Modeling and Controller Design for an Inverted Pendulum System

A thesis submitted in partial fulfillment of the requirements for the degree of

Bachelor of Technology

in

Electrical Engineering

by

Netranjeeb Lenka (107EE027)

Department of Electrical Engineering

National Institute of Technology

Rourkela-769008

2011

Modeling and Controller Design for an Inverted Pendulum System

A thesis submitted in partial fulfillment of the requirements for the degree of

Bachelor of Technology

in

Electrical Engineering

by

Netranjeeb Lenka (107EE027)

Under the guidance of

Prof. Arun Ghosh

Department of Electrical Engineering

National Institute of Technology

Rourkela-769008

2011

National Institute of Technology
Rourkela
2011

CERTIFICATE

This is to Certify that the thesis entitled, "**Modeling and Controller Design for an Inverted Pendulum System**" submitted by Mr. Netranjeeb Lenka in Partial fulfillment of the requirements for the award of Bachelor of Technology Degree in Electrical Engineering at the National Institute of Technology, Rourkela (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/Institute for the award of any Degree or Diploma.

Date: **Professor Arun Ghosh**

ACKNOWLEDGEMENT

My first and foremost regards for **Prof. Arun Ghosh**, Dept. of Electrical Engineering for assigning me the project titled "**Modeling and Controller Design for an Inverted Pendulum System**". His supportive nature, continuous guidance and constructive ideas were really valuable during every stage of this project.

I sincerely convey my gratitude to the senior M.Tech(Research) students and all the staff members of Department of Electrical Engineering(Annexe Lab) for eagerly extending their helping hand whenever I have been encountered with difficulties. I would also like to thank all others who have supported me for the project or in any other aspects related to my project work.

Netranjeeb Lenka

107EE027

Date:

Place: Rourkela

Modeling and Controller Design

for an

Inverted Pendulum System

Abstract

The Inverted Pendulum System is an under actuated, unstable and nonlinear system. Therefore, control system design of such a system is a challenging task. To design a control system, this thesis first obtains the nonlinear modeling of this system. Then, a linearized model is obtained from the nonlinear model about vertical (unstable) equilibrium point. Next, for this linearized system, an LQR controller is designed. Finally, a PID controller is designed via pole placement method where the closed loop poles to be placed at desired locations are obtained through the above LQR technique. The PID controller has been implemented on the experimental set up.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

INTRODUCTION

1. INTRODUCTION

1.1 Introduction

If we remember ever trying to balance a broom-stick on our index finger or the palm of our hand, we had to constantly adjust the position of our hand to keep the object upright. An Inverted Pendulum does basically the same thing. But in case of an Inverted Pendulum the motion is restricted to one dimension only, where as in case of a broom-stick the hand is free to move in any directions [4].

Inverted Pendulum is a very good platform for control engineers to verify and apply different logics in the field of control theory. Most of the modern technologies use the basic concept of Inverted Pendulum, such as attitude control of space satellites and rockets, landing of aircrafts, balancing of ship against tide [2], Seismometer (which monitors motion of the ground due to earthquake and nuclear explosions) etc.

An Inverted Pendulum has its mass above the pivoted point, which is mounted on a cart which can be moved horizontally. The pendulum is stable while hanging downwards, but the inverted pendulum is inherently unstable and need to be balanced. In this case the system has one input - the force applied to the cart, and two outputs - position of the cart and the angle of the pendulum [5], making it as a SIMO system. There are mainly three ways of balancing an inverted pendulum i.e. (i) by applying a torque at the pivoted point (ii) by moving the cart horizontally (iii) by oscillating the support rapidly up and down.

Just like the broom-stick, an Inverted Pendulum is an inherently unstable system. Force must be properly applied to keep the system intact. To achieve this, proper control theory is required [4]. The Inverted Pendulum is a non-linear time variant open loop system. So the standard linear techniques cannot model the non-linear dynamics of the system. This makes the system more challenging for analysis [2]. The dynamics of the actual non-linear system is more complicated. But this non-linearized system can be approximated as a liner system if the operating region is small, i.e. the variation of the angle from the norm.

In an overall way the controllers can be divided in to two parts:

(1) Conventional Controller

Proportional Integral Derivative (PID)

Linear Quadratic Regulator (LQR)

(2) Artificial Intelligence Controller

Fuzzy Logic Controller (FLC)

Artificial Neural Network Controller (ANN) [2]

In this thesis a light is put on the Conventional Controller, i.e. Linear Quadratic Regulator (LQR) and Proportional Integral Derivative (PID) controller. The thesis discusses the choice of state variables which influence the stability properties of the designed closed loop system. First part of the paper describes the experimental setup of the system to be modeled and controlled, second part shows the development of the linear model, third part describes the LQR controller design, fourth part describes the PID controller design and the last part represents about the experimental results [5].

Chapter 2

OBJECTIVES & WORKING METHODOLOGY

2. OBJECTIVES & WORKING METHODOLOGY

2.1 Objectives of the project

1. To find out the mathematical model of the Inverted Pendulum System.
2. To get the transfer functions.
3. Feedback gains are to be obtained from the state-space matrices for LQR controller, and then a Matlab Simulink model is to be designed.
4. Values of Proportional(K_p), Integral(K_i) and Derivative(K_d) gains are obtained using Pole-Placement method for PID controller, then a Matlab Simulink model is to be designed.

2.2 Working Methodology

1. To know the basic concepts of an Inverted Pendulum by literature review.
2. To derive the mathematical model.
3. To take some assumptions for linearizing the model.
4. To design transfer function and state space form.
5. The LQR and PID controller is to be studied, and a model is to be designed for each of them.
6. The theoretically obtained models are then implemented in real time control action.
7. Discussion on the result.
8. Conclusion.

Chapter 3

SYSTEM DESCRIPTION

3. SYSTEM DESCRIPTION

3.1 Digital Pendulum mechanical unit

As shown in the figure 3.1.1, the pendulum setup consists of a cart which can be moved along the 1 meter length track. The cart has a shaft to which two pendulums are attached and are able to rotate freely. The cart can move back and forth hence causing the pendulum to swing.

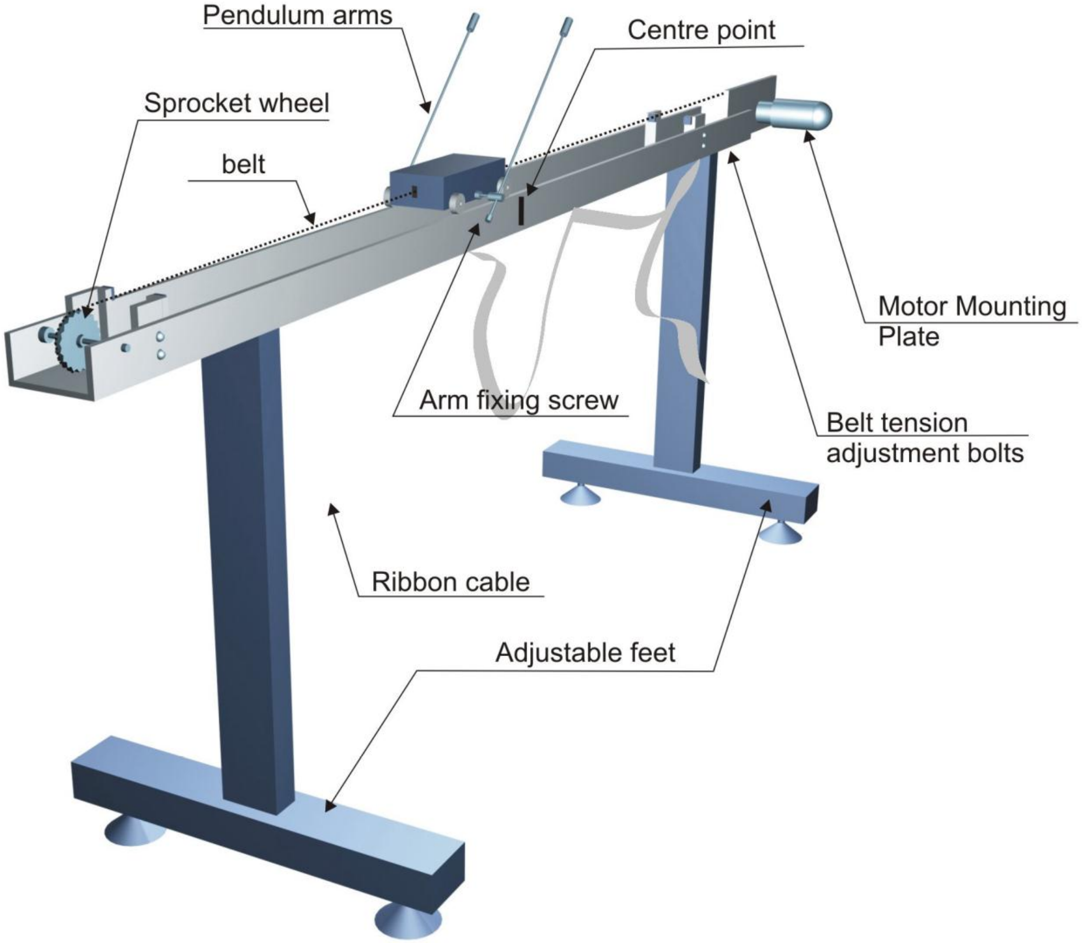

Fig 3.1.1: Digital Pendulum mechanical unit [1]

The movement of the cart is caused by pulling the belt in two directions by the DC motor attached at the end of the rail. By applying a voltage to the motor we control the force with which the cart is pulled. The value of the force depends upon the value of the control voltage. Here the voltage is the control signal. The two parameters that are read from the pendulum (using optical encoders) are the pendulum position (angle) and the position of the cart on the rail. The controller task is to vary the DC motor voltage according to these two variables, in such a way that the control task is fulfilled (balancing in upright position) [1].

3.2 Pendulum control system

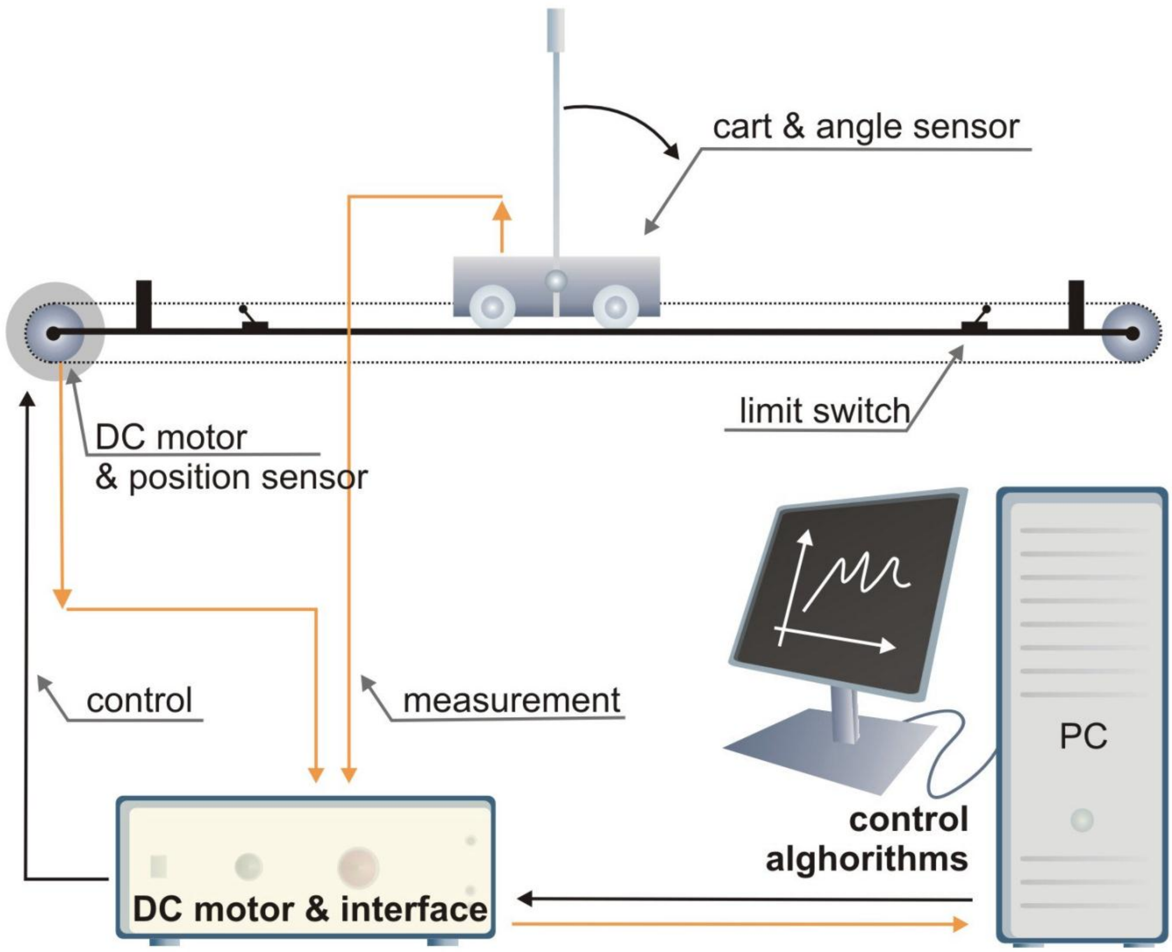

Fig 3.2.1: Pendulum control system [1]

Figure 3.2.1 represents how the control action is organized. In order to design any control algorithms one must understand the physical background behind the whole process and carry out identification experiments. The next section explains the modeling process of the pendulum [1].

Chapter 4

MODELING

4. MODELING

4.1 Pendulum Model

Every control project starts with a plant modeling. The mechanical model of the pendulum is as shown in figure 4.1.1 [1]. There is a mass **m** attached to the end of a bar, which is mounted on a cart of mass **M**. The cart can be moved horizontally. This arrangement is also known as cart and pole system.

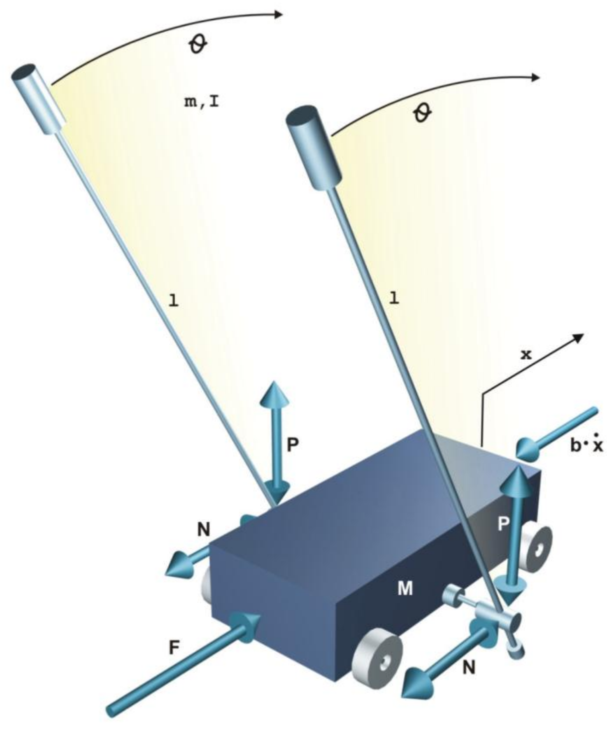

Fig 4.1.1: Pendulum phenomenological model [1]

Referring to the figure 4.1.1, we have,

Θ= Acute angle of the pendulum w.r.t vertical position.

F= Horizontal force to be applied on the cart to maintain the pendulum in vertical position.

N= Horizontal component of the reaction force at the pivoted point.

P= Vertical component of the reaction force at the pivoted point.

x= Distance covered by the cart from the starting point.

b= Cart friction coefficient.

$b\dot{x}$= Friction force on the cart.

I= Moment of inertia of the inverted pendulum.

The phenomenological model of the pendulum is nonlinear, meaning that one of the states(x and its derivative or Θ and its derivative) is an argument of a nonlinear function. For such a model to present in transfer function (a form of linear plant dynamics representation used in control engineering), it has to be linearized [1].

There are certain limitations which are to be kept in mind while preparing a model for this system. In this case the limitations are:

- The distance covered by the cart from the starting point (or from the center of the rail), i.e. x should be in the range of 0.3 meter.
- The acute angle of the pendulum w.r.t vertical position, i.e. Θ should be in the range of 0.2 radian.
- The applied voltage to the DC motor should remain within the range of +2.5V to -2.5V.

4.2 Equations of Motion

There are several methods for finding the dynamics of the Inverted Pendulum system. In this article, we have focused on Newton's second law of motion to find the dynamics. For this we need to have a clear idea what forces are acting on each of the free bodies of the system.

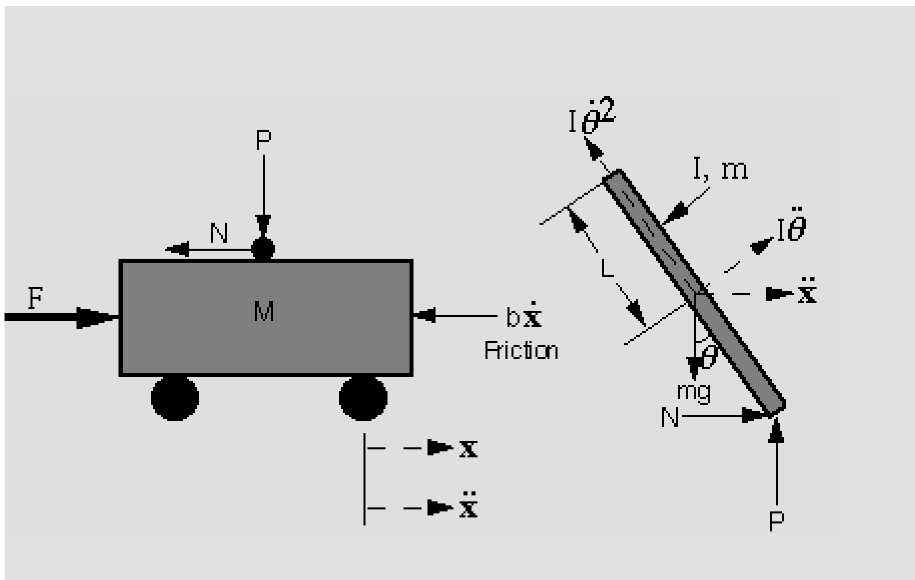

Fig 4.2.1: Free body diagram of the Inverted Pendulum system [2]

To derive the suitable mathematical model for an Inverted Pendulum system, consider Figure 4.2.1

Adding all the forces on the cart in the horizontal direction, we have,

$$M\ddot{x} + b\dot{x} + N = F \qquad (1)$$

Adding all the forces on the pendulum in the horizontal direction, we have,

$$m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = N \qquad (2)$$

Substituting equation (1) in equation (1), we have,

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\Theta}\cos\Theta - ml\ddot{\Theta}^2\sin\Theta = F \quad (3)$$

Adding all the forces along the vertical direction of the pendulum,

$$P\sin\Theta + N\cos\Theta - mg\sin\Theta = ml\ddot{\Theta} + m\ddot{x}\cos\Theta \quad (4)$$

Considering sum of the moments about the center of gravity (C.G) of the pendulum,

$$-Pl\sin\Theta - Nl\cos\Theta = I\ddot{\Theta} \quad (5)$$

Now, from equation (4) & (5)

$$(I+ml^2)\ddot{\Theta} + mgl\sin\Theta = -ml\ddot{x}\cos\Theta \quad (6)$$

The system under consideration is a non-linear system. For ease of modeling and simulation, we have to take a small case approximation such that the system will be a linear one. Let's take the linearization point will be $\Theta = \Pi$.

Say $\Theta = \Pi + \Phi$

Where, Φ is the angle between the pendulum and vertical upward direction. If we choose $\Phi \approx 0$, then $\cos\Theta = -1, \sin\Theta = -\Phi, \frac{d\Theta}{(dt)^2} = 0$.

So, after linearization equation (6) becomes,

$$(I+ml^2)\ddot{\Phi} - mgl\Phi = ml\ddot{x} \quad (7)$$

And equation (3) becomes,

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\Phi} = F \qquad (8)$$

Here, F is the mechanical force to be applied on the moving cart system. But in real time model we have to give input voltage proportional to the force F. If the input voltage is u, then equation (8) becomes,

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\Phi} = u$$ [2]

4.3 Transfer Functions

In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equations

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero. Consider the linear time-invariant system defined by the following differential equation:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bx^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

Where y is the output of the system and x is the input. The transfer function of the system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

Transfer function = G(s) = $\frac{Ł[output]}{Ł[input]}$, at zero initial conditions

$$= \frac{Y(s)}{X(s)} = \frac{bx^{(m)} + b_1 x^{(m-1)} + \dots + b_{m-1}\dot{x} + b_m x}{a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1}\dot{y} + a_n y}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equation in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an nth-order system [3].

Laplace transform of equation (7)

$$(I + ml^2)\Phi(s)s^2 - mgl\Phi(s) = -mlX(s)s^2 \qquad (9)$$

Laplace transform of equation (8)

$$(M + m)X(s)s^2 + bX(s)s - ml\Phi(s)s^2 = U(s) \qquad (10)$$

Solving equation (9) for $X(s)$,

$$X(s) = \left[\frac{(I+ml^2)}{ml} - \frac{g}{s^2}\right]\Phi(s) \qquad (11)$$

Substituting equation (11) in (10),

$$(M + m)\left[\frac{(I+ml^2)}{ml} + \frac{g}{s}\right]\Phi(s)s^2 + b\left[\frac{(I+ml^2)}{ml} + \frac{g}{s}\right]\Phi(s)s - ml\Phi(s)s^2 = U(s)$$

(12)

From the above equation

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 - \frac{b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s}$$

(13)

Where $q = [(M+m)(I+ml^2) - (ml)^2]$

In equation (13), it is clear that one pole and zero is at origin. This leads to cancelation of one pole and zero. So resulting equation will be,

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 - \frac{b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl}{q}s - \frac{bmgl}{q}}$$

(14)

Here in this case the angle from the vertical position (Φ(s)) is taken as the output and the applied force to the cart (u(s)) is taken as the input function.

Again from equation (11)

$$\Phi(s) = \frac{mls^2}{(I+ml^2)s^2 - mgl}X(s) \qquad (15)$$

Now putting the value of Φ(s) in equation (12),

$$\frac{X(s)}{U(s)} = \frac{(I+ml^2)s^2 - mgl}{\left((M+m)(I+ml^2) - ml^2\right)s^4 + \{b(I+mll^2)\}s^3 - \{(M+m)mgl\}s^2 - mglbs}$$

(16) [2]

Here the distance of the cart from the origin is treated as the output function whereas the applied force on the cart is still the input function.

4.4 State-Space

In this section we shall present the introductory material on state-space analysis and the estimation of the state-space matrices of the considered Inverted Pendulum system.

4.4.1 Modern Control Theory

Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity, and easy access to large scale computers, modern control theory, which is a new approach for analysis and design of complex control systems, has been developed since 1960. This new approach is based on the concept of state. The concept of state has been in existence for a long time in the field of classical dynamics and other fields [3].

4.4.2 State-Space Equations

In a state-space system representation, we have a system of two equations: an equation is to determine the state of the system, and the other equation is to determine the output of the system. We will use variable y(t) as the output of the system, x(t) as the state of the system, and u(t) as the input of the system. We use the notation $\dot{x}$ for the first derivative of the state vector of the system, as dependent on the current system and current input. We can write these two equations as:

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad \textbf{[State Equation]}$$

$$y(t) = C(t)x(t) + D(t)u(t) \qquad \textbf{[Output Equation]}$$

If the systems themselves are time-invariant, we can re-write this as follows:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

The State Equation shows the relationship between the system's current state and its input, and the future state of the system. The Output Equation shows the relationship between the system state and its input, and the output. These equations show that in a given system, the current output is dependent on the current input and the current state. The future state is also dependent on the current state and the current input.

Matrices: A B C D

Matrix A: Matrix *A* is called the **system matrix**, and relates how the current state affects the state change $\dot{x}$. If the state change is not dependent on the current state, *A* will be a zero matrix. The dimension of the matrix is in the form $p \times p$.

Matrix B: Matrix *B* is called the **control matrix**, and determines how the system input affects the state change. If the state change is not dependent on the system input, then *B* will be a zero matrix. The dimension of the matrix is in the form $p \times q$.

Matrix C: Matrix *C* is the output matrix, and determines the relationship between the system state and the system output. The dimension of the matrix is in the form $r \times p$.

Matrix D: Matrix *D* is the feed-forward matrix, and allows for the system input for affecting the system output directly. A basic feedback system does not have a feed-forward element, and therefore for most of the systems the *D* matrix is considered to be a zero matrix. If it exists then the dimension should be $r \times q$ [10].

Now, with this basic knowledge choosing the state variables of the above considered Inverted Pendulum,

$x_1 = x;$

$x_2 = \dot{x} = \dot{x}_1;$

$x_3 = \Phi;$

$x_4 = \dot{\Phi} = \dot{x}_3$

$$\dot{x}_1 = x_2 \tag{17}$$

Now equation (10) can be written as:

$$(M + m)\dot{x}_2 - ml\dot{x}_4 + bx_2 = u \tag{18}$$

And equation (9) as:

$$(I+ml^2)\dot{x}_4 - ml\dot{x}_2 = mglx_3 \tag{19}$$

Putting the value of $\dot{x}_4$ from equation (19) in equation (18), we have,

$$\dot{x}_2 = \frac{-b(I + ml^2)}{I(M + m) + Mml^2} x_2 + \frac{m^2 gl^2}{I(M + m) + Mml^2} x_3 + \frac{I + ml^2}{I(M + m) + Mml^2} u \tag{20}$$

$$\dot{x}_3 = x_4 \tag{21}$$

Now putting the value of $\dot{x}_2$ from equation (20) in equation (19), we have,

$$\dot{x}_4 = \frac{-mlb}{I(M+m)+Mml^2}x_2 + \frac{mgl(M+m)}{I(M+m)+Mml^2}x_3 + \frac{mlu}{I(M+m)+Mml^2}$$

(22)

There are two outputs so,

$y_1(t) = \Phi(t)$

$y_2(t) = X(t)$

Now constructing the state space matrix

$\dot{X} = AX + BU$

$Y = CX$, where

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\Phi} \\ \ddot{\Phi} \end{bmatrix} \; ; \quad X = \begin{bmatrix} x \\ \dot{x} \\ \Phi \\ \dot{\Phi} \end{bmatrix}; \quad U = u(t) \; ; \quad Y = \begin{bmatrix} x(t) \\ \Phi(t) \end{bmatrix}$$

Referring from equation 17, 20, 21 and 22, we have,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} ; B = \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{mlu}{I(M+m)+Mml^2} \end{bmatrix} ; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Chapter 5

LQR CONTROLLER DESIGN

5. LQR Controller Design

5.1 Introduction

This chapter describes the Linear Quadratic Regulator (LQR) control technique. This technique uses a state-space approach to analyze a system. This method provides a systematic way of computing the state feedback control gain matrix. In optimal control one attempts to find a controller that provides the best possible performance with respect to some given measure of performance. In general, optimality with respect to some criterion is not the only desirable property for a controller. One would also like stability of the closed-loop system.

5.1.1 Quadratic Optimal Regulator

We shall now consider the optimal regulator problem that, given the system equation

$\dot{x} = Ax + Bu$

Determines the matrix K of the optimal control vector

$u(t) = -Kx(t)$

So as to minimize the performance index

$$J = \int_0^{\infty} (x * Qx + u * Ru)dt \qquad (23)$$

Where Q is a positive-semidefinite and R is a positive-definite matrix. The matrices Q and R determine the relative importance of the error. Here the elements of the matrix K are determined so as to minimize the performance index, then $u(t) = -Kx(t)$ is optimal for any initial state $\boldsymbol{x}(0)$. The block diagram showing the optimal configuration is shown in Figure 5.1.1.1.

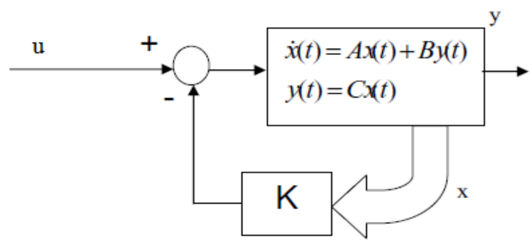

Fig 5.1.1.1: Block diagram for optimal configuration [2]

The equation (23), can be further simplified to,

$$A^{*}P + PA - PBR^{-1}B^{*}P + Q = 0 \qquad (24)$$

Where, P is a positive-definite Hermitian or real symmetric matrix. If the system is stable, there always exists one positive-definite matrix P to satisfy this equation. Equation (24) is called the reduced-matrix Riccati equation. The design steps may be stated as follows:

1. Solve equation (24), the reduced-matrix Riccati equation, for the matrix P.
2. Substitute the matrix P in equation$K = R^{-1}B^{*}P$. The resulting matrix K is the optimal matrix [3].

Another option is to use the LQR function in matlab to obtain the optimal controller. By using LQR function in matlab, two matrices i.e. Q and R are to be chosen which will balance the relative importance of the input and state of the function, for achieving optimization.

5.2 Calculation of State-Space Matrices

For the calculation of state-space matrices we should introduce the values of all parameters of the model present in the laboratory.

Parameter	Value
g- *gravity*	*9.81 m/s²*
l- *pole length*	*0.36 to 0.4 m* *- depending on the configuration*
M- *cart mass*	*2.4 kg*
m- *pole mass*	*0.23 kg*
I- *moment of interia of the pole*	*about 0.099 kg·m²* *- depends on the configuration*
b - *cart friction coefficient*	*0.05 Ns/m*
d - *pendulum damping coefficient*	*although negligible,necessary* *in the model- 0.005 Nms/rad*

Table 5.2.1: Values of all parameters of the Inverted Pendulum model [1]

Now putting the values from above table in matrix A, we have,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.0194 & 0.2204 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.0128 & 6.6321 & 0 \end{bmatrix}$$

Similarly putting the values in B matrix, we have,

$$B = \begin{bmatrix} 0 \\ 0.3888 \\ 0 \\ 0.2570 \end{bmatrix}$$

Using state-space method, it is relatively simple to work with a multi-output system. In this chapter an LQR controller is designed considering both pendulum's angle and cart's position. In section 4.4.2 the four states are taken as $x, \dot{x}, \Phi, \dot{\Phi}$. These four states represent the position, velocity of the cart, angle and angular velocity of the pendulum. The output y contains both the position of the cart and the angle of the pendulum. A controller is to be designed such that, when the pendulum is displaced, it eventually returns to zero angle (i.e. the vertical) and the cart should be moved to a new position according to the controller.

5.3 LQR Control Method

The next step in designing such a control is to determine the feedback gains. In matlab we use the LQR function that will give the optimal controller. Using LQR function, two parameters i.e. R and Q can be chosen, which will balance the relative importance of the input [2].

The element at row 1, column 1 in Q matrix weights to the position of the cart. Similarly the element at row 2, column 2 weights to the velocity of the cart, element at row 3 column 3 weights to the pendulum angle, element at row 4 column 4 weights to the angular velocity of the pendulum. R gives weight to the input voltage. The K matrix can be produced by choosing a suitable value of Q and R using matlab command. For a particular gain matrix the response of cart's position and pendulum's angle can be plotted. Q and R matrix is adjusted by hit and trial method to obtain the desired response, such that $|x|<0.3\ m$, $|\angle\Phi|<0.2\ rad$ and $|u|<2.5\ volts$.

With the help of Q and R, the K matrix i.e. the feedback matrix that will produce a good controller could be found by running the m-file code in Matlab, and hence the response can be plotted easily.

5.4 Simulink Model

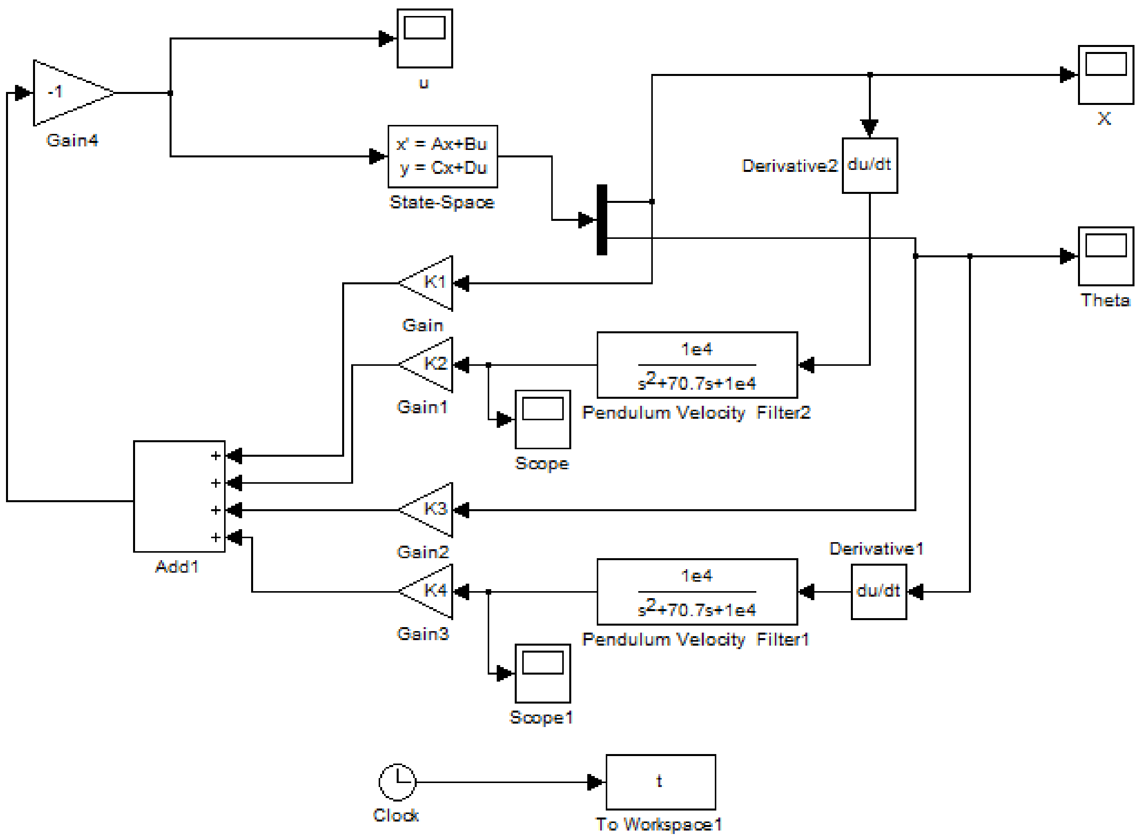

Fig 5.4.1: Schematic diagram of the State feedback controller (using LQR) in Simulink

A Simulink model is shown in Fig 6.4.1 using the feedback gains obtained in Appendix A1. From this Simulink model, the response of cart's position and pendulum's (angle from vertical upward direction) are obtained. According to the condition given earlier, the cart's position should not exceed 0.3 meter from the center of the platform, and the pendulum's angle should not exceed 0.2 radian from vertical upward direction.

As the derivative of a function adds noise in to the system, so to get a smooth response velocity filters are added after the derivative function. Here the system is simulated for 10 seconds. The responses are seen through scopes. The responses for cart's position and pendulum's angle are shown in section 7.2. Here the value of Q and R, which yields the required output, can further be used to design the PID controller using pole placement method, which is described in Chapter 6.

Chapter 6

PID CONTROLLER DESIGN

6. PID Controller Design

6.1 Introduction

PID (Proportional, Integral and Differential) controller is the most common form of feedback. It became the standard tool when process control emerged in 1940s. [7] In PID controller the basic idea is the examination of signals from sensors placed in the system, called feedback signals.

Let’s consider the following unity feedback system

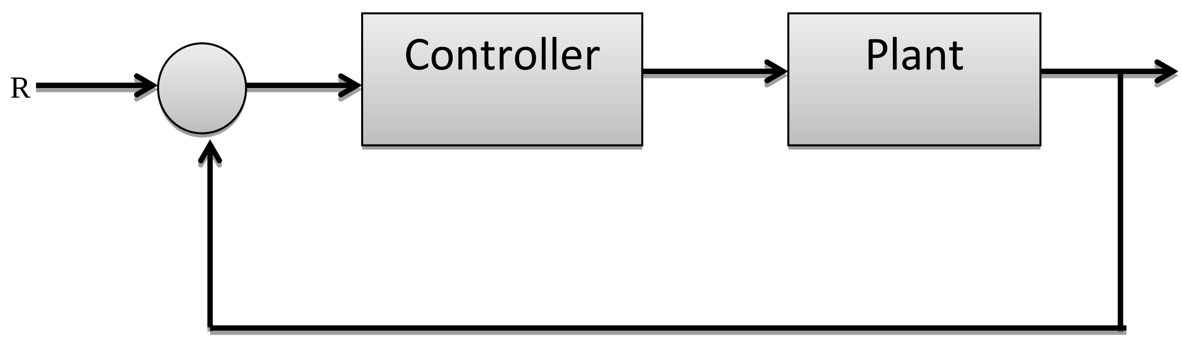

Fig 6.1.1: Schematic diagram of the a feedback control system [11]

Plant: A system is to be controlled.

Controller: Designed to control the overall system behavior as per requirement.

6.1.1 The three-term controller

The transfer function of the PID controller is like,

$$K_P + \frac{K_I}{s} + K_D s = \frac{K_P s + K_I + K_D s^2}{s}$$

- K_P=Proportional gain
- K_I=Integral gain
- K_D=Differential gain

The error signal is sent to the PID controller, and the controller computes both the derivative and integral of the error signal. The signal (u) just past the controller is now equal to the proportional gain (K_P) times the magnitude of the error plus the integral gain (K_I) times the integral of the error plus the derivative gain (K_D) times the derivative of the error.

$$u = K_P e + K_I \int e(t)dt + K_D \frac{de}{dt}$$

The controller takes the new error signal and computes it derivative and integral again. This process goes on and on [11].

6.1.2 The characteristics of P, I, and D controllers

A proportional controller (K_P) will have effect of reducing the rise time, but never eliminates the steady-state error. An integral controller (K_I) will reduce the steady-state error but may make the transient response worse. A derivative controller (K_D) will have an effect on stability of the system, it reduces the overshoot, and improving the transient response. Effects of these three controllers can be summarized as shown in the table 6.1.2.1 [11].

CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S S ERROR
K_P	Decrease	Increase	Small Change	Decrease
K_I	Decrease	Increase	Increase	Eliminate
K_D	Small Change	Decrease	Decrease	Small Change

Table 6.1.2.1: Characteristics of P, I, and D controllers [11]

6.2 PID Control Method

As mentioned in section 4.3, from equation 13,

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 - \frac{b(I + ml^2)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s}$$

If the frictional force is neglected here, then the equation can be written as,

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}}{s^4 - \frac{(M+m)mgl}{q}s^2}$$

Now, cancelling the common poles i.e. s=0, we have,

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}}{s^2 - \frac{(M+m)mgl}{q}} = \frac{b_1}{s^2 - a^2}$$

where $b_1 = \frac{ml}{q}$ and $a = \sqrt{\frac{(M+m)mgl}{q}}$ (25)

Putting the values of the parameters in equation 25 from the table 5.2.1, we have,

$a^2 = 6.6307$

$b_1 = 0.2570$

Similarly from equation 16,

$$\frac{X(s)}{U(s)} = \frac{(I + ml^2)s^2 - mgl}{\left((M+m)(I+ml^2) - ml^2\right)s^4 + \{b(I+mll^2)\}s^3 - \{(M+m)mgl\}s^2 - mglbs}$$

If we neglect the frictional force and consider the mass of the pendulum (m) is negligible compared to the mass of the cart (M), then the above transfer function can be reduced to

$$\frac{X(s)}{U(s)} \simeq \frac{{}^{1}/_{(M+m)}}{s^2} = \frac{b_2}{s^2} \qquad (26)$$

where $b_2 = (M + m)$

Putting the values of the parameters in equation 26 from the table 5.2.1, we have,

$b_2 = 0.3802$

With this transfer functions obtained in equation 25 and 26, the block diagram for a PID controller is shown in Figure 6.2.1.

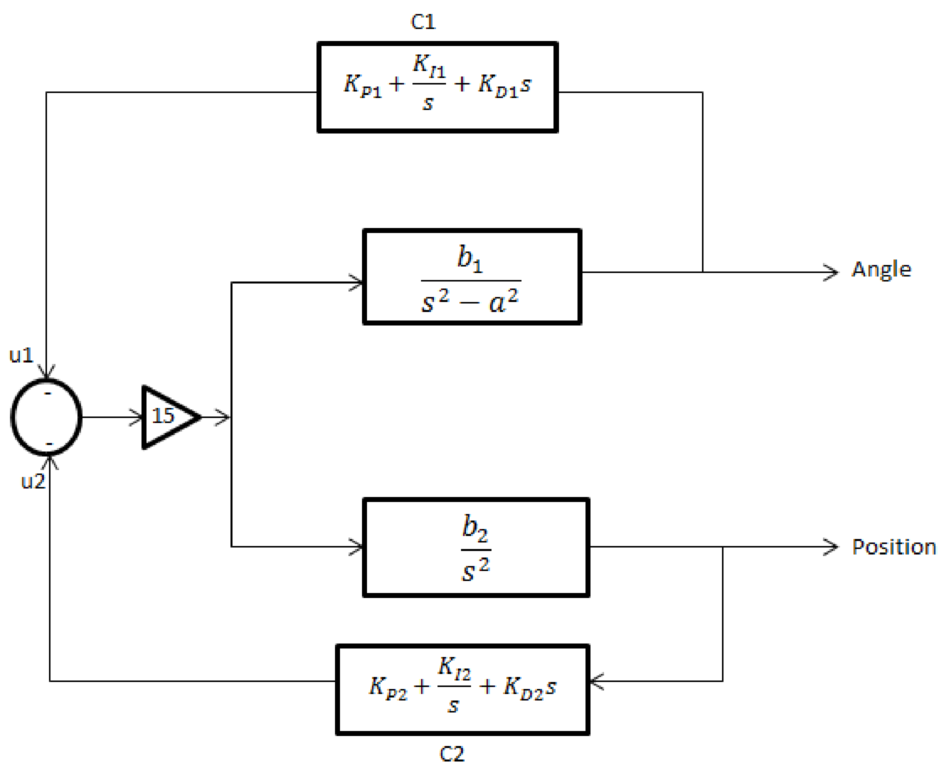

Fig 6.2.1 Schematic diagram of the PID controller

Here the initial condition for the pendulum's angle is $\Phi(0) = 0.2\ rad$ and the initial condition for the cart's position is $x(0) = 0$. The motor which helps to move the cart along the horizontal direction has a gain of 15.

From the Figure 6.2.1, we can write

$$u = -u1 - u2$$

$$=> u = -C1\Phi - C2x$$

$$=> u = -15C1\frac{b_1}{s^2 - a^2}u - 15C2\frac{b_2}{s^2}u$$

$$=> u\left(1 + 15C1\frac{b_1}{s^2 - a^2} + 15C2\frac{b_2}{s^2}\right) = 0 \quad (27)$$

Equation 27 is known as the closed loop equation, where,

$C1 = K_{P1} + \frac{K_{I1}}{s} + K_{D1}s$, and

$C2 = K_{P2} + \frac{K_{I2}}{s} + K_{D2}s$

Solving the closed loop equation,

$$s^5 + 15s^4(b_1K_{D1} + b_2K_{D2}) + s^3(15b_1K_{P1} + 15b_2K_{P2} - a^2) + 15s^2(b_1K_{I1} + b_2K_{I2} - b_2K_{D2}a^2) - 15b_2K_{P2}a^2s - 15b_2K_{I2}a^2 = 0 \quad (28)$$

The next step is to determine the PID gains for both pendulum's angle and cart's position. There are several methods for determining the PID gains. We will adopt here the Pole Placement method.

6.3 Pole Placement Method

The pole-placement method is somewhat similar to the root-locus method. The basic difference is that in the root-locus design we place only dominant closed loop poles (Poles that are close to the imaginary axis) at the desired locations, while in the pole-placement design we place all closed-loop poles at desired locations.

Here we assume that all state variables are measurable and are available for feedback. The necessary and sufficient condition for pole-placement is that the system should be completely state controllable, such that the poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix. [3]

The present design technique begins with the determination of closed-loop poles based on the required transient response. From this closed-loop poles PID gains are obtained. The m-file for obtaining the closed-loop poles is shown in Appendix A2. Running this m-file we get the closed loops poles as

$\lambda_1 =$ -3.1811+3.0466j

$\lambda_2 =$ -3.1811-3.0466j

$\lambda_3 =$ -2.4446+0.4400j and $\lambda_4 =$ -2.4446-0.4400j

Form these four closed-loop poles it is clear that λ_3 and λ_4 are dominant poles. As from equation 28 the system should have five poles, let's assume another pole in such a way that it will not affect the system response and the best way is to choose λ_5 as 6 times the real part of the dominant pole, i.e. $\lambda_5 = -14.6676$

So, the characteristic equation becomes

$$(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)(s-\lambda_5) = 0$$

Putting all the values of λ in the above equation and solving it,

$$s^5 + 25.9190s^4 + 221.6958s^3 + 965.1848s^2 + 2085.8s + 1754.6 = 0 \qquad (29)$$

Comparing equation 28 and 29, and choosing K_{D2} as 5

$K_{P1} = 140.83, K_{P2} = -55.16$

$K_{I1} = 269.9688, K_{I2} = -46.4$

$K_{D1} = 14.1204, K_{D2} = -5$

With this PID gains a PID controller is designed in Simulink.

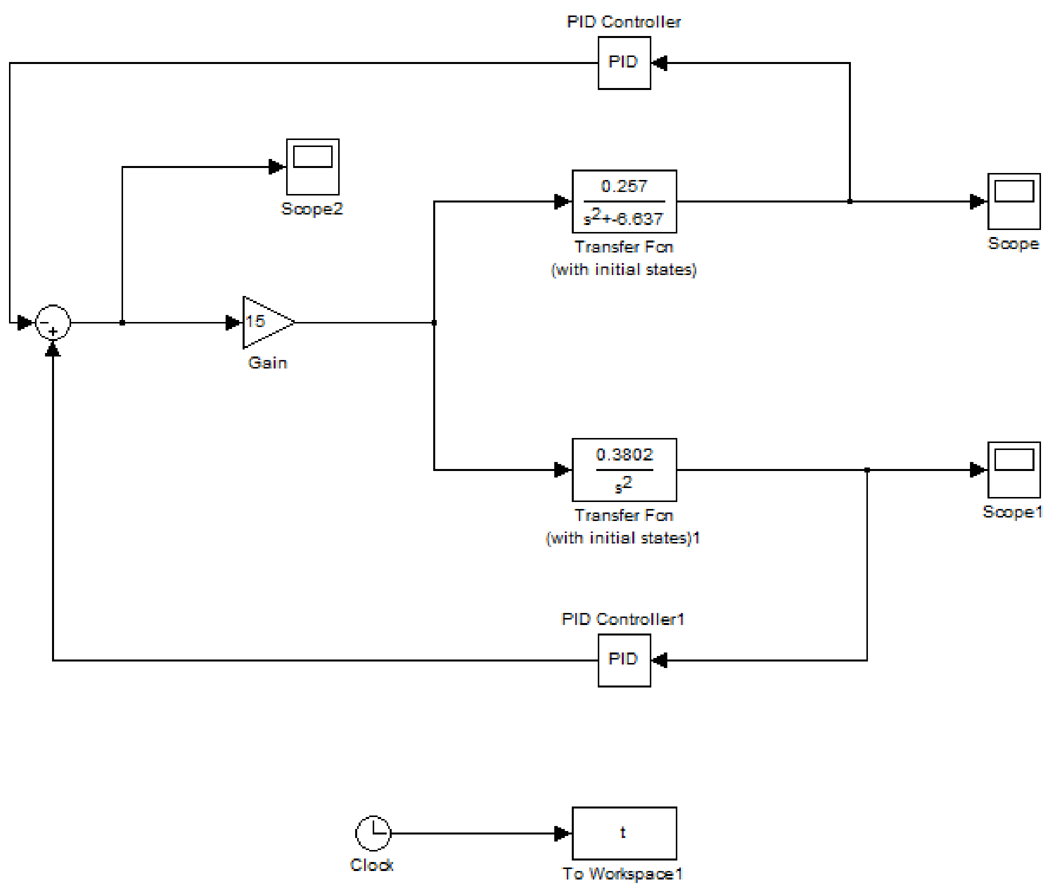

Fig 6.3.1 Schematic diagram of the PID controller in Simulink

Chapter 7

RESULTS

7. Results

7.1 Introduction

This chapter discusses the output responses of the Inverted Pendulum system with LQR and PID controllers.

7.2 LQR Control Method

For a trial basis the m-file in Appendix A1 is run with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And R = 1.

We get the values of feedback gains are

K1 = -1.0000

K2 = -1.3843

K3 = 11.0773

K4 = 4.3167

With this feedback gains the model file is run (as shown in figure 5.4.1)

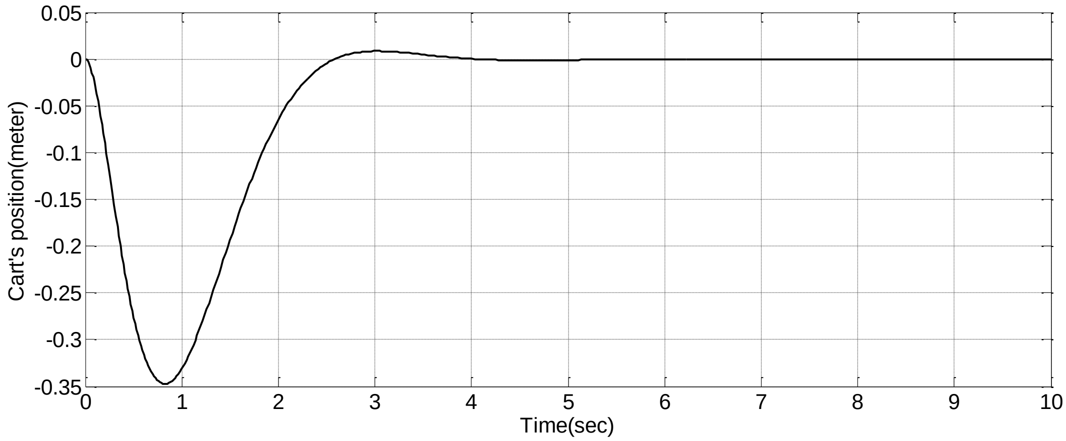

Fig 7.2.1 Response curve for cart’s position

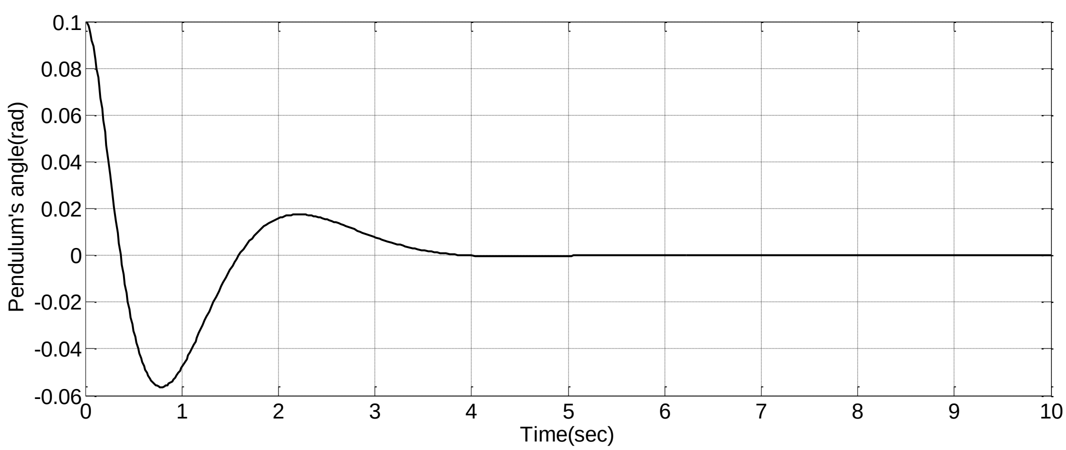

Fig 7.2.2 Response curve for pendulum’s angle

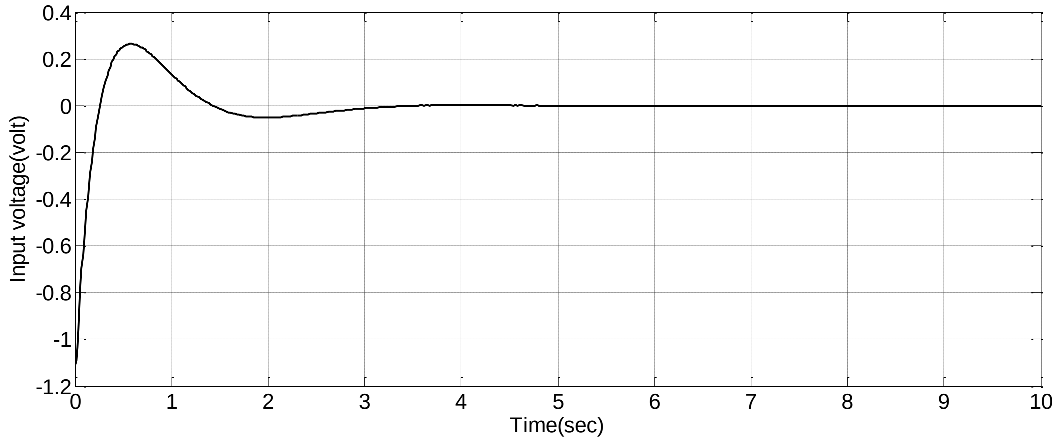

Fig 7.2.3 Response curve for input voltage to the motor

From the response curve it's clear that we need to adjust the values of Q and R because the cart's position is exceeding 0.3 meter (as in figure7.2.1). By trial and error method we get Q and R values as

$$Q = \begin{bmatrix} 15000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And R = 1500

This yields the feedback gains as

K1 = -3.1623

K2 = -3.5469

K3 = 21.2032

K4 = 8.2794

With this feedback gains the model file is run (as shown in figure 5.4.1)

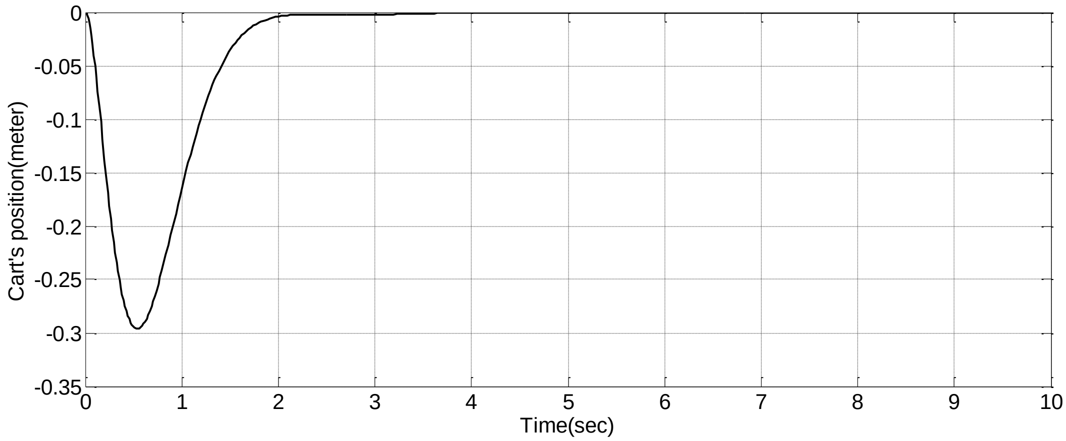

Fig 7.2.4 Response curve for cart's position

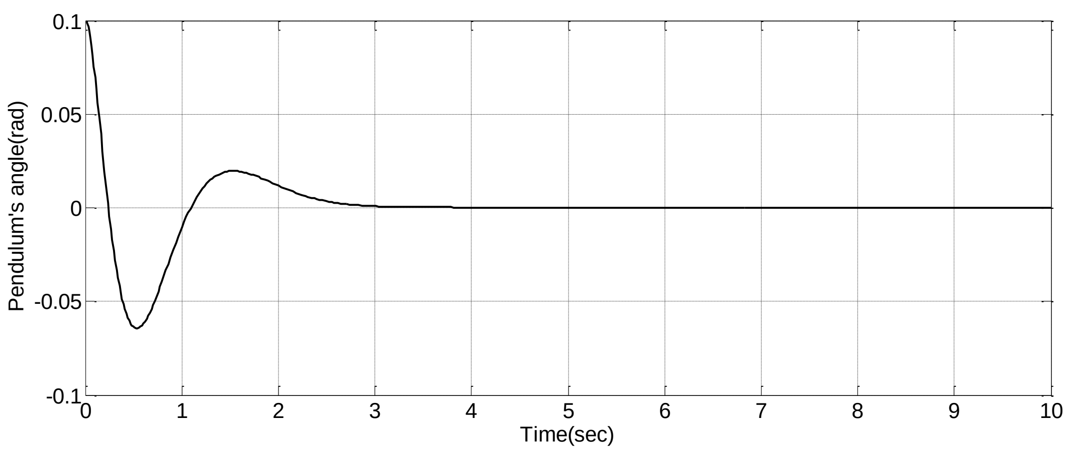

Fig 7.2.5 Response curve for pendulum's angle

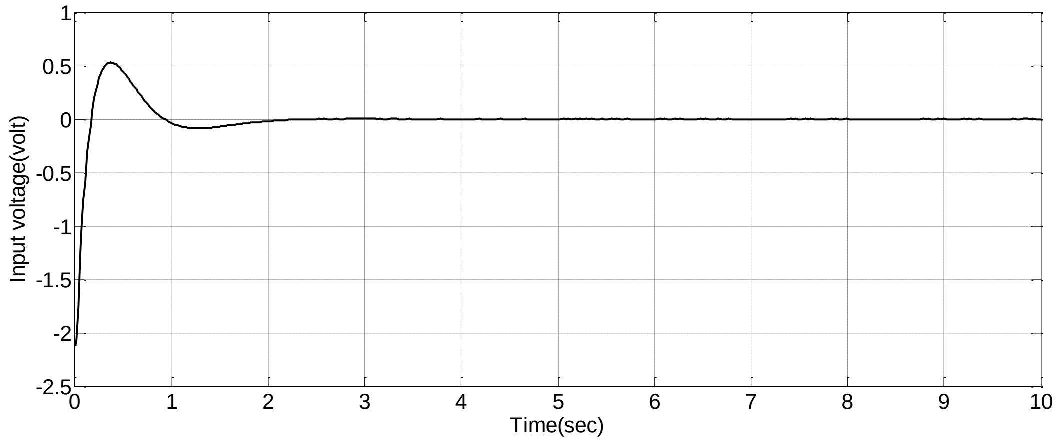

Fig 7.2.6 Response curve for input voltage to the motor

7.3 PID Control Method

The PID gains obtained in section 7.3 are used in the Simulink model (as shown in figure7.3.1).

The response curves for cart's position, pendulum's angle and input voltage are shown below.

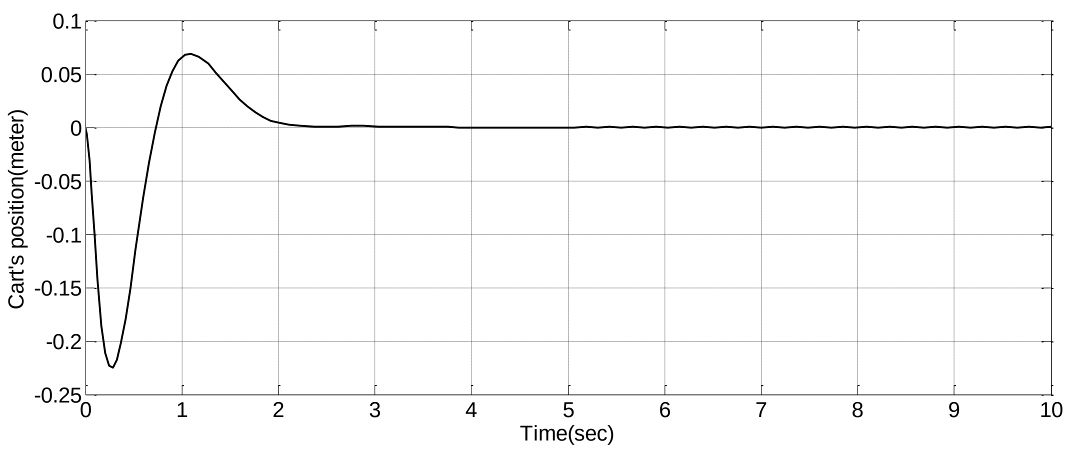

Fig 7.3.1 Response curve for cart's position

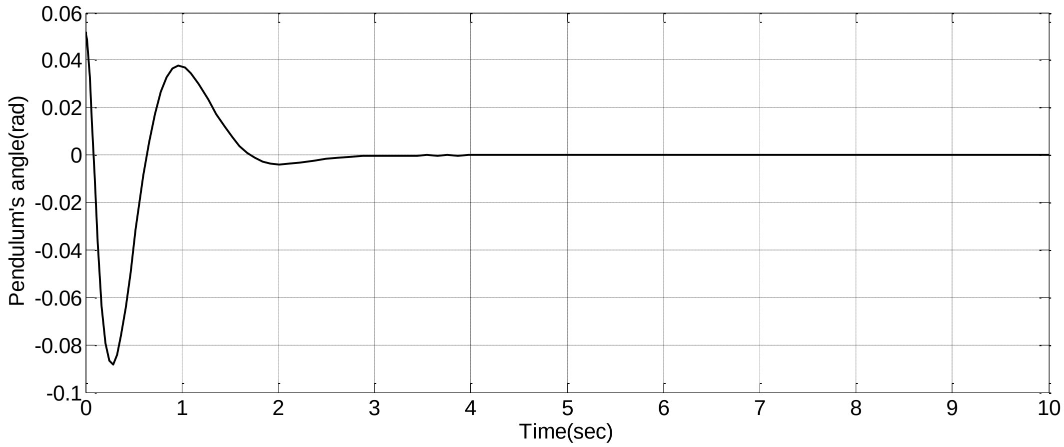

Fig 7.3.2 Response curve for pendulum's angle

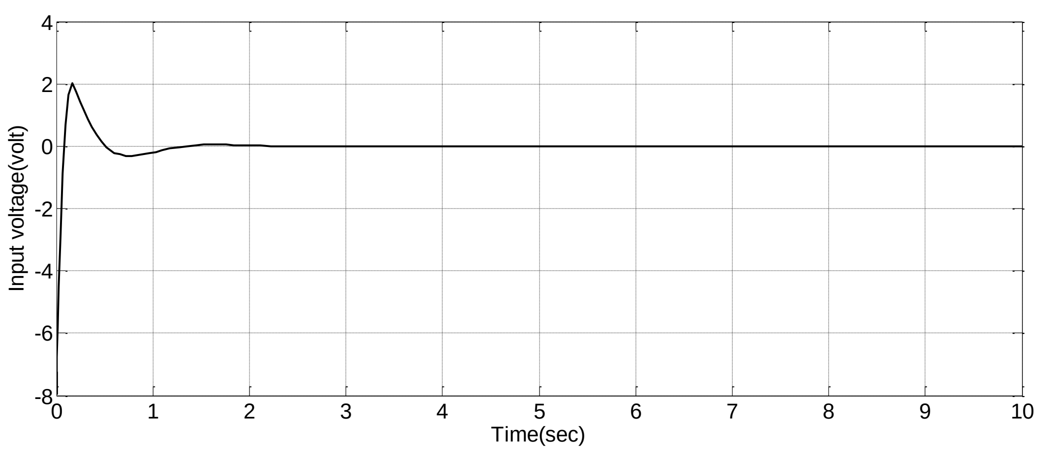

Fig 7.3.3 Response curve for input voltage to the motor

From these figures it's clear that the required output is achieved with these transfer functions.

7.3.1 Real Time Implementation

The real time model is run with these values of PID gains. The responses are shown in figure 7.3.1.1

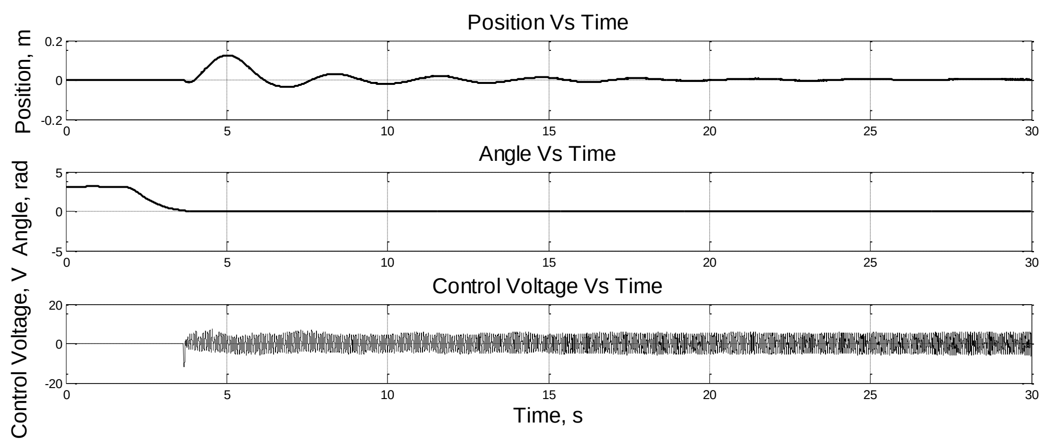

Fig 7.3.1.1 Response curves for PID controller in real time model

Chapter 9

CONCLUSION

8. Conclusion

8.1 Conclusion

From the above discussion, it can be concluded that both the control methods of conventional controllers (LQR and PID) can control the cart's position and the pendulum's angle for the linearized system. The response characteristics satisfied the requirements of the designed criteria.

These two controllers are compared. The PID controller gives a better performance compared to LQR controller. PID controller can control the whole system even if the initial angle of pendulum i.e. $\Phi(0) = 0.2$ rad or in case more than that, where as in LQR controller the control action is limited to initial angle $\Phi(0) = 0.1$ rad. The PID controller is implemented in the real time model present in the lab and satisfactory results are obtained.

8.2 Future Work

There is a lot of scope in control system engineering for the balancing of Inverted Pendulum system. In this thesis only conventional controllers are discussed. There are a lot more controllers, which can be used for the balancing purpose. Artificial Intelligence Controller, like, Fuzzy Logic Controller (FLC) and Artificial Neural Network Controller (ANN) can be used for better and robust control action. Similarly Dynamic Controller can also be used for the control action of the system.

Finally, the proposed model of conventional controller for balancing the Inverted Pendulum system can be used in many real time applications.

9. REFERENCES

[1] User manual, Feedback Instruments Limited on "Digital pendulum control experiments", 33-936s.

[2] A journal by Ahmad Nor Kasruddin Bin Nasir, Universiti Technology Malaysia, "Modelling and controller design for an inverted pendulum system", 2007.

[3] Katsuhiko Ogata, University of Minnesota, A text book of "Modern Control Engineering", ISBN 978-81-317-0311-3.

[4] The "INVERTED PENDULUM, ANALYSIS, DESIGN AND IMPLEMENTATION", a collection of MATLAB functions, scripts, and SIMULINK models, IIEE Visionaries. Document Version 1.0, developed by Khalil Sultan, B.E. in Industrial Electronics, Institute of Industrial Electronics Engineering (IIEE), PCSIR, NEDUET, Karachi, Pakistan.

[5] Balancing an Inverted Pendulum on a Seesaw by Maxim V. Subbotin.

[6] Undergraduate Lecture Notes on LQG/LQR controller design by Jo~ao P. Hespanha, April 1, 2007

[7] PID control system design by Karl Johan Astrom, 2002.

[8] PID Instruction VLT 6000 HVAC, FHT 1, PID Controller Theory.

[9] Sakshat Virtual Labs.

http://iitg.vlab.co.in/?sub=59&brch=163&sim=612&cnt=1086

[10] Wiki books. Control system/state-space Equations.

http://en.wikibooks.org/wiki/Control_Systems/State-Space_Equations#State-Space_Equations

[11] Control Tutorials for Matlab, THE UNIVERSITY OF MICHIGAN, 1817.

http://www.engin.umich.edu/group/ctm/PID/PID.html

[12] Matlab & Simulink- A tutorial by Tom Nguyen.

http://edu.levitas.net/Tutorials/Matlab/Simulink/

APPENDIX

APPENDIX A1

```
A = [    0    1.0000        0             0
         0   -0.0194   0.2204          0
         0         0         0       1.0000
         0    -0.0128  6.6321          0];

B = 15*[0;0.3888;0;0.2570];

C = [    1 0 0 0
         0 0 1 0];

D = [0;0];

Q = diag([15000;0;5000;0]);

R = 1500;
[K,S,e] = lqr(A,B,Q,R);
K1 = K(1)
K2 = K(2)
K3 = K(3)
K4 = K(4)
Output:
     K1 = -3.1623
     K2 = -4.1313
     K3 = 25.6776
      K4 =10.0636
```

A1: Matlab command to find feedback gains of the system

APPENDIX A2

```
A = [      0   1.0000       0          0
           0  -0.0194   0.2204         0
           0        0        0    1.0000
           0  -0.0128   6.6321         0];
B = 15*[0;0.3888;0;0.2570];
C = [      1 0 0 0
           0 0 1 0];
D = [0;0];
Q = diag([15000;0;5000;0]);
R = 1500;
[K,S,e] = lqr(A,B,Q,R);
e(1)
e(2)
e(3)
e(4)
Output:
ans = -3.1811 + 3.0446i
ans = -3.1811 - 3.0446i
ans = -2.4446 + 0.4400i
ans = -2.4446 - 0.4400i
```

A2: Matlab command to find closed-loop poles of the system