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Model Order Reduction in the L2-Gap Metric

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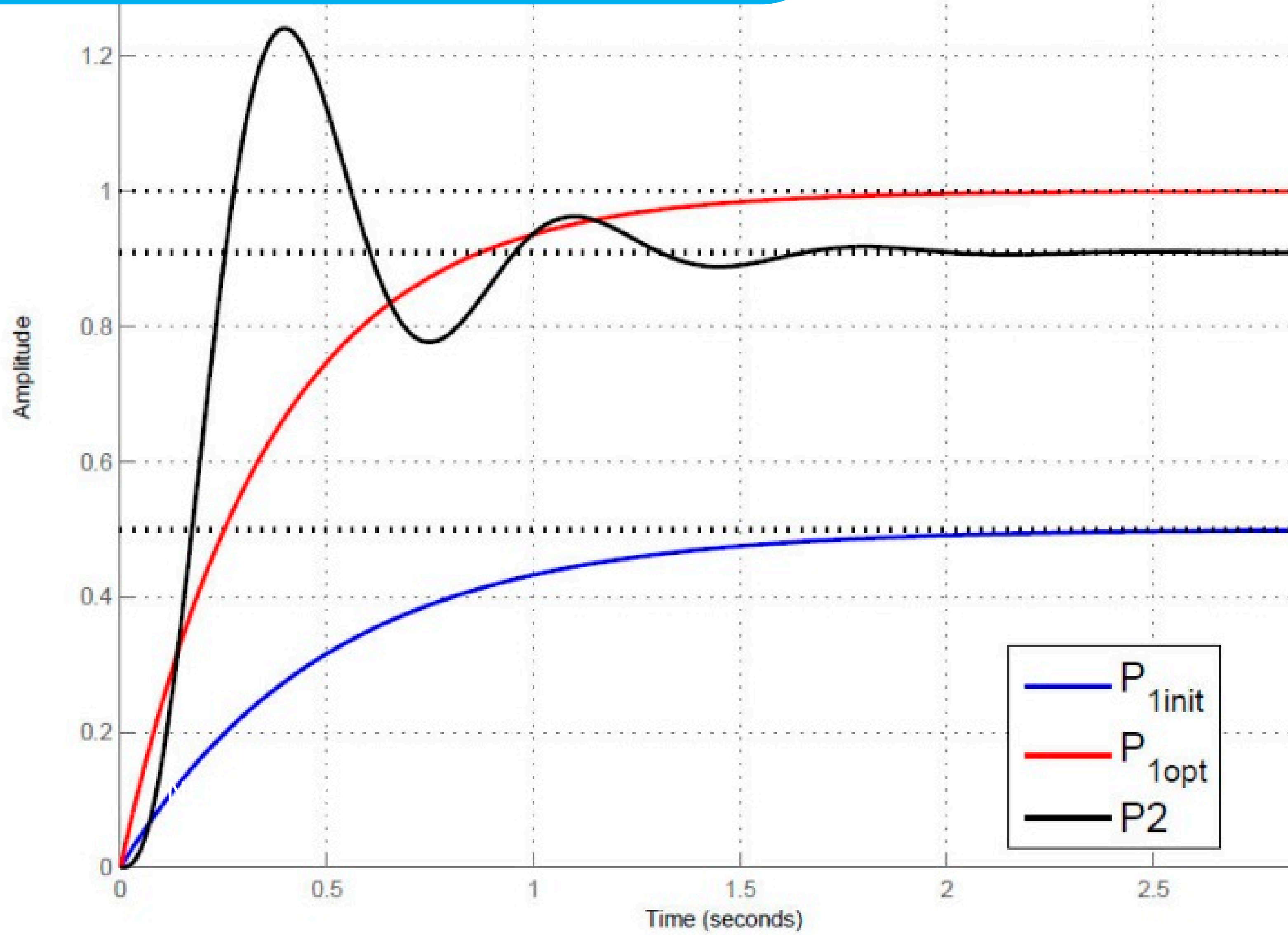

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EXECUTIVE SUMMARY

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Model Order Reduction in the L2-Gap Metric

Problem area

The L_2-gap metric is a central component of the nu-gap metric. The latter represents a good measure of the distance between systems in a closed-loop setting. For two Linear Time-Invariant (LTI) plants P_1 and P_2, the nu-gap is expressed as $dn(P_1,P_2)$. There are two main aspects related to the nu-gap metric. The first one is the so called *Winding Number Condition* (WNC), which is associated with the Nyquist diagram, and for which an efficient computational method already exists. If this WNC does not hold then $dn(P_1,P_2)=1$, whereas if it does hold then we have $dn(P_1,P_2) = dL_2(P_1,P_2)$, with $dL_2(P_1,P_2)$ the L_2-gap metric returning a scalar in the [0–1] range. The purpose of this paper is to first present a novel method to compute the $dL_2(P_1,P_2)$ gap. We show that the computation of this $dL_2(P_1,P_2)$ gap is in fact a convex problem, that can easily be expressed as Linear Matrix Inequalities (LMIs). Next, we show that this result can be used for model order reduction, within a Bilinear Matrix Inequalities (BMIs) framework.

Description of work

We demonstrate that the computation of the $dL_2(P_1,P_2)$ gap, i.e. the analysis problem, is in fact a convex problem. Our method consists in expressing the $dL_2(P_1,P_2)$ gap as Linear Matrix Inequalities (LMIs) — subsequently formulated as a Semi-Definite Programs (SDP)—for which there are several powerful numerical solutions The $dL_2(P_1,P_2)$ gap is computed on the full frequency axis and results in an infinite number of LMIs, emanating from the frequency-dependent structure. Subsequently, through the use of the Kalman-Yakubovich-Popov (KYP) Lemma, we remove this frequency dependence, and hence obtain an optimization problem of finite dimension. Our method does not introduce any approximations or sub-optimalities, and applies equally well to Single-Input Single-Output (SISO) or Multiple-Input Multiple-

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Output (MIMO) systems. Next we show that this result can be used for model (or controller) order reduction. The resulting synthesis problem is non-convex, but can be dealt within a Bilinear Matrix Inequalities (BMIs) framework. We illustrate the practicality of the proposed method on several numerical examples.

Results and conclusions

With regard to the $dL_2(P_1,P_2)$ gap metric, between tow LTI plants, we present a convex approach to solve the analysis side of the problem. We believe that this result may be seen as definitive. On the other hand, with regard to the synthesis side of the problem (i.e. model order reduction), we present what we believe to be a useful approach which, however, does come with some liabilities, namely the optimization is based upon BMIs. These BMIs have been solved using a simple, iterative, nonlinear search, in spirit reminiscent of D-K iteration synthesis. Analogously to D-K iteration convergence—for which convergence towards a global optimum, or even a local one, is not guaranteed—our proposed model order reduction algorithm does not inherit any convergence certificates, however in practice convergence has been achieved within 10 to 125 iterations.

Applicability

Compared to previous results, our model order reduction approach is not based upon frequency gridding, and since LMIs/BMIs intrinsically reflect constraints rather than optimality, our approach tends to offer more flexibility for combining several constraints during the synthesis process.

GENERAL NOTE

This report is based on a presentation to be held at the American Control Conference (ACC), Boston (MA-USA), July 6-8 2016.

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Model Order Reduction in the $\mathscr{L}_2$-Gap Metric

Skander Taamallah

***Abstract*— The $\mathscr{L}_2$-gap metric is a central component of the nu-gap (i.e. ν-gap) metric. The latter represents a good measure of the distance between systems in a closed-loop setting. For two Linear Time-Invariant (LTI) plants P_1 and P_2, the ν-gap is expressed as $\delta_\nu(P_1,P_2)$. There are two main aspects related to the ν-gap metric. The first one is the so-called *Winding Number Condition* (WNC), which is associated with the Nyquist diagram, and for which an efficient computational method already exists. If this WNC does not hold then $\delta_\nu(P_1,P_2)=1$, whereas if it does hold then we have $\delta_\nu(P_1,P_2):=\delta_{\mathscr{L}_2}(P_1,P_2)$, with $\delta_{\mathscr{L}_2}(P_1,P_2)$ the $\mathscr{L}_2$-gap metric returning a scalar in the [0–1] range. The purpose of this paper is to first present a novel method to compute the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap. We show that the computation of this $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap is in fact a convex problem, that can easily be expressed as Linear Matrix Inequalities (LMIs). Next, we show that this result can be used for model order reduction, within a Bilinear Matrix Inequalities (BMIs) framework. We illustrate the practicality of the proposed method on several numerical examples.**

I. INTRODUCTION

Gap and graph metrics [1] have been known to provide a measure of the separation between open-loop systems, in terms of their closed-loop behavior. The first attempt to introduce such a metric, simply known as gap metric, was formulated in [2], [3], whereas an efficient method for computing it was presented in [4], with recent works from a fairly general perspective proposed in [5]. Other significant metrics have also been investigated, such as (i) the T-gap metric [6], (ii) the pointwise gap [7], and (iii) Vinnicombe's popular nu-gap (i.e. ν-gap) metric [8], [9]. Similar to its predecessor gap metrics, the ν-gap also provides a means of quantifying feedback system stability and robustness, while being concurrently less conservative and simpler to compute. Time-varying and nonlinear extensions to both the gap metric [10], [11], [12], [13] and the ν-gap metric [14], [15], [16] have also been researched, although analytical computations of these metrics, in this nonlinear setting, is generally difficult. Over the years the use of these metrics has received much attention. In particular, the ν-gap was extensively studied in the realm of system identification [17], [18], [19], model order reduction [20], [21], [22], [23], [24], [25], and robust control [26], [9].

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If we consider two Linear Time-Invariant (LTI) plants P_1 and P_2, each with dimension $n\times m$, then the ν-gap is denoted by $\delta_\nu(P_1,P_2)$. There are two main aspects related to the ν-gap metric. The first one is the so-called *Winding Number Condition* (WNC), which is associated with the Nyquist diagram, and for which an efficient computational method already exists. The WNC is readily obtained by computing the number of right-half-plane poles of a closed-loop transfer function, involving the interconnection of plants P_1 and P_2, see [8], [9]. If this WNC does not hold then $\delta_\nu(P_1,P_2)=1$, whereas if it does hold then we have $\delta_\nu(P_1,P_2):=\delta_{\mathscr{L}_2}(P_1,P_2)$, with $\delta_{\mathscr{L}_2}(P_1,P_2)$ the $\mathscr{L}_2$-gap metric returning a scalar in the [0–1] range.

The purpose of this paper is to first present a novel method to compute the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap, and then show how this result may be used for model order reduction. We demonstrate that the computation of the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap, i.e. the analysis problem, is in fact a convex problem. Our method consists in expressing the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap as Linear Matrix Inequalities (LMIs) [27]—subsequently formulated as a SDP [28]—for which there are several powerful numerical solutions [29], [30].

The $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap is computed on the full frequency axis and results in an infinite number of LMIs, emanating from the frequency-dependent structure. Subsequently, through the use of the Kalman-Yakubovich-Popov (KYP) Lemma, we remove this frequency dependence, and hence obtain an optimization problem of finite dimension. Our method does not introduce any approximations or sub-optimalities, and applies equally well to Single-Input Single-Output (SISO) or Multiple-Input Multiple-Output (MIMO) systems.

Next we show that this result can be used for model (or controller) order reduction. The resulting synthesis problem is non-convex, but can be dealt within a Bilinear Matrix Inequalities (BMIs) framework. Compared to previous results, our approach is not based upon frequency griding, and since LMIs/BMIs intrinsically reflect constraints rather than optimality, our approach tends to offer more flexibility for combining several constraints during the synthesis process.

The nomenclature is fairly standard. M^* denotes the complex-conjugate transpose of a complex matrix M. Matrix inequalities are considered in the sense of *Löwner*. Further $\lambda(M)$ denotes the zeros of the characteristic polynomial $\det(sI-M)=0$. $\bar{\lambda}(M)$ is the maximum eigenvalue of M. Next, $\mathscr{L}_\infty$ is the *Lebesgue* normed space s.t. $\|G\|_\infty := ess \sup_{\omega\in\mathbb{R}} \bar{\sigma}(G(j\omega)) < \infty$, with $\bar{\sigma}(G)$ the largest singular value of matrix $G(\cdot)$. Similarly, $\mathscr{H}_\infty \subset \mathscr{L}_\infty$ is the *Hardy* normed space s.t. $\|G\|_\infty := \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(G(s))$. $\mathscr{RL}_\infty$ (resp. $\mathscr{RH}_\infty$) represent the subspace of real rational Transfer Functions in $\mathscr{L}_\infty$ (resp. $\mathscr{H}_\infty$). Finally I and 0 will be used to denote the identity and null matrices respectively, assuming appropriate sizes.

II. Preliminaries

This section recalls first the KYP Lemma [31], and subsequently introduces the ν-gap metric.

Lemma 1: Let complex matrices A, B, and a symmetric matrix Θ, of appropriate sizes, be given. Suppose $\lambda(A) \subset \mathbb{C}^- \cup \mathbb{C}^+$, then the following two statements are equivalent.

(i) $\forall\omega \in \mathbb{R}\cup\{\infty\}$

$$\begin{bmatrix} (j\omega-A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega-A)^{-1}B \\ I \end{bmatrix} < 0$$

(ii) There exists a matrix $P=P^*$, and a linear matrix map $L(P)$, such that the following LMI holds

$$L(P)+\Theta<0 \quad \text{with}$$
$$L(P) := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

Proof: See [31]. ■

Remark 1: We have dealt here with the strict version of the KYP lemma, i.e. strict inequalities, since no controllability/stabilizability assumptions become necessary.

Remark 2: If matrices A, B, and Θ are all real, the equivalence still holds when restricting P to be real [32].

There exists several equivalent definitions of the ν-gap metric. The one chosen in this paper is most convenient for our purpose.

Definition 1: The ν-gap metric between two LTI plants P_1 and P_2, having dimensions $n \times m$, with $P_1, P_2 \in \mathscr{RL}_\infty$, is given by [8]

$$\delta_\nu(P_1,P_2) := \begin{cases} \delta_{\mathscr{L}_2}(P_1,P_2) & if\ the\ WNC\ holds \\ 1. & else \end{cases} \tag{1}$$

with

$$\delta_{\mathscr{L}_2}(P_1,P_2) := \|(I+P_2P_2^*)^{-1/2}(P_2-P_1)(I+P_1^*P_1)^{-1/2}\|_\infty \tag{2}$$

and *WNC* the so-called *Winding Number Condition* associated with the Nyquist diagram, for which an efficient computational method already exists. The WNC is readily obtained by computing the number of right-half-plane poles of a closed-loop transfer function, involving the interconnection of plants P_1 and P_2 [8], [9].

III. Computation of the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap

The purpose of this paper is to focus upon the $\delta_{\mathscr{L}_2}(P_1,P_2)$ part, i.e. the metric returning a scalar in the [0–1] range. Now from (2), by using the definition of the $\|\cdot\|_\infty$ norm, and after rearranging terms, we obtain the following definition

Definition 2: The $\delta_{\mathscr{L}_2}(P_1,P_2)$ metric between two LTI plants P_1 and P_2, having dimensions $n\times m$, with $P_1,P_2\in\mathscr{RL}_\infty$, is given by

$$\delta_{\mathscr{L}_2}(P_1,P_2) := \sup_{\omega\in\mathbb{R}}\left[\bar{\lambda}\left((P_2-P_1)^*(I+P_2P_2^*)^{-1}\right.\right.$$
$$\left.\left.(P_2-P_1)(I+P_1^*P_1)^{-1}\right)\right]^{(1/2)} \tag{3}$$

Through the use of [27], expression (3) can be recast into an LMI.

Lemma 2: The $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap between two LTI plants P_1 and P_2, having dimensions $n\times m$, with $P_1,P_2\in\mathscr{RL}_\infty$, is given by $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$, with λ computed as

$$\underset{\omega\in\mathbb{R},\ 0<\lambda<1}{\text{minimize}}\ \lambda \quad \text{subject to}$$
$$(P_2-P_1)^*(I+P_2P_2^*)^{-1}(P_2-P_1)<\lambda(I+P_1^*P_1) \tag{4}$$

Proof: By expressing (3) as a maximum eigenvalue problem in LMI form. ■

Next we transform (4) as follows.

Lemma 3: The $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap between two LTI plants P_1 and P_2, having dimensions $n\times m$, with $P_1,P_2\in\mathscr{RL}_\infty$, is given by $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$, with λ computed as

$$\underset{\omega\in\mathbb{R},\ 0<\lambda<1}{\text{minimize}}\ \lambda \quad \text{subject to} \quad \Omega^*\Delta\Omega<0 \quad \text{with}$$
$$\Delta := \begin{bmatrix} 0 & \frac{1}{2}I & 0 & 0\\ \frac{1}{2}I & 0 & 0 & 0\\ 0 & 0 & -\lambda I & 0\\ 0 & 0 & 0 & -\lambda I\end{bmatrix}$$
$$\Omega := \begin{bmatrix} (I+P_2P_2^*)^{-1}(P_2-P_1)\\ P_2-P_1\\ P_1\\ I\end{bmatrix} \tag{5}$$

Proof: By expanding the right-hand side of (4), and noting that $(I+P_2P_2^*)^{-1}>0$, and by regrouping terms as partitioned matrices we get (5). ■

Now we express (5) in a form amenable to the KYP Lemma.

Lemma 4: The $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap between two LTI plants P_1 and P_2, having dimensions $n\times m$, with $P_1,P_2\in\mathscr{RL}_\infty$, is given by $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$, with λ computed as

$$\underset{\omega\in\mathbb{R},\ 0<\lambda<1}{\text{minimize}}\ \lambda \quad \text{subject to}$$
$$\begin{bmatrix}(sI-A_\Psi)^{-1}B_\Psi\\ I\end{bmatrix}^*\Theta\begin{bmatrix}(sI-A_\Psi)^{-1}B_\Psi\\ I\end{bmatrix}<0,$$
$$\text{with} \quad s=j\omega$$
$$\Psi := \left[\begin{array}{c|c} A_\Psi & B_\Psi\\ \hline C_\Psi & D_\Psi\end{array}\right] = \begin{bmatrix}(I+P_2P_2^*)^{-1}(P_2-P_1)\\ P_2-P_1\\ P_1\end{bmatrix}$$
$$\Theta := \begin{bmatrix} C_\Psi & D_\Psi\\ 0 & I\end{bmatrix}^*\Delta\begin{bmatrix} C_\Psi & D_\Psi\\ 0 & I\end{bmatrix}$$
$$\text{and } \Delta \text{ from Lemma 3} \tag{6}$$

Proof: First by construction we have $\Omega=\begin{bmatrix}\Psi\\ I\end{bmatrix}$. Next, in the optimization problem of Lemma 3, substitute $\Omega^*\Delta\Omega<0$, with $\begin{bmatrix}\Psi\\ I\end{bmatrix}^*\Delta\begin{bmatrix}\Psi\\ I\end{bmatrix}<0$, then replace Ψ by $C_\Psi(sI-A_\Psi)^{-1}B_\Psi+D_\Psi$, and expand and regroup terms. ■

A. Main result

The optimization problem of Lemma 4 involves an infinite number of LMIs, emanating from the frequency-dependent structure (i.e. on ω). The goal is now to remove this frequency dependence, and hence obtain an optimization problem of finite dimension.

Theorem 1: Let two LTI plants P_1 and P_2, having dimensions $n\times m$, with $P_1,P_2\in\mathscr{RL}_\infty$, be given. Let their respective realization be $P_1 := \left[\begin{array}{c|c} A_1 & B_1\\ \hline C_1 & D_1\end{array}\right]$, $P_2 := \left[\begin{array}{c|c} A_2 & B_2\\ \hline C_2 & D_2\end{array}\right]$, then the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap between two plants P_1 and P_2 is given by $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$, with λ computed as

$$\begin{aligned}&\underset{P=P^*,\ 0<\lambda<1}{\text{minimize}}\ \lambda \quad \text{subject to}\\ &\begin{bmatrix} A_\Psi^* P + PA_\Psi & PB_\Psi \\ B_\Psi^* P & 0 \end{bmatrix} + \Theta < 0 \end{aligned} \tag{7}$$

with Θ as given in Lemma 4, and further

$$\Psi := \begin{bmatrix} Z_4 \\ Z_1 \\ P_1 \end{bmatrix} = \left[\begin{array}{ccc|c} A_{Z_4} & 0 & 0 & B_{Z_4} \\ 0 & A_{Z_1} & 0 & B_{Z_1} \\ 0 & 0 & A_1 & B_1 \\ \hline C_{Z_4} & 0 & 0 & D_{Z_4} \\ 0 & C_{Z_1} & 0 & D_{Z_1} \\ 0 & 0 & C_1 & D_1 \end{array}\right]$$

$$\begin{aligned}
Z_4 &:= \left[\begin{array}{c|c} A_{Z_4} & B_{Z_4} \\ \hline C_{Z_4} & D_{Z_4} \end{array}\right] = Z_3 Z_1 \\
&= \left[\begin{array}{cc|c} A_{Z_3} & B_{Z_3}C_{Z_1} & B_{Z_3}D_{Z_1} \\ 0 & A_{Z_1} & B_{Z_1} \\ \hline C_{Z_3} & D_{Z_3}C_{Z_1} & D_{Z_3}D_{Z_1} \end{array}\right] \\
Z_3 &:= \left[\begin{array}{c|c} A_{Z_3} & B_{Z_3} \\ \hline C_{Z_3} & D_{Z_3} \end{array}\right] = Z_2^{-1} \\
&= \left[\begin{array}{c|c} A_{Z_2} - B_{Z_2}D_{Z_2}^{-1}C_{Z_2} & -B_{Z_2}D_{Z_2}^{-1} \\ \hline D_{Z_2}^{-1}C_{Z_2} & D_{Z_2}^{-1} \end{array}\right] \\
Z_2 &:= \left[\begin{array}{c|c} A_{Z_2} & B_{Z_2} \\ \hline C_{Z_2} & D_{Z_2} \end{array}\right] = I + P_2 P_2^* \\
&= \left[\begin{array}{cc|c} A_2 & B_2B_2^* & B_2D_2^* \\ 0 & -A_2^* & -C_2^* \\ \hline C_2 & D_2B_2^* & I + D_2D_2^* \end{array}\right] \\
Z_1 &:= \left[\begin{array}{c|c} A_{Z_1} & B_{Z_1} \\ \hline C_{Z_1} & D_{Z_1} \end{array}\right] = P_2 - P_1 \\
&= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ 0 & A_1 & B_1 \\ \hline C_2 & -C_1 & D_2 - D_1 \end{array}\right]
\end{aligned}$$

Proof: From (6), it is a straightforward application of the KYP Lemma (see Lemma 1).

■

IV. Synthesis Problem: Model Order Reduction

We consider here a model order reduction problem in which the approximation error, between two LTI plants P_1 and P_2, is quantified using the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap metric. We suppose that P_2 is given, with the aim of finding P_1 such that $\delta_{\mathscr{L}_2}(P_1,P_2)$ is either minimized, or such that $\delta_{\mathscr{L}_2}(P_1,P_2)<\beta$, with $0<\beta<1$ a given bound. From (7), we see that this optimization problem is non-convex, due to various cross-product terms (e.g. A_1^*P, $C_1^*D_1$) and quadratic terms (e.g. $C_1^*C_1$), in the decision variables. Hence, we simplify the original problem by having only matrices P, A_1, and B_1 as decision variables (i.e. C_1 and D_1 are fixed). Next, from (7), we see that, for a fixed Lyapunov function P, the problem becomes affine in the unknown A_1 and B_1 matrices, thus bi-convex. The following algorithm summaries the procedure for a reduced-order approximation in the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap metric.

Proposition 1 (Model order reduction): Given a user-defined bound $\varepsilon>0$, and a nominal LTI plant $P_2 := \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$, of order k, having dimension $n\times m$, with $P_2\in\mathscr{RL}_\infty$, then a reduced-order LTI plant $P_1 := \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ can be constructed, of order l ($l<k$), having dimension $n\times m$, with $P_1\in\mathscr{RL}_\infty$, such that $\delta_{\mathscr{L}_2}(P_1,P_2)$ is approximately minimized, in the following way

(A) Fix order l of plant P_1 (l is user-defined)
(B) Obtain an initial value for A_1, B_1, and C_1
(C) Set $D_1 = D_2$
(D) Optimize A_1 and B_1 using the following stepwise method
  (a) In LMI (7), fix A_1 and B_1
  (b) Set $\lambda_{min}=0$ and $\lambda_{max}=1$
    (1) Set $\lambda = (\lambda_{min}+\lambda_{max})/2$
    (2) In LMI (7) solve for P
    (3) If optimization is feasible
      set $\lambda_{max}=\lambda$, otherwise $\lambda_{min}=\lambda$
    (4) Repeat from (1) until $|\lambda_{max}-\lambda_{min}|\le\varepsilon$
  (c) Compute $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$
  (d) Retrieve P, and in LMI (7), fix P
  (e) Set $\lambda_{min}=0$ and $\lambda_{max}=1$
    (i) Set $\lambda = (\lambda_{min}+\lambda_{max})/2$
    (ii) In LMI (7) solve for A_1 and B_1
    (iii) If optimization is feasible
      set $\lambda_{max}=\lambda$, otherwise $\lambda_{min}=\lambda$
    (iv) Repeat from (i)
      until $|\lambda_{max}-\lambda_{min}|\le\varepsilon$
  (f) Compute $\delta_{\mathscr{L}_2}(P_1,P_2)=\lambda^{1/2}$

(g) Retrieve A_1 and B_1
(h) Repeat from (a) until $\delta_{\mathscr{L}_2}(P_1,P_2)$ convergence, or maximum iteration reached

Remark 3: This algorithm is only a heuristic for which convergence towards a global optimum, or even a local optimum, is not guaranteed[1]. This said, in practice, convergence has been achieved within 10 to 125 iterations.

V. Numerical Experiments

For the analysis problem, the purpose is to compute $\delta_{\mathscr{L}_2}(P_1,P_2)$, for known LTI plants P_1 and P_2, using the SDP optimization from Theorem 1, and compare the results to the values obtained from the MATLAB function *gapmetric*[2]. We consider here four examples, for which the state-space data, for P_1 and P_2, are given in Appendix A. The results for $\delta_{\mathscr{L}_2}(P_1,P_2)$ are reported in Table I. We see that both approaches compute gap values[3] which are very close to each other, i.e. the absolute deviations are below 10^{-6} (highest deviation seen on example 3). From a computational cost viewpoint, we see from Table II that the cost for the SDP method is only 2.1 to 2.3 times higher. The results presented in Table II are based upon MATLAB runs on a legacy computer hardware.
For the model order reduction synthesis problem, we illustrate the practicality of algorithm 1 on two numerical examples, also given in Appendix A. In Example 5 and 6, LTI plant P_2 has order 3 and 5 respectively. P_2 will be approximated by a LTI plant P_1 having order 1 and 2 respectively. Plant P_{1init} represents the initial P_1 values for algorithm 1, whereas plant P_{1opt} represents the optimized plant computed by algorithm 1. For the case of example 6, P_{1init} has been obtained after balancing and Hankel-norm model reduction[4] [34]. For example 5 and 6, the results for $\delta_{\mathscr{L}_2}(P_1,P_2)$, before and after the optimization of algorithm 1, are reported in Table III (with $\varepsilon = 0.001$). We see that algorithm 1 provides a substantial decrease in the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap metric. Further, closed-loop step responses, under negative unity feedback, for plant P_2, and for plant P_1 (before and after the optimization of algorithm 1), are visualized in Fig. 1 and Fig. 2. In particular, we see that the reduced-order model produced by the Hankel-norm is closed-loop unstable, whereas our method produces a reduced-order approximation which is closed-loop stable.

TABLE I
Computation of $\delta_{\mathscr{L}_2}(P_1,P_2)$ and comparison with the MATLAB function *gapmetric*

	MATLAB *gapmetric*	Our SDP method from Theorem 1
Example 1	0.972806214684513	0.972806213500936
Example 2	0.699486720564277	0.699486721039127
Example 3	0.543879042696287	0.543879961800073
Example 4	0.852097793847522	0.852097793541100

TABLE II
Comparison of the computational costs (in seconds) with the MATLAB function *gapmetric*

	MATLAB *gapmetric*	Our SDP method from Theorem 1
Example 1	1.053	2.361
Example 2	0.998	2.100
Example 3	0.976	2.227
Example 4	1.011	2.311

VI. Conclusion

With regard to the $\delta_{\mathscr{L}_2}(P_1,P_2)$ gap metric, between tow LTI plants, we have presented a convex approach (Theorem 1 In Section III-A) to solve the analysis side of the problem. We believe that this result may be seen as definitive. On the other hand, with regard to the synthesis side of the problem

TABLE III
Model order reduction

	$\delta_{\mathscr{L}_2}(P_{1init},P_2)$	$\delta_{\mathscr{L}_2}(P_{1opt},P_2)$	Nr. of iterations in algorithm 1
Example 5	0.8200	0.5655	125
Example 6	0.4035	0.1523	75

[1] In algorithm 1 the bisection on λ allows for better control of algorithm convergence (e.g. through the choice of bound ε).
[2] The MATLAB function *gapmetric* returns two outputs, the second one is the ν-gap metric.
[3] All LMI problems are solved in a MATLAB® environment using YALMIP [33] together with the SeDuMi solver [30].
[4] Using the *modred* MATLAB function, together with the *Truncate* method which tends to produce better approximations in the frequency domain.

(i.e. model order reduction), we have presented what we believe to be a useful approach which, however, does come with some liabilities, namely the optimization is based upon BMIs. These BMIs have been solved using a simple, iterative, nonlinear search, in spirit reminiscent of D-K iteration synthesis [35]. Analogously to D-K iteration convergence—for which convergence towards a global optimum, or even a local one, is not guaranteed [36]—our proposed model order reduction algorithm does not inherit any convergence certificates, however in practice convergence has been achieved within 10 to 125 iterations.

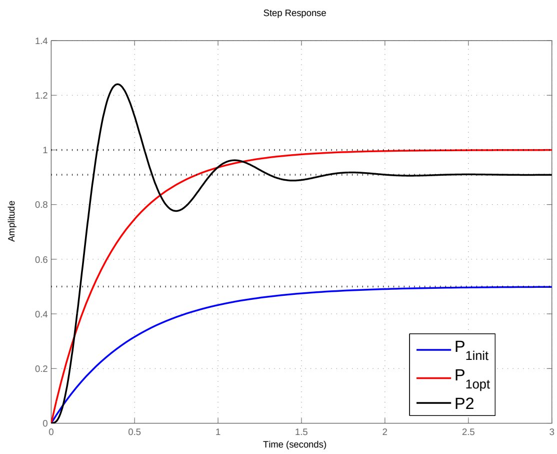

Fig. 1. Model order reduction: example 5 (closed-loop step response under negative unity feedback)

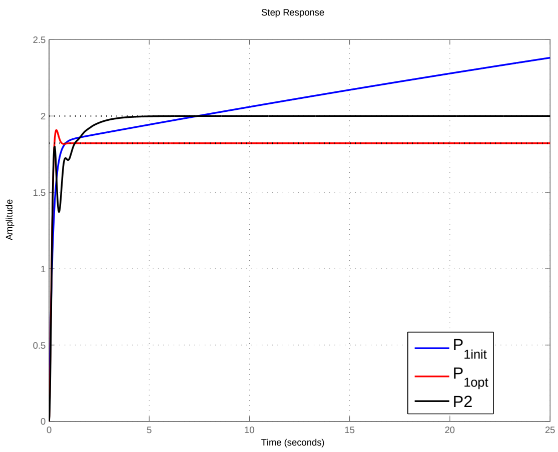

Fig. 2. Model order reduction: example 6 (closed-loop step response under negative unity feedback)

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APPENDIX A: DATA FOR THE NUMERICAL EXPERIMENTS

Example 1: 3rd order, SISO systems

$$P_1 = \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2.5 \\ 1 & -2 & 0 & 3 \\ 1 & 2 & 3 & -1 \\ \hline 2 & -2 & 1 & -2 \end{array}\right]$$

$$P_2 = \left[\begin{array}{ccc|c} 1.5 & 3.5 & 4.5 & 2 \\ 1 & -2.5 & 0.5 & 3 \\ 1 & 2.5 & 2.5 & -1.5 \\ \hline 2 & -1.5 & 1.5 & -1.5 \end{array}\right]$$

Example 2: 2nd order, MISO systems (2 inputs, 1 output)

$$P_1 = \left[\begin{array}{cc|cc} 1 & 1.5 & 2 & 1 \\ 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 0.5 & 1 \end{array}\right] \quad P_2 = \left[\begin{array}{cc|cc} 1.5 & 3 & 2 & 1 \\ 1 & -2 & -1 & 1.5 \\ \hline 1 & 0.5 & 0.5 & 1 \end{array}\right]$$

Example 3: 2nd order, MIMO systems (2 inputs, 2 outputs)

$$P_1 = \left[\begin{array}{cc|cc} -1 & -0.15 & -2.5 & -1 \\ -1 & -1 & -2 & 3 \\ \hline 1 & 6 & 0.5 & 0 \\ 2.5 & -3 & 0 & 0.5 \end{array}\right]$$

$$P_2 = \left[\begin{array}{cc|cc} -1 & -0.15 & -3 & -1 \\ -1 & -1.5 & -3 & 3 \\ \hline 2 & 1 & 0.25 & 0 \\ 2 & -2 & 0.25 & 0.5 \end{array}\right]$$

Example 4: 3rd order, MIMO systems (2 inputs, 3 outputs)

$$P_1 = \left[\begin{array}{ccc|cc} 1 & 3 & 4 & -2.5 & -1 \\ 1 & -2 & 0 & -1 & 3 \\ 1 & 2 & 3 & 3 & 4 \\ \hline 1 & 6 & 6 & 0.5 & 0 \\ 2.5 & -3 & 1 & 0 & 0.5 \\ 2 & 3 & 4 & 2 & 3 \end{array}\right]$$

$$P_2 = \left[\begin{array}{ccc|cc} 1.5 & 3 & 4 & -2.5 & -1 \\ 1 & -2.5 & 0 & -2 & 3 \\ 1 & 2 & 3 & 3 & 4 \\ \hline 1 & 4 & 6 & 0.25 & 0 \\ 2 & -3 & 1 & 0 & 0.5 \\ 2 & 3 & 3 & 2.5 & 3 \end{array}\right]$$

Example 5: **order reduction for** $P_2 = \frac{10}{(s+1)(0.075s+1)^2}$

$$P_{1init} = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right] \quad P_{1opt} = \left[\begin{array}{c|c} -0.0037 & 3.60 \\ \hline 1 & 0 \end{array}\right]$$

Example 6: **order reduction for** $P_2 = \frac{s+2}{(0.0025s+1)^2(0.1s+1)(0.05s^2-1)}$

$$P_{1init} = \left[\begin{array}{cc|c} 4.4721 & 0 & -1.0015 \\ 0 & -0.0373 & 0.1868 \\ \hline -9.7650 & 0.1868 & 0 \end{array}\right]$$

$$P_{1opt} = \left[\begin{array}{cc|c} 3.9175 & -41.6932 & -0.1406 \\ -0.9514 & -18.8116 & 0.5550 \\ \hline -9.7650 & 0.1868 & 0 \end{array}\right]$$

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