and τ' be pseudocompact group topologies on G such that $\tau' \geq \tau$. Then the following conditions are equivalent:*

(a) *$\tau = \tau'$;*
(b) *for every $N \in \Lambda(G, \tau)$ one has $\tau|_N = \tau'|_N$;*
(c) *there exists $N \in \Lambda(G, \tau)$ such that $\tau|_N = \tau'|_N$.*

Proof. (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(a) Let τ_q and τ_q' be the quotient topologies induced on G/N by τ and τ' respectively. Since $\tau \leq \tau'$, $\Lambda(G, \tau) \subseteq \Lambda(G, \tau')$ and so in particular $N \in \Lambda(G, \tau')$. This implies that τ_q and τ_q' are compact group topologies on G/N. Moreover, $\tau_q \leq \tau_q'$ and consequently $\tau_q = \tau_q'$. Since $\tau|_N = \tau'|_N$, Merzon's lemma [29] implies that $\tau = \tau'$ on G. □

Remark 2.12. Let (G,τ) and H be topological groups and $h:(G,\tau)\to H$ a homomorphism. Consider the map $j:G\to\Gamma_h$ such that $j(x)=(x,h(x))$ for every $x\in G$. Observe that j is a homomorphism such that $j(G)=\Gamma_h$, that is j is surjective; moreover j is injective and so j is an isomorphism. If $p_1:G\times H\to H$ is the projection on the first component, p_1 is continuous and also its restriction to Γ_h is continuous; since j is the inverse of $p_1|_{\Gamma_h}$, it follows that j is open. Endow Γ_h with the group topology induced by the product $(G,\tau)\times H$. We define the topology τ_h as *the weakest group topology on G such that $\tau_h\geq\tau$ and for which j is continuous.* Then $j:(G,\tau_h)\to\Gamma_h$ is a homeomorphism.

Note that the topology τ_h so defined is the weakest topology on G such that $\tau_h\geq\tau$ and for which h is continuous. Indeed, if $p_2:G\times H\to H$ is the canonical projection on the second component, then its restriction to Γ_h is continuous. The homomorphism h is the composition of j and p_2, in the sense that $p_2|_{\Gamma_h}\circ j=h:(G,\tau_h)\to H$ and therefore, being j τ_h-continuous, h has to be τ_h-continuous too. Clearly, if h is τ-continuous then $\tau_h=\tau$.

Lemma 2.13. *Let (G,τ) be a topological group and let $h:G\to\mathbb{T}$ be a homomorphism such that (G,τ_h) is pseudocompact. Then the following conditions are equivalent:*

(a) *h is continuous;*
(b) *every $N\in\Lambda(G,\tau)$ is such that $h|_N:N\to\mathbb{T}$ is continuous;*
(c) *there exists $N\in\Lambda(G,\tau)$ such that $h|_N:N\to\mathbb{T}$ is continuous.*

Proof. (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(a) Since $h|_N:N\to\mathbb{T}$ is continuous and $\tau\leq\tau_h$, it follows that $\tau_h|_N=\tau|_N$. As τ and τ_h are pseudocompact by hypothesis, the assertion follows from Lemma 2.11. □

The next property of pseudocompact groups was announced in [6, Remark 2.17] and proved in [21, Lemma 2.3].

Theorem 2.14. *Let G be an infinite pseudocompact abelian group. Then either $r_0(G)\geq\mathfrak{c}$ or G is bounded torsion with $|G|\geq\mathfrak{c}$.*

Proof. If G is torsion, then G is bounded by [11, Lemma 7.4]. The inequality $|G|\geq\mathfrak{c}$ follows from van Douwen's Theorem [31]. Assume G is not torsion. If G is compact, then the assertion follows from [20, Lemma 2.3]. Otherwise, pick a non-torsion element $x\in G$ and consider the cyclic subgroup $C=\langle x\rangle$. Since C is countable, there exists a G_δ-set O around 0 that meets C in $\{0\}$. Find $N\in\Lambda(G)$ contained in O, hence $N\cap C=\{0\}$. Then the quotient group G/N is compact and non-torsion. Hence the compact case applies. □

The following results will be helpful in the sequel to produce G_δ-dense subgroups of given pseudocompact groups. In particular, item (a) of the following lemma has been proved by Comfort and van Mill in [6, Lemma 2.13] and it has been announced with its corollary by Comfort, Gladdines and van Mill in [4, Lemma 4.1].

Lemma 2.15. *Let G be a pseudocompact group.*

(*a*) [6, Lemma 2.13] *If $G = \bigcup_{n=0}^{\infty} A_n$, where A_n is a subgroup of G for every $n \in \mathbb{N}$, then there exist $n \in \mathbb{N}$ and $N \in \Lambda(G)$ such that $A_n \cap N$ is G_δ-dense in N.*

(*b*) [4, Lemma 4.1 (b)] *If $N \in \Lambda(G)$ and D is G_δ-dense in N, then there exists a subgroup E of G with $|E| \leq \mathfrak{c}$ such $D+E$ is a G_δ-dense subgroup of G.*

(*c*) *If G is infinite, then $m(N) = m(G)$ for every $N \in \Lambda(G)$.*

It directly follows from Lemma 2.15 that if G is a pseudocompact abelian group such that $G = \bigcup_{n=0}^{\infty} A_n$, where A_n are subgroups of G, then there exist $n \in \mathbb{N}$, $N \in \Lambda(G)$ and $E \leq G$ with $|E| \leq \mathfrak{c}$ such that $(A_n \cap N) + E$ is G_δ-dense in G. In particular:

Corollary 2.16. *Let G be a pseudocompact abelian group such that $G = \bigcup_{n=0}^{\infty} A_n$, where $A_n \leq G$ for every $n \in \mathbb{N}$. Then there exists a subgroup E of G, with $|E| \leq \mathfrak{c}$, such that $A_n + E$ is G_δ-dense in G.*

3. Extremal pseudocompact abelian groups

3.1. **General properties of extremal groups.**

Lemma 3.1. *Let G be a s (resp. d,c)-extremal pseudocompact abelian group and let N be a pseudocompact subgroup of G.*

(a) *If N is closed, then G/N is s (resp. d,c)-extremal.*

(b) *If N is an algebraic direct summand of G, then N is s (resp. d,c)-extremal. In particular, every divisible pseudocompact subgroup of G is s (resp. d,c)-extremal.*

Proof. (a) Let $\psi : G \to G/N$ be the canonical homomorphism and let H be a G_δ-dense subgroup of G/N. Then $\psi^{-1}(H)$ is a G_δ-dense subgroup of G (cf. [3, Theorem 5.3 (a)]). Hence $G/\psi^{-1}(H) = \{0\}$ (resp. $G/\psi^{-1}(H)$ is divisible, $r_0(G/\psi^{-1}(H)) < \mathfrak{c}$). Since $(G/N)/H \cong G/\psi^{-1}(H)$ we conclude that $(G/N)/H = \{0\}$ (resp. $(G/N)/H$ is divisible, $r_0((G/N)/H) < \mathfrak{c}$).

(b) There exists a subgroup L of G such that $G = N \oplus L$. Let D be a G_δ-dense subgroup of N. Then $D_1 = D \oplus L$ is a G_δ-dense subgroup of G. To see this, let $M \in \Lambda(G)$. As D is G_δ-dense in N, by Lemma 2.7 (a) $M + D \geq N$ and consequently $M + D_1 \geq N \oplus L = G$. Since G is s (resp. d,c)-extremal, one has necessarily $G/D_1 = \{0\}$ (resp. G/D_1 is divisible, $r_0(G/D_1) < \mathfrak{c}$). Since $N/D \cong G/D_1$, N is s (resp. d,c)-extremal. □

Let us recall that stability of s- and r-extremality under taking quotients with respect to closed pseudocompact groups was proved by Comfort and Galindo ([23, Lemma 4.5], [3, Theorem 5.3]):

Lemma 3.2 ([23, 3])**.** *Let G be a pseudocompact abelian group and let N be a closed pseudocompact subgroup of G.*

(a) *If G is r-extremal (resp., s-extremal), then G/N is r-extremal (resp., s-extremal).*

(b) *If G is doubly extremal, then G/N is doubly extremal.*

The G_δ-subgroups are particularly important since the following stability properties hold.

Lemma 3.3. *Let G be a pseudocompact abelian group and let $N \in \Lambda(G)$.*

(a) *If N is s (resp. d,c)-extremal, then G is s (resp. d,c)-extremal.*
(b) [3, Theorem 2.1 (b)] *If N is r-extremal, then G is r-extremal.*
(c) *If N is doubly extremal, then G is doubly extremal.*

Proof. (a) Let H be a G_δ-dense subgroup of G. Then $H \cap N$ is G_δ-dense in N and so $N/H \cap N = \{0\}$ (resp. $N/H \cap N$ is divisible, $r_0(N/H) < \mathfrak{c}$). Since $G = N + H$, the conclusion follows from $G/H = (N+H)/H \cong N/H \cap N$.

(b) Let τ' be a pseudocompact group topology on G finer than τ. Since $N \in \Lambda(G,\tau)$, then $N \in \Lambda(G,\tau')$ and consequently $(N, \tau'|_N)$ is pseudocompact. Then $\tau|_N = \tau'|_N$, since N is r-extremal by hypothesis. By Lemma 2.11 it follows that $\tau = \tau'$, hence G is r-extremal.

(c) It follows directly from (a) and (b). □

Corollary 3.4. *If G is a r (resp. s,d,c)-extremal pseudocompact abelian group and K is a compact metrizable group, then also $G \times K$ is a r (resp. s,d,c)-extremal pseudocompact group.*

Remark 3.5. Note that strong extremality and strong d-extremality are preserved by taking closed, G_δ-subgroups by Corollary 2.6(d).

In the next theorem we extend to c-extremal groups [4, Corollary 4.6] and [23, Theorem 7.3] (proved in [23] and [3, Theorem 5.10 (b)]), where the same result was announced respectively for s- and r-extremal groups. Our proof uses ideas from the proof of [4, Proposition 4.4] and is very similar to that of [3, Theorem 5.10 (b)]. We include it here for the sake of completeness.

Theorem 3.6. *Let G be a pseudocompact abelian group. If G is c-extremal, then $r_0(G) \leq \mathfrak{c}$.*

Proof. Let $\kappa = r_0(G)$. Let M be a maximal independent subset of G consisting of non-torsion elements. Then $|M| = \kappa$ and there exists a partition $M = \bigcup_{n=1}^\infty M_n$ such that $|M_n| = \kappa$ for each n. Let $U_n = \langle M_n \rangle$, $V_n = U_1 \oplus \cdots \oplus U_n$ and $A_n = \{x \in G : n!x \in V_n\}$ for every $n \in \mathbb{N}_+$. Then $G = \bigcup_{n=1}^\infty A_n$. By Corollary 2.16 there exist $n \in \mathbb{N}_+$ and a subgroup E of G such that $\widetilde{A} = A_n + E$ is G_δ-dense in G and $|E| \leq \mathfrak{c}$. Hence $|E/(A_n \cap E)| \leq \mathfrak{c}$. So the isomorphism

$$\widetilde{A}/A_n = (A_n + E)/A_n \cong E/(A_n \cap E)$$

yields $|\widetilde{A}/A_n| \leq \mathfrak{c}$. Since G is c-extremal, $r_0(G/\widetilde{A}) < \mathfrak{c}$, so $r_0(G/A_n) \leq \mathfrak{c}$ by the isomorphism $(G/A_n)/(\widetilde{A}/A_n) \cong G/\widetilde{A}$. On the other hand, $r_0(G/A_n) = \kappa$, as U_{n+1} embeds into G/A_n (note that $A_n \cap U_{n+1} = \{0\}$ since every $x \in A_n \cap U_{n+1}$ satisfies $n!x \in V_n \cap U_{n+1} = \{0\}$, so $x = 0$ as U_{n+1} is a free group), hence $\kappa \leq \mathfrak{c}$. □

3.2. **The dense graph theorem and weakly extremal groups.** The dense graph theorem (see Theorem 3.8 below) was announced in [11]. It gives a sufficient condition for a group to be neither s-extremal nor r-extremal. A lot of useful results in the study of extremality follow from this theorem.

The following lemma has been announced in a similar form in [23] without a proof.

Lemma 3.7. *Let G be a topological abelian group and let H be a compact metrizable abelian group with $|H| > 1$. Let $h : G \to H$ be a surjective homomorphism. Then Γ_h is G_δ-dense in $G \times H$ if and only if* $\ker h$ *is proper and G_δ-dense in G.*

By means of Lemma 3.7 one can prove that for a pseudocompact abelian group G the following statements are equivalent:

(a) there exists a pseudocompact abelian group H with $|H| > 1$ and a homomorphism $h : G \to H$ with Γ_h G_δ-dense in $G \times H$;
(b) there exists a compact metrizable abelian group H with $|H| > 1$ and a surjective homomorphism $h : G \to H$ with Γ_h G_δ-dense in $G \times H$;
(c) there exists a compact metrizable abelian group H with $|H| > 1$ and a surjective homomorphism $h : G \to H$ with $\ker h$ G_δ-dense in G.

Theorem 3.8 (of the dense graph [11, Theorem 4.1])**.** *Let (G, τ) be a pseudocompact abelian group. Suppose that there exist a pseudocompact abelian group H with $|H| > 1$ and $h \in Hom(G, H)$ such that Γ_h is a G_δ-dense subgroup of $G \times H$. Then*

(a) *there exists a pseudocompact group topology τ' on G such that $\tau' > \tau$ and $w(G, \tau') = w(G, \tau)$;*
(b) *there exists a proper dense pseudocompact subgroup D of G such that $w(D) = w(G, \tau)$.*

Corollary 3.9. *Let G be a pseudocompact abelian group. Assume that there exists a surjective homomorphism $h : G \to H$, where H is a non trivial closed subgroup of $\mathbb{T}$. If Γ_h is G_δ-dense in $G \times H$, then G is neither s-extremal nor r-extremal.*

The next corollary directly follows from Lemma 3.7 and Corollary 3.9.

Corollary 3.10. *Let G be a pseudocompact abelian group. Suppose that there exists a surjective homomorphism $h : G \to H$, where H is a closed non-trivial subgroup of $\mathbb{T}$. If* $\ker h$ *is a proper G_δ-dense subgroup of G, then G is neither s-extremal nor r-extremal.*

Remark 3.11. Note that the hypotheses of Theorem 3.8 and Corollary 3.10 imply that the group G cannot be metrizable. Indeed, if G were metrizable, then $\ker h = G$, that is $h \equiv 0$ and so the graph Γ_h of h would not be G_δ-dense in $G \times H$.

Thanks to the previous results it is possible to prove Theorem A of the introduction showing that both s- and r-extremality imply weak extremality (it can be deduced from [3, Theorem 4.4]):

Theorem 3.12. *If a pseudocompact abelian group G is either s-extremal or r-extremal, then G is weakly extremal.*

Proof. Suppose for a contradiction that G is not weakly extremal. Then there exists a G_δ-dense subgroup H of G such that either G/H is not divisible or $r_0(G/H) \geq \mathfrak{c}$. In both cases, H is a proper subgroup of G. Moreover, H is dense and pseudocompact, hence G is not s-extremal. To find a contradiction it remains to prove that G is not even r-extremal.

Let $\psi : G \to G/H$ be the canonical projection.

Case 1. If G/H is not divisible, then there exists a prime p and a non-trivial homomorphism $f : G/H \to \mathbb{Z}(p)$. Composing with the canonical projection $\varphi : G \to G/H$ we get a surjective homomorphism $h = \varphi \circ f : G \to \mathbb{Z}(p)$.

Since $\ker h \supseteq H$, it follows that $\ker h$ is a proper G_δ-dense subgroup of G, then Corollary 3.10 applies to conclude that G is not r-extremal.

Case 2. If $r_0(G/H) \geq \mathfrak{c}$, then there exists a surjective homomorphism $\eta : G/H \to \mathbb{T}$ (as $r_0(G/H) \geq \mathfrak{c} = r_0(\mathbb{T})$). Define the surjective homomorphism $h = \eta \circ \psi : G \to \mathbb{T}$ and observe that $\ker h \supseteq H$. Therefore $\ker h$ is a proper G_δ-dense subgroup of G. As before it follows from Corollary 3.10 that G is not r-extremal. □

4. Singular groups

4.1. **d-extremality.** The next lemma offers an alternative form for singularity of pseudocompact abelian groups (mG is compact metrizable for some $m \in \mathbb{N}_+$). It is useful when checking stability of this property under taking subgroups and quotients.

Lemma 4.1. *Let G be a topological abelian group and $m \in \mathbb{N}_+$.*

(a) *If mG is metrizable, then $G[m] \in \Lambda(G)$.*
(b) *If G is pseudocompact, then $G[m] \in \Lambda(G)$ implies that mG is metrizable (hence compact).*

Proof. Let $\varphi_m : G \to G$ be the continuous homomorphism defined by $\varphi_m(x) = mx$ for every $x \in G$. Then $\ker \varphi_m = G[m]$ and $\varphi_m(G) = mG$. Let $i : G/G[m] \to mG$ be the continuous isomorphism such that $i \circ \pi = \varphi_m$, where $\pi : G \to G/G[m]$ is the canonical homomorphism.

(a) If mG is metrizable, then $\psi(mG) = \omega$ and so $\psi(G/G[m]) = \omega$. This implies that $G[m]$ is a G_δ-set of G; then $G[m] \in \Lambda(G)$.

(b) Suppose that $G[m] \in \Lambda(G)$. Then the quotient $G/G[m]$ is metrizable, hence compact by Theorem 2.5. By the open mapping theorem the isomorphism i is also open and consequently it is a topological isomorphism. Then the group mG is metrizable and so compact. □

Remark 4.2. It immediately follows from Lemma 4.1 that singularity is stable under taking pseudocompact subgroups and quotients w.r.t. closed subgroups.

The next lemma characterizes the singular groups in terms of the free rank of their closed G_δ-subgroups.

Lemma 4.3. *A pseudocompact abelian group G is not singular if and only if $r_0(N) \geq \mathfrak{c}$ for every $N \in \Lambda(G)$. In such a case, $w(G) > \omega$ and $r_0(N) = r_0(G) \geq \mathfrak{c}$ for every $N \in \Lambda(G)$.*

Proof. Assume that G is not singular and let $N \in \Lambda(G)$. Suppose for a contradiction that $r_0(N) < \mathfrak{c}$. Since N is pseudocompact, by Theorem 2.14 N is bounded torsion, i.e., $nN = \{0\}$ for some $n \in \mathbb{N}_+$. In particular, $N \subseteq G[n]$ and therefore Corollary 2.6 implies $G[n] \in \Lambda(G)$, i.e., G is singular, against the hypothesis. The converse implication is immediate.

To prove the second part of the lemma, assume that G is not singular. Then obviously $w(G) > \omega$. Let $N \in \Lambda(G)$. Then the quotient G/N is compact and metrizable, hence $|G/N| \leq \mathfrak{c}$. Since $r_0(N) \geq \mathfrak{c}$, it follows that

$$r_0(G) = r_0(G/N) \cdot r_0(N) \leq \mathfrak{c} \cdot r_0(N) = r_0(N),$$

i.e., $r_0(G) = r_0(N)$. □

We start by an example showing that singular pseudocompact abelian groups need not be d-extremal (see Theorem 4.6 for a general result).

Example 4.4. *Let p be a prime and let H be the subgroup of $\mathbb{Z}(p)^{\mathfrak{c}}$ defined by $H = \sum\{\mathbb{Z}(p)^I : I \subseteq \mathfrak{c}, |I| \leq \omega\}$ (i.e., the Σ-product of $\mathfrak{c}$-many copies of the group $\mathbb{Z}(p)$). If we denote by $\mathbb{T}$ the circle group, then the group $G = \mathbb{T} \times H$ is a singular non-metrizable pseudocompact abelian group with $r_0(G) = \mathfrak{c}$. Thus G is not d-extremal by Theorem 4.6.*

Lemma 4.5. *Let G be a pseudocompact abelian group. If for some $n \in \mathbb{N}_+$ $w(G/\overline{nG}) > \omega$, then G is not d-extremal.*

Proof. Let $H = G/\overline{nG}$. The group H is pseudocompact, non-metrizable and bounded torsion, hence H is not s-extremal by Theorem 1.1. Let H_1 be a proper G_δ-dense subgroup of H. Then H/H_1 is bounded torsion, so H/H_1 cannot be divisible. Thus H is not d-extremal. The subgroup nG of G is pseudocompact as a continuous image of G, hence also its closure $\overline{nG}$ is pseudocompact by Lemma 2.2. Now Lemma 3.1 (a) implies that also G is not d-extremal. □

It was proved by Comfort and Robertson that questions (A) and (B) have positive answer in the case of torsion pseudocompact abelian groups [11, Corollary 7.5]. This can easily be generalized to d-extremal groups replacing "torsion" by a much weaker condition:

Theorem 4.6. *A singular pseudocompact abelian group is d-extremal if and only if it is metrizable.*

Proof. If G is metrizable then it is weakly extremal, hence d-extremal.

Suppose that G is not metrizable, i.e., $w(G) > \omega$. By Lemma 4.1 there exists $m \in \mathbb{N}_+$ such that mG is compact and metrizable. Let us consider the quotient G/mG, that is pseudocompact. Since $w(mG) = \omega$ and $w(G) = w(G/mG) \cdot w(mG)$, it follows that $w(G/mG) = w(G) > \omega$. Then G is not d-extremal by Lemma 4.5. □

It follows from Theorem 4.6 that for a singular pseudocompact abelian group also weak extremality, s-extremality, r-extremality and double extremality are all equivalent to metrizability.

4.2. **c-extremality.** We see now that the impact of singularity on c-extremality, compared to that on d-extremality, is quite different. Let us start by proving the immediate implication singular $\Rightarrow$ c-extremal.

Proposition 4.7. *Every singular pseudocompact abelian group G is c-extremal. In particular, $r_0(G) \leq \mathfrak{c}$.*

Proof. Assume that G is singular. Then there exists a positive integer m such that mG is metrizable by Lemma 4.1. Let H be a G_δ-dense subgroup of G. We have to see that $r_0(G/H) < \mathfrak{c}$. Since the homomorphism $\varphi_m : G \to mG$ is continuous, the subgroup mH of mG is G_δ-dense and so $mH = mG$ as mG is metrizable. Therefore $mG \leq H$, hence the quotient G/H is bounded torsion. In particular $r_0(G/H) = 0$.

The last assertion follows from Theorem 3.6. For a direct argument note that metrizability and compactness of mG yield $r_0(mG) \leq \mathfrak{c}$ and $r_0(G/mG) = 0$. Therefore, $r_0(G) = r_0(mG) \leq \mathfrak{c}$. □

To partially invert the implication (see Theorem 4.13), we need first some preparation.

The construction used in the next lemma follows standard ideas of transfinite induction carried out in similar situations in [6] and [20].

Lemma 4.8. *Let κ be an infinite cardinal, G be an abelian group and $\{H_\alpha : \alpha < \kappa\}$ be a family of subgroups of G such that $r_0(H_\alpha) \geq \kappa$ for every $\alpha < \kappa$. Then for every subset $X = \{x_\alpha : \alpha < \kappa\}$ of G there exists a subgroup L of G such that*

(a) *$L \cap (x_\alpha + H_\alpha) \neq \varnothing$ for every ordinal $\alpha < \kappa$;*
(b) *$r_0(G/L) \geq \kappa$.*

Proof. Our aim will be to build by transfinite recursion two increasing chains $\{L_\alpha : \alpha < \kappa\}$ and $\{L'_\alpha : \alpha < \kappa\}$ of subgroups of G such that the following conditions are fulfilled for every $\alpha < \kappa$:

(a_α) $L_\alpha \cap (x_\alpha + H_\alpha) \neq \varnothing$;
(b_α) $L_\alpha \cap L'_\alpha = \{0\}$;
(c_α) $L'_\alpha / \bigcup_{\beta<\alpha} L'_\beta$ is non-torsion when $\alpha > 0$;
(d_α) $r_0(L_\alpha) \leq |\alpha|$ and $r_0(L'_\alpha) \leq |\alpha|$.

If $\alpha = 0$, take $x_0 = 0$ and $L_0 = L'_0 = \{0\}$. We need the following easy to prove claim for our construction.

Claim. *Let G be an abelian group and $x \in G$. Let H, L and N be subgroups of G such that $L \neq \{0\}$, $r_0(N) \geq \omega$ and $r_0(H) + r_0(L) < r_0(N)$. If $H \cap L = \{0\}$, then there exists a non-torsion element $y \in x + N$ such that $(H + \langle y\rangle) \cap L = \{0\}$ and $\langle y\rangle \cap H = \{0\}$.*

Now suppose that $\alpha > 0$ and L_β and L'_β are built for all $\beta < \alpha$ so that the conditions (a_β)-(d_β) are satisfied for all $\beta < \alpha$. In order to define L_α, L'_α let $M = \bigcup_{\beta<\alpha} L_\beta$ and $M' = \bigcup_{\beta<\alpha} L'_\beta$. Then $r_0(M) + r_0(M') < \kappa$, so by the claim applied with $x = 0$, $N = H_\alpha$, $H = M'$ and $L = M$, there exists a $z_\alpha \in H_\alpha$ such that $M \cap (M' + \langle z_\alpha \rangle) = \{0\}$ and $\langle z_\alpha \rangle \cap M' = \{0\}$. Put $L'_\alpha = M' + \langle z_\alpha \rangle$, hence $r_0(L'_\alpha) = r_0(M') + 1 \leq |\alpha + 1|$ and L'_α/M' is non-torsion. Now $r_0(M) + r_0(L'_\alpha) < \kappa$, so it is possible to apply again the claim with $N = H_\alpha$, $x = x_\alpha$, $H = M$ and $L = L'_\alpha$. Then there exists $y_\alpha \in x_\alpha + H_\alpha$ such that $(M + \langle y_\alpha \rangle) \cap L'_\alpha = \{0\}$. Put $L_\alpha = M + \langle y_\alpha \rangle$ and observe that (a_α)-(d_α) hold. Finally, let $L = \bigcup_{\alpha<\kappa} L_\alpha$ and $L' = \bigcup_{\alpha<\kappa} L'_\alpha$.

Since (a_α) is true for all α, this implies (a). From (b_α) and (c_α) it follows respectively that the subgroup L' satisfies $L \cap L' = \{0\}$ and $r_0(L') \geq \kappa$. Let $\varphi : G \to G/L$ be the canonical homomorphism. Then the restriction $\varphi|_{L'} : L' \to G/L$ is injective, hence $r_0(G/L) \geq r_0(\varphi(L')) = r_0(L') = \kappa$. □

A subgroup L of a group G is said to be a *complement* of H if $G = H + L$. The *co-rank* of L is $r_0(G/L)$. Lemma 4.8 will be used in two different ways. In the proof of Theorem 4.13 we use it to build a G_δ-dense subgroup of a pseudocompact group with large co-rank. In the next corollary we use it to build a complement of a given subgroup with large co-rank.

Corollary 4.9. *Let H be a subgroup of an abelian group G, such that $r_0(H) \geq [G : H]$. Then there exists a subgroup L of G such that $L + H = G$ and $r_0(G/L) \geq r_0(H)$.*

Proof. Let $\kappa = r_0(H)$. Then we can write $G = \bigcup_{\alpha<\kappa}(x_\alpha + H)$. Then by Lemma 4.8 applied to $X = \{x_\alpha : \alpha < \kappa\}$ and to the family $\{H_\alpha : H_\alpha = H,\ \alpha < \kappa\}$, there exists a subgroup L of G with (a) and (b). By (a) $L + H \supseteq x_\alpha + H$ for every α, that is $L + H \supseteq G$, while (b) ensures $r_0(G/L) \geq r_0(H)$. □

Remark 4.10. (a) Obviously, the subgroup L in the above corollary has co-rank precisely $r_0(H)$ as $G/L \cong H/H \cap L$ holds for every complement L of H.

(b) One cannot remove the condition $r_0(H) \geq [G : H]$. Indeed, for every $n \in \mathbb{N}$ the only complement L of the subgroup $H = \mathbb{Z}^n$ of $G = \mathbb{Q}^n$ is $L = G$.

The next theorem shows that c-extremality is stable under taking closed G_δ-subgroups.

Theorem 4.11. *Let G be a c-extremal pseudocompact abelian group. Then every pseudocompact subgroup of G of index $\leq \mathfrak{c}$ is c-extremal. In particular, every closed G_δ-subgroup of G is c-extremal.*

Proof. Aiming for a contradiction, assume that there exists a pseudocompact subgroup H of G with $|G/H| \leq \mathfrak{c}$ such that H is not c-extremal. Then there exists a G_δ-dense subgroup D of H with $r_0(H/D) \geq \mathfrak{c}$, so $|G/H| \leq r_0(H/D)$. Set $G_1 = G/D$ and let $\varphi : G \to G_1$ be the canonical projection. Applying to $\varphi(H) = H/D$ Corollary 4.9, we find a subgroup L of G_1 such that $L + \varphi(H) =$

G_1 and $r_0(G_1/L) \geq r_0(\varphi(H))$. Let $H_0 = \varphi^{-1}(L)$. Then $H + H_0 = G$. Since D is a subgroup of H_0 which is G_δ-dense in H, the closure of H_0 w.r.t. the G_δ-topology contains $H + H_0 = G$ and so H_0 is G_δ-dense in G. Moreover,

$$r_0(G/H_0) = r_0(G_1/L) \geq r_0(\varphi(H)) = r_0(H/D) \geq \mathfrak{c}$$

(the first equality is due to $G/\varphi^{-1}(L) \cong G_1/L$). We have produced a G_δ-dense subgroup H_0 of G with $r_0(G/H_0) \geq \mathfrak{c}$, a contradiction.

Now assume that H is a closed G_δ-subgroup of G. Then G/H is a compact metrizable group, so $|G/H| \leq \mathfrak{c}$. Hence the above argument applies to H. □

Corollary 4.12. *Let G be a c-extremal pseudocompact abelian group of size $\mathfrak{c}$. Then every pseudocompact subgroup of G is c-extremal.*

The next theorem establishes a sufficient condition for a c-extremal pseudocompact abelian group to be singular.

Theorem 4.13. *Let G be a c-extremal pseudocompact abelian group. If $w(G) \leq \mathfrak{c}$ or G is compact, then G is singular.*

Proof. By Theorem 3.6 we have $r_0(G) \leq \mathfrak{c}$. If $r_0(G) < \mathfrak{c}$, then G is bounded torsion by Theorem 2.14. Hence G is singular.

Suppose $r_0(G) = \mathfrak{c}$ and $w(G) \leq \mathfrak{c}$. Assume for a contradiction that G is not singular. By Lemma 4.3 $r_0(N) = r_0(G) = \mathfrak{c}$ for every $N \in \Lambda(G)$. Let $\{x_\alpha + H_\alpha : \alpha < \mathfrak{c}\}$ be a list of all possible cosets of subgroups $H_\alpha \in \Lambda(G)$ (this is possible as $|\Lambda(G)| \leq w(G)^\omega = \mathfrak{c}$). Since $r_0(H_\alpha) = \mathfrak{c}$ for all $\alpha < \mathfrak{c}$, by Lemma 4.8 there exists a subgroup L of G with (a) and (b). Clearly, (a) means that L is a G_δ-dense subgroup of G and (b) is $r_0(G/L) \geq \mathfrak{c}$. This proves that G is not c-extremal, against the hypothesis.

Assume that G is compact. According to [18], for every non-singular compact abelian group G there exists a surjective homomorphism $G \to \prod_{i \in I} K_i$, where I is uncountable and each K_i is a compact non-torsion abelian group. Obviously, such a product cannot be c-extremal, so G is not c-extremal either. □

An alternative proof of the first part of the above theorem ($w(G) \leq \mathfrak{c}$) can be derived from [3, Lemma 6.7] and Lemma 4.3.

Now we can prove as a corollary of Theorem 4.13 a result that generalizes [9, Theorem 3.4 and Remark 4.3] where it is proved that questions (A) and (B) have positive answer in the case of compact groups.

Corollary 4.14. *A compact abelian group G is metrizable if and only if G is weakly extremal.*

Proof. It suffices to note that if G is weakly extremal, then G is singular by Theorem 4.13 and consequently metrizable by Theorem 4.6. □

Using the properties of non singular groups it is possible to prove:

Theorem 4.15. *A pseudocompact abelian group G is metrizable if and only if G is weakly extremal and $w(G) \leq \mathfrak{c}$.*

Proof. If G is metrizable then G is weakly extremal and $w(G) \leq \omega$.

Assume that G is weakly extremal and $w(G) \leq \mathfrak{c}$. Then G is singular by Theorem 4.13, hence Theorem 4.6 applies to conclude that G is metrizable. $\square$

5. Almost connected groups

In this section we study the properties of the connected component and of the subgroups $N \in \Lambda(G)$ of an almost connected group G.

Remark 5.1. Let G be an almost connected, pseudocompact abelian group. Then:

(a) the quotient $G/c(G)$ is metrizable and compact;
(b) the connected component $c(G)$ of G is a pseudocompact group (by Corollary 2.6 (a)); if G is not metrizable, then $w(c(G)) = w(G)$ (by Corollary 2.6 (b)).

Lemma 5.2. *Let G be a pseudocompact abelian group. Then the following conditions are equivalent:*

(a) *G is almost connected;*
(b) *$\widetilde{G}$ is almost connected;*
(c) *if G is dense in a topological group G_1, then G_1 is almost connected.*

Proof. (a)⇒(c) By Lemma 2.2 the group G_1 is pseudocompact and so G is G_δ-dense in G_1. Since G is almost connected, $c(G) \in \Lambda(G)$. As $c(G) \subseteq c(G_1) \cap G$ and $c(G_1) \cap G$ is closed in G, it follows from Corollary 2.6 (c) that $c(G_1) \cap G \in \Lambda(G)$. Now Corollary 2.9 applies to conclude that $c(G_1) \in \Lambda(G_1)$, i.e., G_1 is almost connected.

(c)⇒(b) Is obvious.

(b)⇒(a) Suppose that $\widetilde{G}$ is almost connected. Then $c(\widetilde{G}) \in \Lambda(\widetilde{G})$. Since G is G_δ-dense in $\widetilde{G}$, by Lemma 2.7 (c) $c(\widetilde{G}) \cap G$ is G_δ-dense in $c(\widetilde{G})$, which is connected. Hence $c(\widetilde{G}) \cap G$ is connected too and consequently $c(\widetilde{G}) \cap G \subseteq c(G)$. As $c(\widetilde{G}) \in \Lambda(\widetilde{G})$, it follows that $c(\widetilde{G}) \cap G \in \Lambda(G)$ and so $c(G) \in \Lambda(G)$. $\square$

Corollary 5.3. *If G is an almost connected pseudocompact abelian group and G is dense in a topological group G_1, then $c(G) = c(G_1) \cap G$. In particular, $c(G) = c(\widetilde{G}) \cap G$.*

Proof. The inclusion $c(G) \subseteq c(G_1) \cap G$ holds in general.

Since G is almost connected, Lemma 5.2 yields that also G_1 is almost connected, i.e., $c(G_1) \in \Lambda(G_1)$. Moreover, as G is G_δ-dense in G_1, $c(G_1) \cap G$ is G_δ-dense in $c(G_1)$, that is connected. Hence also $c(G_1) \cap G$ is connected and so $c(G_1) \cap G \subseteq c(G)$. $\square$

Now we prove a remarkable property of the almost connected pseudocompact abelian groups. For a topological group G denote by $q(G)$ the *quasi component* of G (i.e., the intersection of all clopen sets of G [16]). Usually, $c(G) \leq q(G)$, but strict inequality may hold even for pseudocompact abelian groups (see [16] for various levels of the failure of the equality $c(G) = q(G)$).

Corollary 5.4. *Let G be an almost connected pseudocompact abelian group. Then $q(G) = c(G)$.*

Proof. It is known that $q(G) = c(\widetilde{G}) \cap G$ [17]. Hence Corollary 5.3 yields that $c(G) = c(\widetilde{G}) \cap G$. Then $q(G) = c(G)$. □

Next we prove that the connected component of a pseudocompact abelian group can be computed in the same way as in the case of compact abelian groups.

Corollary 5.5. *If G is an almost connected pseudocompact abelian group, then $c(G) = \bigcap_{n=1}^{\infty} \overline{nG}^G$.*

Proof. By Corollary 5.3 $c(G) = G \cap c(\widetilde{G})$. Since for every compact abelian group K one has $c(K) = \bigcap_{n=1}^{\infty} nK$ [19], Lemma 2.10 yields that

$$c(G) = G \cap \bigcap_{n=1}^{\infty} n\widetilde{G} = \bigcap_{n=1}^{\infty} (G \cap n\widetilde{G}) = \bigcap_{n=1}^{\infty} \overline{nG}^G.$$

□

The next proposition shows that for singular (resp. totally disconnected) pseudocompact abelian groups almost connectedness and metrizability are e-quivalent.

Proposition 5.6. *Let G be a pseudocompact abelian group that is either totally disconnected or singular. Then G is almost connected if and only if G is metrizable.*

Proof. If G is metrizable, then G is almost connected.

To prove the converse implication, suppose that G is almost connected. If G is totally disconnected, then $\{0\} = c(G) \in \Lambda(G)$. Hence G is metrizable by Lemma 2.4. If G is singular, then there exists $m \in \mathbb{N}_+$ such that mG is compact and metrizable by Lemma 4.1(b). Since $c(G) \subseteq mG$ by Corollary 5.5, it follows that $w(c(G)) = \omega$. On the other hand, $c(G) \in \Lambda(G)$ by hypothesis, hence $w(G) = w(c(G)) = \omega$, i.e., G is metrizable. □

Lemma 5.7. [6, Theorem 3.4] *Let G be a connected pseudocompact abelian group. Then $c(N) \in \Lambda(G)$ for every $N \in \Lambda(G)$, i.e., N is almost connected for every $N \in \Lambda(G)$.*

The above lemma can be generalized to almost connected pseudocompact abelian groups as follows.

Theorem 5.8. *Let G be a pseudocompact abelian group. Then the following conditions are equivalent:*

(a) *G is almost connected;*
(b) *there exists $N \in \Lambda(G)$ such that N is almost connected;*
(c) *every $N \in \Lambda(G)$ is almost connected;*
(d) *$\overline{nG} \in \Lambda(G)$ for every $n \in \mathbb{N}_+$;*

(e) *there exists a G_δ-dense subgroup H of G such that H is almost connected;*

(f) *every G_δ-dense subgroup H of G is almost connected.*

Proof. (a)⇒(c) Let $N \in \Lambda(G)$ and let $c(N)$ be the connected component of N. Then $c(N) \subseteq c(G)$ so in particular $c(N) \in \Lambda(c(G))$ by Lemma 5.7 applied to $c(G)$. Hence $c(N) \in \Lambda(G)$ by Corollary 2.6 (d).

(c)⇒(b) Is obvious.

(b)⇒(a) If $N \in \Lambda(G)$ is almost connected, then $c(N) \in \Lambda(G)$. Since $c(G)$ is closed in G and $c(N) \subseteq c(G)$, Corollary 2.6 implies $c(G) \in \Lambda(G)$.

(a)⇒(d) If G is almost connected, then by Corollary 5.5 $c(G) = \bigcap_{n=1}^{\infty} \overline{nG} \in \Lambda(G)$. Hence $c(G) \subseteq \overline{nG}$ for every $n \in \mathbb{N}_+$ and consequently $\overline{nG} \in \Lambda(G)$ by Corollary 2.6 (c).

(d)⇒(a) If $\overline{nG} \in \Lambda(G)$ for every $n \in \mathbb{N}_+$, then $\bigcap_{n=1}^{\infty} \overline{nG} \in \Lambda(G)$. The compactness of $\widetilde{G}$ implies $c(\widetilde{G}) = \bigcap_{n=1}^{\infty} n\widetilde{G}$. Therefore

$$c(\widetilde{G}) \cap G = \bigcap_{n=1}^{\infty} n\widetilde{G} \cap G = \bigcap_{n=1}^{\infty} \overline{nG} \in \Lambda(G)$$

and so by Corollary 2.9 one has that $c(\widetilde{G}) \in \Lambda(\widetilde{G})$, i.e., $\widetilde{G}$ is almost connected. Lemma 5.2 applies to conclude that G is almost connected.

(a)⇒(f) Assume that G is almost connected and let H be a G_δ-dense subgroup of G. Since $c(G) \in \Lambda(G)$, it follows that $c(G) \cap H$ is G_δ-dense in $c(G)$. Then $c(G) \cap H$ is connected and so $c(G) \cap H \subseteq c(H)$. Since $c(G) \cap H \in \Lambda(H)$, Corollary 2.6 applies to conclude that $c(H) \in \Lambda(H)$.

(f)⇒(e) Is obvious.

(e)⇒(a) Suppose that H is an almost connected G_δ-dense subgroup of G. Then H is pseudocompact and $c(H) \in \Lambda(H)$. Since $c(H) \subseteq c(G) \cap H$ and $c(G) \cap H$ is closed in H, it follows from Corollary 2.6 that $c(G) \cap H \in \Lambda(H)$. Then Corollary 2.9 implies that $c(G) \in \Lambda(G)$, i.e., G is almost connected. □

Theorem 5.8 can be applied to show that the same conclusion of Proposition 5.6 is true if we suppose that there exists $N \in \Lambda(G)$ such that N is totally disconnected.

Corollary 5.9. *Let G be an almost connected pseudocompact abelian group. If there exists $N \in \Lambda(G)$ such that N is totally disconnected, then G is metrizable.*

Proof. Since G is almost connected, Theorem 5.8 (d) implies that also $N \in \Lambda(G)$ is almost connected. On the other hand, N is totally disconnected, hence metrizable by Proposition 5.6. Then also the group G is metrizable. □

In the next corollary we give two opposite properties of the almost connected groups (as far as the similarity with connected groups is concerned). The first one shows that almost connectedness cannot be destroyed by taking direct products with a compact metrizable group, while the second one is a stability property usually possessed by connected groups.

Corollary 5.10. *Let G be a pseudocompact abelian group.*

(a) *For every compact metrizable group M, the product $G \times M$ is almost connected if and only if G is almost connected.*

(b) *Let N be a closed subgroup of G:*

(b_1) *if G is almost connected, then also G/N is almost connected;*

(b_2) *if both N and G/N are almost connected, then also G is almost connected.*

The next lemma and its corollary are used in the proof of Theorem 5.20 to produce G_δ-dense subgroups of connected pseudocompact groups.

Lemma 5.11. *Let G be a precompact connected group. Then mD is dense in G for every $m \in \mathbb{N}_+$ and for every dense subgroup D of G. In particular, mG is dense in G for every $m \in \mathbb{N}_+$.*

Proof. Since $\widetilde{G}$ is connected and compact, $\widetilde{G}$ is divisible. Then $m\widetilde{G} = \widetilde{G}$ for every $m \in \mathbb{N}_+$. Since D is dense in G, mD is dense in mG. Analogously, the density of G in $\widetilde{G}$, implies that mG is dense in $m\widetilde{G} = \widetilde{G}$. Then also mD is dense in $\widetilde{G}$. As $mD \subseteq G$ one has that mD is dense in G.

The last assertion follows from the density of G in $\widetilde{G}$. □

Corollary 5.12. *Let G be a connected pseudocompact abelian group. Then mA is G_δ-dense in G for every $m \in \mathbb{N}_+$ and for every G_δ-dense subgroup A of G.*

Theorem 5.13. *Let G be an almost connected pseudocompact abelian group of uncountable weight α. Then:*

(a) $r_0(c(G)) = r_0(G) \geq \mathfrak{c}$ *and* $\alpha \leq 2^{r_0(G)}$*;*

(b) $|c(G)| = |G| \leq 2^\alpha \leq 2^{2^{r_0(G)}}$*;*

(c) $m(c(G)) = m(G) \leq r_0(G)$*.*

Proof. (a) Being almost connected and non-metrizable, the group G is not singular (cf. Proposition 5.6), hence $r_0(c(G)) = r_0(G) \geq \mathfrak{c}$ by Lemma 4.3. To prove $\alpha \leq 2^{r_0(G)}$ fix a free subgroup F of size $r_0(G)$ of G. Then $w(F) = w(\overline{F}) \leq 2^{r_0(G)}$. Since G/F is torsion, the pseudocompact group $G/\overline{F}$ is torsion, hence bounded by Theorem 2.14. So there exists $n > 0$ such that $nG \subseteq \overline{F}$. So $w(nG) \leq w(\overline{F}) \leq 2^{r_0(G)}$. Since G is almost connected, $w(nG) = w(G) = \alpha$ by (d) of Theorem 5.8. This proves $\alpha \leq 2^{r_0(G)}$.

(b) If $|G| = \mathfrak{c}$, then by the first part of (a) $\mathfrak{c} \leq r_0(c(G)) = r_0(G) \leq |G| \leq \mathfrak{c}$, hence $|c(G)| = |G| = \mathfrak{c}$. Suppose now that $|G| > \mathfrak{c}$. Since $c(G) \in \Lambda(G)$, the quotient $G/c(G)$ is compact and metrizable (cf. Remark 5.1) and so $|G/c(G)| \leq \mathfrak{c}$. Now $|G| = |G/c(G)| \cdot |c(G)|$ implies $|c(G)| = |G|$.

(c) The equality $m(c(G)) = m(G)$ follows from (c) of Lemma 2.15 as G is almost connected. Moreover, $w(c(G)) = w(G) > \omega$ and $|G/c(G)| \leq \mathfrak{c}$ according by Remark 5.1 and $r_0(c(G)) \geq \mathfrak{c}$ by (a).

Let M be a maximal independent subset of $c(G)$ and let $A = \langle M \rangle$. Then the quotient $c(G)/A$ is torsion, so setting $A_n = \{x \in c(G) : nx \in A\}$ for every

$n \in \mathbb{N}_+$, one has $c(G) = \bigcup_{n=1}^{\infty} A_n$. By Corollary 2.16 there exist $n \in \mathbb{N}_+$ and $E \leq c(G)$, with $|E| \leq \mathfrak{c}$, such that $A_n + E$ is G_δ-dense in $c(G)$. Then by Corollary 5.12 also $n(A_n + E)$ is G_δ-dense in $c(G)$. So for $F := nE$ one has $|F| \leq \mathfrak{c}$ and

$$n(A_n + E) \subseteq nA_n + nE \subseteq A + F \subseteq c(G).$$

Thus $A+F$ is G_δ-dense in $c(G)$. By Lemma 2.15 (b) there exists a subgroup D of G with $|D| \leq \mathfrak{c}$ such that $L = A + F + D$ is G_δ-dense in G. Since $|F + D| \leq \mathfrak{c} \leq |A|$, we have $|L| \leq |A| = r_0(c(G)) = r_0(G)$. Hence $m(G) \leq r_0(G)$. □

The next theorem shows the relation between d-extremality and almost connectedness.

Theorem 5.14. *Every d-extremal pseudocompact abelian group is almost connected. In particular, weakly extremal pseudocompact abelian groups are almost connected.*

Proof. Aiming for a contradiction, assume that G is not almost connected. Then by Theorem 5.8 (d), there exists $n \in \mathbb{N}_+$ such that $\overline{nG} \notin \Lambda(G)$, i.e., $\overline{nG}$ is not a G_δ-set. Hence $w(G/\overline{nG}) > \omega$ and G is not d-extremal by Lemma 4.5, a contradiction. □

Let us see now that for compact groups the above implication can be inverted.

Theorem 5.15. *For a compact abelian group G the following are equivalent:*

(a) *G is almost connected;*
(b) *G is d-extremal;*
(c) *G is strongly d-extremal.*

Proof. The implication (b) $\Rightarrow$ (a) follows from Theorem 5.14. Conversely, if G is almost connected, then $c(G) \in \Lambda(G)$ is compact and connected, hence divisible. In particular, $c(G)$ is d-extremal and consequently G is d-extremal too by Lemma 3.3 (a).

By the equivalence of (a) and (b) and by Theorem 5.8, (b) $\Rightarrow$ (c). □

Remark 5.16. Let us note here that one cannot invert the implication *d-extremal $\Rightarrow$ almost connected* even for connected pseudocompact abelian groups. By [21] there exists a dense pseudocompact subgroup G of $\mathbb{T}^{\mathfrak{c}}$ of size $\mathfrak{c}$. Note that $|G| = w(G) = \mathfrak{c}$. Since $r_0(\mathbb{T}^{\mathfrak{c}}) = 2^{\mathfrak{c}} > \mathfrak{c}$, there exists an infinite cyclic subgroup C of $\mathbb{T}^{\mathfrak{c}}$ such that $C \cap G = \{0\}$. Then the pseudocompact subgroup $G_1 = G + C$ of $\mathbb{T}^{\mathfrak{c}}$ is connected (as $\mathbb{T}^{\mathfrak{c}}$ is connected) and G is G_δ-dense in G_1. Since $G_1/G \cong C$ is not divisible, G_1 is not d-extremal.

In contrast with Remark 5.16 the following corollary shows that for totally disconnected pseudocompact abelian groups d-extremality and almost connectedness are equivalent. Let us recall that Questions (A) and (B) of the introduction have positive answer in the case of zero-dimensional groups [11, Theorem 7.3] and in the case of totally disconnected groups [3]. The following corollary generalizing these results covers Theorem B from the Introduction.

Corollary 5.17. *Let G be a pseudocompact abelian group that is either singular or totally disconnected. Then the following conditions are equivalent:*

(a) *G is almost connected;*
(b) *G is d-extremal;*
(c) *G is metrizable.*

Proof. (c)⇒(b) is obvious, (b)⇒(a) follows from Theorem 5.14 and (a)⇒(c) follows from Proposition 5.6. □

In particular one has:

Corollary 5.18. *A torsion pseudocompact abelian group G is weakly extremal iff G is metrizable.*

Corollary 5.19. *Let G be a pseudocompact abelian group with $w(G/c(G)) > \omega$. Then G is not d-extremal.*

Proof. If G were d-extremal, then by Theorem 5.14 G would be almost connected, i.e., $c(G) \in \Lambda(G)$ and consequently $w(G/c(G)) = \omega$. □

The corollary applies also to pseudocompact abelian groups G with $w(c(G)) < w(G)$ (since then $w(G/c(G)) > \omega$).

Theorem 5.20. *Let G be an almost connected pseudocompact abelian group. Then $c(G)$ is divisible if and only if $c(G)$ is d-extremal (so also G is d-extremal).*

Proof. Since G is almost connected the subgroup $c(G)$ is pseudocompact by Remark 5.1 (b). If $c(G)$ is divisible, then obviously $c(G)$ is d-extremal.

Assume now that $c(G)$ is not divisible. Then there exists $n_0 \in \mathbb{N}_+$ such that $n_0c(G)$ is a proper subgroup of $c(G)$. By Corollary 5.12 $n_0c(G)$ is also G_δ-dense in $c(G)$. Since the quotient $c(G)/n_0c(G)$ is bounded torsion, it cannot be divisible and so $c(G)$ is not d-extremal. □

There is a similar result for strong extremality: for a strongly extremal (strongly d-extremal) pseudocompact abelian group G, $c(G)$ is divisible if and only if $c(G)$ is strongly extremal (strongly d-extremal).

Remark 5.21. For every s-extremal (resp. weakly extremal, d-extremal) pseudocompact abelian group G the subgroup $c(G)$ is divisible if and only if it is s-extremal (resp. weakly extremal, d-extremal). Indeed, by Theorem 5.20, if $c(G)$ is s-extremal, then $c(G)$ is divisible. Vice versa, assume that $c(G)$ is divisible. Note that, $c(G) \in \Lambda(G)$ since G is almost connected by Theorem 5.14. Now Lemma 3.1 (b) applies. The case when $c(G)$ is weakly extremal (resp., d-extremal) is analogous.

Corollary 5.22. *A d-extremal pseudocompact abelian group G is divisible if and only if G is connected.*

Proof. If G is divisible and pseudocompact, then G is connected too [33]. If $G = c(G)$ is connected, Theorem 5.20 applies. □

6. Various characterizations of extremality

6.1. **The closure of the graph of a homomorphism.** If G and H are abelian topological groups and $h : G \to H$ is a homomorphism, the graph Γ_h of h is the subset $\{(x, h(x)) : x \in G\}$ of $G \times H$. Then Γ_h is a subgroup of $G \times H$ such that

(1) if $p_1 : G \times H \to G$ is the canonical projection on the first component, then $p_1(\Gamma_h) = G$;
(2) $\Gamma_h \cap (\{0\} \times H) = \{(0, 0)\}$.

Consequently $G \times H = \Gamma_h \oplus (\{0\} \times H)$. Let $\mathbb{V}_h$ be the vertical component of $\overline{\Gamma_h}$ in $G\times H$, that is $\mathbb{V}_h = \overline{\Gamma_h} \cap (\{0\} \times H)$. Then $\overline{\Gamma_h}$ splits also as $\overline{\Gamma_h} = \Gamma_h \oplus \mathbb{V}_h$.

Since $\mathbb{V}_h$ is a subgroup of $\{0\} \times H$, it is possible to identify $\mathbb{V}_h$ with a closed subgroup of H. The fact that $\mathbb{V}_h$ is a subgroup follows also from the next lemma.

Lemma 6.1. *Let G and H be topological abelian groups and $h : G \to H$ be a homomorphism. Then*

$$\mathbb{V}_h = \{t \in H : \exists \text{ a net } \{x_\alpha\}_{\alpha\in A} \subseteq G \text{ such that } x_\alpha \to 0 \text{ and } h(x_\alpha) \to t\}.$$

Proof. If $t \in H$, then $(0, t) \in \overline{\Gamma_h} \cap (\{0\} \times H)$ if and only if there exists a net $\{(x_\alpha, h(x_\alpha))\}_{\alpha\in A}$ in Γ_h such that $0 = \lim x_\alpha$ and $t = \lim h(x_\alpha)$. □

Remark 6.2. (a) If G and H are abelian topological groups and $h : G \to H$ is a homomorphism, then

(1) Γ_h is closed in $G \times H$ if and only if $\mathbb{V}_h = \{0\}$.
(2) Γ_h is dense in $G \times H$ if and only if $\mathbb{V}_h = H$.

Hence it follows that if H is compact, h is not continuous and Γ_h is not dense in $G \times H$, then $\mathbb{V}_h$ is a proper closed subgroup of H.

(b) If G is a pseudocompact abelian group, H is a compact subgroup of $\mathbb{T}$ and $h : G \to H$ is a non-continuous and surjective homomorphism such that Γ_h is not dense in $G \times H$, then by (a) there exists $n \in \mathbb{N}$, $n > 1$, such that $\mathbb{V}_h = \mathbb{Z}(n)$.

(c) One can easily prove that in the above lemma $\mathbb{V}_h = \bigcap\{\overline{h(U)} : U \in \mathcal{V}_G(0)\}$. For $H = \mathbb{T}$ the latter subgroup was introduced also in [3, Notation 3.3].

Lemma 6.3. *Let G be an abelian topological group and let H be a compact abelian group. Let $h : G \to H$ be a homomorphism. Then for every $k \in \mathbb{N}$:*

(a) *$\mathbb{V}_{kh} = k\mathbb{V}_h$;*
(b) *kh is continuous if and only if $k\mathbb{V}_h = 0$.*

Proof. (a) First we prove that $k\mathbb{V}_h \subseteq \mathbb{V}_{kh}$. Let $t \in \mathbb{V}_h$. Then by Lemma 6.1 there exists a net $\{x_\alpha\}_{\alpha\in A}$ in G such that $0 = \lim x_\alpha$ and $t = \lim h(x_\alpha)$. Hence $kt = \lim k(h(x_\alpha)) = \lim(kh)(x_\alpha)$ and consequently $kt \in \mathbb{V}_{kh}$ by Lemma 6.1.

To prove the converse inclusion consider $t \in \mathbb{V}_{kh}$. In particular, $t \in H$ and by Lemma 6.1 there exists a net $\{x_\alpha\}_{\alpha\in A}$ in G such that $0 = \lim x_\alpha$ and $t = \lim(kh)(x_\alpha)$. Thus $\{h(x_\alpha)\}_{\alpha\in A}$ is a net in the compact group H, hence there exists a subnet $\{h(x_{\alpha_\beta})\}_{\beta\in B}$ of $\{h(x_\alpha)\}_{\alpha\in A}$ which converges to $s \in H$. Since $\{x_{\alpha_\beta}\}_{\beta\in B}$ converges to 0, by Lemma 6.1 $s \in \mathbb{V}_h$. Moreover it follows that $\{kh(x_{\alpha_\beta})\}_{\beta\in B}$ converges to ks and so for the uniqueness of limits $t = ks \in k\mathbb{V}_h$.

(b) It follows directly from (a) that $\overline{\Gamma_{kh}} = \Gamma_{kh} \oplus k\mathbb{V}_h$ for every $k \in \mathbb{N}$. By the closed graph theorem kh is continuous if and only if Γ_{kh} is closed, i.e., $k\mathbb{V}_h = 0$. □

Corollary 6.4. *Let G be a topological abelian group and for $n \in \mathbb{N}_+$ let $h : G \to \mathbb{Z}(n)$ be a homomorphism. Then the following conditions are equivalent:*

(a) *Γ_h is dense in $G \times \mathbb{Z}(n)$;*

(b) *the homomorphism $kh : G \to \mathbb{Z}(n)$ is not continuous for every integer k such that $0 < k < n$;*

(c) *$k\mathbb{V}_h \neq \{0\}$ for every integer k such that $0 < k < n$.*

Proof. (a)⇒(b) If Γ_h is dense in $G \times \mathbb{Z}(n)$ then $\mathbb{V}_h = \mathbb{Z}(n)$. Applying Lemma 6.3 we conclude that the homomorphism kh is not continuous for every $k \in \mathbb{N}_+$ such that $k < n$.

(b)⇔(c) Follows directly from Lemma 6.3 (b).

(c)⇒(a) By hypothesis $\mathbb{V}_h = \mathbb{Z}(n)$, hence Remark 6.2 (2) applies. □

Lemma 6.5. *Let G be a topological abelian group and let H be a closed subgroup of $\mathbb{T}$. Let $h : G \to H$ be a surjective homomorphism such that (G, τ_h) is pseudocompact and $\mathbb{V}_h = \mathbb{Z}(n)$ for some $n \in \mathbb{N}_+$. Let $N = \ker nh$. Then:*

(a) *$N = h^{-1}(\mathbb{V}_h)$;*

(b) *$\Gamma_{h|_N}$ is G_δ-dense in $N \times \mathbb{V}_h$.*

(c) *$\ker h$ is a proper G_δ-dense subgroup of N.*

Proof. (a) Let $x \in N = \ker nh$. Then $(nh)(x) = 0$, that is $n(h(x)) = 0$. It follows that $h(x) \in \mathbb{Z}(n) = \mathbb{V}_h$. This proves that $h(N) \subseteq \mathbb{V}_h$, that is $N \subseteq h^{-1}(\mathbb{V}_h)$.

To prove the opposite inclusion, take $y \in \mathbb{V}_h$. Observe that $\mathbb{V}_h \subseteq H = h(G)$. Then there exists $x \in G$ such that $h(x) = y$. As $y \in \mathbb{V}_h$ and $n\mathbb{V}_h = \{0\}$, in particular $ny = 0$. Consequently $0 = ny = n(h(x)) = (nh)(x)$ and hence $x \in N$, that is $y \in h(N)$. This proves that $h^{-1}(\mathbb{V}_h) \subseteq N$.

(b) By (a) $h|_N : N \to \mathbb{V}_h = \mathbb{Z}(n)$ is a surjective homomorphism. It follows from Lemma 6.3 that the homomorphism kh is not continuous for every $k \in \mathbb{N}$ such that $0 < k < n$. Corollary 6.4 implies that $\Gamma_{h|_N}$ is dense in $N \times \mathbb{V}_h$. Since(G, τ_h) is pseudocompact by hypothesis and nh is continuous, $N \in \Lambda(G, \tau_h)$ and $(N, \tau_h|_N) = (N, (\tau|_N)_{h|_N})$ is pseudocompact. Moreover, $(N, (\tau_h|_N)_{h|_N})$ is homeomorphic to $\Gamma_{h|_N}$ and then also $\Gamma_{h|_N}$ is pseudocompact. Hence $\Gamma_{h|_N}$ is G_δ-dense in $N \times \mathbb{V}_h$.

(c) It follows from (b) that $\Gamma_{h|_N}$ is G_δ-dense in $N \times \mathbb{V}_h$. Then by Lemma 3.7 $\ker h|_N = \ker h$ is proper and G_δ-dense in N. □

Remark 6.6. Let (G, τ) be a topological abelian group and $h : G \to \mathbb{T}$ be a homomorphism such that $H = h(G)$ is a closed subgroup of $\mathbb{T}$.

(a) If $\mathbb{V}_h = \mathbb{Z}(n)$ for some $n \in \mathbb{N}_+$, then Lemma 6.3 implies $\langle h\rangle \cap \widehat{G} = \langle nh\rangle$.

(b) Combining appropriately (a) and the results of this section, it is possible to prove the following theorem, announced in [1, Theorem 5.10] and [23, Lemma 3.6] and proved in [3, Theorem 3.10]. *If* (G,τ) *is pseudocompact and* h *is not* τ*-continuous, then:*

(i) *if* $\widehat{G}\cap\langle h\rangle=\{0\}$*, then* (G,τ_h) *is pseudocompact if and only if* $h(G)$ *is closed in* $\mathbb{T}$ *and* $\ker h$ *is* G_δ*-dense in* (G,τ)*;*

(ii) *if* $\widehat{G}\cap\langle h\rangle=\langle nh\rangle$ *for some* $n\in\mathbb{N}_+$*, then* (G,τ_h) *is pseudocompact if and only if* $h(G)$ *is closed in* $\mathbb{T}$ *and* $\ker h$ *is* G_δ*-dense in* $\ker nh$*.*

Since we are not going to use here this property, the interested reader is invited to consult [3] for a detailed proof.

6.2. **Applications to** r**-extremality.** Now we are able to prove several important results on the relations among different kinds of extremality.

Theorem 6.7. *Let* G *be a pseudocompact abelian group. Then the following conditions are equivalent:*

(a) G *is metrizable;*

(b) *every* $N\in\Lambda(G)$ *is* s*-extremal;*

(c) G *is* s*-extremal and strongly* d*-extremal.*

Proof. (a)⇒(b) is obvious, $(b)\Rightarrow(a)$ is proved in [3].

(b)⇒(c) is obvious.

(c)⇒(b) Let $N\in\Lambda(G)$. Since G is weakly extremal, G is almost connected by Theorem 5.14, hence $c(N)\in\Lambda(G)$. Therefore, $c(N)$ is a connected, d-extremal pseudocompact abelian group and consequently it is divisible by Corollary 5.22. Thus $c(N)$ is s-extremal by Lemma 3.1 (b). Since $c(N)\in\Lambda(N)$ by Theorem 5.8, N is s-extremal too according to Lemma 3.3 (a). □

Here we see that the stronger version of weak extremality, imposed on all $N\in\Lambda(G)$, gives strong extremality.

Theorem 6.8. *A pseudocompact abelian group* G *is strongly extremal if and only if every* $N\in\Lambda(G)$ *is weakly extremal.*

Proof. If G is strongly extremal, then by Theorem 3.12 every $N\in\Lambda(G)$ is weakly extremal.

To prove the converse implication, it suffices to show that if every $N\in\Lambda(G)$ is weakly extremal, then G is r-extremal. Indeed, if this is true, then every $N\in\Lambda(G)$ is r-extremal (cf. Remark 3.5).

Suppose for a contradiction that G is not r-extremal. Then there exists a pseudocompact group topology τ' on G such that $\tau'>\tau$. Since τ and τ' are, in particular, precompact group topologies, there exists a homomorphism $h\in\widehat{(G,\tau')}\setminus\widehat{(G,\tau)}$, i.e., $h:G\to\mathbb{T}$ is τ'-continuous but not τ-continuous. Let $H=h(G)$. Then H is a compact subgroup of $\mathbb{T}$, as the image of the pseudocompact group (G,τ') under the τ'-continuous homomorphism h. Since h is not τ-continuous, Γ_h is not closed in $(G,\tau)\times H$.

There are two cases.

Case 1. Suppose that Γ_h is dense in $(G,\tau)\times H$. Since h is τ'-continuous, $\tau_h\leq\tau'$ and so (G,τ_h) is pseudocompact (as (G,τ') is pseudocompact by hypothesis). Since Γ_h is homeomorphic to (G,τ_h), also Γ_h is pseudocompact. Then Γ_h is G_δ-dense in $(G,\tau)\times H$ and hence $\ker h$ is a G_δ-dense subgroup of G by Lemma 3.7. Moreover, the quotient $G/\ker h$ is algebraically isomorphic to H. Being a compact subgroup of $\mathbb{T}$, the group H is all $\mathbb{T}$ or it is finite. If $H=\mathbb{T}$ then $r_0(H)=r_0(\mathbb{T})=\mathfrak{c}$ and so $r_0(G/\ker h)=r_0(H)=\mathfrak{c}$. If H is finite, then also $G/\ker h$ is finite so in particular it cannot be divisible. In both cases G is not weakly extremal, against the hypothesis.

Case 2. If Γ_h is not dense in $(G,\tau)\times H$, then by Remark 6.2 (b) there exists $n\in\mathbb{N}_+$, $n>1$, such that $\mathbb{V}_h=\mathbb{Z}(n)$. Then nh is τ-continuous by Lemma 6.3 and so $N=\ker nh\in\Lambda(G)$. As observed in Case 1, (G,τ_h) is pseudocompact, hence $\ker h$ is a proper G_δ-dense subgroup of N by Lemma 6.5. Since $N/\ker h\cong\mathbb{Z}(n)$ it cannot be divisible, i.e. N is not weakly extremal, a contradiction. □

Combining with Theorem 4.11 we obtain:

Corollary 6.9. *Let G be a c-extremal, strongly d-extremal pseudocompact abelian group. Then G is strongly extremal.*

Now we can prove Theorem C from the Introduction. We use essentially some ideas from the proof of Theorem 4.4 (b) of [3].

Theorem 6.10. *Let G be a s-extremal pseudocompact abelian group. Then G is doubly extremal if and only if $\overline{G[p]}=\widetilde{G}[p]$ for every $p\in\mathbb{P}$.*

Proof. Assume that $\overline{G[p]}=\widetilde{G}[p]$ for every $p\in\mathbb{P}$ and suppose for a contradiction that (G,τ) is not r-extremal. Then there exist a pseudocompact group topology τ' on G such that $\tau'>\tau$ and a homomorphism $h:G\to\mathbb{T}$ which is τ'-continuous but not τ-continuous (cf. the proof of Theorem 6.8). Take $H=h(G)$; then H is a compact subgroup of $\mathbb{T}$ as h is τ'-continuous. Since h is not τ-continuous, Γ_h is not closed in $(G,\tau)\times H$.

There are two cases:

Case 1. Suppose that Γ_h is dense in $(G,\tau)\times H$. Since h is τ'-continuous, $\tau_h\leq\tau'$ and so (G,τ_h) is pseudocompact. Since Γ_h is homeomorphic to (G,τ_h), also Γ_h is pseudocompact, hence G_δ-dense in $(G,\tau)\times H$. Now Theorem 3.8 applies to conclude that G is not s-extremal, a contradiction.

Case 2. If Γ_h is not dense in $(G,\tau)\times H$, then by Remark 6.2 (b) there exists $n\in\mathbb{N}_+$, $n>1$, such that $\mathbb{V}_h=\mathbb{Z}(n)$. It follows from Lemma 6.3 that nh is continuous.

Let $p\in\mathbb{P}$ such that $p|n$. Take $m=\frac{n}{p}\in\mathbb{N}_+$; we have that $m<n$. Define $h_1:=mh$. Since $m<n$ the homomorphism h_1 is not τ-continuous. On the other hand, h_1 is τ'-continuous (as h is τ'-continuous) and ph_1 is τ-continuous as $ph_1=pmh=nh$. Moreover

$$\mathbb{V}_{h_1}=\mathbb{V}_{mh}=m\mathbb{V}_h=m\mathbb{Z}(n)=\mathbb{Z}(p).$$

Consider now $f = ph_1 : (G, \tau) \to pH \leq H$. Since $f = ph_1 = nh$, one has $f \in \widehat{(G, \tau)}$. Note that $f|_{G[p]} = 0$ and consider a continuous extension $\tilde{f}$ of f to $\widetilde{G}$ (that is, the continuous homomorphism $\tilde{f} : \widetilde{G} \to \mathbb{T}$ such that $\tilde{f}|_G = f$). Since $\tilde{f}|_{G[p]} = f|_{G[p]} = 0$ and $\tilde{f}$ is continuous, we have that $\tilde{f}|_{\overline{G[p]}} = 0$. By hypothesis $\overline{G[p]} = \widetilde{G}[p]$, hence $\tilde{f}|_{\widetilde{G}[p]} = 0$, i.e., $\tilde{f} \in A(\widetilde{G}[p])$. Since $A(\widetilde{G}[p]) = p\widehat{\widetilde{G}}$, there exists a continuous homomorphism $\tilde{g} : \widetilde{G} \to \mathbb{T}$ such that $\tilde{f} = p\tilde{g}$. If we take $g := \tilde{g}|_G$, then $f = pg$.

Since $f = ph_1$ by definition, we have $p(h_1 - g) = 0$ and so $t := h_1 - g$ is a torsion element of $Hom(G, \mathbb{T})$ of order p. Clearly, as h_1 is not τ-continuous, also t is not τ-continuous. Then $t : G \to \mathbb{Z}(p)$ is surjective and Γ_t is not closed. Since by Remark 6.2 (1) $\mathbb{V}_t$ is a non trivial subgroup of $\mathbb{Z}(p)$, it follows that $\mathbb{V}_t = \mathbb{Z}(p)$. Note that (G, τ_t) is a pseudocompact group, since t is τ'-continuous and so $\tau' \geq \tau_t$. We can now apply Lemma 6.5 and conclude that $\ker t$ is a proper G_δ-dense subgroup of $\ker pt = G$. This proves that G is not s-extremal, a contradiction. Hence G is r-extremal, and consequently doubly extremal.

The converse implication follows from Theorem 1.2 (b) since G is s-extremal. □

For a compact abelian group G the *dimension* of G coincides with the free rank $r_0(\widehat{G})$ of the group $\widehat{G}$ of characters of G. In what follows it will be denoted by $\dim G$.

Definition 6.1. *Let G be an almost connected pseudocompact abelian group. For $N \in \Lambda(G)$ let*

$$co\dim N = \dim G/N, \ \ \textit{and}\ \Lambda_n(G) = \{N \in \Lambda(G) : co\dim N = n\}.$$

Let us briefly list some properties of this new invariant:

(a) $\Lambda_0(G)$ consists precisely of those $N \in \Lambda(G)$ that contain $c(G)$ (first note that if $c(G) \in \Lambda(G)$, then for $N \geq c(G)$ the quotient G/N is also a quotient of the compact totally disconnected group $G/c(G)$, so G/N is zero-dimensional);

(b) If $N_1, N_2 \in \Lambda_n$ and $N_1 \leq N_2$, then N_2/N_1 is a compact totally disconnected metrizable group;

(c) Let $\widetilde{\Lambda}_1(G) = \{N \in \Lambda(G) : G/N \cong \mathbb{T}\}$.

Clearly, $\widetilde{\Lambda}_1(G) \subseteq \Lambda_1(G)$ as $\dim \mathbb{T} = 1$. By (b), if $N_1, N_2 \in \widetilde{\Lambda}_1(G)$ and $N_1 \leq N_2$, then N_2/N_1 is finite cyclic (as the only compact totally disconnected subgroups of $\mathbb{T}$ are the finite ones).

The next theorem will make use of the following condition:

($\widetilde{d}$) every $N \in \widetilde{\Lambda}_1(G)$ is d-extremal.

Note that ($\widetilde{d}$) is obviously weaker than strong d-extremality, but implies d-extremality by Lemma 3.3 (a). We prove now Theorem D which shows that

$(\widetilde{d})$ in conjunction with weak extremality is equivalent to r-extremality for connected pseudocompact abelian groups.

Theorem 6.11. *Let G be a connected pseudocompact abelian group. Then G is r-extremal if and only if G is weakly extremal and $(\widetilde{d})$ holds.*

Proof. Since r-extremality implies weak extremality, it suffices to assume that G is weakly extremal and prove that G is r-extremal if and only if $(\widetilde{d})$ holds true. We shall prove that their negations are equivalent. Since G is weakly extremal and connected, we conclude by Corollary 5.22 that G is divisible. Denote by τ the topology of G and assume that (G, τ) is not r-extremal. Then there exists a discontinuous character $h : (G, \tau) \to \mathbb{T}$ such that (G, τ_h) is pseudocompact. Since $h : (G, \tau_h) \to \mathbb{T}$ is continuous and G is divisible, the image $h(G)$ must be a non-trivial divisible compact subgroup of $\mathbb{T}$ (by the pseudocompactness of τ_h). Hence h is surjective. According to Remark 6.6 (b) one has the two alternatives (i) and (ii). In (i) the subgroup $H = \ker h$ of G is G_δ-dense and $G/H \cong \mathbb{T}$ has free rank $\mathfrak{c}$. This contradicts weak extremality of G. Hence we are left with (ii); i.e., there exists $n > 0$ such that nh is τ-continuous and H is G_δ-dense in $N = \ker nh$. Since obviously $N \in \widetilde{\Lambda}_1(G)$ and since N/H is bounded torsion, we conclude that N is not d-extremal, i.e. $(\widetilde{d})$ fails.

Now suppose that some $N \in \widetilde{\Lambda}_1(G)$ is not d-extremal. Then there exists a G_δ-dense subgroup H of N such that N/H is not divisible. It is not restrictive to assume that $N/H \cong \mathbb{Z}(p)$ for some prime p. To prove that G is not r-extremal we shall find a surjective, discontinuous character $h : (G, \tau) \to \mathbb{T}$ such that $\ker ph = N$ and $\ker h = H$. Then τ_h will be pseudocompact by Remark 6.6 and so (G, τ) will be not r-extremal. To build such an h we fix first a continuous surjective homomorphism $l : G \to \mathbb{T}$ with $\ker l = N$ witnessing $N \in \widetilde{\Lambda}_1(G)$. Let $\pi : G \to G/H$ be the canonical homomorphism. Let $N' = \pi(N)$. Then $N' \cong \mathbb{Z}(p)$. Our first aim is to prove now that $G/H \cong \mathbb{T}$.

For simplicity write $X = G/H$ and recall that X is divisible as a quotient of G. Hence the homomorphism $\varphi_p : X \to X$ is surjective and obviously $N' \leq \ker \varphi_p$. Hence φ_p factorizes as $\varphi_p = \gamma \circ \lambda$, where $\lambda : X \to X/N'$ is the canonical homomorphism. As $X/N' \cong G/N \cong \mathbb{T}$ and $\gamma : X/N' \to X$ is surjective, we conclude that X is isomorphic to a quotient of $\mathbb{T}$. Moreover, the homomorphism γ has kernel $K = \lambda(X[p])$, hence $pK = 0$. Since $X/N' \cong \mathbb{T}$, this means that K is finite (cyclic), hence $X \cong (X/N')/K \cong \mathbb{T}$.

Fix an isomorphism $\xi : \mathbb{T} \to X$ and consider the composition $s = \lambda \circ \xi : \mathbb{T} \to \mathbb{T}$. Then $\ker s = \mathbb{Z}(p)$. Let us split $\mathbb{T} = \mathbb{Z}(p^\infty) \oplus T$ and note that $s(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$. Since $End(\mathbb{Z}(p^\infty)) = \mathbb{Z}_p$ is a discrete valuation domain, we can write $s|_{\mathbb{Z}(p^\infty)} = p\eta_1$, where η_1 is an automorphism of $\mathbb{Z}(p^\infty)$. As T has no elements of order p, the restriction

$$s|_T : T \to s(T) \tag{1}$$

is an isomorphism and therefore $s(T) \cap \mathbb{Z}(p^\infty) = \{0\}$. Then we can find an isomorphism $\eta_2 : T \to s(T)$ such that $s|_T = p\eta_2$ (note that $\varphi_p : s(T) \to s(T)$ is an isomorphism as $s(T)$ is divisible). Now let $\eta : \mathbb{T} \to \mathbb{T}$ be the isomorphism defined by $\eta = \eta_1 + \eta_2$. So, setting $h : G \to \mathbb{T}$ to be the composition $\eta \circ \xi^{-1} \circ \pi$, we can claim that $ph = l$ is continuous and $\ker ph = N$. Since $\ker h = H$, this ends up the proof. □

7. The cardinal invariants of extremal pseudocompact groups

Comfort and van Mill [6, Theorem 6.1] proved that $|t(G)| \leq \mathfrak{c}$ for connected s-extremal pseudocompact abelian groups (here $t(G)$ denotes the subgroup of torsion elements of G). Later Comfort, Gladdines and van Mill [4] proved that $r_0(G) \leq \mathfrak{c}$ for an s-extremal pseudocompact abelian group G (given as Theorem 3.6 here). This allowed them to prove that s-extremal pseudocompact abelian groups have size $\leq \mathfrak{c}$ [4, Theorem 4.8]. The next theorem extends these results to *all* weakly extremal pseudocompact abelian groups (as s-extremal groups G obviously satisfy $m(G) = |G|$).

Theorem 7.1. *Let G be an infinite almost connected c-extremal pseudocompact abelian group. Then $|G| \leq \mathfrak{c}$ if and only if $m(G) = |G|$.*

Proof. Since $m(G) \geq \mathfrak{c}$, obviously $|G| \leq \mathfrak{c}$ implies $m(G) = |G|$. Now assume $m(G) = |G|$ holds. By Theorem 3.6 $r_0(G) \leq \mathfrak{c}$. If $r_0(G) < \mathfrak{c}$, then G is bounded torsion according to Theorem 2.14. In particular, G is singular. By Proposition 5.6 G is metrizable, so $|G| \leq \mathfrak{c}$. If $r_0(G) = \mathfrak{c}$ (so in particular G is not torsion), it is sufficient to apply Theorem 5.13 (c). □

Corollary 7.2. *Let G be a weakly extremal pseudocompact abelian group. Then $|G| = \mathfrak{c}$ if and only if $m(G) = |G|$.*

In particular, $r_0(G) = |G| = \mathfrak{c}$ holds true for every non-torsion weakly extremal pseudocompact abelian group G with $m(G) = |G|$.

An elegant application of the inequality $|G| \leq \mathfrak{c}$ for s-extremal groups was found by Comfort and Galindo [3, Theorem 6.1] (announced in [23, Lemma 8.1] and [1, Theorem 5.16 (b)]), namely $(b) \Rightarrow (a)$ of Theorem 6.7 (which shows that for pseudocompact abelian groups the strong s-extremality is equivalent to metrizability).

In the sequel we shall apply Theorem 7.1 to obtain further connections among the cardinal invariants of the extremal pseudocompact groups.

Comfort and van Mill [6, Theorem 5.5] proved that a non-metrizable connected pseudocompact abelian group G with $|G| \geq w(G)^\omega$ is not s-extremal. As an immediate consequence of Theorem 7.1 we obtain the following stronger result.

Corollary 7.3. *An infinite weakly extremal pseudocompact abelian group G is metrizable if and only if $m(G) = |G| \geq w(G)$.*

Proof. By Theorems 7.1 and 2.14 we have $|G| = \mathfrak{c}$. Hence G is weakly extremal with $w(G) \leq \mathfrak{c}$. Now Theorem 4.15 applies to conclude that G is metrizable. □

Corollary 7.4. *Let G be a weakly extremal pseudocompact abelian group. Then $w(G) \leq 2^{\mathfrak{c}}$.*

Proof. If G is singular, then G is metrizable by Proposition 5.6, so $w(G) = \omega$. If G is not singular, then $w(G) > \omega$ and $r_0(G) \geq \mathfrak{c}$ by Lemma 4.3. On the other hand, $r_0(G) = \mathfrak{c}$ by Theorem 3.6, as G is c-extremal. Now Theorem 5.13 (a) applies as G is almost connected. □

Let us recall that according to [17], a group G is *hereditarily pseudocompact* when every closed subgroup of G is pseudocompact. It was proved in [3] that hereditary pseudocompactness has a strong impact on the s-extremal pseudocompact abelian groups:

(a) ([3, Theorem 6.9]) every finitely generated subgroup of G is metrizable.

(b) ([3, Theorem 6.10], under the assumption of Lusin's hypothesis) G itself is metrizable.

Here we strengthen (a) by replacing "s-extremal" by "weakly extremal and $m(G) = |G|$" (Corollary 7.9). Moreover, we show that under this weaker hypothesis one can strengthen further (a) by replacing "finitely generated" by "countable", if G is connected. This implies that hereditary pseudocompactness coincides with ω-boundedness for weakly extremal pseudocompact groups with $m(G) = |G|$ (Corollary 7.11). We were not able to remove Lusin's hypothesis in (b), but in Corollary 7.7 we give a stronger version of (b), with "G s-extremal" replaced by "G weakly extremal and $m(G) = |G|$".

Lemma 7.5. *Let G be a c-extremal, hereditarily pseudocompact abelian group of size $\mathfrak{c}$. Then there exists $m \in \mathbb{N}_+$ such that mH is metrizable for every subgroup H of G with $w(H) \leq \mathfrak{c}$.*

Proof. Let H be a subgroup of G with $w(H) \leq \mathfrak{c}$. Since G is hereditarily pseudocompact, the closure L of H in G is a pseudocompact subgroup of G so L is c-extremal by Corollary 4.12. Since $w(L) = w(H) \leq \mathfrak{c}$, L is singular by Theorem 4.13. Then there exists $m > 0$ such that mL (and so also mH) is metrizable. It remains to see that such an m can be chosen *uniformly* for all subgroups $H \leq G$ with $w(H) \leq \mathfrak{c}$.

Assume the contrary. Then for every $m > 0$ there exists a subgroup H_m of G such that $w(H_m) \leq \mathfrak{c}$ and mH_m is not metrizable. Consider the subgroup $H = \sum_{m=1}^{\infty} H_m$ of G. To see that $w(H) \leq \mathfrak{c}$ it suffices to recall that H is precompact, so $w(H) = |\widehat{H}|$. If $\rho_m : \widehat{H} \to \widehat{H}_m$ is the restriction homomorphism, then the diagonal homomorphism $\rho : \widehat{H} \to \prod_{m=1}^{\infty} \widehat{H}_m$ is injective. Since $|\widehat{H}_m| = w(H_m) \leq \mathfrak{c}$, we conclude that $|\widehat{H}| \leq \mathfrak{c}$. By the first part of the argument there exists $m_0 > 0$ such that $m_0 H$ is metrizable. Then $m_0 H_{m_0}$ is metrizable as well, a contradiction. □

Lemma 7.6. *Let G be a connected pseudocompact abelian group of weight $> \mathfrak{c}$. Then there exists a subgroup H of G of size $\leq \omega_1$ such that mH is not metrizable for any $m > 0$.*

Proof. Following the idea from the proof of [3, Theorem 6.10] one can construct by transfinite induction of length ω_1 a subgroup H of G of size $\leq \omega_1$ such that $r_0(\widehat{H}) \geq \omega_1$. (For every countable subgroup A of G the closure L of A has

$w(L) \leq \mathfrak{c}$, so G/L is a non-trivial connected pseudocompact abelian group. Hence there exists a non-torsion continuous character $\xi : G/L \to \mathbb{T}$, that produces a non-torsion continuous character $\chi : G \to \mathbb{T}$ such that $\chi|_L = 0$. The characters χ_α produced in this way, along with the elements $x_\alpha \in G$ witnessing χ_α is non-torsion give rise to the subgroup $H = \langle x_\alpha : \alpha < \omega_1 \rangle$ of G that has character group $\widehat{H}$ with $r_0(\widehat{H}) > \omega_1$, witnessed by the independent family $\{\chi_\alpha|_H\}$.) Let $m > 0$. Since $\widehat{mH} = \widehat{H}/\widehat{H}[m] \cong m\widehat{H}$, it follows that $w(mH) = |m\widehat{H}| > \omega$, i.e. mH is not metrizable. □

Corollary 7.7. *Under the assumption of the Lusin's hypothesis every weakly extremal hereditarily pseudocompact abelian group G with $m(G) = |G|$ is metrizable.*

Proof. It is not restrictive to assume that G is infinite. According to Theorems 7.1 and 2.14 $|G| = \mathfrak{c}$, hence we can apply Lemma 7.5 to produce an $m > 0$ such that mH is metrizable for every subgroup H of G of weight $\leq \mathfrak{c}$. Assume now that G is not metrizable. Then $w(G) > \mathfrak{c}$ by Theorem 4.15. Since G is almost connected, $w(c(G)) = w(G) > \mathfrak{c}$ according to Remark 5.1. Now by Lemma 7.6 we can find a subgroup H of $c(G)$ with $|H| \leq \omega_1$ and mH non-metrizable. This means that $w(H) > \mathfrak{c}$. Since $w(H) \leq 2^{\omega_1}$, we conclude that the Lusin's hypothesis fails. □

Corollary 7.8. *Let G be a c-extremal, hereditarily pseudocompact abelian group of size $\mathfrak{c}$. Then there exists $m \in \mathbb{N}_+$ such that every countable subgroup of mG is metrizable. In particular, mG is ω-bounded.*

Proof. Let $m \in \mathbb{N}_+$ as in Lemma 7.5 and let D be a countable subgroup of mG. For every $d \in D$ pick an element $g_d \in G$ such that $d = mg_d$ and let D_1 be the subgroup of G generated by the subset $X = \{g_d \in G : d \in D\}$. Then D_1 is countable and $w(D_1) \leq \mathfrak{c}$ so mD_1 is metrizable by Lemma 7.5. Since $D \subseteq mD_1$, D is metrizable as well. The last assertion follows from the fact that mG is hereditarily pseudocompact (as continuous image of G), hence the closure of D in mG is a metrizable pseudocompact (hence compact) subgroup of mG containing D. □

The next corollary generalizes Theorem 6.9 from [3], where s-extremality of G is assumed instead of "c-extremal of size $\mathfrak{c}$".

Corollary 7.9. *Let G be a c-extremal, hereditarily pseudocompact abelian group of size $\mathfrak{c}$. Then every finitely generated subgroup of G is metrizable.*

Proof. Let F be a finitely generated subgroup of G. By Corollary 7.8 there exists $m > 0$ such that the (finitely generated) subgroup mF of mG is metrizable. Then F is metrizable since the quotient F/mF is finite (as every torsion finitely generated abelian group). □

Corollary 7.10. *Let G be a connected, weakly extremal, hereditarily pseudocompact abelian group of size $\mathfrak{c}$. Then every countable subgroup of G is metrizable. In particular, G is ω-bounded.*

Proof. Follows from Corollary 7.8 as G is divisible by Corollary 5.22. □

This corollary implies that for weakly extremal connected pseudocompact group of size $\mathfrak{c}$ hereditary pseudocompactness coincides with ω-boundedness. In the next corollary we isolate this fact for the smaller class of s-extremal groups.

Corollary 7.11. *A connected s-extremal pseudocompact abelian group is hereditarily pseudocompact if and only if it is ω-bounded.*

8. Further comments and open questions

Even if the principal problems (A)-(C) (see Introduction) on metrizability of the extremal pseudocompact abelian groups have not been solved, the class of extremal pseudocompact abelian groups has been restricted to groups with specific properties. For the benefit of the reader we list below the most relevant ones:

(1) $|G| = r_0(G) = \mathfrak{c}$ for every almost connected c-extremal pseudocompact abelian group G with $m(G) = |G|$ (this includes all s-extremal groups); in case G is also hereditarily pseudocompact, then mG must be ω-bounded for some $m > 0$;
(2) if G is weakly extremal and non-metrizable, then $\mathfrak{c} < w(G/N) \leq 2^{\mathfrak{c}}$ for every closed pseudocompact subgroup $N \notin \Lambda(G)$ of G;
(3) if G is d-extremal, then G is almost connected (so non-totally disconnected if G is not metrizable); moreover, if G is connected, then G is divisible;
(4) if G is doubly extremal and non-metrizable, there exists a prime p such that $G[p]$ is dense but not G_δ-dense in $\widetilde{G}[p]$.

The following conjecture, supported by positive evidence in two cases (see Theorem 4.13), plays a central role: it obviously implies (F) (and consequently (A)-(E)).

Main Conjecture. *Every c-extremal pseudocompact abelian group is singular.*

Since weakly extremal groups are c-extremal, the Main Conjecture implies that every weakly extremal pseudocompact abelian group G is singular. Note that this fact is equivalent, by Corollary 5.17, to Question (F), as weakly extremal singular groups are metrizable according to Theorem 4.6.

The Main Conjecture immediately gives "Yes" to the following question (suggested by Corollary 5.22) since connected groups are almost connected.

Question 8.1. *Let G be a connected c-extremal pseudocompact abelian group. Must G be divisible (or, equivalently, d-extremal)?*

Note that "Yes" to this question implies that the following holds true: For an s-extremal pseudocompact abelian group G, $c(G)$ is s-extremal. The same holds true for r-extremality, for double extremality and for d-extremality. Indeed, the subgroup $c(G)$ is c-extremal by Theorem 4.11, so would be divisible.

Now Lemma 3.1 (b) implies that $c(G)$ is s-extremal. The same argument holds for d-extremality.

Along with the Main Conjecture one can consider also its restricted form:

Restricted Main Conjecture. *Every c-extremal pseudocompact abelian group of size $\mathfrak{c}$ is singular.*

By Theorem 7.1 the Restricted Main Conjecture implies that every weakly extremal pseudocompact group G with $m(G) = |G|$ is metrizable. So the Restricted Main Conjecture implies a positive answer to Questions (A), (C) and (D). Note that neither the Restricted Main Conjecture nor (F) follow from a positive answer to all questions (A)–(E), since a singular pseudocompact abelian group of size $\mathfrak{c}$ need not be weakly extremal (nor metrizable).

The following weaker form of Question (E) (as s-extremal groups satisfy $m(G) = |G|$ and has size $\mathfrak{c}$) is open too:

Question 8.2. *Does every r-extremal pseudocompact abelian group G satisfy $m(G) = |G|$?*

Observe that a positive answer to Question 8.2 implies a positive answer to the next question by Theorem 7.1:

Question 8.3. *Is every infinite r-extremal pseudocompact abelian group of size $\mathfrak{c}$?*

On the other hand, a positive answer to Question 8.3, along with the Restricted Main Conjecture, implies a positive answer to (A)-(E).

One can expect that connected weakly extremal pseudocompact abelian groups are r-extremal. A positive answer to this conjecture implies positive answer to (D) in the case of connected groups. In terms of Theorem 6.10, for the latter implication it suffices to check that $\overline{G[p]} = \widetilde{G}[p]$ for every prime p.

Question 8.4. *Does weak extremality coincide with the disjunction of s-extremality and r-extremality?*

If "Yes", then a positive answer to Question 8.3, along with the Restricted Main Conjecture, yields a positive answer to (F).

Note Added December 2005. Recently Comfort and van Mill [8] obtained the following impressive result:

Theorem 8.5. *A pseudocompact abelian group is s-extremal or r-extremal if and only if it is metrizable.*

Clearly, this theorem solves the principal open problems related to extremality: (A)–(E) and 8.2, 8.3, 8.4. Nevertheless, the Restricted Main Conjecture as well as questions 8.1 and (F) are left open by Theorem 8.5. Let us see now that an appropriate modification of the argument from [8] can prove our Main Conjecture, hence all remainig questions formulated in our paper.

Let G be a c-extremal pseudocompact abelian group. Theorem 3.6 yields $r_0(G) \le \mathfrak{c}$. By Theorem 2.14 either G is bounded torsion or $r_0(G) = \mathfrak{c}$. In the former case G is singular. Let $r_0(G) = \mathfrak{c}$ and assume that G is not singular. Let $D = \bigoplus_S \mathbb{Q}$, with $|S| = \mathfrak{c}$, be the divisible hull of the torsion-free quotient $G/t(G)$ and let $\pi : G \to D$ be the composition of the canonical projection $G \to G/t(G)$ and the inclusion $G/t(G) \hookrightarrow D$.

For a subset A of S let $G(A) = \pi^{-1}(\bigoplus_A \mathbb{Q})$ and

$$\mathcal{A} = \{A \subseteq S : G(A) \text{ contains a subgroup } N \in \Lambda(G)\}.$$

Then $\mathcal{A}$ has the countable intersection property and $|A| = \mathfrak{c}$ for all $A \in \mathcal{A}$ as $r_0(N) = \mathfrak{c}$ for every $N \in \Lambda(G)$ by Lemma 4.3. By [8, Lemma 3.2] there exists a partition $\{P_n\}_{n\in\mathbb{N}}$ of S such that $|A \cap P_n| = \mathfrak{c}$ for every $A \in \mathcal{A}$ and for every $n \in \mathbb{N}$. Define $V_n = G(P_0 \cup \cdots \cup P_n)$ for every $n \in \mathbb{N}$ and note that $G = \bigcup_{n=0}^{\infty} V_n$. By Lemma 2.15(a) there exist $m \in \mathbb{N}$ and $N \in \Lambda(G)$ such that $H = V_m \cap N$ is G_δ-dense in N. By Theorem 4.11, to get a contradiction it suffices to show that $r_0(N/H) = \mathfrak{c}$.

Let F be a torsion-free subgroup of N such that $F \cap H = \{0\}$ and maximal with this property. Suppose for a contradiction that $|F| = r_0(N/H) < \mathfrak{c}$. So $\pi(F) \subseteq \bigoplus_{S_1} \mathbb{Q}$ for some $S_1 \subseteq S$ with $|S_1| < \mathfrak{c}$ and $W = P_0 \cup \cdots \cup P_m \cup S_1$ has $|W \cap P_{m+1}| < \mathfrak{c}$. Consequently $W \notin \mathcal{A}$ and so $\Lambda(G) \ni N \not\subseteq G(W)$. Take $x \in N \setminus G(W)$. Since $G/G(W)$ is torsion-free, $\langle x\rangle \cap G(W) = \{0\}$. But $H + F \subseteq G(W)$ and so $\langle x\rangle \cap (H + F) = \{0\}$, that is $(F + \langle x\rangle) \cap H = \{0\}$, this contradicts the maximality of F.

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