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TOLERANCE SYNTHESIS OF MECHANISMS: A ROBUST DESIGN APPROACH

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ABSTRACT
This paper provides a new robust design method to dimension a mechanism and to synthesize its dimensional tolerances. The general issue is to find a robust mechanism for a given task, and to compute its optimal dimensional tolerances. For that purpose, the developed approach follows two consecutive steps, which are independent and complementary. First, the dimensions of the mechanism are computed by means of an appropriate robustness index, which is used to minimize the sensitivity of its performances to variations. These robust dimensions are obtained independently of the amount of variations, and tolerate globally the largest variations. Thus, knowing the acceptable performance error of the mechanism, the second step aims at computing the optimal dimensional tolerances of the mechanism by means of the new tolerance synthesis method. This method is used to find the best distribution of the error between the dimensions of the mechanism. Two serial manipulators are studied to illustrate the theory.

Keywords: robust design, tolerance synthesis, mechanism, variations, sensitivity ellipsoid, optimization.

1 Introduction
Every engineering design is subject to variations that can arise from a variety of sources, including manufacturing operations, variations in material properties, and the operating environment. When variations are ignored, nonrobust designs can result, which are expensive to produce or fail in service. Besides, the robustness of a mechanism is important when calibration is necessary because the lower the sensitivity of the mechanism to dimensional variations, the easier its calibration [1].

The concept of robust design may be first used by Taguchi. He introduced the concept of parameter design to improve the quality of a product whose manufacturing process involves significant variability or noise [2]. Robust design aims at minimizing the sensitivity of performances to variations without controlling the causes of these variations. In the last decades, several authors have contributed to the formulation and the improvement of robust design problems. Kalsi et al. [3] introduced a technique to reduce the effects of uncertainty and incorporate flexibility in the design of complex engineering systems involving multiple decision-makers. Chen et al. [4] studied two broad categories of problems namely, (i) Type 1: minimizing variations in performance caused by variations in noise factors (uncontrollable parameters) and (ii) Type 2: minimizing variations in performance caused by variations in control factors (design variables, DV).

Sundaresan et al. [5] developed a procedure incorporating uncertainties in DV and variations in constraints due to these uncertainties.

The dimensional tolerances of a mechanism are fixed according to various parameters such as the manufacturing process, the performance tolerances, the manufacturing cost. Chase et al. [6, 7] presented the Direct Linearization Method for tolerance analysis of 2-D and 3-D mechanical assemblies. Parkinson [8] used a deterministic method of robust design to determine the optimum nominal dimensions of an assembly in order to improve the assembly quality without tightening tolerances. Moreover, Rajagopalan and Cutkosky [9] used similar methods to analyse the performance errors of mechanisms fabricated in-Situ.

tems based on nonlinear programming, whereas Gadallah and El-
Maraghy [12] presented a method using a system of experimen-
sensitivity distribution to study the sensitivity of the system to
variations, and selected one manipulator among six, by means of
a robustness index. They defined the tolerance box as a contraction
of the circumscribe box of the design sensitivity ellipsoid
of the mechanism. The link between dimensional tolerances and
product’s cost is presented in several works [14].

The paper focuses on mechanisms, which are assemblies of
moving parts performing a complete functional motion.

Here, the study of a mechanism and the calculation of its
dimensional tolerances are conducted in two consecutive steps.
First, its dimensions are computed by means of a robustness in-
dex, which is proposed to minimize the sensitivity of its per for-
mance to variations. For example, a robust dimensioning of the
2R manipulator, whose end effector $E$ has to hit point $P$ with the
highest precision, is depicted in Fig.1. For this dimensioning, the
links of the manipulator are perpendicular when $E$ is supposed to
hit $P$. In this configuration, the maximum positioning error of
$E$, $\varepsilon$, due to dimensional variations, is a minimum.

![Figure 1. A Robust Dimensioning of the 2R Manipulator](image)

Then, knowing the acceptable performance error of the
mechanism, the second step aims at computing the optimal di-

mensional tolerances of the mechanism by means of a new toler-
ance synthesis method. This method is based on the robustness
approach of the first step.

The formulation of a robust design problem is given in sec-
tion 2. Section 3 discusses of an appropriate robustness index
for mechanisms. The new tolerance synthesis method is developed
in section 4. Finally, a 2R manipulator and a 3R manipulator are
studied in section 5 to illustrate the theory.

2 Robust design problem

In a robust design problem, the distinction is made between
three sets: (i) the set of design variables ($DV$) whose nominal
values can be selected between the range of upper and lower
bounds, they are controllable; (ii) the set of design parameters
($DP$) that cannot be adjusted by the designer, they are uncontrol-
able; (iii) the set of performance functions. The $l$-dimensional
vector of design variables is denoted by $x = [x_1, x_2, \ldots, x_l]^T$. The
$m$-dimensional vector of design parameters is denoted by $p =
[p_1, p_2, \ldots, p_m]^T$. Performance functions are grouped into the $n$-
dimensional vector $f = [f_1, f_2, \ldots, f_n]^T$. $DV$ are, however,
subject to uncontrollable variations because of manufacturing er-
rors, wear, or other uncertainties, although their nominal value is
fixed.

For instance, for the slider-crank mechanism depicted by
Fig.2, $f = < N >, x = [l_c, l_r, e]^T$, and $p = [f_p, \mu]$ where $< N >$
is the average side force on piston to be minimized. $l_c$ and $l_r$ are
the lengths of the rod and the crank of the mechanism. $e$ is the
eccentricity between the crank and the piston. $f_p$ is the force on
piston and $\mu$ is the friction coefficient between the piston and the
cylinder.

![Figure 2. Slider-Crank Mechanism](image)

A system is robust when its performance is as little sensitive
as possible to variations. Performance function $f$ depends on $DV$
and $DP$, which are supposed to be independent.

$$f = f(x, p)$$  \hspace{1cm} (1)

Here, the study of the sensitivity of the system to variations is
based on the theory of performance sensitivity distribution.

$$\delta f = [J_f, J_p] [\delta x^T \delta p^T]^T = J \delta X$$  \hspace{1cm} (2)
In this theory, a Jacobian matrix $\mathbf{J}$ describes the effect of the component variations to the system performance, as depicted by eq.(2) where $\mathbf{J}_x = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \mathbf{J}_p = \frac{\partial \mathbf{f}}{\partial \mathbf{p}}, \mathbf{J} = [\mathbf{J}_x \mathbf{J}_p], \mathbf{X}^T = [\mathbf{x}^T \mathbf{p}^T]$. $\delta \mathbf{x}$ and $\delta \mathbf{p}$ are the variations in $\mathbf{DV}$ and in $\mathbf{DP}$, respectively. $\mathbf{J}_x$ and $\mathbf{J}_p$ are the $(n \times 1)$ sensitivity Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ and the $(n \times m)$ sensitivity Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{p}$, respectively. If variations in $\mathbf{DV}$ are not taken into account, then $\mathbf{J} = \mathbf{J}_x$, and $\mathbf{X} = \mathbf{p}$. On the contrary, $\mathbf{J} = \mathbf{J}_s$, and $\mathbf{X} = \mathbf{x}$ when only variations in $\mathbf{DV}$ are considered.

The performance distribution is characterized in the variation space by a set of eigenvalues and eigenvectors, i.e.: by a hyper-ellipsoid. Without loss of generality, assuming that variations in $\mathbf{DV}$ are negligible and that there are only two $\mathbf{DP}$, this design sensitivity hyper-ellipsoid is an ellipse depicted in Fig.3. $\sigma_1$ and $\sigma_2$ are the smallest and the largest singular values of $\mathbf{J}$, respectively, and $\mathbf{q}_1$, $\mathbf{q}_2$ are their corresponding eigenvectors.

Lengths of semi-axes are inversely proportional to singular values of $\mathbf{J}$. Points on the ellipse surface lead to the same norm of variations in its performance, $\|\delta \mathbf{f}\|_2$, where $\|\cdot\|_2$ depicts the Euclidean norm. Moreover, the performance is the least sensitive to variations in the direction of $\mathbf{q}_1$ and the most sensitive to variations in the direction of $\mathbf{q}_2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Design Sensitivity Ellipsoid}
\end{figure}

A mechanism is robust when the sensitivity $S$ of its performances to variations is a minimum. Therefore, $S$ can be defined as the ratio of the Euclidean norm of variations in its performances, $\|\delta \mathbf{f}\|_2$, and the Euclidean norm of variations in $\mathbf{DV}$ and $\mathbf{DP}$, $\|\delta \mathbf{x}\|_2$, $\|\delta \mathbf{p}\|_2$. $S$ represents a variation transmission ratio and means the amount of variations transmitted from the sources to the design. Besides, eq.(3) follows from eq.(2) and means that $S$ is bounded by the smallest singular value, $\sigma_{\text{min}}$, and the largest singular value, $\sigma_{\text{max}}$, of sensitivity Jacobian matrix $\mathbf{J}$.

\begin{equation}
\sigma_{\text{min}} \leq S = \frac{\|\delta \mathbf{f}\|_2}{\|\delta \mathbf{x}\|_2} \leq \sigma_{\text{max}} \tag{3}
\end{equation}

3 Choice of an appropriate robustness index

In order to obtain a robust solution independently of the amount of variations in $\mathbf{DV}$ and $\mathbf{DP}$, a judicious robustness index is required. The robustness indices usually found in the recent literature are the condition number and the Euclidean norm of the sensitivity Jacobian matrix, $\mathbf{J}$, Al-Widyan and Angeles [15], Ting and Long [16] used the condition number of $\mathbf{J}$, Zhu [17] and Hu et al. [17] suggested the use of the Euclidean norm of $\mathbf{J}$. In this section, it is shown that the Euclidean norm of $\mathbf{J}$ is more appropriate for the robust design of mechanisms.

The condition number of a matrix is the ratio of its largest singular value to its smallest singular value. Let $\mathbf{R}_1$ be the condition number of $\mathbf{J}$.

\begin{equation}
\mathbf{R}_1 = \|\mathbf{J}\|_2 \|\mathbf{J}^{-1}\|_2 = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \tag{4}
\end{equation}

According to [12, 15], a design is robust when $\mathbf{R}_1$ is a minimum. Assuming that only variations in $\mathbf{DP}$ are considered, each variation $\delta \mathbf{p}_i$ has the same influence on the norm of variations in performance when $\mathbf{R}_1$ is unitary, i.e.: the sensitivity ellipsoid is a sphere. Although this property is interesting, the previous index is not sufficient because the influence of $\delta \mathbf{p}_i$ on performance is not necessarily a minimum when $\mathbf{R}_1$ is unitary. Indeed, the condition number, $\mathbf{R}_1$, can be small even if the values of $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$ are large.

A singular value of $\mathbf{J}$ corresponds to the error transmission factor in the direction of its corresponding eigenvector and in the space of variations. The ideal solution is the minimization of all the singular values of $\mathbf{J}$, but is not easy to obtain. According to eq.(3), a compromise solution is to minimize the upper bound of $S$, which is the largest singular value of $\mathbf{J}$. Thus, a second robustness index, $\mathbf{R}_2$, is defined by eq.(5).

\begin{equation}
\mathbf{R}_2 = \|\mathbf{J}\|_2 = \sigma_{\text{max}} \tag{5}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Damper}
\end{figure}
The damper shown in Fig. 4 is studied to compare \( R_{I_1} \) and \( R_{I_2} \). The design variables are mass \( M \) and damping coefficient \( C_d \) to be determined with the aim of keeping the magnitude of displacement \( X_0 \) at a nominal value of 0.1m, while the magnitude \( F_0 \) of the excitation force \( F(t) = F_0 \cos(\omega t) \) and its pulsation \( \omega \) undergo considerable variations beyond the control of the designer: \( F_0 = 10 \text{N}, \ \omega = 2\pi \text{ rad/s} \). The displacement is equal to \( X(t) = X_0 \cos(\omega t + \phi) \) where \( \phi \) is the phase. Moreover, the following relations exist:

\[
X_0 = \frac{F_0}{\omega \sqrt{C_d^2 + \omega^2 M^2}}, \quad \phi = \tan^{-1}\left(\frac{\omega M}{C_d}\right)
\]

\[\mathbf{x} = [M \ C_d]^T, \ \mathbf{p} = [F_0 \ \omega]^T, \ \mathbf{f} = [X_0 \ \phi]^T.\] Equation (6) gives the relation between variations in \( \mathbf{f} \) and variations in \( \mathbf{p} \).

\[
\delta \mathbf{f} = \mathbf{J} \delta \mathbf{p}
\]

where the sensitivity Jacobian matrix depends on \( \mathbf{x} \) and \( \mathbf{p} \), and is equal to:

\[
\mathbf{J} = \mathbf{J}_p = \begin{bmatrix} 1 & -1 - \alpha^2 \\ 0 & \alpha \sqrt{1 - \alpha^2} \end{bmatrix}, \ \alpha = \frac{X_0}{F_0} \omega^2 M
\]

and

\[
\delta \mathbf{f} = \begin{bmatrix} \delta X_0 / X_0 \\ \delta \phi \end{bmatrix}, \ \delta \mathbf{p} = \begin{bmatrix} \delta F_0 / F_0 \\ \delta \omega / \omega \end{bmatrix}
\]

Figures 5, 6 depict robustness indices \( R_{I_1} \) and \( R_{I_2} \) of the damper with respect to \( M \), respectively. In Fig. 5, \( R_{I_1} \) is a minimum when \( M/M_{\text{max}} = 0.54 \) with \( M_{\text{max}} = 2.533 \text{kg} \). Fig. 6 shows that \( R_{I_2} \) increases with \( M \). According to Fig. 7, which depicts some design sensitivity ellipses of the damper, plotted for different values of \( M \), the more \( M \) tends towards zero, the larger the size of the ellipse. It means that the design can tolerate globally more variations in \( F_0 \) and \( \omega \), i.e.: it is robust. Besides, the ellipse corresponding to the value of \( M/M_{\text{max}} \) that minimises \( R_{I_1} \) is the smallest one. Therefore, minimizing \( R_{I_1} \) is not equivalent to minimizing the influence of variations in \( DP \) on performance function.

Chen et al. [4] and Parkinson [19] proposed an optimization algorithm to increase the robustness of a design without using robustness indices. However, they need to know the magnitude of source variations to use their algorithm. Assuming that \( \Delta F_0 / F_0 = 0.1, \Delta \omega / \omega = 0.1 \) and \( M/M_{\text{max}} \geq 1/2 \), their algorithm converges on \( M/M_{\text{max}} = 1/2 \) and \( C_d = 13.78 \text{N.s.m}^{-1} \) in order to minimize variations in \( X_0 \) and \( \phi \). So, according to [4] and [19], the mass of the damper is minimized to make it robust, like with robustness index \( R_{I_2} \). In conclusion, the study of the damper confirms that \( R_{I_2} \) is more suitable than \( R_{I_1} \) to evaluate the robustness.
of a mechanism. \( RL_2 \) is used in the following sections of this paper.

**Remark:** It is important not to confuse symbols \( \delta \) and \( \Delta \). \( \delta v \) depicts the variation of variable \( v \) and \( \Delta v \) depicts its tolerance, i.e., \( -\Delta v \leq \delta v \leq \Delta v \).

### 4 An efficient tolerance synthesis method

The dimensional tolerances of a mechanism are usually fixed according to various parameters such as the manufacturing process, the performance tolerances, the manufacturing cost. Some optimization methods for tolerance synthesis exist in the literature. Zhu and Ting [13] defined the tolerance box as a contraction of the circumscribing box of the design sensitivity ellipsoid of the mechanism. However, this Tolerance Box, called Zhu-TB, includes some rejects, cf Fig.3.

Some works in the literature deal with the link between dimensional tolerances and product’s cost [14, 9]. Here, the cost of a mechanism is supposed to decrease when its dimensional tolerances increase. Thus, a new tolerance synthesis method is proposed, which aims at finding the largest tolerance box of a mechanism that does not include rejects. Let \( \tilde{\xi}(C) \) be the design sensitivity ellipsoid of a mechanism corresponding to a norm of variations in its performance equal to \( C \). Assuming that this norm has to be smaller than \( C \), the optimal tolerance box is supposed to be the largest box included in \( \tilde{\xi}(C) \). This tolerance box called Caro-TB and depicted in Fig.8 is smaller than Zhu-TB, but does not include any reject. The choice of the tolerance box depends on the wish of the designer. However, it is always important to know the solution without rejects because the cost of the loss due to rejects can be estimated from this solution.

First, nominal values \( X = [\bar{x}_1 \bar{x}_2 \cdots \bar{x}_l]^T \) of design variables are computed from robustness index \( RL_2 \), presented in section 3.

Then, their optimal dimensional tolerances \( \Delta x_{\text{opt}} \) are computed using the following optimization algorithm:

\[
\left\{ \begin{array}{l}
\max \prod_{l=1}^l |u_i| \\
\text{s.t. } U(u_1, u_2, \ldots, u_l) \in \tilde{\xi}(C) \\
|u_i| \geq \Delta x_{\text{min}}, \quad i = 1, \ldots, l 
\end{array} \right.
\]

\[
\Delta x_{\text{opt}} = |u_i|, \quad \bar{x}_i - \Delta x_{\text{opt}} \leq x_i \leq \bar{x}_i + \Delta x_{\text{opt}}, \quad i = 1, \ldots, l
\]

This algorithm consists in maximizing the hyper-volume of the tolerance box included in \( \tilde{\xi}(C) \). \( V \) is the eigenvector corresponding to the maximum singular value of the sensitivity Jacobian matrix of the mechanism and \( U \) is its \( l^\text{th} \) component. Besides, point \( U \) whose coordinates are \((u_1, u_2, \ldots, u_l)\) belongs to \( \tilde{\xi}(C) \) if and only if \( u^T J^T u = C^2 \) where \( u = [u_1, u_2, \ldots, u_l]^T \). Moreover, each dimensional tolerance \( \Delta x_{i} \) has to be higher than a minimum dimensional tolerance \( \Delta x_{\text{min}} \), depending on the manufacturing process and \( \bar{x}_i \). For instance, Fig.8 depicts all the possible positions of \( U \) when \( l = 2 \) and \( V_1, V_2 \) are negative and positive, respectively.

In the following, a 2R and a 3R manipulators are studied to illustrate the proposed robust design and tolerance synthesis methods. If the positioning error of the end effector has to be smaller than a scalar \( C \) at \( n \) different goal poses, it means that the tolerance box has to be included in \( n \) design sensitivity ellipsoids because each pose of the manipulator is associated with a design sensitivity ellipsoid. However, the tolerance synthesis algorithm works with only one design sensitivity ellipsoid. To cope with this problem, we consider the most restrictive ellipsoid, \( \tilde{\xi}_{\text{min}} \). As for any serial manipulator, a unitary variation in one design variable and no variation in the others lead to a unitary positioning error of its end effector, the design sensitivity ellipsoids intersect at \( 2^n \) points where \( l \) is the number of design variables. Therefore, \( \tilde{\xi}_{\text{min}} \) is the ellipsoid with the smallest small axis among the \( n \) design sensitivity ellipsoids.

### 5 Case studies

#### 5.1 Study of a 2R manipulator

The mechanism studied in this section is a serial 2R manipulator, depicted in Fig.9. It is composed of two revolute joints and two links \( AB \) and \( BE \) of lengths \( l_1 \) and \( l_2 \), respectively. First, the manipulator is designed, so that its end-effector \( E \) can hit all points of a target \( T \), and to be as little sensitive as possible to dimensional variations. Indeed, the lower the sensitivity of the manipulator to dimensional variations, the easier its calibration. Subsequently, the tolerance synthesis method introduced in section 4 is used to compute its optimal dimensional tolerances.
5.1.1 Dimensioning of the 2R manipulator

Let $S_T$ be defined as a set of $n$ points $P_1, P_2, \ldots, P_n$. First, $E$ can hit all points in $S_T$ if and only if $l_1$ and $l_2$ satisfy the following conditions:

$$\left\{ \begin{array}{l}
|l_1 - l_2| \leq r \\
l_1 + l_2 \geq R
\end{array} \right.$$  

with $r = \min_i d(A, P_i)$, $R = \min_i d(A, P_i), i = 1, \ldots, n$ where $d(A, P_i)$ is the distance between $P_i$ and $A$. These conditions bound the feasible design variables space as shown in Fig.8.

The formulation of a robust design problem was given in section 5.1. The robustness of the manipulator with respect to dimensional variations is quantified by robustness index $R_{L2}$, defined in section 5.2. $R_{L2}$ is the maximum singular value of $J_e$ and corresponds to the maximum norm of positioning error of $E$, $||\delta f||_{\text{max}}$, when the norm of dimensional variations is unitary, i.e.: $\delta l_1^2 + \delta l_2^2 = 1$.

Let $S_T$ be made up of four points, $P_1, P_2, P_3, P_4$, whose Cartesian coordinates are $(1,5)$, $(2,7)$, $(3,7)$, $(4,6)$, respectively. Fig.11 shows the isocontours of $R_{L2}$ in the feasible design variable space. We can notice that $R_{L2}$ isocontours form a family of ellipses and that $R_{L2}$ is a minimum when design variables belong to the circle $C_{rob}$. In fact, the algebraic expression of $R_{L2}$ can be derived as shown in eq. (11):

$$R_{L2} = \sqrt{n + \sum_{i=1}^{n} \cos \theta_{2i}} = \sqrt{n + \sum_{i=1}^{n} \frac{x_i^2 + y_i^2 - l_1^2 - l_2^2}{2l_1l_2}}$$  

where $x_i$ and $y_i$ are the Cartesian coordinates of point $P_i$. Thus, the set of solutions $(l_1, l_2)$, satisfying eq. (11) for a fixed $R_{L2}$, is either ellipse $e_1$ or ellipse $e_2$ whose equations are $L_1^2/a_1^2 + L_2^2/b_1^2 = c$ and $L_1^2/a_2^2 + L_2^2/b_2^2 = c$, respectively, where $a_1 = b_2 = 1/R_{L2}$, $a_2 = b_1 = 1/\sqrt{2a - R_{L2}}$. $L_1$ and $L_2$ are the expressions of $l_1$ and $l_2$ in the coordinate frame rotated of 45 deg with respect to the
reference frame of the design variable space. Thus, $\varepsilon_1$ and $\varepsilon_2$, depicted in Fig.11, are the isocontours of robustness index $R_{I2}$.

$$l_1^2 + l_2^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 + y_i^2 = \frac{1}{n} \sum_{i=1}^{n} d_i^2(A, P_i)$$  \hspace{1cm} (12)

According to eq. (11), $R_{I2}$ is a minimum when eq. (12) is satisfied, i.e.: when dimensioning $(l_1, l_2)$ belongs to the circle of radius the square root of the mean of square distances between points $A$ and $P_i$ and centered at the origin of the design space variable. Therefore, this circle corresponds to $C_{rob}$. Its radius is equal to 6.87. Thus, there exists an infinite number of dimensionalizations $(l_1, l_2)$ that minimize $R_{I2}$. Three of them are depicted in Fig.12.

![Figure 11. Design Variables ($l_1, l_2$) corresponding to the same $R_{I2}$](image1)

![Figure 12. Robust Manipulators](image2)

According to eq. (11), the maximum global positioning error of $E$ is a minimum when cosines of angles $\theta_{2i}$ tend towards zero. It means that the links of a robust $2R$ manipulator should be almost perpendicular. That is apparent in Fig.12. The obtained robust dimensions are independent of the amount of variations and tolerate globally the largest variations.

As there are several robust manipulators, the designer can choose another criterion to be optimized. For instance, he can take into account the cost or the complexity of the mechanism. Here, the optimal robust manipulator is supposed to be the one with the best dexterity. This criterion is frequently used in manipulator design. It evaluates the ease of a manipulator to execute motions or arbitrary motions in all directions. It is quantified by the condition number of its kinematic Jacobian matrix, $[20]$. The smaller this condition number, the higher the dexterity. Besides, the manipulator is isotropic when its condition number is equal to one, $[20]$. Let $J_k$ be the kinematic Jacobian matrix of the $2R$ manipulator:

$$J_k = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) - l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \hspace{1cm} (13)$$

For any posture of the manipulator defined by angle $\theta_2$, the condition number of $J_k$ is a minimum if and only if $l_2 = l_1 \sqrt{2}/2$. Let $\ell$ be the line of equation $l_2 = l_1 \sqrt{2}/2$. $D_{opt}$, the intersection of $C_{rob}$ with $\ell$, depicts the optimal robust manipulator, cf Fig.11. In conclusion, the $2R$ manipulator of link lengths $l_1 = l_{1opt} = 5.61$ and $l_2 = l_{2opt} = 3.97$ is the optimal robust one, i.e.: the one with the best dexterity among the least sensitive ones to dimensional variations that allow their end effector $E$ to hit all points in $S_T$. This manipulator is the second one depicted in Fig.12, i.e.: the one whose links are depicted with bold lines.

In conclusion, the robust design method gave the set of all the robust manipulators. The optimal dimensionalization was chosen among this set by means of another criterion, namely the dexterity. In the following section, the tolerance synthesis method presented in section 3 is used to compute the optimal dimensional tolerances of the selected manipulator.

### 5.1.2 Tolerance synthesis of the $2R$ manipulator

In addition to the fact that $E$ has to hit every point of $S_T$, its positioning error has to be smaller than 10 $\mu$m whatever its pose.

Fig.13 depicts the design sensitivity ellipses of the optimal robust manipulator defined in the previous section, when target $S_T$ is defined by $P_1, P_2, P_3, P_4$, plotted in Fig.13. The shape, size and orientation of these ellipses depend only on the second actuated joint variable, $\theta_2$, of the manipulator. $\theta_2$ belongs to interval $[\theta_{2min}, \theta_{2max}]$ in order to $E$ to hit all points of $S_T$. Moreover, points on these ellipses lead to the same positioning error of $E$, equal to 10 $\mu$m.

The optimal dimensional tolerances $\Delta l_{1opt}$ and $\Delta l_{2opt}$ of lengths $l_1$ and $l_2$ are computed from the tolerance synthesis...
method developed in section 3. The ellipse used in the tolerance synthesis algorithm is the one corresponding to $\theta_{2\text{max}}$, i.e.: the one with the smallest semi-axis. Finally, $\Delta l_{1\text{opt}} = \Delta l_{2\text{opt}} = 5.8\mu m$ and the corresponding tolerance box is depicted in Fig. 13.

Points $A_1(1, 0), A_2(0, 1), A_3(-1, 0), A_4(0, -1)$ belong to all design sensitivity ellipses of the manipulator because a unitary variation in $l_i$ and no variation in $l_j, (i \neq j)$, lead to a unitary positioning error of its end effector. As an ellipse is convex, square $A_1A_2A_3A_4$ is included in all the design sensitivity ellipses whatever the bounds of $\theta_2$. It follows that eq. (14) is a sufficient condition for the positioning error of $E$ to be smaller than $10 \mu m$ whatever its pose.

$$\Delta l_1 + \Delta l_2 \leq 10 \mu m \quad (14)$$

Without the tolerance synthesis method developed in section 3, the designer would have chosen dimensional tolerances $\Delta l_1$ and $\Delta l_2$ by means of eq. (14). Here, $\Delta l_{1\text{opt}}$ and $\Delta l_{2\text{opt}}$ do not respect eq. (14) because $\Delta l_{1\text{opt}} + \Delta l_{2\text{opt}} = 11.64\mu m$. However, they allow the positioning error of $E$ to be smaller than $10 \mu m$ on $S_T$. So, knowing the target of the manipulator, the tolerance synthesis method proposed in section 3 is more interesting than the sufficient condition, defined by eq. (14), to synthesize its dimensional tolerances.

The $2R$ manipulator has been studied in order to get graphical interpretations of the results and algebraic expressions of robustness index $R_{2L}$. However, the foregoing methods can be applied to more general mechanisms and $R_{2L}$ may be computed numerically, as for the study of the $3R$ manipulator in the following section.

5.2 Tolerance synthesis of a 3R manipulator

A three-dof serial positioning manipulator with three revolute joints is shown in Fig. 13. Modified D-H parameters are used to describe its geometry [22]. $\theta_1, \theta_2, \theta_3$ are the actuated joint variables of the 3R manipulator and $d_2, r_2, d_3, d_4$ denote its dimensions. Its inverse geometric model was studied in [22]. The positioning error, $\varepsilon_E$, of end effector $E$, has to be smaller than $10\mu m$ at any point $P_i, i = 1, \cdots, n$, of a path. Variations in $\theta_1, \theta_2$ and $\theta_3$ are negligible because the encoders are supposed to be very accurate. So, $\varepsilon_E$ depends on variations in the other D-H parameters. Here, only variations in $d_2, r_2, d_3$ are considered in order to get graphical representation.

The relation between the positioning error of $E$ at $P_i, \varepsilon_E$, and dimensional variations $\delta d_2, \delta r_2$, and $\delta d_3$, is given by eq. (15).

$$\Delta l_{1\text{opt}} = \Delta l_{2\text{opt}} = 5.8\mu m$$

where $\delta d_2 = \delta r_2 = \delta d_3 = 0$. Optimal and dimensional variations $\delta d_2, \delta r_2, \delta d_3$ are considered.

The relation between the positioning error of $E$ at $P_i, \varepsilon_E$, and dimensional variations $\delta d_2, \delta r_2, \delta d_3$, is given by eq. (15).

$$\delta \mathbf{f}_i = \mathbf{J}_i \delta \mathbf{x} \quad \text{with} \quad \mathbf{J}_i = \begin{bmatrix} \cos \theta_{1i} & -\sin \theta_{1i} & \cos \theta_{2i} \\ \sin \theta_{1i} & \cos \theta_{1i} & \sin \theta_{2i} \\ 0 & 0 & -\sin \theta_{2i} \end{bmatrix}$$

Figure 14. 3R Manipulator

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where $\Delta_i$ compute the optimal tolerance box included in the most restrictive ellipsoid. The tolerance synthesis method proposed in section 4 is used to compute the optimal tolerance box included in the most restrictive ellipsoid, i.e.: the following algorithm is used to compute $\Delta d_{2opt}$, $\Delta r_{2opt}$ and $\Delta d_{3opt}$:

$$\max_{u} |u_1u_2u_3|$$

s.t. $U(u_1,u_2,u_3) \in \xi_{mr}$

$u_1 \geq 0$

$u_3 \geq 0$

$|u_i| \geq \Delta x_{imin}, i = 1, \cdots, 3$

where $\Delta x_{1min} = 1 \mu m$, $\Delta x_{2min} = \frac{r_2}{d_2} \Delta x_{1min}$, $\Delta x_{3min} = \frac{d_3}{d_2} \Delta x_{1min}$.

The results of this optimization problem are: $u_1 = 4.08 \mu m$, $u_2 = -5.77 \mu m$ and $u_3 = 4.08 \mu m$. Thus, $\Delta d_{2opt} = 4.08 \mu m$, $\Delta r_{2opt} = 5.77 \mu m$ and $\Delta d_{3opt} = 4.08 \mu m$.

![Figure 15. The Most Restrictive Ellipsoid & the Optimal Tolerance Box](image)

Points $B_1(1,0,0)$, $B_2(-1,0,0)$, $B_3(0,1,0)$, $B_4(0,-1,0)$, $B_5(0,0,1)$ and $B_6(0,0,-1)$ belong to all design sensitivity ellipsoids of the manipulator, because a unit variation of a design variable and no variation of the others lead to a unitary variation of the position of $E$. As an ellipsoid is a convex volume, octahedron $B_1,B_2,B_3,B_4,B_5,B_6$ depicted in Fig. 17 is included in all the design sensitivity ellipsoids of the manipulator such that a point on the surface leads to a positioning error of $E$ equal to $10 \mu m$. It follows that eq. (16) is a sufficient condition for the positioning error of $E$ to be smaller than $10 \mu m$ whatever its pose.

$$\Delta d_2 + \Delta r_2 + \Delta d_3 \leq 10 \mu m \quad (16)$$

Without the tolerance synthesis method proposed in section 4, the designer would have chosen dimensional tolerances $\Delta d_2$, $\Delta r_2$, and $\Delta d_3$ by means of eq. (16). Here, $\Delta d_{2opt}$, $\Delta r_{2opt}$, and $\Delta d_{3opt}$ do not respect eq. (16) because $\Delta d_{2opt} + \Delta r_{2opt} + \Delta d_{3opt} = 13.9 \mu m$. It means that the optimal tolerance box is not included in octahedron $B_1,B_2,B_3,B_4,B_5,B_6$, as depicted by Fig. 17. However, they allow the positioning error of $E$ to be smaller than $10 \mu m$ at each pose $P_i, i = 1, \cdots, 5$. So, knowing the target of the manipulator, the tolerance synthesis method proposed in section 4 is more interesting than the sufficient condition, defined by eq. (16), to synthesize its dimensional tolerances.

In conclusion, the optimal dimensional tolerances of the 3R manipulator are $\Delta d_2 = 4.08 \mu m$, $\Delta r_2 = 5.77 \mu m$ and $\Delta d_3 = 4.08 \mu m$ so that the positioning error of $E$ is less than $10 \mu m$ at any point $P_i, i = 1, \cdots, 5$.

![Figure 16. Validation of the Optimal Tolerance Box](image)

Fig. 16 depicts the values of $\varepsilon_E$ when $\delta d_2$, $\delta r_2$, $\delta d_3$, are between $-\Delta d_{2opt}$ and $\Delta d_{2opt}$, $-\Delta r_{2opt}$ and $\Delta r_{2opt}$, $-\Delta d_{3opt}$ and $\Delta d_{3opt}$, respectively, and for the five poses of the manipulator. $\varepsilon_E$ is always smaller than $10 \mu m$. It means that the positioning error of $E$ is smaller than $10 \mu m$ for any posture of the manipulator when the tolerances of $d_2$, $r_2$, and $d_3$ are $\Delta d_{2opt}$, $\Delta r_{2opt}$, and $\Delta d_{3opt}$, respectively.
6 Conclusions

This paper has provided a new and efficient tolerance synthesis method for mechanisms, based on a robust design approach. The study of the robustness of a mechanism follows two consecutive steps, which are independent and complementary. The first step aims at computing its robust dimensions by means of an appropriate robustness index. The Euclidean norm of the sensitivity Jacobian matrix is such an index. The study of a damper confirmed that the Euclidean norm of its sensitivity Jacobian matrix is more suitable than its condition number, to quantify the robustness of a mechanism. This method yields the set of all the robust manipulators and allows the designer to integrate other criteria. Then, the developed tolerance synthesis method is used to compute the optimal tolerance box of the selected robust manipulator. The theory is illustrated by two serial manipulators. The application of this theory to the robust design and tolerance synthesis of parallel manipulators is one of the next steps in our research work.

REFERENCES