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Stochastic social optimum traffic assignment

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Abstract

This paper formulates a Stochastic Social Optimum (SSO) that relates to the Stochastic User Equilibrium (SUE) in the same way as the Social Optimum (SO) relates to the User Equilibrium (UE) in a deterministic environment. At the SSO solution, the total of the users' perceived costs is minimised. The formulation and analysis is carried out in a general utility-maximising framework, with the probit and logit models being special cases. Conditions for the SSO flow pattern are derived, from which it can be seen that the marginal social costs play the same role in the SSO as the standard costs play in SUE. In particular, it is shown that the SSO solution can be obtained through the use of an algorithm for SUE, but with the marginal costs replacing the standard costs in the stochastic loading and that optimal tolls are the differences between the marginal social costs and the standard costs. For the case of the logit model an explicit path-based objective function is obtained which is of a pleasing symmetrical form when compared with the objective functions for SUE, SO and UE. Additionally, a link-based objective function for the general utility-maximising case is formulated for SSO, which is similar in form to the SUE objective function of Sheffi and Powell.

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Keywords: Traffic assignment; Stochastic user equilibrium; Probit model; Logit model; Optimal tolls; Marginal social costs

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26 1. Introduction

27 In deterministic traffic assignment, there are two different solutions: the User Equilibrium (UE)
28 and the Social (or System) Optimum (SO), corresponding to Wardrop's first and second equilib-
29 rium principles (Wardrop, 1952). The UE flow pattern is how we believe things will be, with driv-
30 ers choosing their routes selfishly, whilst the SO flow pattern is how the traffic engineer might *like*
31 things to be, in that the total network travel cost is minimised under SO. It is well-known that the
32 SO solution can be found by using the marginal social cost-flow functions $m(x)$ in place of the unit
33 link cost-flow functions $t(x)$ in an algorithm to produce the UE solution. It is also known that we
34 can make the congestion-minimising SO flow pattern into a UE solution by imposing the toll
35 $(m_a(x_a^*) - t_a(x_a^*))$ on link a , where x^* is the SO solution.

36 Here, we aim to formulate the same principles but in a stochastic environment. The Stochastic
37 User Equilibrium (SUE) solution corresponds to the UE solution with drivers choosing the route
38 which minimises their personal perceived travel cost, and so we seek to define a Stochastic Social
39 Optimum (SSO) which relates to the SUE solution in the same way as the SO solution relates to
40 the UE solution. The SSO solution therefore is that flow pattern which minimises the total of the
41 travel costs perceived by drivers. Just as the SO solution generally requires some drivers travelling
42 on paths which are not the minimum cost paths for that OD pair, so the SSO solution generally
43 requires some drivers to be assigned to paths that are not their minimum perceived cost path. As
44 will be seen later, Yang (1999) has characterised the SSO solution as that which maximises con-
45 sumer surplus.

46 We also investigate whether there are similar results for (i) finding the SSO solution by use of an
47 algorithm to produce the SUE solution, and (ii) whether there is a corresponding result about the
48 tolls required to make the SSO solution into a SUE solution.

49 2. Notation and assumptions

50 For convenience, we set out here the notation for the principal variables and parameters used in
51 the analysis to follow in the rest of the paper. This notation largely follows that of Sheffi (1985).

52	x_a	flow on link a
53	t_a	$t_a(x_a)$ = cost of travel along link a , a function of x_a only
54	q_{rs}	demand between OD pair rs
55	h_k^{rs}	flow on path k between OD pair rs
56	c_k^{rs}	mean perceived travel cost on path k between OD pair rs
57	m_a	$m_a(x_a)$ = marginal social travel cost on link $a = t_a + x_a \frac{dt_a}{dx_a}$
58	δ_{ak}^{rs}	1 if link a is on path k between OD pair rs , and 0 otherwise
59	S_{rs}	expected minimum perceived travel cost between OD pair rs
60	τ_a	value of the toll charged on link a

61
62 In addition to the separability of the link performance function $t_a(x_a)$, it is assumed throughout
63 the paper that this function is positive, strictly increasing, and convex. Under these conditions, as
64 Sheffi (1985) shows, the UE and SO solutions are unique. The demands q_{rs} are assumed to be con-

65 tinuous and therefore infinitely divisible, so that in calculating expected perceived travel costs, a
66 limiting Weak Law of Large Numbers applies.

67 3. Defining the SSO

68 In stochastic assignment different drivers have different perceptions of the costs on the links and
69 paths, and we use a distribution of perceived costs to describe these differences. Whereas the SO
70 flow pattern is that which minimises the total network travel cost, the SSO is defined as that flow
71 pattern that minimises the total of the *perceived* travel costs in the network.

72 To illustrate the concepts, let us first consider the case of a two-path network between a single
73 O-D pair with a fixed demand q . The flows on the paths are denoted by h_1 and h_2 ($h_1 + h_2 = q$), a
74 driver's perceived values of the path costs are denoted by u_1 and u_2 and the probability density
75 function of the drivers' perceived costs is $f(u_1, u_2)$. Firstly, given path flows of h_1 and h_2 we need
76 to allocate the drivers to the paths so as to minimise the total perceived cost. Generally, this will
77 require some drivers to be assigned to paths that are not their minimum perceived cost paths. See
78 Fig. 1: drivers whose perceived costs lie within the region R_1 (above the line BC) will be assigned
79 to path 1; those whose perceived costs lie below BC will be assigned to path 2. The boundary be-
80 tween R_1 and R_2 is the line BC with equation $u_2 = u_1 + d_2$ where the value of d_2 is such that the
81 probability mass contained within R_1 is $p_1 = h_1/q$. Note that those drivers whose perceived costs
82 fall between the lines BC and OA (the $u_1 = u_2$ line) are those who, for the benefit of the population
83 as a whole, are assigned to their non-minimum cost path.

84 Therefore, d_2 must be found such that

$$\int_{R_1} f(u_1, u_2) du_1 du_2 = p_1 = \frac{h_1}{q} \tag{1}$$

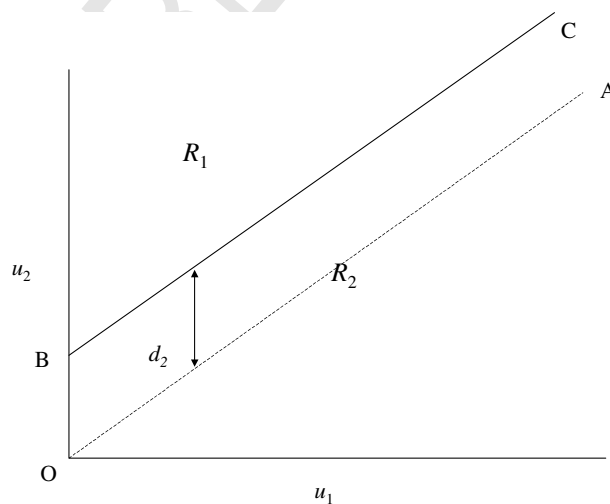


Fig. 1. Sample space of perceived costs u_1, u_2 divided into regions R_1 and R_2 .

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88 With this assignment, the total expected perceived cost is

$$z(h_1, h_2) = q \left(\int_{R_1} u_1 f(u_1, u_2) du_1 du_2 + \int_{R_2} u_2 f(u_1, u_2) du_1 du_2 \right) \quad (2)$$

92 Note that the value of d_2 and hence the regions R_1 and R_2 depend on the path flows h_1, h_2 . Also,
93 the mean perceived path costs c_1 and c_2 are also functions of the path flows, through the cost-flow
94 relations. However, we make the assumption throughout this paper that it is *only* the means c_1, c_2
95 that are affected by the path flows; the variances and covariances remain fixed. Therefore, the fol-
96 lowing condition holds for the bivariate density function of perceived path costs

$$f(u_1 + d_1, u_2 + d_2; c_1, c_2) = f(u_1, u_2; c_1 - d_1, c_2 - d_2) \quad \forall d_1, d_2 \quad (3)$$

100 The choice model is assumed to be a utility-maximising model, including both the logit and probit
101 models. In the logit model, the perception errors are independent Gumbel variates, with fixed
102 variances. In the probit model, the perception errors are multivariate Normal and we assume
103 the (co)variances to be constant (possibly at values related to the free-flow mean costs, as sug-
104 gested by Sheffi (1985)[p. 313] in connection with the Sheffi and Powell objective function for
105 SUE).

106 Hence, from (2) and (3), the total perceived network cost, for flow pattern h is

$$\begin{aligned} z(h_1, h_2) &= q \left(\int_{u_1 < u_2 - d_2} u_1 f(u_1, u_2; c_1, c_2) du_1 du_2 + \int_{u_1 > u_2 - d_2} u_2 f(u_1, u_2; c_1, c_2) du_1 du_2 \right) \\ &= q \left(\int_{u_1 < u_2} u_1 f(u_1, u_2; c_1, c_2 - d_2) du_1 du_2 + \int_{u_1 > u_2} (u_2 + d_2) f(u_1, u_2; c_1, c_2 - d_2) du_1 du_2 \right) \\ &= q \left(\int_{u_1 < u_2} u_1 f(u_1, u_2; c_1, c_2 - d_2) du_1 du_2 + \int_{u_1 > u_2} u_2 f(u_1, u_2; c_1, c_2 - d_2) du_1 du_2 \right) \\ &\quad + q d_2 \int_{u_1 > u_2} f(u_1, u_2; c_1, c_2 - d_2) du_1 du_2 \end{aligned}$$

109 Hence

$$z(h_1, h_2) = q(S(c_1, c_2 - d_2) + d_2 p_2) = qS(c_1, c_2 - d_2) + h_2 d_2 \quad (4)$$

113 where S denotes the “satisfaction” or composite travel cost, given for the logit case by the familiar
114 “logsum” formula

$$S(c_1, c_2) = -\frac{1}{\theta} \log(\exp(-\theta c_1) + \exp(-\theta c_2))$$

117 The SSO flow pattern is then defined as that flow pattern h_1, h_2 that minimises the total perceived
118 travel cost $z(h_1, h_2)$. Note that the decision rule for assigning users to paths can be expressed in the
119 form: assign a user to path 1 if his perceived costs are such that $u_1 - d_1 < u_2 - d_2$ for any pair of
120 values of d_1, d_2 that satisfy the condition in (1): that is, it is only the relative values of the d 's that
121 matters.

122 3.1. A numerical example

123 To illustrate these ideas, consider a simple example with two parallel paths between a single O–
 124 D pair. We assume that the perceived path costs are independent and Gumbel distributed, with
 125 means c_1 and c_2 and a value of the sensitivity parameter θ of 0.1. The two paths have BPR-style
 126 cost-flow functions so that the mean path costs are given by, $c_1 = 10 + 0.02h_1$ and
 127 $c_2 = 15 + 0.005h_2$. The demand $q = 1000$.

128 Since u_1 and u_2 are Gumbel distributed with means c_1 and c_2 the proportion p_1 of drivers for
 129 whom $u_1 < u_2 - d_2$ is the same as the proportion for whom $u_1 < u_2$ when the means are c_1 and
 130 $c_2 - d_2$; that is

$$p_1 = \frac{\exp(-\theta c_1)}{\exp(-\theta c_1) + \exp(-\theta(c_2 - d_2))}$$

133 so that the value of d_2 required to give the correct probability mass $p_1 = h_1/q$ is

$$d_2 = -\frac{1}{\theta} \log\left(\frac{h_1}{h_2}\right) - c_1 + c_2$$

136 Hence the SSO objective function in this two-path logit case is

$$z(h_1, h_2) = -\frac{q}{\theta} \log(\exp(-\theta c_1(h_1)) + \exp(-\theta(c_2(h_2) - d_2))) \\ - \left(\frac{1}{\theta} \log\left(\frac{h_1}{h_2}\right) - c_1(h_1) + c_2(h_2)\right) h_2$$

139 For this example, we can plot the value of this SSO objective function against h_1 . For comparison,
 140 we also show in Fig. 2 the plots of the UE, SO and SUE objective functions against h_1 . The posi-

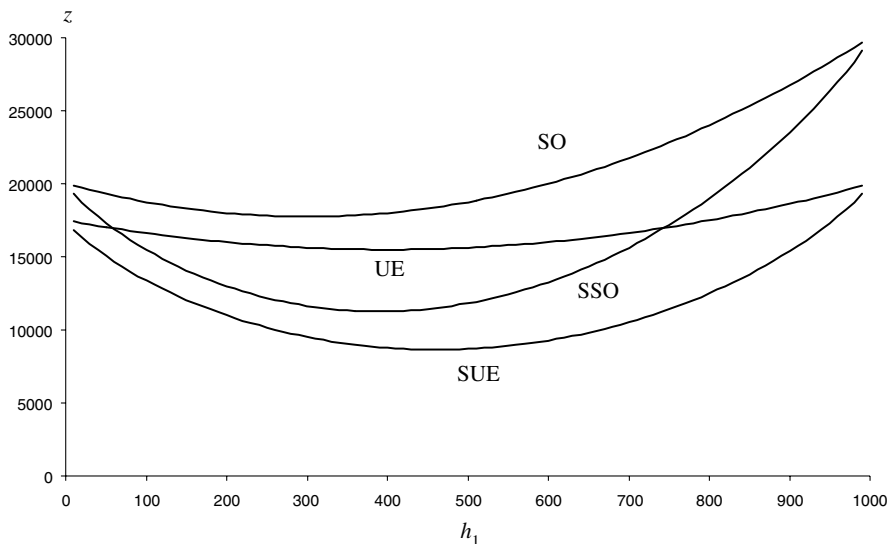


Fig. 2. Plots of z_{SSO} , z_{SO} , z_{UE} and z_{SUE} against h_1 , the flow on path 1.

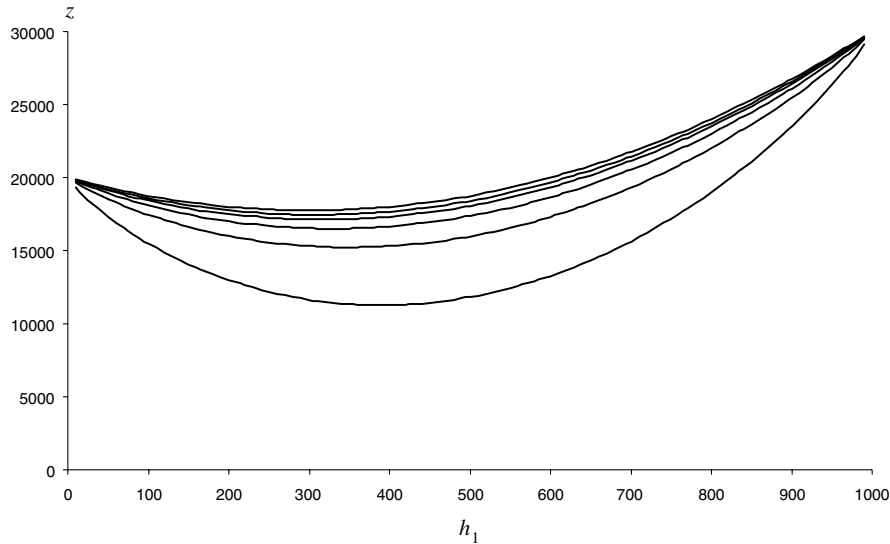


Fig. 3. Plots of z_{SSO} for values of $\theta = 0.1, 0.25, 0.5, 1$ and 2 , and z_{SO} against h_1 the flow on path 1 (from bottom to top).

141 tions of the minima show that the solution for UE is $\mathbf{h} = (400, 600)$, that for SO is $\mathbf{h} = (300, 700)$,
 142 that for SUE is $\mathbf{h} = (462, 538)$ and that for SSO is $\mathbf{h} = (390, 610)$ (We note in passing that it can be
 143 verified that, in this case, this SSO solution is the same as the SUE solution that is obtained by
 144 replacing the unit cost-flow functions by the marginal social cost-flow functions.). In Fig. 3, we
 145 plot the SSO objective function for several values of the sensitivity parameter θ
 146 ($=0.1, 0.25, 0.5, 1$ and 2) and it can be seen that as θ increases, the plot of the SSO objective func-
 147 tion steadily approaches that of the SO objective function, as would be expected, since the degree
 148 of variation in the perceived costs is steadily reducing towards zero.

149 We now aim to extend the formulation in (4) to a more general case, with many (generally over-
 150 lapping) paths between an O-D pair and, subsequently, to multiple O-D pairs. Now, whereas in
 151 the SUE case, the user chooses the path for which the perceived cost is minimum, for the SSO case
 152 we extend the decision rule for the two-path case to one that states that a driver should be as-
 153 signed to path j if its “augmented cost” $u_j - d_j$ is smaller than the augmented costs $u_k - d_k$ for
 154 all other paths k . The values of the d_j must then be set such that the proportion assigned by this
 155 process to path j is h_j/q . To justify this form of decision rule, we consider in the next section a dis-
 156 crete version of the problem, before returning to the continuous case thereafter.

157 4. A discrete version of the problem

158 We consider here a discrete version of the problem: that is, we assume that there is a (large)
 159 number N of users each of which has the same J paths to choose from. The users have ran-
 160 domly-drawn and independent sets of values of the perceived path costs u_{ij} which we take to
 161 be set out in an $N \times J$ matrix. These costs can be thought of being made up a mean value μ_j which
 162 depends on the flow(s) on that path and a perception error e_{ij} that is drawn from some distribution

163 (Gumbel or Normal) with zero mean and constant variances (so that as the mean of any path
164 changes through congestion effects so the perceived costs for all users on that path change by
165 the same amount).

166 The problem is how to assign users to paths so as to minimise the total perceived cost, whilst
167 ensuring that the numbers assigned to each path are as given. That is, given that we are to assign a
168 total of n_j ($j = 1, \dots, J$) users to path j ($n_1 + n_2 + \dots + n_J = N$), we are to find the optimal values of
169 the variables y_{ij} (where $y_{ij} = 1$ if user i is to be assigned to path j , and zero otherwise) so as to min-
170 imise the total perceived cost $z = \sum_{ij} y_{ij} u_{ij}$. As each user is to be assigned to just one path we must
171 have $\sum_j y_{ij} = 1$ and since we must satisfy the constraint on the numbers assigned to each path, we
172 must have $\sum_i y_{ij} = n_j$. This is a special case of the “classical transportation problem” (special in
173 that the row totals are all 1).

174 It is well-known (see, for example, Taha, 1976) that, for such a problem, a basic solution con-
175 sists of exactly $(N + J - 1)$ of the NJ cells being used. Of these it is clear that exactly N will take
176 the value 1 (one per row). The other $J - 1$ must be zeroes (but still be basic). These zero-valued
177 basic cells must therefore appear in at most $(J - 1)$ rows (they could all be in a single row, or at
178 the other extreme could each be in a different row). The optimal solution is a basic solution and
179 the standard solution algorithm iterates through a sequence of basic solutions until the optimum
180 is reached, with the value of the objective function z reducing at each iteration.

181 It is known that the following conditions hold for any basic solution at any iteration. For each
182 basic cell (whether its y_{ij} value is 0 or 1)

$$u_{ij} = \alpha_i + \beta_j$$

185 and for each non-basic cell a negative value of

$$v_{ij} = u_{ij} - \alpha_i - \beta_j$$

188 indicates that if this cell were to be brought into the basis (in exchange for one of the current
189 basics) the z value would reduce. The condition for a basic solution to be optimal is that, for
190 all the non-basic cells, the v_{ij} are ≥ 0 (an equality indicates the existence of an equally-optimal
191 solution). The values of the β_j are determined from the (at most) $(J - 1)$ rows that contain the
192 zero-valued basics. Once they have been found, it is trivial to determine the values of the α_i for
193 all other rows.

194 With an optimal assignment of users to paths, then, a user i is assigned to that path j for which
195 $y_{ij} = 1$. Therefore $\alpha_i = u_{ij} - \beta_j$ and for any other path k , $v_{ik} \geq 0$ so that $u_{ik} - \alpha_i - \beta_k \geq 0$. Hence
196 $u_{ik} - \beta_k \geq u_{ij} - \beta_j$ for all other paths k in that row. That is, at the optimal solution, the β_j values
197 are such that each user i is assigned to that path that has minimal value of $(u_{ij} - \beta_j)$ and the total
198 number assigned to path j is the required value n_j ($j = 1, \dots, J$).

199 The β_j play the role, then, of the d_j in the continuous case (where the form of the decision rule
200 was previously assumed by extrapolation from the two-path case). Since we can make the discrete
201 case as close as we like to the underlying continuous case, by making the number of rows (users) N
202 as large as we like, we deduce that the same result applies in the continuous case: that is, to find
203 the optimal assignment of users to paths such that the proportions so assigned should be con-
204 strained to take the values h_j/q , we need to determine values d_j such that each user is assigned
205 to that path j for which his value of $(u_j - d_j)$ is minimum.

206 **5. The general case**

207 For a single O–D pair, then, with a demand q and with given path flows h_j , we must assign users
 208 to paths so as to minimise their total perceived travel cost, by seeking to partition the whole space
 209 of perceived costs \mathbf{u} into mutually exclusive and exhaustive regions $\{R_j\}$ in an optimal manner.
 210 From the previous section we have seen that the region R_j is that within which $u_j - d_j < u_k - d_k$
 211 for all other k . Therefore,

$$\int_{R_j} f(u_1, u_2, \dots) du_1 du_2 \dots = p_j = \frac{h_j}{q} \quad (5)$$

214 With this assignment, by extension of the expression in (2) the total perceived cost is

$$z(h_1, h_2, \dots) = q \sum_j \int_{R_j} u_j f(u_1, u_2, \dots) du_1 du_2 \dots$$

217 which, setting $w_j = u_j - d_j$ and denoting by C_j the set of perceived path costs for which path j is the
 218 optimum

$$\begin{aligned} z(h_1, h_2, \dots) &= q \sum_j \int_{C_j} (w_j + d_j) f(w_1, w_2, \dots; c_1 - d_1, c_2 - d_2, \dots) dw_1 dw_2 \\ &= q \sum_j \int_{C_j} w_j f(w_1, w_2, \dots; c_1 - d_1, c_2 - d_2, \dots) dw_1 dw_2 \dots + q \sum_j d_j p_j \\ &= qS(c_1 - d_1, c_2 - d_2, \dots) + \sum_j d_j h_j \end{aligned}$$

221 This is for a single O–D pair. Suppose we now have multiple O–D pairs, identified by rs , and with
 222 path flows denoted by h_j^{rs} . The mean travel cost on path j between O–D pairs rs is denoted by c_j^{rs} .
 223 Then the objective function is the total perceived travel cost, taken over all rs

$$z_{SSO}(\mathbf{h}) = \sum_{rs} q_{rs} S^{rs}(\mathbf{c}^{rs} - \mathbf{d}^{rs}) + \sum_{rs} \sum_j d_j^{rs} h_j^{rs} \quad (6)$$

226 which is to be minimised w.r.t. the h_j^{rs} subject to $q_{rs} = \sum_j h_j^{rs} \forall rs$. The path travel costs c_j^{rs} are
 227 potentially functions of *all* path flows (that is, from other O–D pairs as well as from rs), whilst
 228 the d_j^{rs} that are required to satisfy the constraints on the proportions assigned to each path are
 229 functions of the path flows for that rs pair only. Hence the Lagrangian is

$$L(\mathbf{h}) = \sum_{rs} q_{rs} S^{rs}(\mathbf{c}^{rs}(\mathbf{h}) - \mathbf{d}^{rs}(\mathbf{h}^{rs})) + \sum_{rs} \sum_j d_j^{rs} h_j^{rs} + \sum_{rs} \lambda_{rs} (q_{rs} - \sum_j h_j^{rs}) \quad (7)$$

233 Then, differentiating (7) partially w.r.t. a typical path flow, and using the well-known result (see,
 234 for example, Sheffi, 1985) that $\frac{\partial S^{rs}}{\partial \mu_j^{rs}} = p_j^{rs}$, we get

$$\frac{\partial L}{\partial h_j^{rs}} = \sum_{od} q_{od} \sum_k p_k^{rs} \frac{\partial c_k^{od}}{\partial h_j^{rs}} - q_{rs} \sum_k p_k^{rs} \frac{\partial d_k^{rs}}{\partial h_j^{rs}} + d_j^{rs} + \sum_k \frac{\partial d_k^{rs}}{\partial h_j^{rs}} h_k^{rs} - \lambda_{rs} \quad (8)$$

237 Since for all feasible flow patterns h_j^{rs} , the choice of the d_j^{rs} ensures that $q_{rs}p_k^{rs} = h_k^{rs}$, the second and
 238 fourth terms in the above expression cancel, and by setting the derivative above to zero, we obtain
 239 the following condition at the SSO solution

$$d_j^{rs} = - \sum_{od} q_{od} \sum_k p_k^{rs} \frac{\partial c_k^{od}}{\partial h_j^{rs}} + \lambda_{rs} \quad (9)$$

242 Now the travel cost on a path is the sum of the costs on the links that make up that path. Hence
 243 $c_k^{od} = \sum_a \delta_{ak}^{od} t_a$ where $\delta_{ak}^{od} = 1$ if link a is part of path k between O–D pair od , and zero otherwise.
 244 Also, we note that it is only the relative values of the d_j^{rs} for any rs pair that matter, so that the
 245 term λ_{rs} on the right hand side can be dropped. Therefore, at the SSO solution

$$d_j^{rs} = - \sum_{od} q_{od} \sum_k p_k^{od} \sum_a \delta_{ak}^{od} \frac{dt_a}{dx_a} \delta_{aj}^{rs} = - \sum_a \left(\sum_k \sum_{od} \delta_{ak}^{od} q_{od} p_k^{od} \right) \delta_{aj}^{rs} \frac{dt_a}{dx_a} = - \sum_a x_a \delta_{aj}^{rs} \frac{dt_a}{dx_a}$$

248 Hence, the condition at the SSO solution is that the mean augmented travel cost on path j between
 249 O–D pair rs is given by

$$c_j^{rs} - d_j^{rs} = c_j^{rs} + \sum_a x_a \delta_{aj}^{rs} \frac{dt_a}{dx_a} \quad (10)$$

253 which is the marginal social cost for that path.

254 This condition in (10), taken together with the conditions $q_{rs}p_j^{rs} = h_j^{rs}$ (for all j, r, s), implies that,
 255 at the SSO solution \mathbf{h}_{SSO}^* when the marginal social path costs are used as the mean path costs in a
 256 stochastic loading to provide the path choice proportions $\{p_j^{rs}\}$ and hence the auxiliary flow pat-
 257 tern, this auxiliary flow pattern is \mathbf{h}_{SSO}^* .

258 By comparison, we note that at the SUE flow pattern \mathbf{h}_{SUE}^* when the standard path costs c_j^{rs} are
 259 used as the mean path costs in a stochastic loading to provide the path choice proportions and the
 260 auxiliary flow pattern, this auxiliary flow pattern is \mathbf{h}_{SUE}^* .

261 This suggests that whereas in finding the SUE solution by an iterative process such as the Meth-
 262 od of Successive Averages (MSA), it is the standard path costs c_j^{rs} that are used to calculate the
 263 auxiliary flow pattern at each iteration, it is the *marginal social costs* that should be used instead to
 264 find the SSO solution.

265 The finding in (10) shows that there is the same relationship between the SSO and SUE solu-
 266 tions as there is between the SO and UE solutions. Just as an algorithm for solving the UE prob-
 267 lem can be used to find the SO problem, by replacing the standard costs by the marginal social
 268 costs, so an algorithm for solving the SUE problem can be used to find the SSO solution by
 269 replacing the standard path costs by the marginal social costs.

270 Furthermore, it should be noted that at the SSO solution the difference between the marginal
 271 social path cost and standard path cost consists of a sum over the links a that make up that path,
 272 and that the contribution from a link is the same for all O–D pairs. Therefore, the result for the
 273 deterministic case about optimal tolls holds also for the stochastic case; that is, if the tolls
 274 $\tau_a = x_a \frac{dt_a}{dx_a}$ are applied (with values of the x_a taken at the SSO solution), the resulting SUE solution
 275 is the SSO flow pattern. Of course, this marginal social cost (MSC) toll set is not the *only* toll set

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276 capable of making the SSO solution into a SUE solution. Any (non-negative) toll set $\{\tau_a - \Delta_a\}$
277 that satisfies the constraints

$$\sum_a \delta_{ak}^{rs} \Delta_a - \gamma_{rs} = 0 \quad \forall k, r, s \quad \text{and} \quad \Delta_a \leq \tau_a \quad \forall a \quad (11)$$

281 will maintain the same relative values of the mean path costs between any O–D pair rs , and pro-
282 vide non-negative tolls. The values of the Δ_a and the γ_{rs} can then be found according to any chosen
283 criterion, such as that of minimum revenue, by solving the linear programming problem

$$\text{Maximise } z = \sum x_a^* \Delta_a \quad (12)$$

287 (where the x_a^* are the SSO link flows) subject to the constraints in (11).

288 6. The logit case

289 In the logit case, we can additionally derive a closed-form expression for the SSO objective
290 function $z_{\text{SSO}}(\mathbf{h})$, for a given set of path flows $\{h_j^{rs}\}$.

$$p_j^{rs} = \frac{\exp(-\theta(c_j^{rs} - d_j^{rs}))}{\sum_k \exp(-\theta(c_k^{rs} - d_k^{rs}))}$$

293 so that the values of any pair of d_j^{rs} and d_k^{rs} can be found from

$$\log \left(\frac{h_j^{rs}}{h_k^{rs}} \right) = -\theta(c_j^{rs} - d_j^{rs} - c_k^{rs} + d_k^{rs})$$

296 so that

$$d_j^{rs} - d_k^{rs} = c_j^{rs} - c_k^{rs} + \frac{1}{\theta} \log \left(\frac{h_j^{rs}}{h_k^{rs}} \right)$$

299 Without any loss of generality, any one of the d_j^{rs} may be set to zero. Here we shall set $d_1^{rs} = 0$ so
300 that for all other j ($\neq 1$) the necessary value of d_j^{rs} is given by

$$d_j^{rs} = c_j^{rs} - c_1^{rs} + \frac{1}{\theta} \log \left(\frac{h_j^{rs}}{h_1^{rs}} \right)$$

303 so that the objective function for any one O–D pair rs is

$$z_{\text{SSO}}(\mathbf{h}^{rs}) = -\frac{q_{rs}}{\theta} \log \left(\sum_j \exp(-\theta(c_j^{rs} - d_j^{rs})) \right) + \sum_{j \neq 1} h_j^{rs} \left[c_j^{rs} - c_1^{rs} + \frac{1}{\theta} \log \left(\frac{h_j^{rs}}{h_1^{rs}} \right) \right] \quad (13)$$

306 After substitution for the d_j^{rs} followed by some simplification, and then summing over all O–D
307 pairs, the expression for the SSO objective function in the logit case is

$$z_{\text{SSO}}(\mathbf{h}) = \sum_{rs} \sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) - \sum_{rs} \frac{q_{rs}}{\theta} \log q_{rs} + \frac{1}{\theta} \sum_{rs} \sum_j h_j^{rs} \log h_j^{rs} \quad (14)$$

310 The middle term can be omitted, as it is constant, so we can write

$$z_{\text{SSO}}(\mathbf{h}) = \sum_{rs} \sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) + \frac{1}{\theta} \sum_{rs} \sum_j h_j^{rs} \log h_j^{rs} \quad (15)$$

314 This is identical to the objective function derived by Yang (1999) that uses the expected indirect
315 utility received by a randomly sampled individual as the benefit measure (consumer surplus).
316 For comparative purposes, the expressions for the SUE objective functions in the case of logit
317 loading, and expressed in terms of a mixture of path flows \mathbf{h} , link flows \mathbf{x} and link costs \mathbf{t} , is (Fisk,
318 1980):

$$z_{\text{SUE}}(\mathbf{h}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega + \frac{1}{\theta} \sum_{rs} \sum_j h_j^{rs} \log h_j^{rs} \quad (16)$$

322 For completeness, the objective functions for SO and UE are

$$z_{\text{SO}}(\mathbf{h}) = \sum_{rs} \sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) = \sum_a x_a t_a(x_a) \quad (17)$$

$$z_{\text{UE}}(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega \quad (18)$$

329 There is a clear connection or symmetry between the four objective functions (15)–(18).

330 7. A link-based objective function for the general case

331 As it has been established in Section 5 that at the SSO solution, a stochastic loading using the
332 marginal path costs produces an auxiliary flow pattern that is identical to the current flow pattern,
333 this enables us to write down a link-based objective function for the general, utility-maximising
334 case.

335 Sheffi and Powell (1982) showed that the general SUE problem is equivalent to the uncon-
336 strained minimisation, with respect to the link flows \mathbf{x} of the following objective function

$$z_{\text{SUE}}(\mathbf{x}) = - \sum_a \int_0^{x_a} t_a(u) du + \sum_a x_a t_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{t}(\mathbf{x})] \quad (19)$$

339 where t_a is the travel cost on link a , and S_{rs} is the “satisfaction” function, the expected value of the
340 minimum perceived travel cost for users travelling between OD pair rs . Sheffi and Powell (1982)
341 shows that, under the conditions set out earlier for the link performance functions, the uniqueness
342 of the SUE solution is guaranteed. Sheffi (1985) shows that the derivative of this function with
343 respect to a link flow is given by

$$\frac{\partial z_{\text{SUE}}}{\partial x_a} = (x_a - y_a) \frac{dt_a(x_a)}{dx_a}$$

346 It follows that at the SUE solution, the auxiliary flows $\{y_a\}$ are equal to the current flows $\{x_a\}$.
 347 That is, when a stochastic loading is carried out using mean link costs based on the current link
 348 flows, the resulting auxiliary flow pattern is identical to the current flow pattern

$$y_a(\mathbf{t}(\mathbf{x})) = x_a \quad \forall a$$

351 In Section 4, it was shown that at the SSO solution, if the marginal costs are used in place of the
 352 unit costs, the auxiliary flow pattern produced from a stochastic loading is equal to the current
 353 solution. It follows therefore that we need merely replace the unit link costs t_a in the Sheffi and
 354 Powell objective function for SUE by the marginal link costs $m_a = t_a + x_a(dt_a/dx_a)$ in order to
 355 give an objective function for SSO

$$z_{SSO}(\mathbf{x}) = - \sum_a \int_0^{x_a} m_a(u) du + \sum_a x_a m_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{m}(\mathbf{x})] \quad (20)$$

358 At the SSO solution, we therefore have

$$y_a(\mathbf{m}(\mathbf{x})) = x_a \quad \forall a$$

361 Because of the correspondence between the SSO and SUE objective functions, the uniqueness of
 362 the SSO solution is guaranteed by the conditions placed on the link performance functions.

363 8. An illustrative example

364 To illustrate and confirm the results of section 4 consider the five-link network shown in Fig. 4
 365 (that has the same topology as that in Yang, 1999). There is a single O-D pair from node A to
 366 node D (with a demand of 1000 vph), and three paths: 1-4, 1-3-5, and 2-5. The link cost-flow
 367 functions are

$$\begin{aligned} t_1 &= 5 + 0.01x_1 \\ t_2 &= 10 + 0.01x_2 \\ t_3 &= 3.5 + 0.005x_3 \\ t_4 &= 8 + 0.01x_4 \\ t_5 &= 5 + 0.01x_5 \end{aligned}$$

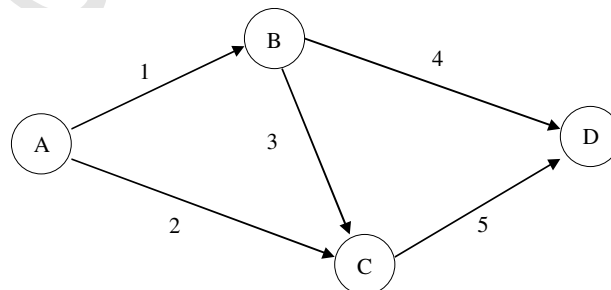


Fig. 4. Five-link, three-path network.

370 We consider the probit case: that is, Normally distributed perception errors with link variances
 371 being fixed and equal to β times the mean free-flow cost. In carrying out the stochastic loading
 372 at any iteration, the algorithm of [Donnolly \(1973\)](#) is used for calculating probabilities from the
 373 bivariate Normal distribution. For more general, larger networks, a variety of approximate meth-
 374 ods can be used to calculate the p_j^{rs} values (see [Rosa and Maher, 2002](#)).

375 The SUE and SSO solutions are found by using respectively the standard path costs c_j^{rs} or the
 376 marginal social path costs $c_j^{rs} + \sum_a x_a \delta_{aj}^{rs} \frac{dc_a}{dx_a}$ at each iteration when calculating the path choice pro-
 377 portions p_j^{rs} to produce the auxiliary path flow pattern. The results in [Table 1](#) have been obtained,
 378 to a high degree of convergence (measured directly by the difference between the current and aux-
 379 iliary solutions), by an iterative process that optimises the step length along the search direction
 380 ($y-x$) at each iteration (see, for example, [Maher and Hughes \(1997\)](#)).

381 The results are shown in [Table 1](#). It is clear that, as the variance-to-mean ratio β is steadily re-
 382 duced in value, the SUE solution moves towards the UE solution and the SSO solution moves
 383 towards the SO solution, as would be expected.

384 For a value of $\beta = 1$, then, the SSO link flows are $x_{SSO} = (578.340, 421.660, 119.275,$
 385 $459.066, 540.934)$ and hence the MSC tolls are $\tau = (5.783, 4.217, 0.596, 4.591, 5.409)$. By setting
 386 up the linear programming problem specified in [\(11\)](#) and [\(12\)](#), the minimal revenue toll set can
 387 be found

$$\text{Maximise } z = 578.34\Delta_1 + 421.66\Delta_2 + 119.28\Delta_3 + 459.07\Delta_4 + 540.93\Delta_5$$

subject to :

$$\Delta_1 + \Delta_4 - \gamma = 0$$

$$\Delta_1 + \Delta_3 + \Delta_5 - \gamma = 0$$

$$\Delta_2 + \Delta_5 - \gamma = 0$$

390 and $\Delta_1 \leq 5.783, \Delta_2 \leq 4.217, \Delta_3 \leq 0.596, \Delta_4 \leq 4.591, \Delta_5 \leq 5.409$

Table 1
Path flows assigned by SUE and SSO for various β values

	h_1	h_2	h_3
SUE $\beta = 1$	463.318	144.990	391.692
SUE $\beta = 0.1$	500.046	89.525	410.429
SUE $\beta = 0.01$	519.977	56.640	423.383
SUE $\beta = 0.001$	528.571	41.773	429.656
SUE $\beta = 0.0001$	531.750	36.155	432.094
SUE $\beta = 0.00001$	532.824	34.244	432.933
UE	533.333	33.333	433.333
SSO $\beta = 1$	471.275	99.277	429.448
SSO $\beta = 0.1$	496.446	54.406	449.148
SSO $\beta = 0.01$	508.804	31.535	459.660
SSO $\beta = 0.001$	513.897	21.936	464.167
SSO $\beta = 0.0001$	515.749	18.416	465.835
SSO $\beta = 0.00001$	516.372	17.230	466.399
SO	516.667	16.667	466.667

391 An optimal solution is found to be $\Delta = (5.035, 4.217, -0.818, 4.591, 5.409)$ so that the minimal
392 revenue tolls $\tau - \Delta = (0.748, 0, 1.414, 0, 0)$, raising a total revenue of 601 compared with a revenue of
393 10,227 raised from the MSC tolls.

394 9. Summary

395 In this paper, we have formulated the SSO (Stochastic Social Optimum) traffic assignment
396 problem to complement the well-known UE, SO and SUE problems and investigated the relation-
397 ships and similarities between them. The formulation is for a general utility-maximisation frame-
398 work, which includes as special cases both logit and probit. Under SSO assignment, the total
399 perceived travel cost is minimised with, generally, some users being assigned to paths that are
400 not their personal minimum perceived cost paths. The analysis was developed in two stages.
401 The first stage involved the optimal assignment of users to paths for any given set of path flows
402 \mathbf{h} , with the idea of augmented path costs $\mathbf{c} - \mathbf{d}$ in which the d_j^{rs} values were such that the flow pattern
403 produced by the stochastic loading matched the required flow pattern, and produced an expres-
404 sion $z_{SSO}(\mathbf{h})$ for the minimum total perceived cost. The second stage then investigated the condi-
405 tions for the minimisation of $z_{SSO}(\mathbf{h})$ with respect to the path flows \mathbf{h} . It was shown that under a
406 general utility-maximising framework that includes the two most important cases of logit and pro-
407 bit loading, the augmented path costs at the SSO solution were the marginal social costs, and
408 hence the relationship of the SSO solution to the SUE is the same as that of SO to UE. In par-
409 ticular, the SSO solution can be found by means of an SUE algorithm, by replacing the standard
410 path costs by the marginal social path costs in the stochastic loading; and the toll set that is opti-
411 mal in the stochastic case has the same form as that which is optimal in the deterministic case, but
412 evaluated at the SSO flow values instead of the SO flow values. Additionally, for the logit case, an
413 expression for the objective function $z_{SSO}(\mathbf{h})$ has been derived which has a pleasing symmetry with
414 those for UE, SO and SUE. Finally, a link-based objective function has been formulated for the
415 general utility-maximising case (that includes probit as well as logit), which is similar in form to
416 the SUE objective function of Sheffi and Powell.

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