Three solutions for a perturbed Dirichlet problem

Giuseppe Cordaro\textsuperscript{a,}* , Giuseppe Rao\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, University of Messina, 98166 Sant’Agata-Messina, Italy
\textsuperscript{b} Department of Mathematics, University of Palermo, via Archirafi, 34 - 90123 Palermo, Italy

Received 2 April 2007; accepted 18 April 2007

Abstract

In this paper we prove the existence of at least three distinct solutions to the following perturbed Dirichlet problem:

\[
\begin{aligned}
-\Delta u &= f(x, u) + \lambda g(x, u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary $\partial \Omega$ and $\lambda \in \mathbb{R}$. Under very mild conditions on $g$ and some assumptions on the behaviour of the potential of $f$ at 0 and $+\infty$, our result assures the existence of at least three distinct solutions to the above problem for $\lambda$ small enough. Moreover such solutions belong to a ball of the space $W^{1,2}_0(\Omega)$ centered in the origin and with radius not dependent on $\lambda$.

\copyright 2007 Elsevier Ltd. All rights reserved.

MSC: 35J20

Keywords: Weak solutions; Critical points; Weakly sequentially lower semicontinuity

1. Introduction and statement of the result

In this paper we present a multiplicity result for the following perturbed problem:

\[
\begin{aligned}
-\Delta u &= f(x, u) + \lambda g(x, u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  \hspace{1cm} (P_{\lambda})

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$, with boundary $\partial \Omega$ smooth enough, $\lambda$ is a real number, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following growth conditions:

(f) There exist $q > \frac{N}{2}, a_1 \in L^q(\Omega), a_2 > 0$ and $s > 1$, with $s < \frac{N+2}{N-2}$ if $N > 2$, such that:

When $N \geq 2$:

\[
|f(x, t)| \leq a_1(x) + a_2|t|^s, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.
\]  \hspace{1cm} (P_{\lambda})

* Corresponding author.
E-mail addresses: cordaro@dipmat.unime.it (G. Cordaro), rao@math.unipa.it (G. Rao).

0362-546X/S - see front matter \copyright 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2007.04.027
When $N = 1$:
\[
\sup_{|t| \leq M} |f(\cdot, t)| \in L^1(\Omega), \quad \text{for every } M > 0 \text{ and } t \in \mathbb{R}.
\]

(g) There exist $a_3 \in L^{2N/(N-2)}(\Omega)$, $a_4 > 0$ and $p > 1$, with $p < \frac{N+2}{N-2}$ if $N > 2$, such that:

When $N \geq 2$:
\[
|g(x, t)| \leq a_3(x) + a_4|t|^p, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.
\]

When $N = 1$:
\[
\sup_{|t| \leq M} |g(\cdot, t)| \in L^1(\Omega), \quad \text{for every } M > 0 \text{ and } t \in \mathbb{R}.
\]

The above growth conditions allow us to introduce the following functionals:
\[
\Psi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left( \int_0^{u(x)} f(x, t) \, dt \right) \, dx
\]
and
\[
\Phi(u) = -\int_\Omega \left( \int_0^{u(x)} g(x, t) \, dt \right) \, dx
\]
defined on the Sobolev space $W^{1,2}_0(\Omega)$, endowed with the norm of gradient $\| \cdot \| = \int_\Omega |\nabla(\cdot)|^2$. By standard results, it is well known that such functionals are well defined, continuously differentiable and weakly sequentially lower semicontinuous on $W^{1,2}_0(\Omega)$. The critical points of $\Psi + \lambda \Phi$ are the weak solutions of problem $(P_\lambda)$. We recall that a weak solution of $(P_\lambda)$ in $W^{1,2}_0(\Omega)$ is any $u \in W^{1,2}_0(\Omega)$ such that
\[
\int_\Omega \nabla u(x) \nabla v(x) \, dx - \int_\Omega (f(x, u(x)) + \lambda g(x, u(x)))v(x) \, dx = 0,
\]
for every $v \in W^{1,2}_0(\Omega)$.

Let us denote by $\lambda_1$ the first eigenvalue of the Dirichlet problem:
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We recall the variational characterization of $\lambda_1$:
\[
\lambda_1 = \inf_{u \in W^{1,2}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.
\]

Our result is the following theorem:

**Theorem 1.1.** Assume that, besides (f) and (g), the following conditions are satisfied:

(i) $\limsup_{|t| \to +\infty} \frac{\int_0^t f(x,s) \, ds}{|t|^2} < \frac{1}{2} \lambda_1$ uniformly in $x \in \Omega$.

(ii) $\limsup_{|t| \to 0} \frac{\int_0^t f(x,s) \, ds}{|t|^2} < \frac{1}{2} \lambda_1$, uniformly in $x \in \Omega$.

(iii) There exists $u_1 \in W^{1,2}_0(\Omega)$ such that $\Psi(u_1) < 0$.

Then, there exist $\lambda^* > 0$ and $r > 0$ such that, for each $\lambda \in ]-\lambda^*, \lambda^*[$, problem $(P_\lambda)$ has at least three distinct weak solutions whose norms are less than $r$.

The above theorem belongs to the class of multiplicity results for perturbed problems with minimal assumptions on the perturbation term $g$. To the best of our knowledge the first paper where the authors proposed a result of this type is [6]. In that paper, Li and Liu obtained the existence of multiple solutions for problem $(P_\lambda)$ where $g$ is supposed...
to be only continuous on \( \overline{\Omega} \times \mathbb{R} \) and \( f \) is required to be odd in the second variable \( t \) uniformly in \( x \). The possibility of considering functions \( f \) with no symmetric properties has been already widely investigated; see for instance [1–5,8].

**Theorem 1.1** gives a contribution in this direction. We propose some assumptions on the non-perturbed term \( f \) of the nonlinearity in order to obtain the existence of at least three distinct solutions to \((P_\lambda)\), for \( \lambda \) small enough. It is worth noticing that such solutions satisfy a stability property because they belong to a fixed ball centered at the origin when the parameter \( \lambda \) varies in a suitable interval.

### 2. Proof of Theorem 1.1

The first step of the proof is to apply Theorem 3.8 of [4] which is a consequence of the more general results established in [7].

From (i), choosing \( \gamma \in \mathbb{R} \) with

\[
\limsup_{|t| \to +\infty} \frac{\int_0^t f(x, s)ds}{|t|^2} < \gamma < \frac{1}{2} \lambda_1,
\]

(2.1)

there exists \( M > 0 \) such that, for all \( |t| > M \) and a.e. \( x \in \Omega \), one has

\[
\int_0^t f(x, s)ds < \gamma t^2.
\]

Define \( \{|u| \leq M\} = \{x \in \Omega : |u(x)| \leq M\} \) and denote by \( \{|u| > M\} \) its complement in \( \Omega \).

Hence it results that

\[
\Psi(u) = \frac{1}{2} \|u\|^2 - \int_{\{|u| \leq M\}} \left( \int_0^u f(x, s)ds \right) dx + \int_{\{|u| > M\}} \left( \int_0^u f(x, s)ds \right) dx
\]

\[
\geq \frac{1}{2} \|u\|^2 - \gamma \int_\Omega |u(x)|^2 dx - c
\]

\[
\geq \left( \frac{1}{2} - \frac{\gamma}{\lambda_1} \right) \|u\|^2 - c,
\]

(2.2)

for all \( u \in W^{1,2}_0(\Omega) \). The existence of a constant \( c > 0 \) such that

\[
\left| \int_{\{|u| \leq M\}} \left( \int_0^u f(x, s)ds \right) dx \right| \leq c,
\]

follows from growth condition (f).

From (2.1) and (2.2) it follows that \( \lim_{\|u\| \to +\infty} \Psi(u) = +\infty \).

Now we prove that \( u_0 \equiv 0 \) is a strict local minimum of \( \Psi \).

By (ii), we can choose \( \beta \in \mathbb{R} \) and \( \delta > 0 \) such that

\[
\frac{\int_0^t f(x, s)ds}{|t|^2} < \beta < \frac{1}{2} \lambda_1,
\]

(2.3)

for all \( 0 < |t| < \delta \) and a.e. \( x \in \Omega \). So, for each \( u \in W^{1,2}_0(\Omega) \), one has

\[
\Psi(u) = \frac{1}{2} \|u\|^2 - \int_{\{|u| < \delta\}} \left( \int_0^u f(x, s)ds \right) dx + \int_{\{|u| \geq \delta\}} \left( \int_0^u f(x, s)ds \right) dx
\]

\[
\geq \left( \frac{1}{2} - \frac{\beta}{\lambda_1} \right) \|u\|^2 - \int_{\{|u| \geq \delta\}} \left( \int_0^u f(x, s)ds \right) dx.
\]

(2.4)

In order to estimate the last term in a suitable neighbourhood of zero in \( W^{1,2}_0(\Omega) \), we distinguish the cases \( N = 1 \) and \( N > 1 \).
In the first case, exploiting the compact embedding of $C(\overline{\Omega})$ into $W^{1,2}_0(\Omega)$, one can find $r_\delta > 0$ such that
\[
\max_{x \in \overline{\Omega}} |u(x)| < \delta,
\]
for all $u \in W^{1,2}_0(\Omega)$ with $\|u\| < r_\delta$. So, if $\|u\| < r_\delta$ it follows that
\[
\int_{\{|u| \geq \delta\}} \left( \int_0^{|u(x)|} f(x,s)ds \right) dx = 0.
\]
(2.5)
In the case $N > 1$, from (f), we have
\[
\left| \int_{\{|u| \geq \delta\}} \left( \int_0^{|u(x)|} f(x,s)ds \right) dx \right| \leq \int_{\{|u| \geq \delta\}} a_1(x)|u(x)| + \int_{\{|u| \geq \delta\}} \frac{a_2}{s+1}|u(x)|^{s+1} dx.
\]
(2.6)
If $N > 2$, since $q > \frac{N}{2}$, there exists $m \in \mathbb{R}$ with $\frac{2a}{q-1} < m < \frac{2N}{N-q}$. When $N = 2$, it is enough to choose $m > \frac{2a}{q-1}$.

Setting $l = \frac{m(q-2)}{q}$, one has
\[
\int_{\{|u| \geq \delta\}} a_1(x)|u(x)| \leq \int_{\{|u| \geq \delta\}} \frac{a_1(x)}{\delta^{q-1}} |u(x)|^q dx
\leq \frac{1}{\delta^{q-1}} \left( \int_{\Omega} a_1(x)|u|^q dx \right) \frac{q-1}{q} \left( \int_{\Omega} |u|^m dx \right)^{\frac{q-1}{q}}
\leq C_1 \|u\|^q.
\]
(2.7)
Here $C_1 > 0$, constant with respect to $u$, exists because of the embedding theorems. Analogously there exists $C_2 > 0$, not dependent on $u$, such that
\[
\int_{\{|u| \geq \delta\}} \frac{a_2}{s+1}|u(x)|^{s+1} dx \leq C_2 \|u\|^{s+1}.
\]
(2.8)
From (2.4)–(2.7) and (2.8) it follows that
\[
\Psi(u) \geq \left( \frac{1}{2} - \frac{\beta}{\lambda_1} \right) \|u\|^2 - C_1 \|u\|^q - C_2 \|u\|^{s+1},
\]
for all $u \in W^{1,2}_0(\Omega)$, with $\|u\| < r_\delta$. Since $l > 2$, $s + 1 > 2$ and $\frac{\beta}{\lambda_1} < \frac{1}{2}$, $\Psi$ has a strict local minimum at $u_0 \equiv 0$.

By condition (iii), $u_0$ is not a point of global minimum for $\Psi$.

At this point we apply Theorem 3.8 of [4] twice, taking as the perturbing term $\Phi$ and $-\Phi$. So, choose $r_1 > 0$ such that $u_0 \equiv 0$ is a strict global minimum of $\Psi$ in $B(0,r_1)$, where $B(0,r_1)$ is the open ball in $W^{1,2}_0(\Omega)$ centered at the origin and with radius $r_1$. For any $\rho_1, \rho_2 \in \mathbb{R}$ with $\inf_{W^{1,2}_0(\Omega)} \Psi < \rho_1 < 0$ and $\rho_2 > 0$, there exists $\tilde{\lambda} > 0$ such that $\Psi + \lambda \Phi$ has two distinct local minima $u_1^{(\lambda)} \in \Psi^{-1}([-\infty, \rho_1])$ and $u_2^{(\lambda)} \in \Psi^{-1}([\rho_2, \infty)) \cap B(0,r_1)$, for all $\lambda \in [-\tilde{\lambda}, \tilde{\lambda}]$.

Hence, arguing as in the proof of Theorem 4.2 in [4] and applying a mountain pass lemma without a (P.S.) condition, Theorem 2.8 of [9], there exist $r > 0$ and $\lambda^* \in \mathbb{R}$ with $0 < \lambda^* < \tilde{\lambda}$, such that $\Psi + \lambda \Phi$ has a third critical point $u_3^{(\lambda)}$ distinct from $u_1^{(\lambda)}$ and $u_2^{(\lambda)}$, and $u_1^{(\lambda)}, u_2^{(\lambda)}, u_3^{(\lambda)} \in B(0,r)$, for all $\lambda \in [-\lambda^*, \lambda^*]$. So the theorem is proved.

References