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Asset and liability management for insurance products with minimum guarantees: The UK case

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Abstract

Modern insurance products are becoming increasingly complex, offering various guarantees, surrender options and bonus provisions. A case in point are the with-profits insurance policies offered by UK insurers. While these policies have been offered in some form for centuries, in recent years their structure and management have become substantially more involved. The products are particularly complicated due to the wide discretion they afford insurers in determining the bonuses policyholders receive. In this paper, we study the problem of an insurance firm attempting to structure the portfolio underlying its with-profits fund. The resulting optimization problem, a non-linear program with stochastic variables, is presented in detail. Numerical results show how the model can be used to analyze the alternatives available to the insurer, such as different bonus policies and reserving methods.

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1. Introduction

In recent times there has been a dramatic move of households' assets towards higher return investment vehicles. By far the vast majority of these assets have entered the mutual fund market. Insurance companies have attempted to compete for a portion of households' funds by introducing innovative policies offering various bonus provisions, minimum rates of guarantee and surrender options. This paper analyzes the problem of optimally structuring the portfolio of an insurer that offers accumulating "with-profits" policies with minimum guarantees, similar to those offered in the UK. These products offer policyholders both a minimum guaranteed rate of return, and the ability to participate in the returns of a portfolio with a high equity content. The insurer also provides a smoothing of these returns to the policyholders, so that they do not experience the full volatility of the underlying portfolio. In this paper, we present the insurer's portfolio selection problem as a non-linear mathematical program with stochastic variables, which is analyzed to highlight the different features of the policies.

1.1. The insurance products

Insurance products in which investors participate in the profits of the company, referred to as "with-profits" policies, have been offered in the UK since the 18th century. The modern products have become significantly more complicated than the originals, offering numerous guarantees and surrender options. The policies have also recently come under a great deal of scrutiny due to the wide discretion they afford the firm in terms of structuring the bonuses received by policyholders. Ironically, the first company to offer with-profits policies, Equitable Life (the world's oldest mutually owned assurance company), had to close its fund to new business after suffering substantial losses and losing a high profile decision by the House of Lords (Equitable v Hyman). The case arose when Equitable attempted to exercise the discretion that it believed it had under the policies.

A useful description of the nature and types of with-profits business in the UK is provided by the Financial Services Authority (2001). In particular, the following key features of with-profits policies are identified:

- Premiums are pooled and invested in a portfolio with significant proportions in equity and property (high return, but high risk investments).
- The insurance company provides some "smoothing" of the portfolio returns, so that investors do not experience the full volatility of the portfolio.
- Investors may participate in the profits and losses of the insurer, including mortality and expense risks.

- Certain guarantees are offered. For example, the payment of a guaranteed amount on retirement or death.

The guarantees mentioned in the final point are often stated in terms of a prespecified minimum rate of return. Thus, investors participate in the returns of the insurance company's portfolio, subject to a minimum floor on their return. Investors also have the ability to surrender the policy before maturity, possibly subject to a penalty.

1.2. Bonuses

In addition to the guaranteed rate of return, policyholders receive bonuses. These bonuses are meant to reflect the overall performance of the firm's portfolio, and to correspond to "Policyholders' Reasonable Expectations", based on such things as the firm's past performance, market practice, and any promotional material or communications to policyholders. Policyholders receive two types of bonus: regular and terminal bonuses. Regular bonuses are declared each year. Once a regular bonus has been declared, the bonus becomes guaranteed and the minimum rate of return now applies to the original amount plus the declared bonus. The terminal bonus is awarded upon maturity or surrender of the policy or upon occurrence of the insured event.

Bonuses typically reflect:

- The return achieved by the firm's portfolio.
- General level of performance of the market.
- Policyholders' expectations of bonus levels.
- A target level of terminal bonus (often expressed as a percentage of total policyholder benefits).

1.3. Existing literature

There is now a substantial and growing literature on the *pricing* of insurance products with guarantees. The first papers date back to the 1970s, including Brennan and Schwartz (1976) and Boyle and Schwartz (1977), which analyzed unit-linked maturity guarantees. More recently, a complete analysis of the policies has been given by Grosen and Jørgensen (2000), who decompose the liability into a risk-free bond (the minimum guarantee), a bonus option and a surrender option. Since the ability of the insurance firm to meet any prespecified guarantee is greatly dependent on interest rates, pricing the policies in the context of an appropriate stochastic interest rate model is particularly important. Miltersen and Persson (1999) address the pricing of insurance products with minimum guarantees in the interest rate framework of Heath et al. (1992). It is important to note that the literature on pricing the policies assumes that the company's asset portfolio is given exogenously, and does not address the problem of structuring this portfolio optimally.

Research on with-profits policies has mainly focussed on methods for reserving and valuing the policies. Chadburn (1997) analyzes the impact of different reserving methods for addressing insurance company insolvency. Boyle and Hardy (1997) compare a traditional reserving method with one based on an option pricing approach.

Comparatively little research has been undertaken on the issue of the optimal structure of the insurance company's asset portfolio when guarantees are offered. In recent work, Iwaki and Yumae (2002) consider the problem of optimally structuring the portfolio of a firm offering a one-off maturity guarantee in a continuous time economy. Jensen and Sørensen (2002) ask whether products with minimum guarantees really serve investors' interests, with interesting conclusions. The study of the asset and liability management of "plain-vanilla" minimum guarantee products such as those offered in the Italian industry has been undertaken by Consiglio et al. (2001). Among the results in their paper, they demonstrated that the firm could substantially increase shareholder value by considering the integrated asset and liability management problem of structuring the firm's portfolio optimally. In particular, it was shown that firms could increase their profits and offer higher guarantees by investing a higher proportion of their assets in an (optimally structured) equity portfolio. Consiglio et al. (2003a,b) show that for these products, the portfolio optimization problem can be solved very efficiently using algorithms for generalized geometric programming. Booth et al. (1997) study the utility maximization of asset allocations for insurers offering non-profit policies in the UK, without guarantees. Consiglio, Saunders and Zenios (2002) present a comparison of the Italian and UK minimum guarantee policies.

The remainder of the paper is structured as follows. Section 2 presents the scenario optimization model for management of an insurance company offering accumulating with-profits policies. Section 3 presents results based on employing the model with scenarios generated using the Wilkie (1995) stochastic asset model. Section 4 presents results on the sensitivity of the model to different parameters and insurance policy features. Section 5 summarizes the paper and presents conclusions.

2. The scenario optimization model

In this section, we present the model for asset and liability management for insurance products with guarantees. The model is a non-linear mathematical program, which models stochastic variables using discrete scenarios. All portfolio decisions are made at time $t = 0$ in anticipation of an uncertain future. At the end of the planning horizon, the impact of these portfolio decisions is evaluated and risk aversion is introduced through a utility function. Portfolio decisions optimize expected utility over the specified horizon.

2.1. Features of the model

We let Ω denote the index set of scenarios $l = 1, 2, \ldots, N$, indicating realizations of random variables, $\mathcal{U}$ the universe of available asset classes, and $t = 1, 2, \ldots, T$, discrete points in time from today ($t = 0$) until the maturity date T.

We consider four accounts:

1. The liability account L, which grows according to the guaranteed rate and is augmented by any declared bonuses. L_t^l is the liability value at time t in scenario l.
2. The asset account A, which grows according to the portfolio returns, net of any payments due to death or policy surrender. We denote by A_t^l the asset value at time t in scenario l.
3. The "reduced" asset account RA, which is used for the purpose of calculating regular bonuses. The value of the policyholders' reduced asset share at time t in scenario l is denoted by RA_t^l.
4. The equity account E, which tracks the present value of all funds invested by shareholders. The total equity at time t in scenario l is denoted by E_t^l.

The multi-period dynamics of these accounts are conditioned on discrete scenarios of realized asset returns and the composition of the asset portfolio.

In this paper, we consider a proprietary company operating a fund on a 90/10 basis (i.e. 90% of the benefits go to the policyholders and 10% go to the shareholders). We assume that the company has a unique cohort of single-premium endowment policies (i.e. there is only one generation of policyholders, paying a single upfront premium) and attempt to determine the investment strategy that maximizes shareholders' utility.

2.2. Model parameters

The parameters of the model are as follows:

- r_{it}^l, rate of return of asset i during the period $t-1$ to t in scenario l.
- r_{ft}^l, risk-free rate during the period $t-1$ to t in scenario l.
- g, minimum guaranteed rate of return.
- ρ, regulatory equity to debt ratio.
- B_t^l, benchmark rate, usually taken to be the yield on consols.
- Λ_t^l, rate of abandon of the policy due to lapse or death at period t in scenario l.
- γ, policyholders' rate of participation in the profits of the firm (usually taken to be 90%).
- β, target terminal bonus rate.

2.3. Decision variables

The decision variables of the model are defined as follows:

- x_i, fraction of initial capital invested in the ith asset.
- A_t^l, value of the assets at time t under scenario l.
- L_t^l, value of the guaranteed liability at time t under scenario l.

- E_t^l, value of the equity at time t under scenario l.
- RA_t^l, value of the "reduced asset share" (used in calculating regular bonuses in some approaches) at time t under scenario l.
- y_{At}^l, expenses due to lapse or death at time t in scenario l.
- z_t^l, amount of equity provided by shareholders at time t in scenario l.
- RB_t^l, declared rate of regular bonus at time t in scenario l.
- ΔRB_t^l, change in regular bonus between time $t-1$ and time t.
- TB^l, policyholders' terminal bonus under scenario l.
- R_{Pt}^l, portfolio rate of return during the period $t-1$ to t in scenario l, with positive and negative parts R_t^{+l} and R_t^{-l}.
- y_t^{+l}, y_t^{-l}, positive and negative parts of the policyholders' shortfall under g at time t in scenario l.

2.4. Variable dynamics and constraints

We invest the premium collected (L_0) and the equity required by the regulators (E_0) in the asset portfolio. This initial amount A_0 is allocated to assets in proportion x_i such that $\sum_{i\in\mathscr{U}} x_i = 1$, and the dynamics of the portfolio return are given by

$$R_{Pt}^l = \sum_{i\in\mathscr{U}} x_i r_{it}^l, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{1}$$

We observe that the above equation assumes that the portfolio is rebalanced to the initial asset mix x_i, $i \in \mathscr{U}$ at the end of each year (a so-called "fixed-mix" strategy). The strategy therefore entails some transactions costs, which could be introduced into the optimization model without difficulty. Their effects tend to be secondary, and we ignore them in this paper. Shortselling is disallowed, and hence the investment variables are constrained to be non-negative.

The liability account grows at the guaranteed rate, plus any additional rate due to the declared regular bonus. Therefore the dynamics of the liability are given by

$$L_t^l = (1 - \Lambda_t^l) L_{t-1}^l (1 + g)(1 + \max[RB_t^l, 0]). \tag{2}$$

We shall examine different methods for determining the regular bonus (and therefore different equations for RB_t^l) in Section 2.5.

Shortfalls are funded through the infusion of additional equity by shareholders. The dynamics of the equity therefore become

$$E_t^l = E_{t-1}^l (1 + r_{ft}^l) + z_t^l, \tag{3}$$

where z_t^l is the amount of equity infused to fund shortfall at time t under scenario l. We shall investigate different methods of funding shortfall (and therefore different equations for E_t^l) in Section 2.6.

Any equity provided by the shareholders is immediately invested in the asset portfolio. This leads to the following equation for the dynamics of the assets.

$$A_t^l = A_{t-1}^l (1 + R_{Pt}^l) + z_t^l - y_{At}^l. \tag{4}$$

Upon surrender or occurrence of the insured event, policyholders receive the guaranteed amount:

$$y^l_{At} = \Lambda^l_t L^l_{t-1}(1+g)(1+RB^l_{t-1}). \tag{5}$$

For the results reported in this paper, we have chosen to ignore lapse, and set $\Lambda \equiv 0$, in order to focus on other policy features.

2.5. Bonus policy

In this section, we describe two different alternatives for the structure of a bonus policy, reflecting two views on bonuses that have been put forth in the UK. The first is derived from work done by an Institute of Actuaries Working Party, as presented by Chadburn (1997). The second is a more traditional actuarial approach, based on Ross (1989), which assumes that assets and liabilities grow at fixed rates and aims for a target level of terminal bonus.

2.5.1. The working party approach

This bonus policy is based on Chadburn (1997), which is in turn based on work done by an Institute of Actuaries Working Party. The bonus policy has been somewhat simplified in order to bring out its most salient features. Chadburn states (with our notation):

> The main features of the assumed bonus philosophy are to declare a regular bonus RB^l_t... which, together with any guaranteed rate of fund increase, broadly reflects the yield on consols, subject to policyholders' funds remaining lower than the value of a 'reduced policy asset share' (RA^l_t). RA^l_t accumulates at approximately 75% of the total rate of return on assets... Should L^l_t exceed RA^l_t, pressure to cut the bonus rate will be generated in the model. Hence the difference between RA^l_t and the full policy asset share (A^l_t) effectively represents a minimum value for the terminal bonus payable under the contract.

There is also some smoothing of the reversionary bonus rates over time. This smoothing (representing policyholders' reasonable expectations that bonuses will not fluctuate wildly from year to year) is reflected in the autoregressive nature of the bonus equation:

$$RB^l_t = \frac{1}{2}RB^l_{t-1} + \Delta RB^l_t. \tag{6}$$

The change in the regular bonus ΔRB^l_t reflects the yield on consols, and the fact that the liabilities should not exceed the policyholders' reduced asset share:

$$\Delta RB^l_t = p\max\left(\frac{B^l_t - g}{1+g}, 0\right) - q\max\left(\frac{L^l_{t-1} - RA^l_{t-1}}{RA^l_{t-1}}, 0\right), \tag{7}$$

where p and q are parameters representing the relative contributions from the market benchmark and the reduction for solvency. We usually take $p = \frac{1}{2}$ and $q = \frac{1}{4}$, as in Chadburn (1997). The reduced asset share is given by

$$RA_t^l = RA_{t-1}^l \left(1 + \frac{3}{4} R_{Pt}^{+l} - \frac{4}{3} R_{Pt}^{-l}\right), \tag{8}$$

where R_{Pt}^{+l} and R_{Pt}^{-l} are the positive and negative parts of the portfolio return:

$$R_{Pt}^l = R_{Pt}^{+l} - R_{Pt}^{-l} \tag{9}$$

with $R_{Pt}^{+l}, R_{Pt}^{-l} \geqslant 0$ and $R_{Pt}^{+l} \cdot R_{Pt}^{-l} = 0$ for all $t = 1, 2, \ldots, T$ and all $l \in \Omega$.

The bonus policy reflects policyholders' expectations to earn at least the yield of consols on their investment. Assume that the consol rate is constant $B_t^l \equiv B$ and that $B > g$ and $RA_t^l > L_t^l$. Then assuming that $RB_{t-1}^l = (B - g)/(1 + g)$ (for reasons that will soon become apparent), the regular bonus becomes:

$$RB_t^l = \frac{1}{2} RB_{t-1}^l + \Delta RB_t^l \tag{10}$$

$$= \frac{1}{2}\left(\frac{B-g}{1+g}\right) + \frac{1}{2}\left(\frac{B-g}{1+g}\right) + 0 \tag{11}$$

$$= \frac{B-g}{1+g} \tag{12}$$

$$= RB_{t-1}^l. \tag{13}$$

In this case the liability account (assuming no mortality) grows at the consol rate B:

$$(1+g)\left(1 + \left(\frac{B-g}{1+g}\right)\right) = 1 + g + B - g = 1 + B. \tag{14}$$

2.5.2. *Aiming for a target terminal bonus*

In this approach, based partly on Ross (1989), the firm wishes the policyholders' terminal benefit to be a fixed portion of the total benefit. For the purpose of calculating bonuses, it is assumed that assets will grow at a constant benchmark rate. This results in the following terminal asset value:

$$A_T^l = A_t^l (1 + B_t^l)^{T-t}. \tag{15}$$

It is also assumed that the future level of regular bonus will remain constant so that the guaranteed liabilities will also grow at a fixed rate:

$$L_T^l = (1+g)^{T-t} (1 + RB_t^l)^{T-t}. \tag{16}$$

The terminal bonus received by policyholders is $TB^l = \gamma(A_T^l - L_T^l)$. In order for this to constitute β% of the total payout to policyholders, we must have

$$\frac{TB^l}{TB^l + L_T^l} = \beta. \tag{17}$$

Substituting (15) and (16) into (17), and solving for RB_t^l yields:

$$1 + RB_t^l = \frac{1 + B_t^l}{1 + g} \cdot \left(\frac{\gamma(1-\beta)}{\beta + \gamma(1-\beta)} \cdot \frac{A_t^l}{L_t^l} \right)^{\frac{1}{T-t}}. \tag{18}$$

Note the role of the "solvency ratio" A/L. When this ratio is higher, regular bonuses increase, while when it is lower bonuses decrease. This should correspond to policyholders' expectations of the firm's practice when it faces insolvency. It is intuitively obvious, and easy to show, that regular bonuses are a decreasing function of β.

2.6. Reserving methods

Recall the basic equations for the asset and equity accounts

$$A_t^l = A_{t-1}^l (1 + R_{Pt}^l) + z_t^l - y_{At}^l, \tag{19}$$

$$E_t^l = E_{t-1}^l (1 + r_{ft}^l) + z_t^l. \tag{20}$$

In this section, we examine different methods for specifying the term z_t^l.

2.6.1. Reserving for underperformance

The first specification is the one used in Consiglio et al. (2001), and reflects the strict requirements applied in the Italian industry. With this method, funds are required from shareholders whenever the asset portfolio underperforms the guarantee (even when there already exists a surplus of assets over liabilities). This policy results in the following specification:

$$z_t^l = y_t^{-l} L_{t-1}^l, \tag{21}$$

$$\gamma R_{Pt}^l - g = y_t^{+l} - y_t^{-l}, \tag{22}$$

where $y_t^{+l}, y_t^{-l} \geqslant 0$ and $y_t^{+l} \cdot y_t^{-l} = 0$ for all $t = 1, 2, \ldots, T$ and for all $l \in \Omega$.

2.6.2. Reserving for solvency

In this method, which is similar to one considered for UK insurers in Booth et al. (1997), one asks whether the assets would be sufficient to cover the current surrender value of all the policies, plus some additional margin (to account for the possibility of a future decline in assets). If not, then capital is infused so that this coverage does exist. With a required regulatory margin of ρ (in this paper, we take $\rho = 0.04$), this leads to the following equation:

$$z_t^l = \max((1+\rho)L_t^l - (A_{t-1}^l(1 + R_{Pt}^l) - y_{At}^l), 0). \tag{23}$$

Thus, the prescription for determining whether any equity is added to the portfolio at time t is as follows.

- Is the value of the assets after accounting for the return achieved, and any payouts due to actuarial events, sufficient to meet the solvency ratio $A/L \geqslant 1.04$?
- If yes, then do not obtain equity from shareholders.

- If no, then obtain sufficient equity from shareholders in order to meet the minimum solvency requirement $A/L \geqslant 1.04$.

2.7. Objective function

With both the bonus policies and reserving methods specified the complete dynamics for all the accounts, and all trading constraints, are known. It remains to specify the objective function to be maximized in structuring the firm's optimal portfolio. Since return on equity is scenario dependent, we maximize the expected value of the utility of excess return. For ease of reference, in the exhibits this expected value is converted into a certainty equivalent by applying the inverse of the utility function. The objective function of the model is to compute the maximal Certainty Equivalent Excess Return on Equity (CEexROE) given by

$$\text{CEexROE} = U^{-1}\left(\max_x \frac{1}{N}\sum_{l\in\Omega} U\left(\frac{(1-\gamma)(A_T^l - L_T^l) + \gamma E_T^l}{E_T^l}\right)\right). \tag{24}$$

Here $U(\cdot)$ is the shareholders' utility function and $(1-\gamma)(A_T^l - L_t^l) + \gamma E_T^l$ is their share of terminal wealth. If we had only $(1-\gamma)(A_T^l - L_T^l)$, then we would not be treating shareholders fairly. The returns on their invested equity would be handed over to policyholders, even in the event of no underperformance below the guaranteed rate. This formulation arises from setting the shareholders' final wealth to $(1-\gamma)((A-E)-L)+E$. Policyholders then receive $\gamma((A-E)-L)+L$.

2.8. The non-linear programming problem

The resulting non-linear program is

$$\max_x \quad \frac{1}{N}\sum_{l\in\Omega} U\left(\frac{(1-\gamma)(A_T^l - L_T^l) + \gamma E_T^l}{E_T^l}\right). \tag{25}$$

$$\text{s.t.} \quad \sum_{i\in\mathscr{U}} x_i = 1, \quad x_i \geqslant 0, \tag{26}$$

$$A_t^l = A_{t-1}^l(1+R_{Pt}^l) + z_t^l - y_{At}^l, \tag{27}$$

$$L_t^l = (1-\Lambda_t^l)L_{t-1}^l(1+g)(1+\max[RB_t^l, 0]), \tag{28}$$

$$E_t^l = E_{t-1}^l(1+r_{ft}^l) + z_t^l, \tag{29}$$

where the equations hold for $t = 1, 2, \ldots, T$ and for all $l \in \Omega$, with $A_0 = L_0 + E_0$, $E_0 = \rho L_0$, $RA_0 = L_0$, and RB_t^l and z_t^l are determined using one of the prescriptions above (full specifications of all the models solved in the paper are given in the appendices). In all cases, the above optimization problem is a non-linearly constrained non-linear programming problem. In general, the problem is not convex. It is therefore possible for the optimizer to halt at a local, rather than a global maximum. The results for this paper were checked by starting the optimization procedure with different initial points. The model solution is insensitive to the choice of starting point.

We shall refer to the following specifications as the "base model": a logarithmic utility function, reserving for underperformance, bonuses declared using the working party approach, and parameter values $\rho = 0.04$, $p = \frac{1}{2}$, $q = \frac{1}{4}$, $\Lambda \equiv 0$, $\gamma = 0.9$.

3. Model testing and validation

In this section, we discuss the performance of the base model. These results will serve as a benchmark for the analysis of different policy features and parameter sensitivities carried out in the following sections.

Scenarios are generated using the Wilkie (1995) asset model. This results in five asset classes: Cash (invested at the short-term risk-free rate), Consols (irredeemable government bonds), Index-linked stock (inflation linked government bonds), Shares, and Property. (The classes are ordered in terms of increasing expected return.) The Wilkie model has been the subject of much scrutiny in the actuarial literature. Criticisms include the fact that it fits stationary models to certain economic time series which may be non-stationary. This can have a dramatic effect on long-term investment returns. For a very thorough presentation of the model see Wilkie (1995). A brief description of the model and its equations is available online.[1] For a critical evalution, see Huber (1997).

Five hundred random scenarios were produced using the parameters recommended in Wilkie (1995) and starting with "neutral" conditions (essentially, starting the processes at their long-run means, see Wilkie, 1995). The model uses a yearly frequency between time points t, and we consider a time horizon of 10 years. For each model run, we determine the net annualized CEexROE

$$(\sqrt[T]{\mathrm{CEexROE}} - 1). \tag{30}$$

The effects of taxation (particularly differential taxation across asset classes), while important, are ignored in the current study.

3.1. Cost of the guarantee

With our model it is possible to compute the "expected cost of the guarantee" (as distinct from the arbitrage-free price). This is the expected present value of reserves required to fund shortfalls due to portfolio performances below the guarantee. The dynamic variable E_t^l models the time t value of the total required funds, future-valued at the risk-free rate. However, this variable also includes the initial amount of equity ρL_0 in addition to premiums required by the regulators. This is not a cost and should be deducted from E_t^l. Thus, the cost of the guarantee is given as the expected present value of the final equity E_T^l adjusted by the regulatory equity, that is,

$$\bar{O}_G = \frac{1}{N} \sum_{l \in \Omega} \left(\frac{E_T^l}{\prod_{t=1}^T (1 + r_{ft}^l)} - \rho L_0 \right). \tag{31}$$

[1] http://www.inqa.com/GlobalWilkieMainFormulae.htm.

3.2. Results with the base model

Fig. 1 shows the tradeoff between shareholders' net CEexROE and the level of the guarantee that the firm is able to offer. This is a delicate decision. The firm will be able to attract more customers by offering higher and higher levels of the guarantee. However, this increase will come at a reduced *per policy* benefit to shareholders. Our model does not directly address the issue of finding the optimal level of guarantee to offer, but rather allows the user to examine various sensitivities with respect to policy features when making this decision. We remark that the results may be sensitive to the chosen simulation methodology. The results show a steady decrease in shareholders' expected utility with increasing levels of the guarantee.

Figs. 2 and 3 examine the cost (in the sense of (31)) that shareholders pay in order to fund the guarantee. Fig. 2 shows the cost incurred by shareholders at each level of

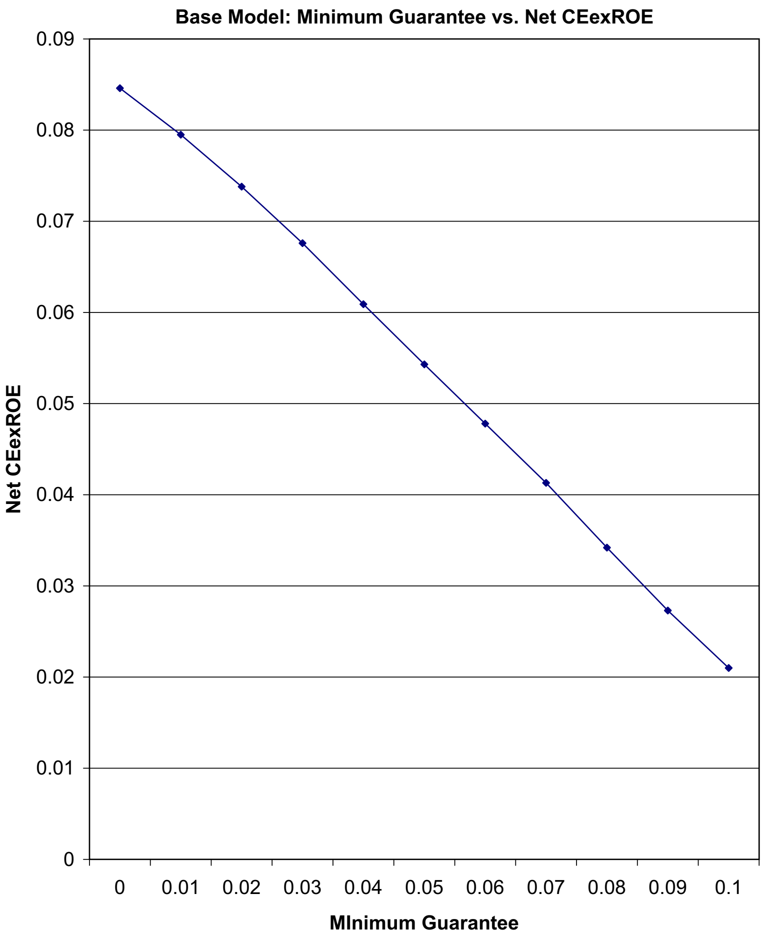

Fig. 1. Base model: shareholders' utility for different levels of minimum guarantee.

minimum guarantee. As expected, this function is strictly increasing, and becomes increasingly steep as unrealistically high levels of guarantee are offered. Fig. 3 plots CEexROE vs. Cost for the optimal portfolios at each level of minimum guaranteed return. This curve is analogous to a mean–variance efficient frontier in that it displays the tradeoff between shareholder benefit (Net CEexROE) and risk (expected cost to fund the guarantee).

Fig. 4 examines the optimal asset allocation strategy (weights for cash and consols are nearly zero; recall that we optimize over fixed-mix strategies) for different levels of the minimum guarantee. Observe that it is generally optimal to pursue aggressive portfolios (high proportions in property – a high risk and high return asset class). For reasonable levels of the minimum guarantee, this aggressive portion of the portfolio is supplemented by a significant position in index-linked stock (it is this investment that hedges the guaranteed liability, while the aggressive portfolio component

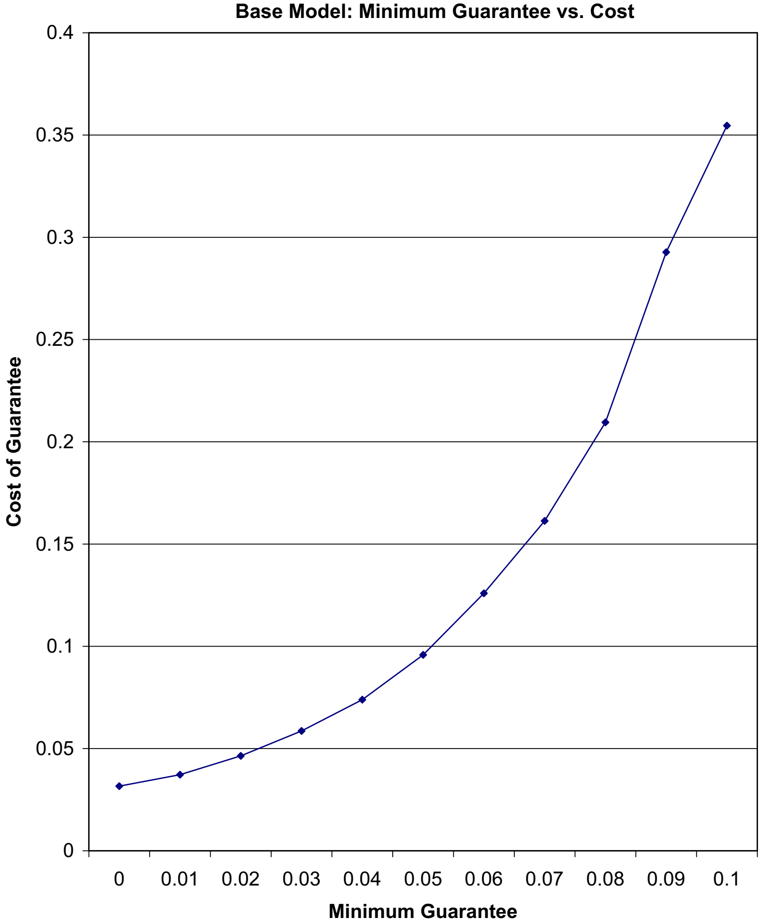

Fig. 2. Base model: cost of funding different levels of guarantee.

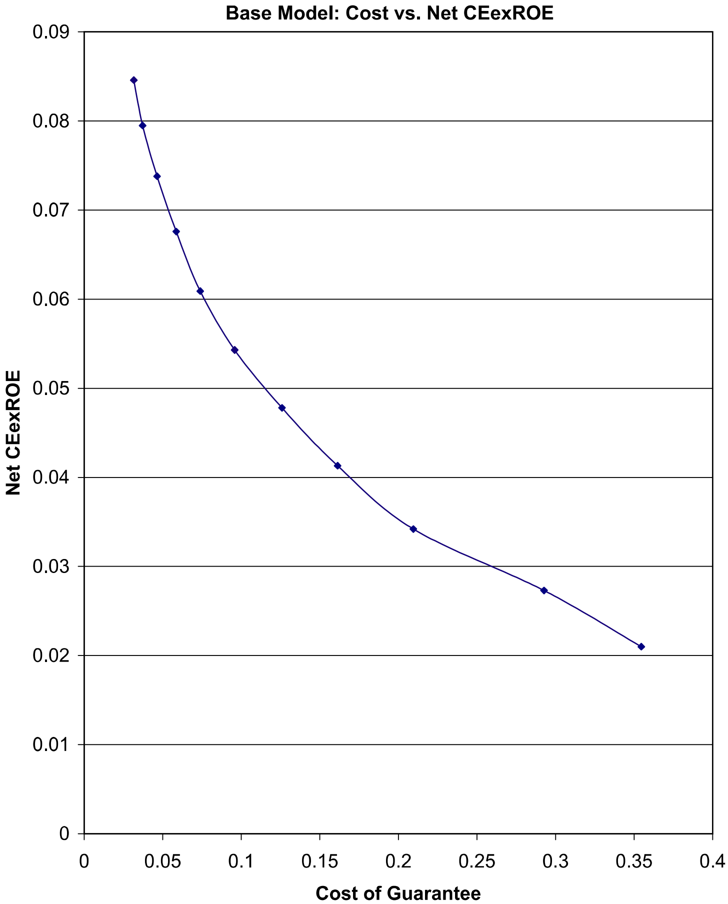

Fig. 3. Base model: tradeoff between CEexROE and cost of the guarantee.

seeks to maximize returns). For high levels of the minimum guarantee, the company is forced to invest all the funds in the highest returning asset classes. This is the only way it can hope to meet the guarantee (of course, when property loses money, as this high risk asset class often will, the liability must be covered with new funds from the shareholders, producing a correspondingly high cost). This asset allocation is consistent with other experiments involving optimal portfolio problems based on the Wilkie model (see the comments section in Wilkie, 1995).

4. Comparison of policy features

We examine all the possible combinations of reserving strategies and bonus policies as described above. In the figures, these strategies are referenced as follows:

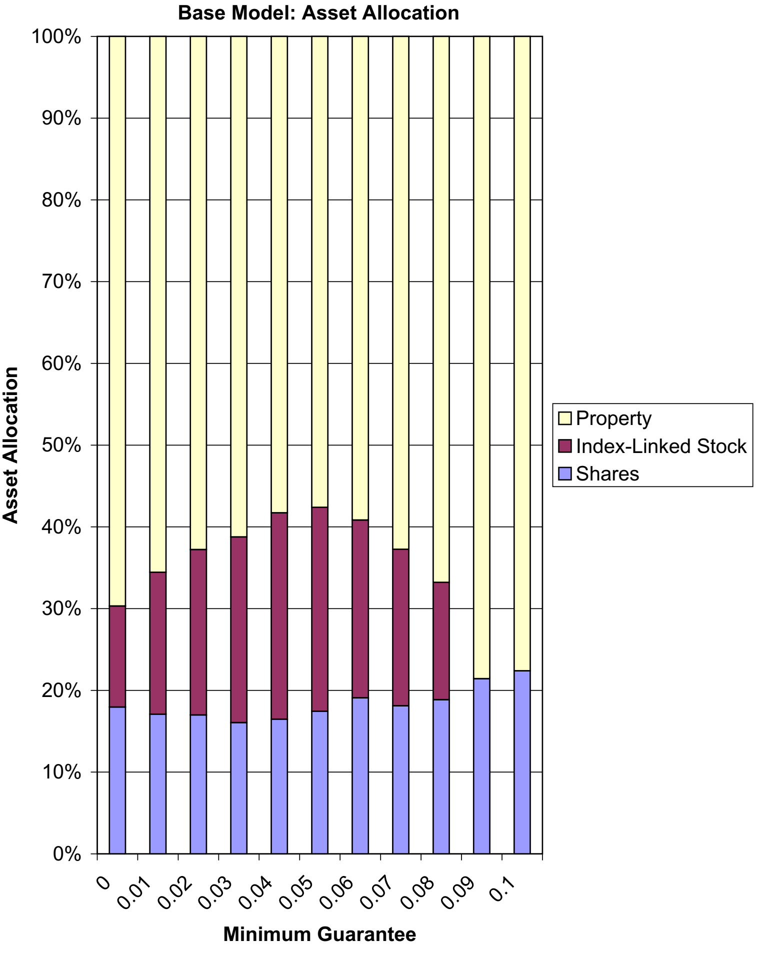

Fig. 4. Base model: asset allocation for different levels of the minimum guarantee.

- **Targ-Solv:** Aiming for a target level of terminal bonus and reserving for solvency.
- **Targ-Und:** Aiming for a target level of terminal bonus and reserving for underperformance.
- **WP-Sol:** Declaring regular bonuses using the working party approach and reserving for solvency.
- **WP-Und:** Declaring regular bonuses using the working party approach and reserving for underperformance.

Fig. 5 presents the CEexROE for different levels of the minimum guarantee for each of these bonus/reserving strategies.

The first observation is that regardless of bonus strategy, reserving for solvency outperforms reserving for underperformance. This is to be expected, as reserving

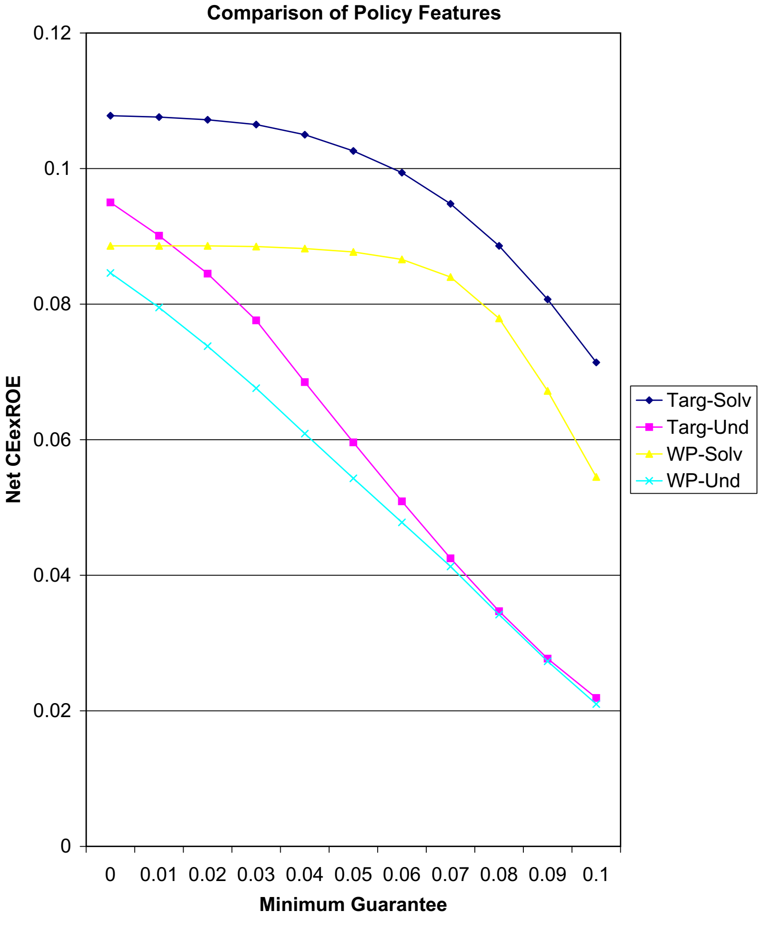

Fig. 5. Base model: comparison of performance of policies with different features for different levels of the guarantee.

for underperformance places strict requirements on shareholders to provide funds whenever the portfolio does worse than the guarantee rate. This substantially increases the size of the investment, and does not produce a correspondingly large increase in return. Thus the model accords with our intuition that the stricter reserving method is worse for shareholders. The model also provides a way of quantifying the damage to shareholders caused by more stringent regulations (such as those in the Italian industry) requiring reserving for underperformance.

Bonus policies based on aiming at a target level of terminal bonus tend to outperform those which follow the working party strategy. This is because the target level of terminal bonus calculations assume a rate of asset growth that is in general lower than that realized. The lower regular bonus rate leads to greater freedom on the part of the insurer (as to how it can invest the surplus $A - L$), as well as lower (non-terminal) liabilities, and therefore a smaller probability of needing to request further funds from shareholders (thus hurting shareholder returns). If higher rates of asset growth were assumed in the target calculation, results would be correspondingly different.

Fig. 6 shows the cost of providing different levels of the guarantee for each of the strategies. Not surprisingly, reserving for underperformance costs significantly more than reserving for solvency. This also stems from the reduced freedom insurance companies have when required to use this reserving method. Once again, the working party method of declaring bonuses slightly underperforms the method where bonuses are declared using a fixed target terminal bonus rate, and this is due to the unrealistic growth rates assumed in this bonus calculation (which are biased towards the insurance company and away from policyholders).

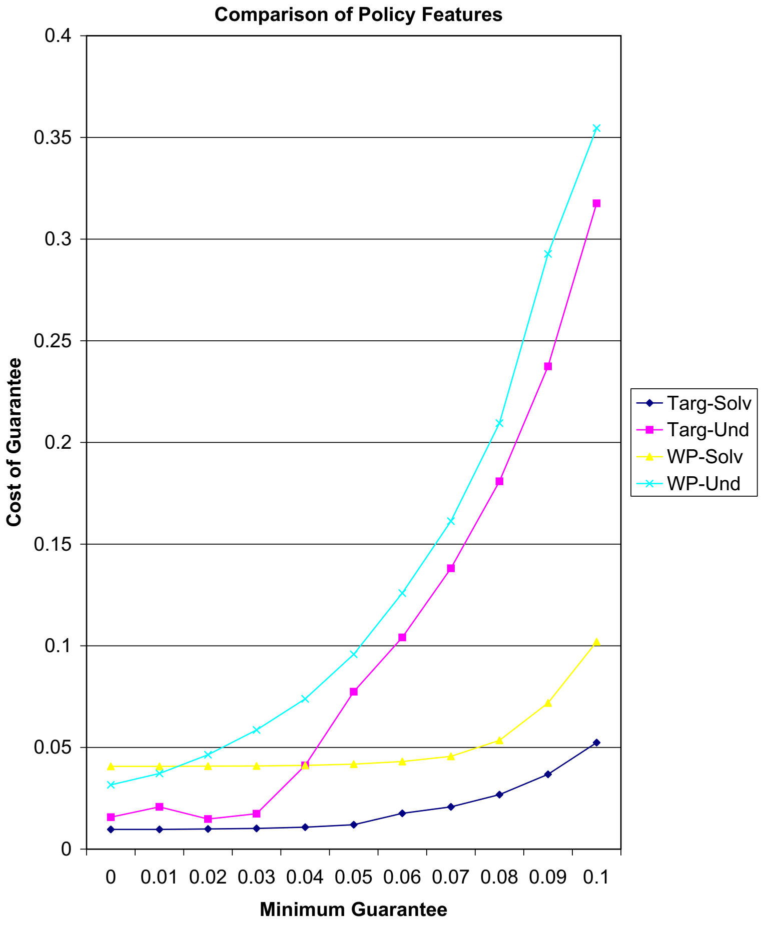

Fig. 6. Base model: comparison of cost of providing the guarantee for different policy features.

Fig. 7 plots the Cost vs. Net CEexROE frontier for each of the strategies discussed above. The results are generally consistent with those present above. In this graph, more efficient strategies lie above and to the right of less efficient strategies. Therefore, the results agree with the general ranking of policy features, from best to worst, that can be arrived at from looking at any of the above graphs:

1. Bonuses using target level of terminal bonus and reserving for solvency.
2. Bonuses using the working party approach and reserving for solvency.
3. Bonuses using target level of terminal bonus and reserving for underperformance.
4. Bonuses using the working party approach and reserving for underperformance.

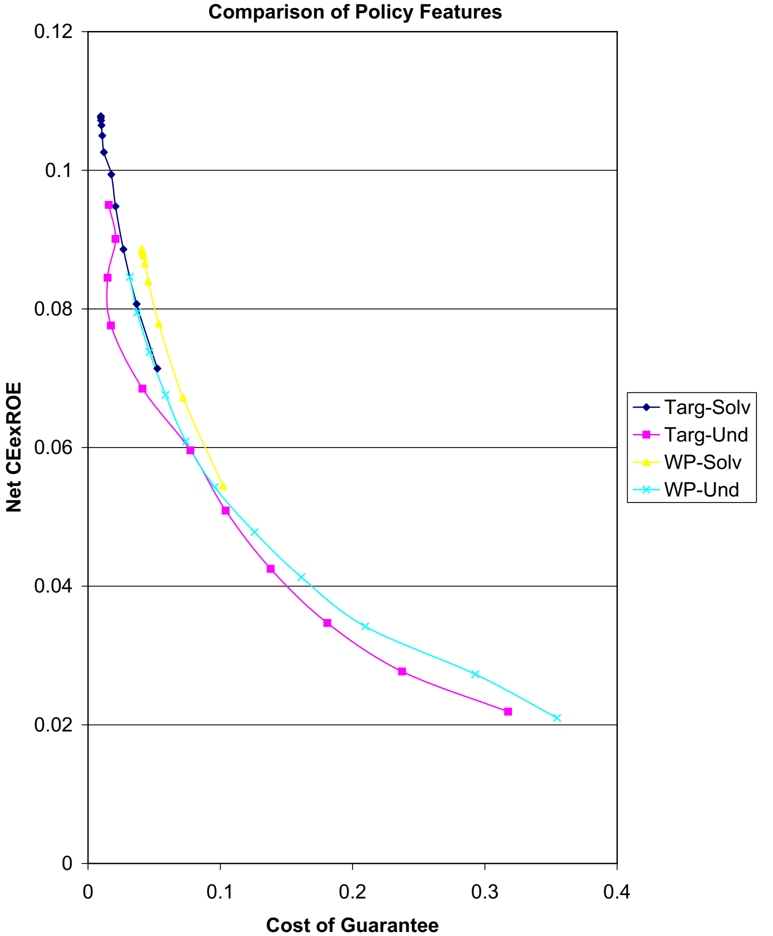

Fig. 7. Base model: comparison of performance of policies with different features for different levels of the guarantee.

Some observations regarding practical implementations of the model are in order. The first is that since it is a non-convex problem, great care should be taken to ensure that the local optima returned by a non-linear programming solver. The second is our optimal portfolios contain significant investments in property. The model for property in Wilkie (1995) is based on a short time-series, and the author warns that it should be used with caution. In practice, we recommend comparing the results presented herein with those generated with an upper bound (possibly zero) on the allowable level of investment in property. A large portion of the funds would then be transferred from property to shares. We note that the basic qualitative features of the results (shapes of curves, relative ranking of bonus and reserving policies, etc.) remain unchanged when property investment is disallowed and a simple brute force grid search is employed for the optimization.

5. Conclusions

This paper has presented a model for analyzing the investment decisions made by insurance firms that offer policies with minimum guarantee provisions. The model allows the insurance company to compare different policy features in order to determine how best to structure its policies. Numerical results demonstrate how this can be done in order to compare different bonus policies and different methods for reserving against insolvency.

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Appendix A. Complete model specifications

In the appendix, we present the full form of all the optimization models solved in the paper. All variables indexed by l are over $l \in \Omega$; those over i are for $i \in \mathscr{U}$ and those indexed by t are for $t = 0, \ldots, T$. For instance r_{it}^{l} is the return of asset i at time t in scenario l. Constraints that are set apart after a space are those that vary between the models.

A.1. Base case: Working party bonus approach and reserving for solvency

The parameters are γ, L_0, r^l_{it}, ρ, Λ^l_t, g, r^l_{ft}, B^l_t, β, p, q, $l \in \Omega$, $i \in \mathscr{U}$, $t = 1,\ldots,T$.

$$\max_{x,y^l_{At},y^{-l}_t,y^{+l}_t,A^l_t,E^l_t,L^l_t,z^l_t,RA^l_t,RB^l_t,\Delta RB^l_t,R^l_{Pt}} \frac{1}{N}\sum_{l\in\Omega} U\left(\frac{(1-\gamma)(A^l_T - L^l_T) + \gamma E^l_T}{E^l_T}\right),$$

$$\sum_{i\in\mathscr{U}} x_i = 1,$$

$$x_i \geqslant 0 \quad (i \in \mathscr{U}),$$

$$R^l_{Pt} = \sum_{i\in\mathscr{U}} x_i r^l_{it}, \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$A^l_t = A^l_{t-1}(1 + R^l_{Pt}) + z^l_t - y^l_{At}, \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$A^l_0 = (1+\rho)L_0, \quad l \in \Omega,$$

$$L^l_t = (1 - \Lambda^l_t)(1+g)(1 + \max[RB^l_t, 0]), \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$y^l_{At} = \Lambda^l_t L^l_{t-1}(1+g)(1 + RB^l_{t-1}), \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$E^l_t = E^l(1 + r^l_{ft}) + z^l_t, \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$E^l_0 = \rho L_0, \quad l \in \Omega.$$

$$RA^l_t = RA^l_{t-1}\left(1 + \frac{3}{4}\max(R^l_{Pt}, 0) - \frac{4}{3}\max(-R^l_{Pt}, 0)\right), \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$RA^l_0 = L_0, \quad l \in \Omega,$$

$$RB^l_t = \frac{1}{2}RB^l_{t-1} + \Delta RB^l_t, \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$\Delta RB^l_t = p\max\left(\frac{B^l_t - g}{1+g}, 0\right) - q\max\left(\frac{L^l_{t-1} - RA^l_{t-1}}{RA^l_{t-1}}, 0\right), \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$z^l_t = \max((1+\rho)L^l_t - (A^l_{t-1}(1 + R^l_{Pt}) - y^l_{At}), 0), \quad t = 1,\ldots,T, \ \ l \in \Omega.$$

A.2. Target level of terminal bonus and reserving for solvency

The parameters are γ, L_0, r^l_{it}, ρ, Λ^l_t, g, r^l_{ft}, B^l_t, β, $l \in \Omega$, $i \in \mathscr{U}$, $t = 1,\ldots,T$.

$$\max_{x,y^l_{At},A^l_t,E^l_t,L^l_t,z^l_t,RB^l_t,R^l_{Pt}} \frac{1}{N}\sum_{l\in\Omega} U\left(\frac{(1-\gamma)(A^l_T - L^l_T) + \gamma E^l_T}{E^l_T}\right),$$

$$\sum_{i\in\mathscr{U}} x_i = 1,$$

$$x_i \geqslant 0 \quad (i \in \mathscr{U}),$$

$$R^l_{Pt} = \sum_{i\in\mathscr{U}} x_i r^l_{it}, \quad t = 1,\ldots,T, \ \ l \in \Omega,$$

$$
\begin{aligned}
&A_t^l = A_{t-1}^l(1 + R_{Pt}^l) + z_t^l - y_{At}^l, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&A_0^l = (1 + \rho)L_0, \quad l \in \Omega,\\
&L_t^l = (1 - \Lambda_t^l)(1 + g)(1 + \max[RB_t^l, 0]), \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&y_{At}^l = \Lambda_t^l L_{t-1}^l (1 + g)(1 + RB_{t-1}^l), \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&E_t^l = E^l(1 + r_{ft}^l) + z_t^l, \quad t = 1, \ldots, T, \quad l \in \Omega,\\
&E_0^l = \rho L_0, \quad l \in \Omega.
\end{aligned}
$$

$$
\begin{aligned}
&RB_t^l = \frac{1 + B_t^l}{1 + g} \cdot \left(\frac{\gamma(1 - \beta)}{\beta + \gamma(1 - \beta)} \cdot \frac{A_t^l}{L_t^l}\right)^{\frac{1}{T-t}} - 1, \quad t = 0, \ldots, T, \;\; l \in \Omega,\\
&z_t^l = \max((1 + \rho)L_t^l - (A_{t-1}^l(1 + R_{Pt}^l) - y_{At}^l), 0), \quad t = 1, \ldots, T, \;\; l \in \Omega.
\end{aligned}
$$

A.3. Target level of terminal bonus and reserving for underperformance

The parameters are γ, L_0, r_{it}^l, ρ, Λ_t^l, g, r_{ft}^l, B_t^l, β, $l \in \Omega$, $i \in \mathscr{U}$, $t = 1, \ldots, T$.

$$
\max_{x, y_{At}^l, y_t^{-l}, y_t^{+l}, A_t^l, E_t^l, L_t^l, z_t^l, RB_t^l, R_{Pt}^l} \frac{1}{N} \sum_{l \in \Omega} U\left(\frac{(1 - \gamma)(A_T^l - L_T^l) + \gamma E_T^l}{E_T^l}\right),
$$

$$
\begin{aligned}
&\sum_{i \in \mathscr{U}} x_i = 1,\\
&x_i \geqslant 0 \quad (i \in \mathscr{U}),\\
&R_{Pt}^l = \sum_{i \in \mathscr{U}} x_i r_{it}^l, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&A_t^l = A_{t-1}^l(1 + R_{Pt}^l) + z_t^l - y_{At}^l, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&A_0^l = (1 + \rho)L_0, \quad l \in \Omega,\\
&L_t^l = (1 - \Lambda_t^l)(1 + g)(1 + \max[RB_t^l, 0]), \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&y_{At}^l = \Lambda_t^l L_{t-1}^l (1 + g)(1 + RB_{t-1}^l), \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&E_t^l = E^l(1 + r_{ft}^l) + z_t^l, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&E_0^l = \rho L_0, \quad l \in \Omega.
\end{aligned}
$$

$$
\begin{aligned}
&RB_t^l = \frac{1 + B_t^l}{1 + g} \cdot \left(\frac{\gamma(1 - \beta)}{\beta + \gamma(1 - \beta)} \cdot \frac{A_t^l}{L_t^l}\right)^{\frac{1}{T-t}} - 1, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&z_t^l = y_t^{-l} L_{t-1}^l, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&\gamma R_{Pt}^l - g = y^{+l} - y_t^{-l}, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&y_t^{-l} \cdot y_t^{+l} = 0, \quad t = 1, \ldots, T, \;\; l \in \Omega,\\
&y_t^{-l}, y_t^{+l} \geqslant 0, \quad t = 1, \ldots, T, \;\; l \in \Omega.
\end{aligned}
$$

A.4. Working party bonus approach and reserving for underperformance

The parameters are γ, L_0, r^l_{it}, ρ, Λ^l_t, g, r^l_{ft}, B^l_t, β, p, q, $l \in \Omega$, $i \in \mathscr{U}$, $t = 1, \ldots, T$.

$$\max_{x, y^l_{At}, y^{-l}_t, y^{+l}_t, A^l_t, E^l_t, L^l_t, z^l_t, RA^l_t, RB^l_t, \Delta RB^l_t, R^l_{Pt}} \frac{1}{N} \sum_{l \in \Omega} U\left(\frac{(1-\gamma)(A^l_T - L^l_T) + \gamma E^l_T}{E^l_T}\right),$$

$$\sum_{i \in \mathscr{U}} x_i = 1,$$

$$x_i \geqslant 0 \quad (i \in \mathscr{U}),$$

$$R^l_{Pt} = \sum_{i \in \mathscr{U}} x_i r^l_{it}, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$A^l_t = A^l_{t-1}(1 + R^l_{Pt}) + z^l_t - y^l_{At}, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$A^l_0 = (1+\rho)L_0, \quad l \in \Omega,$$

$$L^l_t = (1 - \Lambda^l_t)(1+g)(1 + \max[RB^l_t, 0]), \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$y^l_{At} = \Lambda^l_t L^l_{t-1}(1+g)(1 + RB^l_{t-1}), \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$E^l_t = E^l(1 + r^l_{ft}) + z^l_t, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$E^l_0 = \rho L_0, \quad l \in \Omega.$$

$$RA^l_t = RA^l_{t-1}\left(1 + \frac{3}{4}\max(R^l_{Pt}, 0) - \frac{4}{3}\max(-R^l_{Pt}, 0)\right), \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$RA^l_0 = L_0, \quad l \in \Omega,$$

$$RB^l_t = \frac{1}{2} RB^l_{t-1} + \Delta RB^l_t, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$\Delta RB^l_t = p \max\left(\frac{B^l_t - g}{1+g}, 0\right) - q \max\left(\frac{L^l_{t-1} - RA^l_{t-1}}{RA^l_{t-1}}, 0\right), \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$z^l_t = y^{-l}_t L^l_{t-1}, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$\gamma R^l_{Pt} - g = y^{+l} - y^{-l}_t, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$y^{-l}_t \cdot y^{+l}_t = 0, \quad t = 1, \ldots, T, \ \ l \in \Omega,$$

$$y^{-l}_t, y^{+l}_t \geqslant 0, \quad t = 1, \ldots, T, \ \ l \in \Omega.$$

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