Equilibrium and Guaranteeing Solutions in Evolutionary Nonzero Sum Games

Kryazhimskiy, A.V. and Tarasyev, A.M.

IIASA Interim Report
January 1998
http://pure.iiasa.ac.at/5643/

Interim Reports on work of the International Institute for Applied Systems Analysis receive only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute, its National Member Organizations, or other organizations supporting the work. All rights reserved. Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage. All copies must bear this notice and the full citation on the first page. For other purposes, to republish, to post on servers or to redistribute to lists, permission must be sought by contacting repository@iiasa.ac.at
Equilibrium and Guaranteeing Solutions in Evolutionary Nonzero Sum Games

A. V. Kryazhimskii (kryazhim@iiasa.ac.at, kryazhim@mi.ras.ru),
A. M. Tarasyev (tarasiev@iiasa.ac.at)

Approved by
Gordon MacDonald (macdon@iiasa.ac.at)
Director, IIASA
Abstract

Advanced methods of theory of optimal guaranteeing control and techniques of generalized (viscosity, minimax) solutions of Hamilton-Jacobi equations are applied to nonzero sum game interaction between two large groups (coalitions) of agents (participants) arising in economic and biological evolutionary models. Random contacts of agents from different groups happen according to a control dynamical process which can be interpreted as Kolmogorov’s differential equations in which coefficients describing flows are not fixed a priori and can be chosen on the feedback principle. Payoffs of coalitions are determined by the functionals of different types on infinite horizon. The notion of a dynamical Nash equilibrium is introduced in the class of control feedbacks. A solution based on feedbacks maximizing with the guarantee the own payoffs (guaranteeing feedbacks) is proposed. Guaranteeing feedbacks are constructed in the framework of the theory of generalized solutions of Hamilton-Jacobi equations. The analytical formulas are obtained for corresponding value functions. The equilibrium trajectory is generated by guaranteeing feedbacks and its properties are investigated. The considered approach provides new qualitative results for the equilibrium trajectory in evolutionary models. The first striking result consists in the fact that the designed equilibrium trajectory provides better (in some bimatrix games strictly better) index values for both coalitions than trajectories which converge to static Nash equilibria (as, for example, trajectories of classical models with the replicator dynamics). The second principle result implies both evolutionary and revolutionary properties of the equilibrium trajectory: evolution takes place in the characteristic domains of Hamilton-Jacobi equations and revolution at switching curves of guaranteeing feedbacks. The third specific feature of the proposed solution is the “positive” nature of guaranteeing feedbacks which maximize the own payoff unlike the “negative” nature of punishing feedbacks which minimize the opponent payoff and lead to static Nash equilibrium. The fourth concept takes into account the foreseeing principle in constructing feedbacks due to the multiterminal character of payoffs in which future states are also evaluated. The fifth idea deals with the venturous factor of the equilibrium trajectory and prescribes the risk barrier surrounding it. These results indicate promising applications of theory of guaranteeing control for constructing solutions in evolutionary models.
# Contents

**Introduction** 1

1 Nonzero Sum Evolutionary Game. Dynamical Nash Equilibrium 2
   1.1 Dynamics, Payoff Functionals ................................... 2
   1.2 Definition of Dynamical Nash Equilibria ...................... 4
   1.3 The Auxiliary Zero-Sum Games .................................... 5
   1.4 Construction of Nash Equilibria .................................. 6

2 The Analytical Solution of the Differential Game with the Terminal Functional 7
   2.1 Value Functions and Generalized Solutions of Hamilton-Jacobi Equations 7
   2.2 The Description of the Analytical Solution for the Terminal Boundary Value Problem ........................................ 9
   2.3 Verification of Differential Inequalities in the Terminal Boundary Value Problem ........................................ 11

3 The Lower Envelope of Terminal Value Functions and the Value Function of the Game with the Multiterminal Payoff Functional 15
   3.1 The Differential Game with the Multiterminal Functional ....... 15
   3.2 Description of a Solution to the Game with the Multiterminal Functional ........................................ 18
   3.3 Testing $u$— and $v$— Stability for the Multiterminal Game ........................................ 22

4 Flexible “Positive” Feedback Controls Generated by Value Functions of Multiterminal Games 25
   4.1 Description of Optimal Feedback Controls .......................... 25
   4.2 The Value Provided by Optimal Feedbacks for Multiterminal Payoffs ........................................ 28

5 A Nash Equilibrium with Flexible “Positive” Feedbacks of Multiterminal Games 30
   5.1 The Structure of a Nash Equilibrium ................................ 30
   5.2 Trajectories Generated by Flexible “Positive” Feedbacks ........ 32

6 Quasioptimal Feedbacks in Games with Integral Payoff Functionals 33
   6.1 Nash Equilibria in the Nonzero-Sum Differential Game with the Averaged Integral Functionals .......................... 33
   6.2 Two-step Optimal Control Problems ................................ 34
   6.3 The Solution of the Two-step Optimal Control Problem ........ 36
   6.4 Values Guaranteed by Optimal Feedbacks in the Problem with the Integral Payoffs ........................................ 39

7 Quasioptimal Feedback Controls in Games with the Coordinated “Long-Term” and “Short-Term” Interests of Populations and Individuals 41
   7.1 Three-Step Optimal Control Problems .............................. 41
   7.2 Solution of the Three-step Optimal Control Problem ........... 45
7.3 The Guaranteed Value in the Three-Step Optimal Control Problem . . . . . 48

Conclusion 50

References 51
Equilibrium and Guaranteeing Solutions in Evolutionary Nonzero Sum Games

A. V. Kryazhimskii * (kryazhim@iiasa.ac.at, kryazhim@mi.ras.ru),
A. M. Tarasyev ** (tarasiev@iiasa.ac.at)

Introduction

We consider a model of evolutionary nonzero sum game between two coalitions of participants in the framework of differential games theory (see [Krasovskii, Subbotin, 1988], [Krasovskii, 1985]) using, especially, some ideas of the approach for nonantagonistic problems (see [Kleimenov, 1993]), statements and methods of analysis of evolutionary games proposed in [Kryazhimskii, 1994]). We concentrate our attention on constructing dynamical Nash equilibria and guaranteeing feedbacks which maximize corresponding payoffs. We obtain resolving trajectories which give better results than solutions of classical models.

The dynamics of game interaction is related to differential (see [Isaacs, 1965]) and evolutionary game-theoretical models (see [Friedman, 1991], [Young, 1993], [Nelson, Winter, 1982], [Intriligator, 1971] [Hofbauer, Sigmund, 1988]), [Basar, Olsder, 1982] [Vorobyev, 1985], [Kaniovskii, Young, 1994]. Random contacts of participants are represented by a control dynamical process in which corresponding probabilities form the phase vector and informational signals play the role of control parameters. This dynamics can be interpreted as generalization of the well-known Kolmogorov’s equations which arise in some stochastic models of mathematical economics and queueing theory. The generalization consists in introducing control parameters instead of fixed coefficients which describe incoming and outgoing flows within coalitions. The process evolves on the infinite interval of time. Payoffs of participants are specified by payoff matrixes. Payoffs of coalitions are defined as average payoffs of participants (payoff mean values). We consider different types of these mean values: terminal - for a fixed time; multiterminal - for a time interval; global terminal - for a limit on the infinite time interval; global integral - for a limit of integral payoffs on the infinite time interval. Note that non-zero sum games with discounting integral payoffs were analyzed in [Tarasyev, 1994]. The global functionals are connected with the foreseeing concept which takes into account not only local terminal interests of coalitions but is oriented also on the global future change.

We introduce the notion of a dynamical Nash equilibrium in the class of control feedbacks. Note that feedbacks generated by classical “punishing” solutions of static bimatrix games give the natural and elementary example of a Nash equilibrium in the dynamical sense. The nature of these feedbacks is antagonistic: they minimize the opponent payoff and don’t maximize the own one.

* The work is partially supported by the Russian Fund for Fundamental Research (97-01-00161).
** The work is partially supported by the Russian Fund for Fundamental Research (96-01-00219, 96-15-96245, 97-01-00161).
We propose another approach which is based on the “guaranteeing” concept and provides better results than classical solutions. This new solution is generated by constructions of the theory of positional differential games and involves guaranteeing feedbacks of auxiliary zero sum games (see [Krasovskii, Subbotin, 1988], [Kleimenov, 1993]). We consider these zero sum games in the framework of the theory of viscosity (minimax) solutions of Hamilton-Jacobi equations (see [Subbotin, 1980, 1991, 1995], [Crandall, Lions, 1983, 1984]). We construct analytically value functions and optimal guaranteeing feedbacks and verify the corresponding necessary and sufficient conditions for them formulated in terms of conjugate derivatives [Subbotin, Tarasyev, 1985]. The synthesis of guaranteeing feedbacks is determined by switching curves (generated by value functions) for control signals.

Stress once again that guaranteeing feedbacks maximize with the guarantee payoffs of coalitions in contrast to classical static strategies which punish each other by minimizing opponent payoffs. We generate equilibrium trajectories using these switching curves. Equilibrium trajectories have evolutionary properties as well as revolutionary one. An evolution takes place when equilibrium trajectories develop along the characteristics of Hamilton-Jacobi equations and a revolution happens on switching curves where coalitions change their behaviors. We consider the venturous factor of equilibrium trajectories and prescribe the risk barrier surrounding it.

The behavior of new equilibrium solutions generated by bang-bang control synthesis differs qualitatively from the evolution of trajectories of classical models with the replicator dynamics. Remind that these trajectories converge a fortiori to a static Nash equilibrium or circulate in its neighborhood (see [Young, 1993], [Nelson and Winter, 1982], [Hofbauer, Sigmund, 1988]). The new equilibrium solutions are disposed in the intersection of domains in which the payoffs values are better than the corresponding values calculated at a static Nash equilibrium. Examples of “almost antagonistic” games show that these trajectories converge to the points of intersection of switching curves - to the “new” points of equilibrium with better index values.

1 Nonzero Sum Evolutionary Game. Dynamical Nash Equilibrium

1.1 Dynamics, Payoff Functionals

Let us consider the system of differential equations which describes behavioral dynamics for two coalitions (populations)

\[
\begin{align*}
\frac{dx}{dt} &= -x + u \\
\frac{dy}{dt} &= -y + v
\end{align*}
\]

(1.1)

Assume that parameter \( x, 0 \leq x \leq 1 \) is the probability of the fact that a randomly taken individual of the first coalition holds the first strategy (respectively, \( 1-x \) is the probability of playing the second one). Parameter \( y, 0 \leq y \leq 1 \) is the probability of choosing the first strategy by an individual of the second coalition (respectively, \( 1-y \) is the probability of playing the second strategy). Control parameters \( u \) and \( v \) satisfy the restrictions \( 0 \leq u \leq 1, 0 \leq v \leq 1 \) and can be interpreted as signals for individuals to change their strategies. For example, the value \( u = 0 (v = 0) \) corresponds to the signal: “change the first strategy for the second one”, the value \( u = 1 (v = 1) \) corresponds to the signal: “change the second strategy for the first one” and the value \( u = x (v = y) \) corresponds to the signal: “keep previous strategies".
Let us note that the ground for dynamics (1.1) and its properties were developed in [Kryazhimskii, 1994], [Tarasyev, 1994]. In this dynamics we generalized Kolmogorov’s differential equations assuming that coefficients of incoming and outgoing flows inside coalitions are not given a priori and can be designed in the control process on the feedback principle.

For interpretations of dynamics (1.1) let us consider the game interaction of two large group of firms (or their capital investments) on two markets. Let \( x \) be a part of facilities which firms of the first coalition (it may be a financial or industrial group) invest into the first market (it may be a market of currencies, goods or new technologies [Vorobyev, 1985], [Kaniovskii, Young, 1994]) and \((1 - x)\) - into the second market respectively. Let \( y \) be a part of facilities which the second coalition invests into the first market and \((1 - y)\) - into the second one. Assume that activity of coalitions on the markets can be regulated by managing councils. Managing councils using control parameters \( u \) and \( v \) can influence on the distribution of facilities \( x \) and \( y \). The dynamics of this influence is described by the system (1.1) and provides some inertness (or independence) of firms with respect to control signals \( u, v \) since velocities \( \dot{x}, \dot{y} \) of changing of distributions \( x, y \) are not proportional directly to signals but depend on distributions. For example, the first equation in (1.1) means that according to the signal \( u = 0 \) distribution \( x \) decreases with the diminishing velocity \( \dot{x} = -x \).

We assume that the payoff of a participant from the first coalition is described by the payoff matrix \( A = \{a_{ij}\} \), and the payoff of a participant from the second one - by the payoff matrix \( B = \{b_{ij}\} \).

Specifically, for the game of firms on two markets we can consider the following situation. Assume that the first market is more profitable for investments than the second one and so the following relations hold for payoff matrixes \( A \) and \( B \)

\[
a_{11} > a_{2j}, \quad a_{12} > a_{21}
\]
\[
b_{12} > b_{2j}, \quad b_{11} > b_{22}, \quad j = 1, 2
\]

Let us suppose also that firms of the first coalition are stronger than firms of the second one. They try to conquer both markets and hence the payoff matrix \( A \) has the dominating main diagonal

\[
a_{ii} > a_{ij}, \quad i \neq j, \quad i, j = 1, 2
\]

Firms of the second coalition try to avoid interactions on the same market with firms of the first coalition and, therefore, the payoff matrix \( B \) has the dominating secondary diagonal

\[
b_{ij} > b_{ii}, \quad i \neq j, \quad i, j = 1, 2
\]

The terminal payoff functionals of coalitions are defined as mathematical expectations corresponding to payoff matrixes \( A, B \) and can be interpreted as “local” interests of coalitions (populations)

\[
g_A(x(T), y(T)) = a_{11}x(T)y(T) + a_{12}x(T)(1 - y(T)) + a_{21}(1 - x(T))y(T) + a_{22}(1 - x(T))(1 - y(T)) = C_Ax(T)y(T) - a_1x(T) - a_2y(T) + a_{22}
\]

\[
g_B(x(T), y(T)) = b_{11}x(T)y(T) + b_{12}x(T)(1 - y(T)) + b_{21}(1 - x(T))y(T) + b_{22}(1 - x(T))(1 - y(T)) = C_Bx(T)y(T) - b_1x(T) - b_2y(T) + b_{22}
\]
at a given instant $T$. Here parameters $C_A$, $\alpha_1$, $\alpha_2$ and $C_B$, $\beta_1$, $\beta_2$ are determined according to the classical theory of bimatrix games (see, for example, [Vorobyev, 1985])

$$
\begin{align*}
C_A &= a_{11} - a_{12} - a_{21} + a_{22} \\
\alpha_1 &= a_{22} - a_{12} \\
\alpha_2 &= a_{22} - a_{21} \\
C_B &= b_{11} - b_{12} - b_{21} + b_{22} \\
\beta_1 &= b_{22} - b_{12} \\
\beta_2 &= b_{22} - b_{21}
\end{align*}
$$

(1.4)

We define “global” interests $J^\infty_A$, $J^\infty_B$ of coalitions (populations) as multifunctions generated by lower and upper limits of mean values

$$
\begin{align*}
J^\infty_A &= [J^-_A, J^+_A] \\
J^+_A &= J_A(x(\cdot), y(\cdot)) = \liminf_{t \to \infty} g_A(x(t), y(t)) \\
J^-_A &= J^-_A(x(\cdot), y(\cdot)) = \limsup_{t \to \infty} g_A(x(t), y(t)) \\
J^\infty_B &= [J^-_B, J^+_B] \\
J^+_B &= J_B(x(\cdot), y(\cdot)) = \liminf_{t \to \infty} g_B(x(t), y(t)) \\
J^-_B &= J^-_B(x(\cdot), y(\cdot)) = \limsup_{t \to \infty} g_B(x(t), y(t))
\end{align*}
$$

(1.7)

calculated on the trajectories $(x(\cdot), y(\cdot))$ of the system (1.1).

Consider an evolutionary nonzero sum game with the dynamics (1.1) and payoffs (1.6),(1.7). There is an approach (see, for example, [Krasovskii, Subbotin, 1988], [Kleimenov, 1993]) in the differential games theory for constructing equilibrium solutions in the class of feedback strategies $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ for nonzero sum problems. This approach is based on solving auxiliary zero sum games. In connection with our statements of the problem (see functionals (1.6),(1.7)) we consider zero sum games with the functionals $J^+_A$, $J^-_A$, $J^+_B$, $J^-_B$. It is known that zero sum problems can be solved and resolving feedbacks can be constructed on the principle of dynamical programming. This principle requires finding value functions. Below we obtain corresponding solutions - value functions and optimal feedbacks for all considered problems in the framework of the theory of generalized (minimax, viscosity) solutions of Hamilton-Jacobi equations.

### 1.2 Definition of Dynamical Nash Equilibria

Following [Kleimenov, 1993], [Kryazhimskii, 1994] we introduce the notion of a dynamical Nash equilibrium in the class of closed-loop strategies (feedbacks) $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ for the nonzero sum game with the dynamics (1.1) and multivalued payoff functionals (1.6),(1.7).

**Definition 1.1** Let $\varepsilon > 0$ and $(x_0, y_0) \in [0, 1] \times [0, 1]$. A pair of feedbacks $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ is called a Nash equilibrium for an initial position $(x_0, y_0)$ if for any other feedbacks $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ the following condition holds: for all trajectories

$$
\begin{align*}
(x^0(\cdot), y^0(\cdot)) &\in X(x_0, y_0, U^0, V^0), \\
(x_1(\cdot), y_1(\cdot)) &\in X(x_0, y_0, U, V^0) \\
(x_2(\cdot), y_2(\cdot)) &\in X(x_0, y_0, U^0, V)
\end{align*}
$$
the inequalities
\[ J_A(x^0(\cdot), y^0(\cdot)) \geq J_A^*(x_1(\cdot), y_1(\cdot)) - \varepsilon \quad (1.8) \]
\[ J_B(x^0(\cdot), y^0(\cdot)) \geq J_B^*(x_2(\cdot), y_2(\cdot)) - \varepsilon \quad (1.9) \]
are valid.

1.3 The Auxiliary Zero-Sum Games

To construct desired equilibrium feedbacks \( U^0, V^0 \) we use the approach of [Kleimenov, 1993]. According to this approach, we compose an equilibrium with the help of optimal feedbacks constructed for zero-sum differential games \( \Gamma_A = \Gamma^+_A \cup \Gamma^-_A \) and \( \Gamma_B = \Gamma^+_B \cup \Gamma^-_B \) with the payoffs \( J_A^\infty \) (1.6) and \( J_B^\infty \) (1.7). In the game \( \Gamma_A \) the first coalition maximizes with the guarantee the functional \( J_A(x(\cdot), y(\cdot)) \) using a feedback \( U = u(t, x, y, \varepsilon) \), and the second coalition attempts, on the contrary, to minimize the functional \( J_A^*(x(\cdot), y(\cdot)) \) using a feedback \( V = v(t, x, y, \varepsilon) \). Conversely, in the game \( \Gamma_B \) the second coalition maximizes with the guarantee the functional \( J_B^*(x(\cdot), y(\cdot)) \), and the first coalition minimizes the functional \( J_B^*(x(\cdot), y(\cdot)) \).

Let us introduce the following notations. By \( u_A^0 = u_A^0(t, x, y, \varepsilon) \) and \( v_B^0 = v_B^0(t, x, y, \varepsilon) \) denote feedbacks solving, respectively, the problem of guaranteeing maximization of the long term coalitions benefits and, therefore, can be called “positive” ones. By \( u_B^0 = u_B^0(t, x, y, \varepsilon) \) and \( v_A^0 = v_A^0(t, x, y, \varepsilon) \) we denote feedbacks mostly unfavorable for the opposite coalitions; namely, those minimizing the payoff functionals \( J_B^+, J_A^+ \) of the opposite coalitions respectively. These feedbacks can be called “punishment” feedbacks.

Let us note that inflexible solutions of the indicated problems can be obtained in the framework of the classical theory of bimatrix games. Really, assume for the definiteness that

\[
C_A > 0, \quad C_B < 0 \\
0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < x_B = \frac{\beta_2}{C_B} < 1 \\
0 < y_A = \frac{\alpha_1}{C_A} < 1, \quad 0 < y_B = \frac{\beta_1}{C_B} < 1
\]

One can prove the following statement.

**Proposition 1.1** Differential games \( \Gamma_A, \Gamma_A^+ \) have equal values

\[ v_A^- = v_A^+ = v_A = \frac{a_{22}C_A - \alpha_1\alpha_2}{C_A} \quad (1.10) \]

and differential games \( \Gamma_B, \Gamma_B^+ \) have equal values

\[ v_B^- = v_B^+ = v_B = \frac{b_{22}C_B - \beta_1\beta_2}{C_B} \quad (1.11) \]

for any initial position \((x_0, y_0) \in [0, 1] \times [0, 1]\). These values may be guaranteed, for example, by “positive” feedbacks \( u_A^0, v_B^0 \) corresponding to classical static solutions \( x_A, y_B \)

\[
u_A^0 = u_A^d = u_A^d(x, y) = \begin{cases} 
0 & \text{if } x_A < x \leq 1 \\
1 & \text{if } 0 \leq x < x_A \\
0,1 & \text{if } x = x_A
\end{cases} \quad (1.12)\]
\[ v_B^0 = v_B^c = v_B^c(x, y) = \begin{cases} 
0 & \text{if } y_B < y \leq 1 \\
1 & \text{if } 0 \leq y < y_B \\
0, 1 & \text{if } y = y_B 
\end{cases} \tag{1.13} \]

"Punishment" feedbacks are determined by formulas

\[ u_B^0 = u_B^c = u_B^c(x, y) = \begin{cases} 
0 & \text{if } x_B < x \leq 1 \\
1 & \text{if } 0 \leq x < x_B \\
0, 1 & \text{if } x = x_B 
\end{cases} \tag{1.14} \]

\[ v_A^0 = v_A^c = v_A^c(x, y) = \begin{cases} 
0 & \text{if } y_A < y \leq 1 \\
1 & \text{if } 0 \leq y < y_A \\
0, 1 & \text{if } y = y_A 
\end{cases} \tag{1.15} \]

and correspond to classical static solutions \( x_B, y_A \) which generate a static Nash equilibrium \( NE = (x_B, y_A) \).

**Remark 1.1** Note that "positive" feedbacks (1.12), (1.13) are rather inflexible because they are obtained in the static model of bimatrix games and don’t take into account information about dynamics (1.1). Our main goal is to construct flexible "positive" feedbacks which essentially use information about dynamics.

**Remark 1.2** Values of payoff functions \( g_A(x, y), g_B(x, y) \) coincide at points \( (x_A, y_B), (x_B, y_A) \)

\[ g_A(x_A, y_B) = g_A(x_B, y_A) = v_A, \quad g_B(x_A, y_B) = g_B(x_B, y_A) = v_B \tag{1.16} \]

But the point \( NE = (x_B, y_A) \) is a “mutually punishing” Nash equilibrium and the point \( x_A, y_B \) does not have equilibrium properties in the corresponding static game.

### 1.4 Construction of Nash Equilibria

Let us construct now a Nash equilibrium pair of feedbacks by pasting together “positive” feedbacks \( u_A^0, v_B^0 \) and “punishment” feedbacks \( u_B^0, v_A^0 \).

Let us choose an initial position \( (x_0, y_0) \in [0, 1] \times [0, 1] \) and an accuracy parameter \( \varepsilon > 0 \). Choose a trajectory \( (x^0(.), y^0(.)) \in X(x_0, y_0, u_A^0(.), v_B^0(.)) \) generated by “positive” feedbacks \( u_A^0 = u_A^0(t, x, y, \varepsilon) \) and \( v_B^0 = v_B^0(t, x, y, \varepsilon) \). Let \( T_\varepsilon > 0 \) be such that

\[ g_A(x^0(t), y^0(t)) > \int_A^0(x^0(.), y^0(.)) - \varepsilon \]

\[ g_B(x^0(t), y^0(t)) > \int_B^0(x^0(.), y^0(.)) - \varepsilon \]

\[ t \in [T_\varepsilon, +\infty) \]

Denote by \( u_A^0(t) : [0, T_\varepsilon) \to [0, 1], v_B^0(t) : [0, T_\varepsilon) \to [0, 1] \) step-by-step realizations of strategies \( u_A^0, v_B^0 \) such that the corresponding step-by-step motion \((x_\varepsilon(.), y_\varepsilon(.))\) satisfies the condition

\[ \max_{t \in [0, T_\varepsilon]} \| (x^0(t), y^0(t)) - (x_\varepsilon(t), y_\varepsilon(t)) \| < \varepsilon \]

Using the approach of [Kleimenov, 1993] one can prove the following statement.

**Proposition 1.2** The pair of feedbacks \( U^0 = u^0(t, x, y, \varepsilon), V^0 = v^0(t, x, y, \varepsilon) \) pasting together “positive” feedbacks \( u_A^0, v_B^0 \) and “punishment” feedbacks \( u_B^0, v_A^0 \) in accordance with formulas

\[ U^0 = u^0(t, x, y, \varepsilon) = \begin{cases} 
\frac{u_A^0(t)}{v_B^0(x, y)} & \text{if } \| (x, y) - (x_\varepsilon(t), y_\varepsilon(t)) \| < \varepsilon \\
\frac{u_A^0}{v_B^0(x, y)} & \text{otherwise} 
\end{cases} \tag{1.17} \]
\[ V^0 = v^0(t, x, y, \varepsilon) = \begin{cases} v_B^0(t) & \text{if } \|(x, y) - (x(t), y(t))\| < \varepsilon \\ v_A^0(x, y) & \text{otherwise} \end{cases} \] 

is a dynamical Nash \( \varepsilon \)-equilibrium.

**Remark 1.3** Let us note that the number \( \varepsilon \) can be interpreted as a parameter of “reliance” of coalitions to each other or a level of “risk” which the coalitions admit in the game. This parameter determines the risk barrier surrounding the equilibrium trajectory \((x_i(\cdot), y_i(\cdot))\). Coalitions either follow the equilibrium trajectory not leaving the prescribed risk barrier and then they obtain better index values or they violate it and then “punishment” strategies give worse results.

**Remark 1.4** Consider trajectories which can be generated by a dynamical Nash equilibrium (1.17), (1.18) with inflexible classical feedbacks (1.12)-(1.15). If trajectory \((x^{\text{pos}}(\cdot), y^{\text{pos}}(\cdot))\) evolves according to “positive” strategies \(u_A^0, v_B^0\) (1.12), (1.13) then it converges to the new equilibrium point \((x_A, y_B)\). If coalitions punish each other by strategies \(u_B^1, v_A^1\) (1.14),(1.15) then trajectory \((x^{\text{pun}}(\cdot), y^{\text{pun}}(\cdot))\) converges to a static Nash equilibrium \((x_B, y_A)\). But values of functionals (1.6), (1.7) are equal in both cases (see Remark 1.3).

\[ J_A^\infty(x^{\text{pos}}(\cdot), y^{\text{pos}}(\cdot)) = J_A^\infty(x^{\text{pun}}(\cdot), y^{\text{pun}}(\cdot)) = v_A \]

\[ J_B^\infty(x^{\text{pos}}(\cdot), y^{\text{pos}}(\cdot)) = J_B^\infty(x^{\text{pun}}(\cdot), y^{\text{pun}}(\cdot)) = v_B \]

Below we construct flexible “positive” feedbacks which generate trajectories \((x^{fl}(\cdot), y^{fl}(\cdot))\) converging to the “better” positions than inflexible dynamical equilibria \((x_B, y_A), (x_A, y_B)\) by both criteria \(J_A^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_A, J_B^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_B\). To this end in Section 2 we consider auxiliary zero sum games with terminal payoffs and obtain for them complete solutions. In Section 3 we construct lower envelopes of value functions of terminal games and deduce solutions for auxiliary multiterminal games. In Section 4 we derive flexible “positive” feedbacks from the structure of value functions of multiterminal games.

## 2 The Analytical Solution of the Differential Game with the Terminal Functional

### 2.1 Value Functions and Generalized Solutions of Hamilton-Jacobi Equations

In this section we consider auxiliary zero sum terminal differential games with dynamics (1.1) and payoff functionals (1.2) and (1.3) respectively. Further solutions of terminal differential games will be used for constructing foreseeing feedbacks by calculating low envelopes for multiterminal functionals. The value functions \(w_i(T, t, x, y), i = 1, 2\) of terminal games are determined as values of corresponding maximin (minimax) operations

\[ w_1(T, t_0, x_0, y_0) = \max_{u(t, x, y)} \min_{(x_1(\cdot), y_1(\cdot))} g_A(x_1(T), y_1(T)) = \min_{v(t, x, y)} \max_{(x_2(\cdot), y_2(\cdot))} g_A(x_2(T), y_2(T)) \]  

\[ w_2(T, t_0, x_0, y_0) = \max_{v(t, x, y)} \min_{(x_2(\cdot), y_2(\cdot))} g_B(x_2(T), y_2(T)) = \min_{u(t, x, y)} \max_{(x_1(\cdot), y_1(\cdot))} g_B(x_1(T), y_1(T)) \]  

\(\cdot\)
for every initial position \((t_0, x_0, y_0)\). Here trajectories \((x_1(\cdot), y_1(\cdot))\) are generated by feedback controls \(u(t, x, y, \varepsilon)\) and arbitrary behaviors \(v(t)\), trajectories \((x_2(\cdot), y_2(\cdot))\) are generated by feedback controls \(v(t, x, y, \varepsilon)\) and arbitrary behaviors \(u(t)\) from the initial position \((t_0, x_0, y_0)\).

The value functions \(w_i(T, t, x, y), i = 1, 2\) satisfy the principle of dynamical programming which implies the existence of nondecreasing and nonincreasing directions accessible for the dynamical system at every current position (the so-called properties of \(u\) and \(v\) stability of the value function). At points where the value functions are differentiable these properties turn into the first order partial differential equations of the Hamilton-Jacobi type

\[
\frac{\partial w_1}{\partial t} - \frac{\partial w_1}{\partial x} x - \frac{\partial w_1}{\partial y} y + \max_{0 \leq u \leq 1} \frac{\partial w_1}{\partial x} u + \min_{0 \leq v \leq 1} \frac{\partial w_1}{\partial y} v = 0 \tag{2.3}
\]

\[
\frac{\partial w_2}{\partial t} - \frac{\partial w_2}{\partial x} x - \frac{\partial w_2}{\partial y} y + \min_{0 \leq u \leq 1} \frac{\partial w_2}{\partial x} u + \max_{0 \leq v \leq 1} \frac{\partial w_2}{\partial y} v = 0 \tag{2.4}
\]

The value functions \(w_i(T, t, x, y), i = 1, 2\) satisfy also the terminal boundary condition when \(t = T\)

\[
w_1(T, T, x, y) = C_A xy - \alpha_1 x - \alpha_2 y + a_{22} = g_A(x, y) \tag{2.5}
\]

\[
w_2(T, T, x, y) = C_B xy - \beta_1 x - \beta_2 y + b_{22} = g_B(x, y) \tag{2.6}
\]

We consider now the terminal boundary-value problems (2.3),(2.5) and (2.4),(2.6) for value functions \(w_1(T, t, x, y), w_2(T, t, x, y)\). We turn our attention to the first problem (2.3),(2.5). It is known (see [Crandall, Lions, 1983, 1985], [Subbotin, 1980, 1991]) that the value function \(w_1(T, t, x, y)\) coincides with the generalized (minimax, viscosity) solution of this problem which exists, is unique and determined by the terminal boundary value condition (2.5) and the pair of differential inequalities for conjugate derivatives \(D^*w_1\) and \(D_*w_1\) corresponding to the Hamilton-Jacobi equation (2.3)

\[
D^*w_1(T, t, x, y)|_{(s)} \geq H(x, y, s) \tag{2.7}
\]

\[
D_*w_1(T, t, x, y)|_{(s)} \leq H(x, y, s) \tag{2.8}
\]

\((t, x, y) \in [t_0, T] \times (0, 1) \times (0, 1), \quad s = (s_1, s_2) \in R^2\)

Conjugate derivatives \(D^*w_1\) and \(D_*w_1\) and the Hamiltonian \(H\) are given by formulas (see [Subbotin, Tarasyev, 1985])

\[
D^*w_1(T, t, x, y)|_{(s)} = \sup_{h \in R^2} (\langle s, h \rangle - \partial_- w_1(T, t, x, y)|_{(1, h)}) \tag{2.9}
\]

\[
D_*w_1(T, t, x, y)|_{(s)} = \inf_{h \in R^2} (\langle s, h \rangle - \partial_+ w_1(T, t, x, y)|_{(1, h)}) \tag{2.10}
\]

\[
H(x, y, s) = -s_1 x - s_2 y + \max_0 \min_0 s_1 u + \min_0 s_2 v \tag{2.11}
\]

Here symbol \(\langle s, h \rangle\) denotes the usual inner product of vectors \(s\) and \(h\), symbols \(\partial_- w_1(T, t, x, y)|_{(1, h)}, \partial_+ w_1(T, t, x, y)|_{(1, h)}\) denote Dini directional derivatives of the value function \(w_1\) at a point \((t, x, y)\) in a direction \((1, h), h = (h_1, h_2) \in R^2\)

\[
\partial_- w_1(T, t, x, y)|_{(1, h)} = \lim_{\delta \downarrow 0} \inf_0 \frac{w_1(T, t + \delta, x + \delta h_1, y + \delta h_2) - w_1(T, t, x, y)}{\delta} \tag{2.12}
\]

\[
\partial_+ w_1(T, t, x, y)|_{(1, h)} = \lim_{\delta \downarrow 0} \sup_0 \frac{w_1(T, t + \delta, x + \delta h_1, y + \delta h_2) - w_1(T, t, x, y)}{\delta} \tag{2.13}
\]
For the piecewise smooth value function $w_1$ directional derivatives and conjugate derivatives can be calculated in the framework of nonsmooth and convex analysis. Let us assume that in some neighborhood $O_{\varepsilon}(t^*, x^*, y^*)$ of a point $(t^*, x^*, y^*) \in [t_0, T] \times (0, 1) \times (0, 1)$ the function $w_1$ is given by formulas

\begin{equation}
\begin{aligned}
  w_1(T, t, x, y) &= \min_{i \in I} \max_{j \in J} \varphi_{ij}(T, t, x, y) = \max_{j \in J} \min_{i \in I} \varphi_{ij}(T, t, x, y) \quad (2.14) \\
  w_1(T, t^*, x^*, y^*) &= \varphi_{ij}(T, t^*, x^*, y^*), \quad i \in I, \quad j \in J
\end{aligned}
\end{equation}

Directional derivatives are determined in this case by relations

\begin{equation}
\begin{aligned}
  \partial_{-}w_1(T, t^*, x^*, y^*)|(1, h) &= \partial_{+}w_1(T, t^*, x^*, y^*)|(1, h) = \partial w_1(T, t^*, x^*, y^*)|(h) = \\
  \min_{i \in I} \max_{j \in J} (a_{ij} + \langle b_{ij}, h \rangle) &= \max_{j \in J} \min_{i \in I} (a_{ij} + \langle b_{ij}, h \rangle) \quad (2.15)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
  a_{ij} &= \frac{\partial \varphi_{ij}}{\partial t} \\
  b_{ij} &= \left( \frac{\partial \varphi_{ij}}{\partial x}, \frac{\partial \varphi_{ij}}{\partial y} \right)
\end{aligned}
\end{equation}

Assume

\begin{equation}
\begin{aligned}
  C = \bigcap_{i \in I} B_i, \quad B_i = co\{b_{ij} : j \in J\} \\
  D = \bigcap_{j \in J} B_j, \quad B_j = co\{b_{ij} : i \in I\}
\end{aligned}
\end{equation}

Conjugate derivatives are determined by relations

\begin{equation}
\begin{aligned}
  D^w_1(T, t^*, x^*, y^*)|(s) &= \begin{cases} 
    \max_{i \in I} \min_{j \in J} \left\{ -\sum_{j \in J} \lambda_j(s)a_{ij} \right\} 
    & \text{if } s \in C \\
    +\infty & \text{otherwise}
  \end{cases} \quad (2.16) \\
  D^w_1(T, t^*, x^*, y^*)|(s) &= \begin{cases} 
    \min_{j \in J} \max_{i \in I} \left\{ -\sum_{i \in I} \lambda_i(s)a_{ij} \right\} 
    & \text{if } s \in D \\
    -\infty & \text{otherwise}
  \end{cases} \quad (2.17)
\end{aligned}
\end{equation}

Here coefficients $\lambda_j(s)$ and $\lambda_i(s)$ satisfy the relations

\begin{equation}
\begin{aligned}
  \sum_{j \in J} \lambda_j(s)b_{ij} &= s, \quad \lambda_j(s) \geq 0, \quad \sum_{j \in J} \lambda_j(s) = 1 \\
  \sum_{i \in I} \lambda_i(s)b_{ij} &= s, \quad \lambda_i(s) \geq 0, \quad \sum_{i \in I} \lambda_i(s) = 1
\end{aligned}
\end{equation}

\section{The Description of the Analytical Solution for the Terminal Boundary Value Problem}

The terminal boundary value problem (2.3), (2.5) has the analytic solution. The corresponding value function $w_1(T, t, x, y)$ is piecewise smooth and consists of five smooth functions $\varphi_k(T, t, x, y)$, $k = 1, \ldots, 5$ which are pasted by operations of maximum and minimum. Analytic formulas for smooth components $\varphi_k(T, t, x, y)$, $k = 1, \ldots, 5$ can be obtained via methods of characteristics for corresponding linear Hamilton-Jacobi equations which
arise from the nonlinear one (2.3) by substituting different combinations of extremal values 0 and 1 into $\max$ and $\min$ expressions. Let us give formulas for these functions

$$\varphi_1(T, t, x, y) = C_A e^{2(t-T)} xy - \alpha_1 e^{(t-T)} x - \alpha_2 e^{(t-T)} y + a_{22}$$

$$\varphi_2(T, t, x, y) = C_A e^{2(t-T)} xy - \alpha_1 e^{(t-T)} x - (C_A e^{2(t-T)} + (\alpha_2 - C_A) e^{(t-T)}) y + \alpha_1 e^{(t-T)} + a_{12}$$

$$\varphi_3(T, t, x, y) = C_A e^{2(t-T)} xy - (C_A e^{2(t-T)} + (\alpha_1 - C_A) e^{(t-T)}) x - (C_A e^{2(t-T)} + (\alpha_1 + \alpha_2 - 2C_A) e^{(t-T)}) y + C_A e^{2(t-T)} + \alpha_1 - (-\alpha_2 e^{(t-T)} y + \alpha_2 e^{(t-T)} + a_{21})$$

$$\varphi_4(T, t, x, y) = \frac{a_{22} C_A - \alpha_1 \alpha_2}{C_A} = \frac{a_{11} a_{22} - a_{12} a_{21}}{C_A} = \frac{D_A}{C_A} = v_A$$

Here $v_A$ is the value of the static matrix game. Functions $\varphi_k$, $k = 1, ..., 5$ are continuously pasted together on four lines $L_m = L_m(T, t), m = 1, ..., 4$

$$L_1 = \{(x, y) : x_1(T, t) \leq x \leq 1, \ y = y_2(T, t)\}$$

$$L_2 = \{(x, y) : x = x_1(T, t), \ y_1(T, t) \leq y \leq 1\}$$

$$L_3 = \{(x, y) : 0 \leq x \leq x_2(T, t), \ y = y_1(T, t)\}$$

$$L_4 = \{(x, y) : x = x_2(T, t), \ 0 \leq y \leq y_2(T, t)\}$$

Here

$$x_1(T, t) = \max\{0, 1 - (1 - \frac{\alpha_2}{C_A}) e^{(T-t)}\}$$

$$x_2(T, t) = \min\{1, \frac{\alpha_2}{C_A} e^{(T-t)}\}$$

$$y_1(T, t) = \max\{0, 1 - (1 - \frac{\alpha_1}{C_A}) e^{(T-t)}\}$$

$$y_2(T, t) = \min\{1, \frac{\alpha_1}{C_A} e^{(T-t)}\}$$

Let us give the description of the value function $w_1$.

**Proposition 2.1** The value function $w_1(T, t, x, y)$ is determined by the formula

$$w_1(T, t, x, y) = \varphi_k(T, t, x, y), \text{ if } (x, y) \in D_k(T, t), \ k = 1, ..., 5$$

Here domains $D_k = D_k(T, t), k = 1, ..., 5$ are given by inequalities

$$D_1(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ x_2(T, t) \leq x \leq 1, \ 0 \leq y \leq y_2(T, t)\}$$

$$D_2(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ x_1(T, t) \leq x \leq 1, \ y_2(T, t) \leq y \leq 1\}$$

$$D_3(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ 0 \leq x \leq x_1(T, t), \ y_1(T, t) \leq y \leq 1\}$$

$$D_4(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ 0 \leq x \leq x_2(T, t), \ 0 \leq y \leq y_1(T, t)\}$$

$$D_5(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ x_1(T, t) \leq x \leq x_2(T, t), \ y_1(T, t) \leq y \leq y_2(T, t)\}$$
Remark 2.1 Some of domains $D_k$, $k = 1, \ldots, 5$ can be empty. In the case when
\[
0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < y_A = \frac{\alpha_1}{C_A} < 1
\] (2.30)
all domains $D_k$, $k = 1, \ldots, 5$ have nonempty interior during the finite interval of time $(T_f, T)$,
\[
T_f = \max\{t_{x_1}, t_{x_2}, t_{y_1}, t_{y_2}\} \quad (2.31)
\]
\[
t_{x_1} = \max\{t : x_1(T, t) = 0\}
\]
\[
t_{x_2} = \max\{t : x_2(T, t) = 1\}
\]
\[
t_{y_1} = \max\{t : y_1(T, t) = 0\}
\]
\[
t_{y_2} = \max\{t : y_2(T, t) = 1\}
\]
Then interior parts of domains $D_1, D_2, D_3, D_4$ disappear at the corresponding moments $t_{x_2}, t_{y_2}, t_{x_1}, t_{y_1}$ and the value function $w_1(T, t, x, y)$ becomes equal to the constant
\[
w_1(T, t, x, y) = \varphi_5 = \frac{D_A}{C_A} = v_A
\] (2.32)
\[
\quad (t \leq T_d, \quad (x, y) \in [0, 1] \times [0, 1])
\]
\[
T_d = \min\{t_{x_1}, t_{x_2}, t_{y_1}, t_{y_2}\}
\] (2.33)
On Fig.1 lines $L_m$, $m = 1, \ldots, 4$ and domains $D_k$, $k = 1, \ldots, 5$ are shown for the case when $x_A = 0.6, y_A = 0.4, e^{(T-t)} = 1.5$.

2.3 Verification of Differential Inequalities in the Terminal Boundary Value Problem

Let us prove that the necessary and sufficient conditions (2.5), (2.7), (2.8) are valid for the function $w_1(T, t, x, y)$ determined by formulas (2.28), (2.29).

Proof.

It is obvious that the terminal boundary value condition (2.5) is fulfilled for the function $w_1(T, t, x, y)$. Let us verify that the function $w_1(T, t, x, y)$ satisfies differential inequalities (2.7), (2.8). It is not difficult to convince oneself that functions $\varphi_k(T, t, x, y)$, $k = 1, \ldots, 5$ satisfy Hamilton-Jacobi equation (2.3) at all internal points $D_0^k = \text{int}\{D_k\}$ of domains $D_k$, $k = 1, \ldots, 5$. It remains to verify inequalities (2.7), (2.8) at points of boundaries $\partial D_k$ of these domains, more precisely, at points of lines $L_m(T, t)$, $m = 1, \ldots, 4$. Let us consider, for example, a part
\[
L_{12}^1 = L_{12}^1(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \quad x_2(T, t) < x \leq 1, \quad y = y_2(T, t)\}
\]
of the line $L_1$ at points of which functions $\varphi_1$ and $\varphi_2$ are pasted together (see Fig.1). We calculate derivatives $\partial \varphi_i / \partial t, \partial \varphi_i / \partial x, \partial \varphi_i / \partial y$, $i = 1, 2$ on $L_{12}^1$
\[
\frac{\partial \varphi_1}{\partial t} = 2C_A e^{2(t-T)}xy - \alpha_1 e^{(t-T)}x - \alpha_2 e^{(t-T)}y = \alpha_1 e^{(t-T)}x - \frac{\alpha_1 \alpha_2}{C_A}
\]
\[
\frac{\partial \varphi_1}{\partial x} = C_A e^{2(t-T)}y - \alpha_1 e^{(t-T)} = 0
\]
\[
\frac{\partial \varphi_1}{\partial y} = C_A e^{2(t-T)}x - \alpha_2 e^{(t-T)}
\]
Figure 1: The structure of the value function $w_1$ in the terminal problem.
One can see that partial derivatives \( \partial \phi_1/\partial y \) and \( \partial \phi_2/\partial y \) are connected at points of the set \( L_1^2 \) by inequalities

\[
0 \leq \frac{\partial \phi_1}{\partial y} \leq \frac{\partial \phi_2}{\partial y}
\]

Therefore, functions \( \varphi_1 \) and \( \varphi_2 \) are pasted on the line \( L_1^2 \) by the operation of the maximum type. Thus, for function \( w_1(T, t, x, y) \) the following relation

\[
w_1(T, t, x, y) = \max\{\varphi_1(T, t, x, y), \varphi_2(T, t, x, y)\}
\]

is valid in some neighborhood of the set \( L_1^2 \). Besides that, one can obtain the equalities

\[
\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} = 0
\]

Hence, the directional derivative in a direction \((1, h) = (1, h_1, h_2)\) is determined by the formula

\[
\partial w_1(T, t, x, y)|((1, h) = \max\{\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_1}{\partial y} h_2, \frac{\partial \phi_2}{\partial t} + \frac{\partial \phi_2}{\partial y} h_2\}
\]

Then for the conjugate derivatives we obtain relations

\[
D^*w_1(T, t, x, y)|s = \begin{cases} 
-\lambda a_1 - (1 - \lambda)a_2 & \text{if } s_1 = 0 \text{ and } s_2 = \lambda b_1 + (1 - \lambda)b_2 \\
+\infty & \text{otherwise}
\end{cases}
\]

\[
D_s w_1(T, t, x, y)|s = -\infty
\]

Here \( 0 \leq \lambda \leq 1 \), \( a_i = \partial \varphi_i/\partial t \), \( b_i = \partial \varphi_i/\partial y \), \( i = 1, 2 \).

Let us remind that we need to verify the pair of conditions

\[
D^*w_1(T, t, x, y)|s \geq H(x, y, s) \\
D_s w_1(T, t, x, y)|s \leq H(x, y, s)
\]

It is obvious that the second inequality is fulfilled. Let us verify the first one. For the vectors

\[
s = (s_1, s_2), \quad s_1 = 0, \quad s_2 = \lambda \frac{\partial \phi_1}{\partial y} + (1 - \lambda)\frac{\partial \phi_2}{\partial y}
\]

we have relations

\[
D^*w_1(T, t, x, y)|s = (1 - \lambda)\alpha_1(e^{(t-T)} - 1) - \alpha_1 e^{(t-T)} x + \frac{\alpha_1 \alpha_2}{C_A}
\]

\[
H(x, y, s) = \max_{0 \leq u \leq 1} s_1(-x + u) + \min_{0 \leq v \leq 1} s_2(-y + v) = -s_2 y
\]

\[
(1 - \lambda)\alpha_1(e^{(t-T)} - 1) - \alpha_1 e^{(t-T)} x + \frac{\alpha_1 \alpha_2}{C_A}
\]
Hence, we obtain

\[ D^*w_1(T, t, x, y)(s) = H(x, y, s) \]

for \( s \in \text{dom}\{D^*w_1\} \). Thus, differential inequalities (2.7),(2.8) are valid for the function \( w_1 \) on the set \( L_{12}^1 \).

Let us verify conditions (2.7),(2.8) at points of one more typical pasting line. Consider the set

\[ L_{15}^{15} = L_{14}^{15}(T, t) = \{(x, y) \in [0, 1] \times [0, 1] : \ x = x_2(T, t), \ y_1(T, t) \leq y \leq y_2(T, t)\} \]

where functions \( \varphi_1 \) and \( \varphi_5 \) are pasted together (see Fig.1). It is obvious that partial derivatives of the function \( \varphi_5 \) are equal to zero

\[ \frac{\partial \varphi_5}{\partial t} = \frac{\partial \varphi_5}{\partial x} = \frac{\partial \varphi_5}{\partial y} = 0 \]

As is easily seen the following inequality takes place

\[ \frac{\partial \varphi_1}{\partial x} = C_Ae^{2(t-T)}y - \alpha_1e^{t-T} \leq 0 = \frac{\partial \varphi_5}{\partial x} \]

on the set \( L_{14}^{15} \) since \( y \leq (\alpha_1e^{(T-t)})/C_A \) for points \( (x, y) \in L_{14}^{15} \). Therefore, functions \( \varphi_1 \) and \( \varphi_5 \) are pasted with the help of the minimum type operation

\[ w_1(T, t, x, y) = \min\{\varphi_1(T, t, x, y), \varphi_5(T, t, x, y)\} \]

on the set \( L_{14}^{15} \). Taking into account that

\[ \frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_5}{\partial y} = 0 \]

we obtain the following relations for the directional derivative on the line \( L_{14}^{15} \)

\[ \partial w_1(T, t, x, y)(1, h) = \min\{0, \frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_2}{\partial x}h_1\} \]

The conjugate derivatives are determined by formulas

\[ D^*w_1(T, t, x, y)(s) = +\infty \]

\[ D_*w_1(T, t, x, y)(s) = \begin{cases} -\lambda a_1 - (1 - \lambda)a_5 & \text{if } s_2 = 0 \text{ and } s_1 = \lambda b_1 + (1 - \lambda)b_5 \\ -\infty & \text{otherwise} \end{cases} \]

Here \( a_i = \partial \varphi_i/\partial t \), \( b_i = \partial \varphi_i/\partial x \), \( i = 1, 5 \).

It is obvious that

\[ D^*w_1(T, t, x, y)(s) \geq H(x, y, s), \ s \in R^2 \]

Let us calculate the lower conjugate derivative \( D_*w_1(T, t, x, y)(s) \) and the Hamiltonian \( H(x, y, s) \) on vectors

\[ s = (s_1, s_2), \ s_2 = 0, \ s_1 = \lambda \frac{\partial \varphi_1}{\partial x} + (1 - \lambda)\frac{\partial \varphi_5}{\partial x} \]

We have relations

\[ D_*w_1(T, t, x, y)(s) = -\lambda\alpha_2e^{(t-T)}y + \lambda\frac{\alpha_1\alpha_2}{C_A} \]
the differential game with the multiterminal payoff functional

\[ H(x, y, s) = \max_{0 \leq u \leq 1} s_1(-x + u) + \min_{0 \leq v \leq 1} s_2(-y + v) = -s_1 x - \lambda \alpha_2 e^{(t-T)} y + \frac{\alpha_1 \alpha_2}{C_A}. \]

It is obvious that

\[ D_s w_1(T, t, x, y)(s) = H(x, y, s) \]

for \( s \in \text{dom}\{D_s w_1\} \). Thus, differential inequalities (2.7), (2.8) are also fulfilled for the function \( w_1 \) on the set \( L_{15} \).

The conditions (2.7), (2.8) on other parts of lines \( L_m, m = 1, ..., 4 \) can be verified analogously. Thus, it is proved that the function \( w_1(T, t, x, y) \) determined by formulas (2.28), (2.29) is the generalized (viscosity, minimax) solution of the terminal boundary value problem (2.3), (2.5) and, hence, it coincides with the value function of the corresponding differential game (1.1), (1.2).

3 The Lower Envelope of Terminal Value Functions and the Value Function of the Game with the Multiterminal Payoff Functional

3.1 The Differential Game with the Multiterminal Functional

In the previous section we have obtained the solution for the auxiliary terminal boundary value problem (2.3), (2.5). The solution of this problem (the value function) \( w_1(T, t, x, y) \) depends on the terminal instant \( T \). Of course, such a solution is not appropriate in the evolutionary sense because we obtain a “good” result at a single moment \( T \) but not at other times including infinity. Therefore, in this section we will construct the value function for the differential game with the multiterminal payoff functional

\[ G_A(x(\cdot), y(\cdot)) = \inf_{t_0 \leq t < +\infty} g_A(x(t), y(t)) \quad (3.1) \]

The functional (3.1) determines the foreseeing principle since it takes into account future states \( g_A(x(t), y(t)) \) from time \( t_0 \) till infinity \( +\infty \).

Using results of differential games theory (see [Krasovskii, Subbotin, 1988]) and viability theory (see [Aubin, 1990]) one can prove that the zero-sum differential game with the dynamics (1.1) and the payoff (3.1) has the value

**Theorem 3.1** There exists the saddle point determining the stationary value function

\[ \sup_{u(t, x, y, \varepsilon)} \inf_{x(\cdot), y(\cdot)} g_A(x_1(s), y_1(s)) = \inf_{u(t, x, y, \varepsilon)} \sup_{x(\cdot), y(\cdot)} g_A(x_2(s), y_2(s)) = \]

\[ \lim_{T \to +\infty} \min_{u(t, x, y, \varepsilon)} \max_{x(\cdot), y(\cdot)} g_A(x_2(s), y_2(s)) = \lim_{T \to +\infty} \max_{u(t, x, y, \varepsilon)} \min_{x(\cdot), y(\cdot)} g_A(x_1(s), y_1(s)) = \]

\[ w_A(t_0, x_0, y_0) = w_A(x_0, y_0) \quad (3.2) \]

Here trajectories \((x_1(\cdot), y_1(\cdot)), (x_2(\cdot), y_2(\cdot))\) are generated from the initial position \((t_0, x_0, y_0)\) by feedback controls \(u(t, x, y, \varepsilon), v(t, x, y, \varepsilon)\) of maximizing and minimizing players respectively and arbitrary controls of their opponents.
Proposition 3.3
The function $w(t,x,y)$ is subjected to the principle of dynamical programming. The corresponding properties of $u(t,x,y)$ are carried out via the notion of viability kernel (see [Aubin, 1990], [Sonnevend, 1981]).

The scheme of the proof is the following.

In the general case inequalities take place

\[
\sup_{u(t,x,y)} \inf_{x,y} g_A(x_1(s), y_1(s)) \leq \inf_{v(t,x,y)} \sup_{x,y} g_A(x_2(s), y_2(s)) \leq \lim_{T \to +\infty} \min_{v(t,x,y)} \max_{x,y} g_A(x_2(s), y_2(s)) = \lim_{T \to +\infty} \max_{v(t,x,y)} \min_{x,y} g_A(x_1(s), y_1(s)) = w_A(t,x,y)
\]

(3.3)

One can verify the following properties of the function $w_A(t,x,y)$ (3.3).

**Proposition 3.1** The function $w_A$ is a stationary one (it does not depend on $t$). More precisely,

\[
w_A(t,x,y) = w_A(s,x,y) = w_A(x,y)
\]

(3.4)

for all $(x,y) \in [0,1] \times [0,1]$, $t \in R$, $s \in R$.

**Proposition 3.2** The function $w_A$ satisfies the Lipschitz condition

\[
|w_A(x_1,y_1) - w_A(x_2,y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)
\]

(3.5)

for all $(x_i,y_i) \in [0,1] \times [0,1]$, $i = 1,2$.

**Proposition 3.3** The function $w_A$ is majorized by the payoff $g_A$

\[
w_A(x,y) \leq g_A(x,y), \quad (x,y) \in [0,1] \times [0,1]
\]

(3.6)

**Proposition 3.4** The function $w_A$ is the maximal function which satisfies inequality (3.6) and is subjected to the principle of dynamical programming. The corresponding properties of $u$-stability and $v$-stability can be represented in the infinitesimal form as

\[
\min_{0 \leq s \leq 1} \max_{0 \leq v \leq 1} \partial_+ w_A(x,y)(-x + u, -y + v) \geq 0
\]

(3.7)

for all $(x,y) \in (0,1) \times (0,1)$ and

\[
\max_{0 \leq s \leq 1} \min_{0 \leq v \leq 1} \partial_- w_A(x,y)(-x + u, -y + v) \leq 0
\]

(3.8)

for all $(x,y) \in (0,1) \times (0,1)$ where $w_A(x,y) < g_A(x,y)$. Here the directional derivatives $\partial_- w_A(x,y)|(h_1,h_2)$, $\partial_+ w_A(x,y)|(h_1,h_2)$ of the function $w_A$ at a point $(x,y) \in (0,1) \times (0,1)$ along a direction $h = (h_1,h_2)$ are determined by

\[
\partial_- w_A(x,y)|(h_1,h_2) = \lim_{\delta \downarrow 0} \inf_{\delta} \frac{w_A(x + \delta h_1, y + \delta h_2) - w_A(x,y)}{\delta}
\]

(3.9)

\[
\partial_+ w_A(x,y)|(h_1,h_2) = \lim_{\delta \downarrow 0} \sup_{\delta} \frac{w_A(x + \delta h_1, y + \delta h_2) - w_A(x,y)}{\delta}
\]

(3.10)
Proposition 3.5 Properties of $u$-stability (3.7) and $v$-stability (3.8) can be rewritten (see, for example, [Subbotin, Tarasyev, 1985]) in terms of conjugate derivatives

\[ D_u w_A(x, y)(s) \leq H(x, y, s), \quad (x, y) \in (0, 1) \times (0, 1), \quad s = (s_1, s_2) \in R^2 \quad (3.11) \]

\[ D^* w_A(x, y)(s) \geq H(x, y, s), \quad (x, y) \in (0, 1) \times (0, 1), \quad w_A(x, y) < g_A(x, y), \quad s = (s_1, s_2) \in R^2 \quad (3.12) \]

Here the conjugate derivatives $D^* w_A, D_* w_A$ and the Hamiltonian $H$ are determined by

\[ D^* w_A(x, y)(s) = \sup_{h \in R^2} (s, h) - \partial_- w_A(x, y)(h) \quad (3.13) \]

\[ D_* w_A(x, y)(s) = \inf_{h \in R^2} (s, h) - \partial_+ w_A(x, y)(h) \quad (3.14) \]

\[ H(x, y, s) = -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\} \quad (3.15) \]

Taking into account stability (viability) properties and using the concept of “extremal shift” strategy (see [Krasovskii, 1985]) one can prove that corresponding trajectories provide the value of functional $G_A$ (3.1) on $[t_0, +\infty)$ equal to the value of function $w_A$ (3.2). Hence, all inequalities in (3.3) turn into equalities and this fact proves Theorem 3.1.

The inverse result is also valid.

Proposition 3.6 Differential inequalities (3.7),(3.8) (or, equivalently, (3.11),(3.12)) together with the boundary condition (3.6) determine the value function $w_A$ uniquely. More precisely, there exists the single maximal function which satisfies these conditions, and this function coincides with $w_A$.

Thus, either the set of Propositions 3.3,3.4,3.6 or the set of Propositions 3.3,3.5,3.6 provide necessary and sufficient conditions for the value function $w_A$.

Stability properties (3.7),(3.8) (or, equivalently, (3.11),(3.12)) are connected with the stationary Hamilton-Jacobi equation.

Proposition 3.7 At points $(x, y) \in (0, 1) \times (0, 1)$ where $w_A(x, y) < g_A(x, y)$ and $w_A$ is differentiable, inequalities (3.7),(3.8) (or (3.11),(3.12)) turn into the stationary Hamilton-Jacobi equation

\[ -\frac{\partial w_A}{\partial x} x - \frac{\partial w_A}{\partial y} y + \max\{0, \frac{\partial w_A}{\partial x}\} + \min\{0, \frac{\partial w_A}{\partial y}\} = 0 \quad (3.16) \]

For piecewise smooth functions we have the following stationary version of stability properties.

Proposition 3.8 If the value function $w_A$ is piecewise smooth then conjugate derivatives (3.13),(3.14) are determined by

\[ D^* w_A(x_*, y_*)(s) = \begin{cases} 0 & \text{if } s \in C \\ +\infty & \text{otherwise} \end{cases} \quad (3.17) \]

\[ D_* w_A(x_*, y_*)(s) = \begin{cases} 0 & \text{if } s \in D \\ -\infty & \text{otherwise} \end{cases} \quad (3.18) \]
Here

\[ C = \bigcap_{i \in I} B_i, \quad B_i = \text{co}\{b_{ij} : j \in J\} \]

\[ D = \bigcap_{j \in J} B_j, \quad B_j = \text{co}\{b_{ij} : i \in I\} \]

\[ b_{ij} = \left( \frac{\partial \varphi_{ij}}{\partial x}, \frac{\partial \varphi_{ij}}{\partial y} \right) \]

\[ w_A(x, y) = \min_{i \in I} \max_{j \in J} \varphi_{ij}(x, y) = \max_{j \in J} \min_{i \in I} \varphi_{ij}(x, y) \]

\[ w_A(x_s, y_s) = \varphi_{ij}(x_s, y_s), \quad i \in I, \quad j \in J \]

\[ (x, y) \in O_{\varepsilon}(x_s, y_s) \]

And differential inequalities (3.11), (3.12) turn into formulas

\[ 0 \leq -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\} \]

\[ (x, y) \in (0, 1) \times (0, 1), \quad s = (s_1, s_2) \in D \]

\[ 0 \geq -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\} \]

\[ (x, y) \in (0, 1) \times (0, 1), \quad w_A(x, y) < g_A(x, y), \quad s = (s_1, s_2) \in C \]

### 3.2 Description of a Solution to the Game with the Multiterminal Functional

Before describing an analytic solution of the game in question, we introduce the envelopes of the smooth components \( \varphi_1, \varphi_3 \) (2.18), (2.20) of the value function \( w_1 \) (2.28) parametrized by \( s = t - T \). Low envelopes provide considering of multiterminal interests and introducing foreseeing principle for designing feedbacks. To construct the envelope \( \psi_A^1 \) of \( \varphi_1 \) it is necessary to calculate the derivative with respect to \( s \), set it equal to zero, find the root of the obtained equation, and substitute the root into \( \varphi_1 \). Namely, we have

\[ \frac{\partial \varphi_1}{\partial s} = 2C_A e^{2s} x y - \alpha_1 e^s x - \alpha_2 e^s y = 0 \]

\[ e^s = \frac{\alpha_1 x + \alpha_2 y}{2C_A x y} \]

\[ \psi_A^1(x, y) = \varphi_1(x, y) = a_{22} - \frac{(\alpha_1 x + \alpha_2 y)^2}{4C_A x y} \]  \hspace{1cm} (3.19)

Making the same with \( \varphi_3 \) we obtain its lower envelope \( \psi_A^2 \) with respect to \( s \)

\[ \frac{\partial \varphi_3}{\partial s} = 2C_A e^{2s} x y - \]

\[ (C_A e^{2s} + (\alpha_1 - C_A) e^s) x - (C_A e^{2s} + (\alpha_2 - C_A) e^s) y + C_A e^{2s} + (\alpha_1 + \alpha_2 - 2C_A) e^s = 0 \]

\[ e^s = \frac{(\alpha_1 - C_A)(x - 1) + (\alpha_2 - C_A)(y - 1)}{2C_A (x - 1)(y - 1)} \]

\[ \psi_A^2(x, y) = \varphi_3(x, y) = a_{11} - \frac{(C_A - \alpha_1)(1 - x) + (C_A - \alpha_2)(1 - y))^2}{4C_A (1 - x)(1 - y)} \]  \hspace{1cm} (3.20)
Let us pass to the analytic description of the value function $w$. Here the domains $E_i$ respectively following lines $K^i$, $i = 1, ..., 4$ are continuously pasted together along the following lines $K_A^j$, $j = 1, ..., 5$

$$K_A^1 = \{(x, y) \in [0, 1] \times [0, 1]: \ x = \frac{\alpha_2}{C_A}, \ 0 \leq y \leq 1\}$$

$$K_A^2 = \{(x, y) \in [0, 1] \times [0, 1]: \ \alpha_2 C_A \leq x \leq 1, \ \frac{\alpha_1}{C_A} \leq y \leq 1, \ y = \frac{\alpha_1}{\alpha_2}\}$$

$$K_A^3 = \{(x, y) \in [0, 1] \times [0, 1]: \ 0 \leq x \leq \frac{\alpha_2}{C_A}, \ 0 \leq y \leq \frac{\alpha_1}{C_A}, \ y = -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)}(1 - x) + 1\}$$

$$K_A^4 = \{(x, y) \in [0, 1] \times [0, 1]: \ \frac{\alpha_2}{C_A} \leq x \leq 1, \ 0 \leq y \leq \frac{\alpha_1}{C_A}, \ y = -\frac{\alpha_1 x}{2C_A x - \alpha_2}\}$$

$$K_A^5 = \{(x, y) \in [0, 1] \times [0, 1]: \ 0 \leq x \leq \frac{\alpha_2}{C_A}, \ \frac{\alpha_1}{C_A} \leq y \leq 1, \ y = -\frac{(C_A - \alpha_1)(1 - x)}{2C_A(1 - x) - (C_A - \alpha_2) + 1}\}$$

Let us pass to the analytic description of the value function $w_A$.

**Proposition 3.9** In the case when $C_A > 0$ the value function $(x, y) \rightarrow w_A(x, y)$ is determined by

$$w_A(x, y) = \psi^i_{A}(x, y), \ \text{if} \ (x, y) \in E_A^i, \ i = 1, ..., 4 \quad (3.23)$$

Here the domains $E_A^i$, $i = 1, ..., 4$ are defined as follows

$$E_A^1 = \{(x, y) \in [0, 1] \times [0, 1]: \ \alpha_2 C_A \leq x \leq 1, \ \frac{\alpha_1 x}{2C_A x - \alpha_2} \leq y \leq \frac{\alpha_1}{\alpha_2}\}$$

$$E_A^2 = \{(x, y) \in [0, 1] \times [0, 1]: \ 0 \leq x \leq \frac{\alpha_2}{C_A}, \ \frac{\alpha_1}{C_A} \leq y \leq \frac{(C_A - \alpha_1)(1 - x) + 1}{(C_A - \alpha_2) + 1}\}$$

$$E_A^3 = E_A^{31} \cup E_A^{32}$$

$$E_A^{31} = \{(x, y) \in [0, 1] \times [0, 1]: \ \frac{\alpha_2}{C_A} \leq x \leq 1, \ 0 \leq y \leq \frac{\alpha_1 x}{2C_A x - \alpha_2}\}$$

$$E_A^{32} = \{(x, y) \in [0, 1] \times [0, 1]: \ 0 \leq x \leq \frac{\alpha_2}{C_A}, \ -\frac{(C_A - \alpha_1)(1 - x)}{2C_A(1 - x) - (C_A - \alpha_2) + 1} + 1 \leq y \leq 1\}$$

Similarly we define the lower envelopes $\psi_A^3$, $\psi_A^4$ of smooth components $\phi_2$, $\phi_4 (2.19),(2.21)$, respectively

$$\psi_A^3(x, y) = C_A xy - \alpha_1 x - \alpha_2 y + a_{22} \quad (3.21)$$

$$\psi_A^4(x, y) = \frac{a_{22} C_A - \alpha_1 \alpha_2}{C_A} = v_A \quad (3.22)$$
Remark 3.1 In the domain $E^4_A$ the following relations hold

$$g_A(x, y) \geq w_A(x, y) = v_A$$

(3.25)

Remark 3.2 The positional strategy $U^0_A = u_0^A(x, y)$ corresponding to the value function $w_A$ (see below relations (4.1)) provides viability of trajectories $(x(\cdot), y(\cdot))$ of the system (1.1) in the domain $E^4_A$.

Remark 3.3 In the case when $C_A < 0$ the value function $(x, y) \rightarrow w_A(x, y)$ is determined by relations

$$w_A(x, y) = \psi^i_A(x, y), \quad \text{if} \quad (x, y) \in E^i_A, \quad i = 1, \ldots, 4$$

(3.26)

$$\psi^1_A(x, y) = a_{21} + \frac{(C_A - \alpha_1)x + a_2(1-y))^2}{4C_A(1-y)}$$

$$\psi^2_A(x, y) = a_{12} + \frac{(\alpha_1(1-x) + (C_A - \alpha_2)y)^2}{4(C_A(1-x)y}$$

$$\psi^3_A(x, y) = C_Axy - \alpha_1x - \alpha_2y + a_{22}$$

$$\psi^4_A(x, y) = v_A$$

Here the domains $E^i_A$, $i = 1, \ldots, 4$ are defined as follows

$$E^1_A = \{(x, y) \in [0, 1] \times [0, 1] : \frac{\alpha_2}{C_A} \leq x \leq 1,$$

$$-\frac{(C_A - \alpha_1)x + \alpha_2(1-y)}{2C_A} \leq y \leq \frac{(C_A - \alpha_1)x}{2C_A(1-x)} + 1\}$$

$$E^2_A = \{(x, y) \in [0, 1] \times [0, 1] : \quad 0 \leq x \leq \frac{\alpha_2}{C_A},$$

$$\frac{\alpha_1(1-x)}{2C_A(1-x)} - (C_A - \alpha_2) \leq y \leq \frac{\alpha_1}{(C_A - \alpha_2)(1-x)}\}$$

$$E^3_A = E^3_A \cup E^4_A$$

$$E^4_A = \{(x, y) \in [0, 1] \times [0, 1] : \quad \frac{\alpha_2}{C_A} \leq x \leq 1, \quad$$

$$-\frac{(C_A - \alpha_1)x + 1}{2C_Ax - \alpha_2} \leq y \leq 1\}$$

$$E^3_A = \{(x, y) \in [0, 1] \times [0, 1] : \quad 0 \leq x \leq \frac{\alpha_2}{C_A},$$

$$-\frac{(C_A - \alpha_1)x}{2C_Ax - \alpha_2} + 1 \leq y \leq 1\}$$
\[ 0 \leq y \leq \frac{\alpha_1(1-x)}{2C_A(1-x)-(C_A-\alpha_2)} + 1 \]

\[ E_A^1 = E_A^{41} \cup E_A^{32} \]
\[ E_A^{41} = \{(x,y) \in [0,1] \times [0,1] : \frac{\alpha_2}{C_A} \leq x \leq 1, \]
\[ 0 \leq y \leq -\frac{(C_A-\alpha_1)}{\alpha_2}x + 1 \}
\[ E_A^{42} = \{(x,y) \in [0,1] \times [0,1] : \ 0 \leq x \leq \frac{\alpha_2}{C_A}, \]
\[ \frac{\alpha_1}{(C_A-\alpha_2)}(1-x) \leq y \leq 1 \}\]

Remark 3.4 For the matrix \( B \) the value function \( w_B \) can be determined analogously. In the case when \( C_B > 0 \) the value function \((x,y) \rightarrow w_B(x,y)\) is determined by relations

\[ w_B(x,y) = \psi_B^i(x,y), \text{ if } (x,y) \in E_B^i, \ i = 1,\ldots,4 \]  
\[ \psi_B^1(x,y) = b_{22} - \frac{(\beta_1 x + \beta_2 y)^2}{4C_B xy} \]
\[ \psi_B^2(x,y) = b_{11} - \frac{((C_B-\beta_1)(1-x) + (C_B-\beta_2)(1-y))^2}{4C_B(1-x)(1-y)} \]
\[ \psi_B^3(x,y) = C_B xy - \beta_1 x - \beta_2 y + b_{22} \]
\[ \psi_B^4(x,y) = v_B = \frac{b_{22}C_B - \beta_1 \beta_2}{C_B} \]

Here domains \( E_B^i, i = 1,\ldots,4 \) are defined as follows

\[ E_B^1 = \{(x,y) \in [0,1] \times [0,1] : \frac{\beta_1}{C_B} \leq y \leq 1, \]
\[ \frac{\beta_2 y}{2C_B y - \beta_1} \leq x \leq \frac{\beta_2 y}{\beta_1} \} \]
\[ E_B^2 = \{(x,y) \in [0,1] \times [0,1] : 0 \leq y \leq \frac{\beta_1}{C_B}, \]
\[ -\frac{(C_B-\beta_2)}{(C_B-\beta_1)}(1-y) + 1 \leq x \leq -\frac{(C_B-\beta_2)(1-y)}{2C_B(1-y)-(C_B-\beta_1)} + 1 \} \]
\[ E_B^3 = E_B^{31} \cup E_B^{32} \]
\[ E_B^{31} = \{(x,y) \in [0,1] \times [0,1] : \frac{\beta_1}{C_B} \leq y \leq 1, \]
\[ 0 \leq x \leq \frac{\beta_2 y}{2C_B y - \beta_1} \} \]
\[ E_B^{32} = \{(x,y) \in [0,1] \times [0,1] : 0 \leq y \leq \frac{\beta_1}{C_B}, \]
\[ -\frac{(C_B-\beta_2)(1-y)}{2C_B(1-y)-(C_B-\beta_1)} + 1 \leq x \leq 1 \} \]
\[ E_B^4 = E_B^{41} \cup E_B^{42} \]
\[ E_B^{41} = \{(x,y) \in [0,1] \times [0,1] : \frac{\beta_1}{C_B} \leq y \leq 1, \]
\[ \frac{\beta_2}{\beta_1} y \leq x \leq 1 \} \]
\begin{equation}
E_{B}^{12} = \{(x, y) \in [0, 1] \times [0, 1] : \quad 0 \leq y \leq \frac{\beta_1}{C_B}, \quad 0 \leq x \leq \frac{(C_B - \beta_2)}{(C_B - \beta_1)}(1 - y) + 1\}
\end{equation}

In the case when \(C_B < 0\) the value function \((x, y) \rightarrow w_B(x, y)\) is determined by relations
\begin{equation}
w_B(x, y) = \psi_i^B(x, y), \quad \text{if} \quad (x, y) \in E_i^B, \quad i = 1, ..., 4
\end{equation}
\begin{align*}
\psi_1^B(x, y) &= b_{12} + \frac{(\beta_1(1-x) + (C_B - \beta_2)y)^2}{4C_B(1-x)y} \\
\psi_2^B(x, y) &= b_{21} + \frac{(C_B - \beta_1)x + \beta_2(1-y)^2}{4C_Bx(1-y)} \\
\psi_3^B(x, y) &= C_Bxy - \beta_1x - \beta_2y + b_{22} \\
\psi_4^B(x, y) &= v_B
\end{align*}

Here domains \(E_i^B, i = 1, ..., 4\) are defined as follows
\begin{align*}
E_1^B &= \{(x, y) \in [0, 1] \times [0, 1] : \quad \frac{\beta_1}{C_B} \leq y \leq 1, \quad -\frac{(C_B - \beta_2)}{\beta_1}y + 1 \leq x \leq -\frac{(C_B - \beta_2)y}{2C_By - \beta_1} + 1\} \\
E_2^B &= \{(x, y) \in [0, 1] \times [0, 1] : \quad 0 \leq y \leq \frac{\beta_1}{C_B}, \quad \frac{\beta_2(1-y)}{2C_B(1-y) - (C_B - \beta_1)} \leq x \leq \frac{\beta_2}{(C_B - \beta_1)}(1-y)\} \\
E_3^B &= E_1^B \cup E_2^B \\
E_4^B &= \{(x, y) \in [0, 1] \times [0, 1] : \quad 0 \leq y \leq \frac{\beta_1}{C_B}, \quad 0 \leq x \leq \frac{\beta_2(1-y)}{2C_B(1-y) - (C_B - \beta_1)}\} \\
E_5^B &= E_4^B \cup E_2^B \\
E_6^B &= \{(x, y) \in [0, 1] \times [0, 1] : \quad \frac{\beta_1}{C_B} \leq y \leq 1, \quad 0 \leq x \leq -\frac{(C_B - \beta_2)1}{\beta_1}(1-y) \leq x \leq 1\}
\end{align*}

\subsection{3.3 Testing \(u\)– and \(v\)– Stability for the Multiterminal Game}

In this section we prove that for function \(w_A\) defined by (3.23),(3.24) the necessary and sufficient conditions for being the value of the multiterminal game, i.e. the boundary condition (3.6) and the differential inequalities (3.11),(3.12), are fulfilled.
Proof.

The boundary condition is obviously fulfilled because functions \( \psi^i_A, i = 1, ..., 4 \) are the lower envelopes of the terminal solution \( w_i(T, t, x, y) \) and, hence,

\[
\psi^i_A(x, y) \leq \phi_i(t, t, x, y) \leq g_A(x, y), \quad (i = 1, ..., 4, \quad (x, y) \in [0, 1] \times [0, 1])
\]

Let us verify now that differential inequalities (3.11),(3.12) are valid for the function \( w_A \). It is not difficult to prove that functions \( \psi^i_A, i = 1, 2, 4, (3.19),(3.20),(3.22) \) satisfy the Hamilton-Jacobi equation (3.16) at internal points of domains \( E^i_A, i = 1, 2, 4 \). One can verify that function \( \psi^3_A (3.21) \) coincides with the boundary function \( g_A \) and satisfies the inequality

\[
-\frac{\partial \psi^3_A}{\partial x} x - \frac{\partial \psi^3_A}{\partial y} y + \max\{0, \frac{\partial \psi^3_A}{\partial x}\} + \min\{0, \frac{\partial \psi^3_A}{\partial y}\} \geq 0
\]

at internal points of the domain \( E^3_A \).

It remains to check differential inequalities (3.11),(3.12) on lines \( K^j_A, j = 1, ..., 5 \). Let us do it on lines \( K^2_A, K^3_A \) (see Fig.2). At points of the line \( K^2_A \) functions \( \psi^1_A \) and \( \psi^4_A \) are continuously pasted together. Let us calculate partial derivatives of these functions

\[
\frac{\partial \psi^1_A}{\partial x} = \frac{\alpha^1 x^2 - \alpha^2 y^2}{4C_A x^2 y}, \quad \frac{\partial \psi^1_A}{\partial y} = \frac{\alpha^1 x^2 - \alpha^2 y^2}{4C_A x y^2}, \\
\frac{\partial \psi^4_A}{\partial x} = 0, \quad \frac{\partial \psi^4_A}{\partial y} = 0
\]

As is easily seen these derivatives are equal to zero on line \( K^3_A \)

\[
\frac{\partial \psi^1_A}{\partial x} = 0, \quad \frac{\partial \psi^1_A}{\partial y} = \frac{\partial \psi^4_A}{\partial x} = \frac{\partial \psi^4_A}{\partial y} = 0
\]

In other words functions \( \psi^1_A \) and \( \psi^4_A \) are pasted together smoothly and continuously. Hence, relations (3.11),(3.12) on \( K^3_A \) are fulfilled as equality (3.16) (see Proposition (3.7)). Analogously one can prove smooth pasting of functions \( \psi^3_A, \psi^3_A \) which leads to (3.11),(3.12) in form (3.16) on line \( K^3_A \).

Consider now line \( K^4_A \) (see Fig.2) where functions \( \psi^1_A \) and \( \psi^3_A \) are pasted together. It can be verified that pasting is smooth. For partial derivatives calculated on line \( K^4_A \) we have

\[
\frac{\partial \psi^1_A}{\partial x} = \frac{\partial \psi^3_A}{\partial x} = \frac{\alpha_1 (\alpha_2 - C_A x)}{2C_A x - \alpha_2}, \\
\frac{\partial \psi^1_A}{\partial y} = \frac{\partial \psi^3_A}{\partial y} = C_A x - \alpha_2
\]

Smooth pasting means that function \( w_A \) satisfies Hamilton-Jacobi equation on line \( K^4_A \). One can similarly verify smoothness of function \( w_A \) on line \( K^5_A \).

Along the last line \( K^1_A \) (see Fig.2) functions \( \psi^1_A \) and \( \psi^4_A \) are pasted together. Their derivatives on \( K^1_A \) are determined by

\[
\frac{\partial \psi^3_A}{\partial x} = C_A y - \alpha_1, \quad \frac{\partial \psi^4_A}{\partial x} = 0 \\
\frac{\partial \psi^3_A}{\partial y} = \frac{\partial \psi^4_A}{\partial y} = 0
\]
Figure 2: The structure of the value function $w_A$ in the multiterminal problem.
Note that since \( w_A = \varphi_A^3 = \varphi_A^4 = g_A \) on line \( K_A^1 \), we need to verify only condition (3.11).
As it is easily seen, in a small neighborhood of line \( K_A^1 \) we have
\[
w_A(x, y) = \min\{\varphi_A^3(x, y), \varphi_A^4(x, y)\}
\]
Hence for points \( (x, y) \in K_A^1 \) we obtain (see (3.10),(3.14))
\[
\partial w_A(x, y)\big|[(h_1, h_2)] = \min\{0, C_{Ay} - \alpha_1\} h_1
\]
\[
D_x w_A(x, y)\big|[(s_1, s_2)] = \begin{cases} 0 & \text{if } s_1 = \lambda(C_{Ay} - \alpha_1) \text{ and } s_2 = 0 \\ -\infty & \text{otherwise} \end{cases}
\] (3.32)
Here \( 0 \leq \lambda \leq 1 \). For points \( (x, y) \in K_A^1 \) and vectors \( s = (s_1, s_2) \), \( s_1 = \lambda(C_{Ay} - \alpha_1) \), \( s_2 = 0 \) the Hamiltonian \( H(x, y, s) \) is determined by
\[
H(x, y, s) = -s_1 x + \max\{0, s_1\} = \begin{cases} -s_1 x & \text{if } s_1 \leq 0 \\ s_1(1 - x) & \text{otherwise} \end{cases}
\]
It is clear that for these values the Hamiltonian is larger or equal than the lower conjugate derivative (3.32). Hence, inequality (3.11) on line \( K_A^1 \) is proved.

We have verified conditions (3.11),(3.12) for the function \( w_A \) at all points of the square \([0, 1] \times [0, 1] \). Thus, we have proved that the function \( w_A \) (3.23),(3.24) is the value function in the game with the multiterminal functional.

4 Flexible “Positive” Feedback Controls Generated by Value Functions of Multiterminal Games

4.1 Description of Optimal Feedback Controls

Let us give the description of the flexible “positive” feedback control \( u_A^0 = u_A^{fl} = u_A^{fl}(x, y) \) which solves the problem of guaranteeing maximization for the multiterminal functional \( G_A(x_1(\cdot), y_1(\cdot)) \) (3.1) on the trajectories \((x_1(\cdot), y_1(\cdot))\) of the system (1.1). This maximizing feedback is constructed via “extremal shift” principle in the direction of the gradient (generalized gradient) of the value function \( w_A \) and takes into account its foreseeing nature.

It is not difficult to verify that partial derivative \( \partial w_A/\partial x \) of the value function \( w_A \) changes its sign on lines \( K_A^2 \) and \( K_A^3 \) (see Fig.2). Therefore, the optimal feedback control \( u_A^0 \) has the following structure (see, for example, [Krasovskii, Subbotin, 1988]). The control parameter \( u_A^{fl} = u_A^{fl}(x, y) \) is equal to zero if the current position \((x, y) = (x_1(t), y_1(t))\) lies on the right from the line \( K_A = K_A^2 \cup K_A^3 \), it is equal to unit if the current position lies on the left from this line and it can take arbitrary values at points of line \( K_A^1 \). Namely, if \( C_A > 0 \) then
\[
u_A^0 = u_A^{fl} = u_A^{fl}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in D_A^1 \\ 1 & \text{if } (x, y) \in D_A^2 \\ \in [0, 1] & \text{if } (x, y) \in K_A \end{cases}
\] (4.1)
\[
D_A^1 = D_A^{11} \cup D_A^{12}
\]
\[
D_A^{11} = \{(x, y) \in [0, 1] \times [0, 1] : y < \frac{\alpha_1}{C_A} x, y \geq \frac{\alpha_1}{C_A}\}
\]
\[
D_A^{12} = \{(x, y) \in [0, 1] \times [0, 1] : y < -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)}(1 - x) + 1, y \leq \frac{\alpha_1}{C_A}\}
\]
\[D_A^2 = D_A^{21} \cup D_A^{22}\]
\[D_A^{21} = \{(x, y) \in [0, 1] \times [0, 1]: \ y > \frac{\alpha_1}{\alpha_2} x, \ y \geq \frac{\alpha_1}{C_A}\}\]
\[D_A^{22} = \{(x, y) \in [0, 1] \times [0, 1]: \ y > -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)} (1 - x) + 1, \ y \leq \frac{\alpha_1}{C_A}\}\]
\[K_A = K_A^2 \cup K_A^3\]
\[K_A^2 = \{(x, y) \in [0, 1] \times [0, 1]: \ y = \frac{\alpha_1}{\alpha_2} x, \ y \geq \frac{\alpha_1}{C_A}\}\]
\[K_A^3 = \{(x, y) \in [0, 1] \times [0, 1]: \ y = -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)} (1 - x) + 1, \ y \leq \frac{\alpha_1}{C_A}\}\] (4.2)

On Fig.3 the switch line \(K_A\) and domains \(D_A^1, D_A^2\) are depicted. Directions of velocity \(\dot{x}\) in domains \(D_A^1, D_A^2\) are shown by arrows.

If \(C_A < 0\) then the flexible “positive” feedback \(u_A^{fl}\) has the structure (4.1) where the switching line \(K_A\) and the domains \(D_A^1, D_A^2\) are determined by

\[D_A^1 = D_A^{11} \cup D_A^{12}\]
\[D_A^{11} = \{(x, y) \in [0, 1] \times [0, 1]: \ y > \frac{\alpha_1}{C_A} (1 - x), \ y \geq \frac{\alpha_1}{C_A}\}\]
\[D_A^{12} = \{(x, y) \in [0, 1] \times [0, 1]: \ y < -\frac{(C_A - \alpha_1)}{\alpha_2} x + 1, \ y \leq \frac{\alpha_1}{C_A}\}\]
\[D_A^2 = D_A^{21} \cup D_A^{22}\]
\[D_A^{21} = \{(x, y) \in [0, 1] \times [0, 1]: \ y < \frac{\alpha_1}{C_A} (1 - x), \ y \geq \frac{\alpha_1}{C_A}\}\]
\[D_A^{22} = \{(x, y) \in [0, 1] \times [0, 1]: \ y < -\frac{(C_A - \alpha_1)}{\alpha_2} x + 1, \ y \leq \frac{\alpha_1}{C_A}\}\]
\[K_A = K_A^2 \cup K_A^3\]
\[K_A^2 = \{(x, y) \in [0, 1] \times [0, 1]: \ y = \frac{\alpha_1}{C_A} (1 - x), \ y \geq \frac{\alpha_1}{C_A}\}\]
\[K_A^3 = \{(x, y) \in [0, 1] \times [0, 1]: \ y = -\frac{(C_A - \alpha_1)}{\alpha_2} x + 1, \ y \leq \frac{\alpha_1}{C_A}\}\] (4.3)

The problem of guaranteeing optimization of the multiterminal functional \(G_B(x_2(\cdot), y_2(\cdot))\) for the second coalition is solved analogously. The flexible “positive” feedback control \(v_B^{0} = v_B^{fl} = v_B^{fl}(x, y)\) has the structure similar to (4.1). Namely, if \(C_B > 0\) then the optimal feedback \(v_B^{fl}\) is described by

\[v_B^{0} = v_B^{fl} = v_B^{fl}(x, y) \begin{cases} 0 & \text{if } (x, y) \in D_B^1 \\ 1 & \text{if } (x, y) \in D_B^2 \\ \in [0, 1] & \text{if } (x, y) \in K_B \end{cases} \] (4.4)

\[D_B^1 = D_B^{11} \cup D_B^{12}\]
\[D_B^{11} = \{(x, y) \in [0, 1] \times [0, 1]: \ y > \frac{\beta_1}{\beta_2} x, \ x \geq \frac{\beta_1}{C_B}\}\]
\[D_B^{12} = \{(x, y) \in [0, 1] \times [0, 1]: \ y > -\frac{(C_B - \beta_1)}{(C_B - \beta_2)} (1 - x) + 1, \ x \leq \frac{\beta_2}{C_B}\}\]
\[D_B^2 = D_B^{21} \cup D_B^{22}\]
\[D_B^{21} = \{(x, y) \in [0, 1] \times [0, 1]: \ y < \frac{\beta_1}{\beta_2} x, \ x \geq \frac{\beta_2}{C_B}\}\]
Figure 3: The synthesis of the flexible “positive” feedback control $u^f_A$. 
\[
D_{B}^{11} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y > -\frac{(C_{B} - \beta_{1})}{\beta_{2}} (1 - x) + 1, \quad x \leq \frac{\beta_{2}}{C_{B}} \} \\
K_{B} = K_{B}^{2} \cup K_{B}^{3} \\
K_{B}^{2} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y = \frac{\beta_{1}}{\beta_{2}} x, \quad x \geq \frac{\beta_{2}}{C_{B}} \} \\
K_{B}^{3} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y = -\frac{(C_{B} - \beta_{1})}{(C_{B} - \beta_{2})} (1 - x) + 1, \quad x \leq \frac{\beta_{2}}{C_{B}} \} \tag{4.5} \\
\]

If \( C_{B} < 0 \) then the optimal feedback \( v_{B}^{fl} \) is determined by (4.4) where the switching line \( K_{B} \) and the domains \( D_{B}^{1}, D_{B}^{2} \) are given by

\[
D_{B}^{1} = D_{B}^{11} \cup D_{B}^{12} \\
D_{B}^{11} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y > -\frac{(C_{B} - \beta_{1})}{\beta_{2}} x + 1, \quad x \geq \frac{\beta_{2}}{C_{B}} \} \\
D_{B}^{12} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y > \frac{\beta_{1}}{C_{B} - \beta_{2}} (1 - x), \quad x \leq \frac{\beta_{2}}{C_{B}} \} \\
D_{B}^{2} = D_{B}^{21} \cup D_{B}^{22} \\
D_{B}^{21} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y < \frac{(C_{B} - \beta_{1})}{\beta_{2}} x + 1, \quad x \geq \frac{\beta_{2}}{C_{B}} \} \\
D_{B}^{22} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y < \frac{\beta_{1}}{C_{B} - \beta_{2}} (1 - x), \quad x \leq \frac{\beta_{2}}{C_{B}} \} \\
K_{B} = K_{B}^{2} \cup K_{B}^{3} \\
K_{B}^{2} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y = -\frac{(C_{B} - \beta_{1})}{\beta_{2}} x + 1, \quad x \geq \frac{\beta_{2}}{C_{B}} \} \\
K_{B}^{3} = \{ (x, y) \in [0, 1] \times [0, 1] : \quad y = \frac{\beta_{1}}{C_{B} - \beta_{2}} (1 - x), \quad x \leq \frac{\beta_{2}}{C_{B}} \} \tag{4.6} \\
\]

\section{4.2 The Value Provided by Optimal Feedbacks for Multiterminal Payoffs}

The optimal feedback \( u_{A}^{fl}(x, y) \) (4.1) guarantees that the current payoff to the first coalition becomes in the long term no worse than the value \( v_{A} = D_{A}/C_{A}, C_{A} > 0 \) of the zero-sum matrix game for the matrix \( A \) (call it the \( A \)-matrix game). Note that this result is not obviously worse than the value of the function \( w_{A} \)

\[ v_{A} \geq w_{A}(x, y), \quad \forall (x, y) \in [0, 1] \times [0, 1] \]

More precisely, the following statement is true.

\begin{proposition}
For any initial position \( (x_{0}, y_{0}) \in [0, 1] \times [0, 1] \) and any trajectory

\[ (x_{1}(\cdot), y_{1}(\cdot)) \in X(x_{0}, y_{0}, u_{A}^{fl}), \quad x_{1}(t_{0}) = x_{0}, \quad y_{1}(t_{0}) = y_{0}, \quad t_{0} = 0 \]

generated by the optimal feedback control \( u_{A}^{fl} = u_{A}^{fl}(x, y) \) there exists a finite time \( t_{s} \in [0, T_{A}] \)

\[ T_{A} = \ln(\max\left\{ \frac{C_{A}}{\alpha_{2}}, \frac{C_{A}}{C_{A} - \alpha_{2}} \right\}) \]

such that at this time the trajectory \( (x_{1}(\cdot), y_{1}(\cdot)) \) enters into the domain \( E_{A}^{4} \) (see (3.24) or (3.27))

\[ (x_{1}(t_{s}), y_{1}(t_{s})) \in E_{A}^{4} \]

\end{proposition}
where the value function $w_A$ is equal to the the value $v_A$ of the $A$-matrix game

$$w_A(x_1(t_s), y_1(t_s)) = v_A,$$

and remains in the domain $E^4_A$ on the time interval $[t_s, +\infty)$ (and, hence, on the time interval $[T_A, +\infty)$) (see (3.23),(3.24) and Remark (3.2)). Therefore, according to the definition (3.2) of the value function $w_A$ the following inequality takes place

$$g_A(x_1(t), y_1(t)) \geq v_A, \quad (t \geq t_s) \quad (4.7)$$

and, in particular,

$$\liminf_{t \to +\infty} g_A(x_1(t), y_1(t)) \geq v_A \quad (4.8)$$

**Proof.** Note that the domain $E^4_A$ (see (3.24) or (3.27) has nonempty intersections with all lines $L_\lambda, 0 \leq \lambda \leq 1$

$$L_\lambda = \{(x, y) \in (0, 1) \times (0, 1) : \ y = \lambda\}$$

Hence, any possible trajectory $(x_1(\cdot), y_1(\cdot))$ generated by the optimal feedback control $u^{fl}_A$ (the values of which are equal to zero or to unit) crosses this domain $E^4_A$ because the projection of velocity on lines $L_\lambda$ for such trajectories is not equal to zero and conserves the sign till the moment of intersection of the trajectory with the domain $E^4_A$.

The analogous statement can be formulated for $B$-matrix game.

**Proposition 4.2** For any initial position $(x_0, y_0) \in [0, 1] \times [0, 1]$ and any trajectory

$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, v^{fl}_B), \quad x_2(t_0) = x_0, \quad y_2(t_0) = y_0, \quad t_0 = 0$$

generated by the optimal feedback control $v^{fl}_B = v^{fl}_B(x, y) (4.4)$ there exists a finite time $t_s \in [0, T_B]$

$$T_B = \ln(\max\{\frac{C_B}{\beta_1}, \frac{C_B}{C_B - \beta_1}\})$$

such that at this time the trajectory $(x_2(\cdot), y_2(\cdot))$ enters into the domain $E^4_B$ (see (3.29) or (3.31))

$$(x_2(t_s), y_2(t_s)) \in E^4_B$$

where the value function $w_B$ is equal to the the value $v_B$ of the $B$-matrix game

$$w_B(x_2(t_s), y_2(t_s)) = v_B,$$

and remains in the domain $E^4_B$ on the time interval $[t_s, +\infty)$ (and, hence, on the time interval $[T_B, +\infty)$). Therefore, according to the definition of the value function $w_B$ the following inequality takes place

$$g_B(x_2(t), y_2(t)) \geq v_B, \quad (t \geq t_s) \quad (4.9)$$

and, in particular,

$$\liminf_{t \to +\infty} g_B(x_2(t), y_2(t)) \geq v_B \quad (4.10)$$

Propositions 4.1, 4.2 imply the following statement.
Proposition 4.3 The intersection $E^0$ of sets $E_A^4$ and $E_B^4$ is not empty

$$E^0 = E_A^4 \cap E_B^4 \neq \emptyset$$  \hspace{1cm} (4.11)

and, hence, optimal strategies $u_A^{fl}, v_B^{fl}$ generate the trajectory $(x^{fl}(\cdot), y^{fl}(\cdot))$ which enters the intersection $E^0$ and remains in it on the time interval $[T^0, +\infty)$, $T^0 = \max\{T_A, T_B\}$. In the set $E^0$ the inequalities

$$g_A(x^{fl}(t), y^{fl}(t)) \geq v_A, \quad g_B(x^{fl}(t), y^{fl}(t)) \geq v_B, \quad t \in [T^0, +\infty)$$  \hspace{1cm} (4.12)

are fulfilled. Therefore, the set $E^0$ can be called the favorable domain for both coalitions.

On Fig.4 domains $E_A^4$, $E_B^4$, $E^0$ are shown for the game with the following payoff matrices

$$A = \begin{pmatrix} 5 & 3 \\ 2 & 5 \end{pmatrix}$$  \hspace{1cm} (4.13)

$$C_A = a_{11} - a_{12} - a_{21} + a_{22} = 5$$
$$a_1 = a_{22} - a_{12} = 2$$
$$a_2 = a_{22} - a_{21} = 3$$
$$x_A = \frac{a_2}{C_A} = 0.6, \quad y_A = \frac{a_1}{C_A} = 0.4$$

$$B = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$$  \hspace{1cm} (4.14)

$$C_B = b_{11} - b_{12} - b_{21} + b_{22} = -5$$
$$\beta_1 = b_{22} - b_{12} = -4$$
$$\beta_2 = b_{22} - b_{21} = -2$$
$$x_B = \frac{\beta_2}{C_B} = 0.4, \quad y_B = \frac{\beta_1}{C_B} = 0.8$$

Directions of velocity $\dot{x}$ for the first coalition are depicted by horizontal arrows and directions of velocity $\dot{y}$ for the second one - by vertical arrows.

Note that matrices $A$ (4.13) and $B$ (4.14) correspond to the game of firms interacting on two markets which was described in Section 1.

5 A Nash Equilibrium with Flexible “Positive” Feedbacks of Multiterminal Games

5.1 The Structure of a Nash Equilibrium

Let us construct now a Nash equilibrium pair of feedbacks by pasting together flexible “positive” feedbacks $u_A^{fl}, v_B^{fl}$, $v_A^{fl} = v_B^{fl}$ (4.1), (4.4) and “punishment” feedbacks $u_B^0 = u_B^{fl}, v_B^0 = v_B^{fl}$ (1.14), (1.15). Let us choose an initial position $(x_0, y_0) \in [0, 1] \times [0, 1]$ and an accuracy parameter $\varepsilon > 0$. Choose a trajectory $(x^{fl}(\cdot), y^{fl}(\cdot)) \in X(x_0, y_0, u_A^{fl}(\cdot), v_B^{fl}(\cdot))$ generated by flexible “positive” feedbacks $u_A^{fl}$ and $v_B^{fl}$. Let $T_\varepsilon > 0$ be such that

$$g_A(x^{fl}(t), y^{fl}(t)) > J_A^-(x^{fl}(\cdot), y^{fl}(\cdot)) - \varepsilon$$
$$g_B(x^{fl}(t), y^{fl}(t)) > J_B^-(x^{fl}(\cdot), y^{fl}(\cdot)) - \varepsilon$$
$$t \in [T_\varepsilon, +\infty)$$
Figure 4: The portrait of guaranteeing feedbacks $u_A^{fl}$, $v_B^{fl}$ and the Nash equilibrium trajectory $TR = (x^{fl}(\cdot), y^{fl}(\cdot))$. 
Denote by $u^{f^1}_A(t): [0, T_\varepsilon) \to [0, 1]$, $v^{f^1}_B(t): [0, T_\varepsilon) \to [0, 1]$ step-by-step realizations of strategies $u^{f^1}_A$, $v^{f^1}_B$ such that the corresponding step-by-step motion $(x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ satisfies the condition
\[
\max_{t \in [0, T_\varepsilon]} \|(x^{f^1}_\varepsilon(t), y^{f^1}_\varepsilon(t)) - (x^{f^1}_\varepsilon(t), y^{f^1}_\varepsilon(t))\| < \varepsilon
\]

**Proposition 5.1** A pair of feedbacks $U^0 = u^0(t, x, y, \varepsilon), V^0 = v^0(t, x, y, \varepsilon)$ pasting together flexible “positive” feedbacks $u^{f^1}_A$, $v^{f^1}_B$ (4.1), (4.4) and “punishment” feedbacks $u^{f^1}_B$, $v^{f^1}_A$ (1.14), (1.15) in accordance with Proposition 1.2
\[
\begin{align*}
U^0 &= u^0(t, x, y, \varepsilon) = \begin{cases} 
    u^{f^1}_A(t) & \text{if } \|(x, y) - (x^{f^1}_\varepsilon(t), y^{f^1}_\varepsilon(t))\| < \varepsilon \\
    u^{f^1}_B(x, y) & \text{otherwise}
\end{cases} \\
V^0 &= v^0(t, x, y, \varepsilon) = \begin{cases} 
    v^{f^1}_A(t) & \text{if } \|(x, y) - (x^{f^1}_\varepsilon(t), y^{f^1}_\varepsilon(t))\| < \varepsilon \\
    v^{f^1}_B(x, y) & \text{otherwise}
\end{cases}
\end{align*}
\]
is a dynamical Nash $\varepsilon$-equilibrium.

Let us remind that the trajectory $(x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ constitutes the core of the dynamical Nash equilibrium. Therefore, it can be called the equilibrium trajectory. It is generated by guaranteeing feedbacks $u^{f^1}_A$ and $v^{f^1}_B$ and so provides better index values than at a static Nash equilibrium with the guarantee. This trajectory is surrounded by the risk $\varepsilon$-barrier which gives the possibility for trusting and venturing.

### 5.2 Trajectories Generated by Flexible “Positive” Feedbacks

The qualitative behavior of trajectories generated by flexible “positive” feedbacks which form the basis of the dynamical Nash equilibrium (5.1), (5.2) is the question of great interest. The complete classification of possible limit points, attractors, cycles or chaotic circulation could form the theme of the future analysis. Preliminary, we can propose the following statement.

**Proposition 5.2** The values of the payoff functionals $J_A$, $J_B$ on an arbitrary trajectory $(x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ generated by flexible “positive” feedback controls $u^{f^1}_A$, $v^{f^1}_B$ (4.1), (4.4) are no worse than the values of these functionals on any trajectory converging to the static Nash equilibrium $(x_B, y_A) = (\beta_2/C_B, \alpha_1/C_A)$ whose components have the sense of coalitions’ distributions mostly unfavorable for the opposite coalition. According to Propositions 4.1-4.3 trajectories $(x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ enter the favorable domain $E^0$ and stay in it on the infinite time interval. There are the following possible qualitative behaviors of the trajectory $(x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ in the favorable domain $E^0$: - it can converge to an intersection point of switching lines $K_A$, $K_B$; - it can tend to points situated on the bound of the square (in the case when the intersection of lines $K_A$, $K_B$ is empty); - it can tend to nonantagonistic static Nash equilibria (in the case when such equilibria exist); - it can simply circulate in the favorable domain $E^0$.

For example, on Fig.4 one of the cases mentioned in Proposition 5.2 is shown for the game with the payoff matrices $A$ (4.13) and $B$ (4.14). The trajectory $TR = (x^{f^1}_{\varepsilon}(\cdot), y^{f^1}_{\varepsilon}(\cdot))$ generated by flexible “positive” feedbacks $u^{f^1}_A$, $v^{f^1}_B$ starts from the initial position $IP = (0.7, 0.1)$ and moves at first along the straight line to the corner $(0, 1)$ of the unit square $[0, 1] \times [0, 1]$. Then it contacts the switching line $K_A$. Next it moves along the line $K_A$ and further to the corner $(1, 1)$ till the meeting with the switching line $K_B$. After that
it evolves along the line $K_B$ on the border of the domain $E^0$ altering velocity directions from the corner $(1,1)$ to the corner $(1,0)$ till arriving to the point $I S = K_A \cap K_B$ of intersection of the switching lines $K_A, K_B$. Note that here we give description of the trajectories behavior in the domain $E^0$ up to the value of the accuracy parameter $\varepsilon$.

**Remark 5.1** Note that the “punishment” feedbacks $u^B_{cl}, v^A_{cl}$ \(1.14),\(1.15) being components of the dynamical Nash equilibrium \((5.1),(5.2) lead trajectories to the unfavorable static Nash equilibrium \((x_B,y_A)\).

**Remark 5.2** Using constructions of flexible “positive” feedbacks we obtain the unexpected striking result: the equilibrium trajectory $TR = (x^R(\cdot),y^R(\cdot))$ provides better (in the given example strictly better) index values for both coalitions than index values at static Nash equilibrium \((x_B,y_A)\). Hence, the equilibrium trajectory has better properties than trajectories of classical models with replicator and best response dynamics which converge or circulate around static Nash equilibrium \((x_B,y_A)\).

6 Quasioptimal Feedbacks in Games with Integral Payoff Functionals

6.1 Nash Equilibria in the Nonzero-Sum Differential Game with the Averaged Integral Functionals

In this section we consider the nonzero-sum differential game of two coalitions with the averaged integral payoff functionals

\[
JI^\infty_A = [JI^_,A, JI^+,A] \tag{6.1}
\]

\[
JI^_A = JI^+_A(x(\cdot),y(\cdot)) = \liminf_{T \to \infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x(t), y(t)) dt
\]

\[
JI^+_A = JI^+_A(x(\cdot),y(\cdot)) = \limsup_{T \to \infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x(t), y(t)) dt
\]

\[
JI^\infty_B = [JI^-,B, JI^+,B] \tag{6.2}
\]

\[
JI^-_B = JI^-_B(x(\cdot),y(\cdot)) = \liminf_{T \to \infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_B(x(t), y(t)) dt
\]

\[
JI^+_B = JI^+_B(x(\cdot),y(\cdot)) = \limsup_{T \to \infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_B(x(t), y(t)) dt
\]

determined on the trajectories $(x(\cdot), y(\cdot))$ of the system (1.1).

The averaged integral functionals (6.1), (6.2) are traditional for problems of evolutionary economics and biological evolution. Unlike the multiterminal payoffs (3.1) they allow to loose on some intervals in order to win on others and to obtain the better summarized result. This property provides another character of switching curves in guaranteeing feedbacks which give the opportunity for longer staying in profitable domains where $g_A(x,y) > v_A, g_B(x,y) > v_B$.

We introduce the notion of dynamical Nash equilibrium for the evolutionary game with the integral functionals (6.1), (6.2) in the same context as for the game with functionals (1.6), (1.7)
Definition 6.1 A dynamical Nash equilibria \((U^0, V^0)\), \(U^0 = u^0(t, x, y, \varepsilon), V^0 = v^0(t, x, y, \varepsilon)\) in the class of feedback controls \(U = u(t, x, y, \varepsilon), V = v(t, x, y, \varepsilon)\) for this problem are defined analogously to Definition 1.1 by inequalities

\[
J_{I_A}(x^0(\cdot), y^0(\cdot)) \geq J_{I_A}(x_1(\cdot), y_1(\cdot)) - \varepsilon \quad (6.3)
\]
\[
J_{I_B}(x^0(\cdot), y^0(\cdot)) \geq J_{I_B}(x_2(\cdot), y_2(\cdot)) - \varepsilon \quad (6.4)
\]

\[
(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0), \quad (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0)
\]
\[
(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V)
\]

Proposition 6.1 A dynamical Nash equilibrium \((U^0, V^0)\) can be constructed according to Proposition 1.2. Namely, in Sections 6.2, 6.3 we find solutions of auxiliary two-step optimal control problems and obtain switching curves \(M_A (6.20),(6.21), M_B (6.23),(6.24)\) for “positive” dual-step feedbacks \(u_A^0 = u_A^{ds}(x, y), v_B^0 = v_B^{ds}(x, y)\) maximizing integral payoff functionals \((6.1),(6.2)\). Then we paste these “positive” feedbacks \(u_A^0, v_B^0\) with “punishment” feedbacks \(u_A^0, v_A^0 (1.14),(1.15)\) according to the scheme \((1.17),(1.18)\) and obtain a Nash equilibrium \((U^0, V^0)\).

6.2 Two-step Optimal Control Problems

In order to construct “positive” feedbacks \(u_A^0 = u_A^{ds}(x, y), v_B^0 = v_B^{ds}(x, y)\) we consider in this section the auxiliary two-step optimal control problem with the integral payoff functional for the first coalition when actions of the second coalition are the most unfavorable. More precisely, we analyze the optimal control problem for the dynamical system (1.1)

\[
\begin{align*}
\dot{x} &= -x + u, \quad x(0) = x_0 \\
\dot{y} &= -y + v, \quad y(0) = y_0
\end{align*}
\]

with the payoff functional

\[
J_A^f = \int_0^{T_f} g_A(x(t), y(t)) dt
\]

We determine the terminal instant \(T_f = T_f(x_0, y_0)\) later.

Let us assume for definiteness that the following conditions hold

\[
C_A > 0, \quad v_A = \frac{D_A}{C_A} = 0, \quad 0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < y_A = \frac{\alpha_1}{C_A} < 1
\]

We consider the case when initial conditions \((x_0, y_0)\) for the system (1.1) satisfy relations

\[
x_0 = x_A, \quad y_0 > y_A
\]

(6.7)

Let us suppose that actions of the second coalition are the most unfavorable for the first coalition. For the trajectories of the system (1.1) starting from the initial positions \((x_0, y_0)\) (6.7) these actions \(v_A^0 = v_A^{cl}(x, y)\) are rather simple

\[
v_A^{cl}(x, y) \equiv 0
\]

(6.8)

The optimal actions \(u_A^0 = u_A^{ds}(x, y)\) of the first coalition with respect to the payoff functional \(J_A^f\) in this situation can be represented as the two-step bang-bang control: it is
equal to unit from the initial instant \( t_0 = 0 \) till the optimal switch instant \( s \) and then it is equal to zero till the final instant \( T_f \)
\[
 u_A^0(t) = u_A^d(x(t), y(t)) = \begin{cases} 
 1 & \text{if } t_0 \leq t < s \\
 0 & \text{if } s \leq t < T_f 
\end{cases}
\] (6.9)

The parameter \( s \) here is the parameter of optimization. The final instant \( T_f \) is determined by the condition that a trajectory \((x(\cdot), y(\cdot))\) of the system (1.1) starting from the line where \( x(t_0) = x_A \) returns to this line \( x(T_f) = x_A \).

Thus, we consider two aggregates of characteristics. The first one is described by the system of differential equations
\[
\dot{x} = -x + 1 \\
\dot{y} = -y 
\] (6.10)
solutions of which are determined by the Cauchy formula
\[
x(t) = (x_0 - 1)e^{-t} + 1 \\
y(t) = y_0e^{-t} 
\] (6.11)
where initial positions \((x_0, y_0)\) satisfy conditions (6.7) and time parameter \( t \) satisfies inequality \( 0 \leq t < s \).

The second aggregate of characteristics is given by the system of differential equations
\[
\dot{x} = -x \\
\dot{y} = -y 
\] (6.12)
solutions of which are determined by the Cauchy formula
\[
x(t) = x_1e^{-t} \\
y(t) = y_1e^{-t} 
\] (6.13)
where initial positions \((x_1, y_1) = (x_1(s), y_1(s))\) are determined by relations
\[
x_1 = x_1(s) = (x_0 - 1)e^{-s} + 1 \\
y_1 = y_1(s) = y_0e^{-s} 
\] (6.14)
and time parameter \( t \) satisfies inequality \( 0 \leq t < p \). Here the final moment of time \( p = p(s) \) and the final position \((x_2, y_2) = (x_2(s), y_2(s))\) of the whole trajectory \((x(\cdot), y(\cdot))\) are given by formulas
\[
x_1e^{-p} = x_A, \quad p = p(s) = \ln \frac{x_1(s)}{x_A} \\
x_2 = x_A \\
y_2 = y_1e^{-p} 
\] (6.15)

The optimal control problem consists in finding such moment of time \( s \) and corresponding switching point \((x_1, y_1) = (x_1(s), y_1(s))\) on the trajectory \((x(\cdot), y(\cdot))\) that the integral \( I = I(s) \) attains the maximum value
\[
I(s) = I_1(s) + I_2(s) 
\] (6.16)
\[
I_1(s) = \int_0^s (CA((x_0 - 1)e^{-t} + 1)y_0e^{-t} - \alpha_1((x_0 - 1)e^{-t} + 1) - \alpha_2y_0e^{-t} + a_{22})dt 
\] (6.17)
\[
I_2(s) = \int_0^{p(s)} (C_Ax_1(s)y_1(s)e^{-2t} - \alpha_1x_1(s)e^{-t} - \alpha_2y_1(s)e^{-t} + a_{22})dt 
\] (6.18)
Figure 5: Two families of characteristics $CH = (x(\cdot), y(\cdot))$, $CH1 = (x_1(\cdot), y_1(\cdot))$ in the dual-step problem.

On Fig.5 initial position $IP = (x_0, y_0) = (0.6, 0.9)$, characteristics $CH = (x(\cdot), y(\cdot))$, $CH1 = (x_1(\cdot), y_1(\cdot))$, switching point $SP = (x_1, y_1)$ and final point $FP = (x_2, y_2)$ are shown.

6.3 The Solution of the Two-step Optimal Control Problem

We obtain the solution of the two-step optimal control problem (6.10)-(6.18) if calculate the derivative $dI/ds$, express it as the function of the optimal switching points $(x, y) = (x_1, y_1)$, equate this derivative to zero $dI/ds = 0$ and find the equation $F(x, y) = 0$ for the curve consisting of optimal switching points $(x, y)$.

Let us calculate at first the integrals $I_1, I_2$

$$
I_1 = I_1(s) = C_A(x_0 - 1)y_0\frac{(1 - e^{-2s})}{2} + C_A y_0 (1 - e^{-s}) - \alpha_1((x_0 - 1)(1 - e^{-s}) + s) - \alpha_2 y_0 (1 - e^{-s}) + a_{22}s
$$
\[ I_2 = I_2(s) = CAx_1(s)y_1(s) \frac{(1 - e^{-2p(s)})}{2} - \alpha_1 x_1(s)(1 - e^{-p(s)}) - \alpha_2 y_1(s)(1 - e^{-p(s)}) + a_{22}p(s) \]

Let us calculate derivatives \( dI_1/ds, dI_2/ds \) and express them as functions of optimal switching points \((x,y) = (x_1, y_1)\)

\[
\frac{dI_1}{ds} = CA(x_0 - 1)y_0e^{-2s} + CAy_0e^{-s} - \\
\alpha_1((x_0 - 1)e^{-s} + 1) - \alpha_2 y_0 e^{-s} + a_{22} = \\
CAxy - \alpha_1 x - \alpha_2 y + a_{22}
\]

\[
\frac{dI_2}{ds} = CA(\frac{dx}{ds} \frac{(1 - e^{-2p})}{2} + \\
x \frac{dy}{ds} \frac{(1 - e^{-2p})}{2} + xye^{-2p} \frac{dp}{ds}) - \\
\alpha_1 \frac{dx}{ds} (1 - e^{-p}) - \alpha_1 xe^{-p} \frac{dp}{ds} - \\
\alpha_2 \frac{dy}{ds} (1 - e^{-p}) - \alpha_2 ye^{-p} \frac{dp}{ds} + a_{22} \frac{dp}{ds} = \\
(C^2A^2x^2y - \frac{\alpha^2_2 y}{2A} - 2C^2Ax^3 y - \\
2\alpha_1 CAx^2 + 2\alpha_1 CAx^3 + 2\alpha_2 CAx^2 y + \\
2CAa_{22} x - 2CAa_{22}x^2)/(2CAx^2)
\]

In the last equation we took into account the following expressions for derivatives \( dx/ds, dy/ds, dp/ds \) and exponentials \( e^{-p}, e^{-2p}, (1 - e^{-p}), (1 - e^{-2p}) \) as functions of variables \((x, y)\)

\[
\frac{dx}{ds} = 1 - x \\
\frac{dy}{ds} = -y \\
\frac{dp}{ds} = \frac{1 - x}{x} \\
e^{-p} = \frac{\alpha_2}{CAx} \\
e^{-2p} = \frac{\alpha_2}{C^2A^2x^2} \\
1 - e^{-p} = \frac{CAx - \alpha_2}{CAx} \\
1 - e^{-2p} = \frac{C^2Ax^2 - \alpha_2}{C^2Ax^2}
\]

Summarizing derivatives \( dI_1/ds, dI_2/ds \), carrying the expression to the common denominator and equating the sum to zero we obtain the following equation for the switching curve

\[
\frac{C^2Ax^2y - 2\alpha_1 CAx^2 - \alpha_2 y + 2CAa_{22} x}{2CAx^2} = 0
\]

Taking into account that \( v_A = 0 \) (see (6.6)) we come to the final expression for the switching curve \( M^1_A \)

\[
y = \frac{2\alpha_1 x}{CAx + \alpha_2} \quad (6.19)
\]
The curve $M_A^1$ is a hyperbola which passes through the points $(0,0), (x_A, y_A)$ and has the horizontal asymptote

$$y = \frac{2\alpha_1}{C_A}$$

**Remark 6.1** The derivative of the curve $M_A^1$ at the point $(x_A, y_A)$ is determined by the formula

$$\frac{dy}{dx} = \left. \frac{2\alpha_1 \alpha_2}{(C_A x + \alpha_2)^2} \right|_{x=x_A} = \frac{\alpha_1}{2\alpha_2}$$

It is less than the derivative $dy/dx = \alpha_1/\alpha_2$ of the switching line $K_A^2$, $y = (\alpha_1/\alpha_2)x$ (see (4.2)) in the game with the multiterminal functional. Hence, the curve $M_A^1$ is situated below the line $K_A^2$. And we can conclude that in the game with the integral functional coalitions have more opportunities for maneuver (optimal feedback controls are more flexible) since individuals can keep for a longer time strategies (not switch them) which coincide with their “local” interests determined by the gradient $(\partial g_A/\partial x, \partial g_A/\partial y)$ of the payoff function $g_A(x,y)$ and so stay longer in preferable domains.

In order to construct the complete switching curve $M_A$ for the optimal strategy of the first coalition in the game with the integral payoff we should add to the curve $M_A^1$ the similar curve $M_A^2$ in the domain where $y \leq y_A$

$$M_A = M_A^1 \cup M_A^2$$

$$M_A^1 = \{(x, y) \in [0, 1] \times [0, 1] : y = \frac{2\alpha_1 x}{C_A x + \alpha_2}, \ y \geq \frac{\alpha_1}{C_A} \}$$

$$M_A^2 = \{(x, y) \in [0, 1] \times [0, 1] :$$

$$y = -\frac{2(C_A - \alpha_1)(1-x)}{C_A(1-x) + (C_A - \alpha_2)} + 1, \ y \leq \frac{\alpha_1}{C_A} \} \quad (6.20)$$

In the case when $C_A < 0$ the curves $M_A^1$ and $M_A^2$ are described by formulas

$$M_A^1 = \{(x, y) \in [0, 1] \times [0, 1] : y = \frac{2\alpha_1(1-x)}{C_A(1-x) + (C_A - \alpha_2)}, \ y \geq \frac{\alpha_1}{C_A} \}$$

$$M_A^2 = \{(x, y) \in [0, 1] \times [0, 1] : y = -\frac{2(C_A - \alpha_1)x}{C_A x + \alpha_2} + 1, \ y \leq \frac{\alpha_1}{C_A} \} \quad (6.21)$$

The curve $M_A$ divides the unit square $[0, 1] \times [0, 1]$ into two parts - the upper part

$$D_A^u \supset \{(x, y) : x = x_A, \ y > y_A \}$$

and the lower part

$$D_A^l \supset \{(x, y) : x = x_A, \ y < y_A \}$$

The “positive” feedback $u_A^{ds}$ has the following structure

$$u_A^{ds} = u_A^{ds}(x,y) = \begin{cases} 
\max\{0, -\text{sgn}(C_A)\} & \text{if } (x,y) \in D_A^u \\
\max\{0, \text{sgn}(C_A)\} & \text{if } (x,y) \in D_A^l \\
\in [0,1] & \text{if } (x,y) \in M_A
\end{cases} \quad (6.22)$$

For the second coalition one can obtain the similar switching curves $M_B$ for the optimal control problem with the integral functional. Precisely, in the case when $C_B > 0$ the
switching curve $M_B$ is given by relations

$$M_B = M^1_B \cup M^2_B$$

$$M^1_B = \{(x, y) \in [0, 1] \times [0, 1] : \ y = \frac{\beta_1 x}{2\beta_2 - C_B x}, \ x \geq \frac{\beta_2}{C_B}\}$$

$$M^2_B = \{(x, y) \in [0, 1] \times [0, 1] : \ y = -\frac{(C_B - \beta_1)(1 - x)}{2(C_B - \beta_2) - C_B(1 - x)} + 1, \ x \leq \frac{\beta_2}{C_B}\} \quad (6.23)$$

In the case when parameter $C_B$ is negative $C_B < 0$ the curves $M^1_B$ and $M^2_B$ are determined by formulas

$$M^1_B = \{(x, y) \in [0, 1] \times [0, 1] : \ y = -\frac{(C_B - \beta_1)x}{2\beta_2 - C_B x}, \ x \geq \frac{\beta_2}{C_B}\}$$

$$M^2_B = \{(x, y) \in [0, 1] \times [0, 1] : \ y = \frac{\beta_1(1 - x)}{2(C_B - \beta_2) - C_B(1 - x)}, \ x \leq \frac{\beta_2}{C_B}\} \quad (6.24)$$

The curve $M_B$ divides the unit square $[0, 1] \times [0, 1]$ into two parts - the left part

$$D^l_B \ni \{(x, y) : \ x < x_B, \ y = y_B\}$$

and the right part

$$D^r_B \ni \{(x, y) : \ x > x_B, \ y = y_B\}$$

The “positive” feedback $v^d_B$ has the following structure

$$v^d_B = v^d_B(x, y) = \begin{cases} 
\max\{0, -\text{sgn}(C_B)\} & \text{if } (x, y) \in D^l_B \\
\max\{0, \text{sgn}(C_B)\} & \text{if } (x, y) \in D^r_B \\
[0, 1] & \text{if } (x, y) \in M_B 
\end{cases} \quad (6.25)$$

On Fig.6 switching curves $M_A$, $M_B$ are shown for matrixes $A$ (4.13), $B$ (4.14). Directions of velocities $\dot{x}$ are depicted by horizontal (left and right) arrows and directions of velocities $\dot{y}$ - by vertical (up and down) arrows.

6.4 Values Guaranteed by Optimal Feedbacks in the Problem with the Integral Payoffs

Let us formulate the statement which affirms that the “positive” optimal feedback control $u^d_A(x, y)$ (6.22) with the switching curve $M_A$ determined by formulas (6.20),(6.21) guarantees that the averaged value of the integral functional is more or equal than the value $v_A = D_A/C_A$ of the static matrix game.

**Proposition 6.2** For any initial position $(x_0, y_0) \in [0, 1] \times [0, 1]$ and for any trajectory

$$(x^d(\cdot), y^d(\cdot)) \in X(x_0, y_0, u^d_A), \ x^d(t_0) = x_0, \ y^d(t_0) = y_0, \ t_0 = 0$$

generated by the optimal feedback control $u^d_A = u^d_A(x, y)$ there exists a finite moment $t_* \in [0, T_A]$ such that at this moment the trajectory $(x^d(\cdot), y^d(\cdot))$ comes to the line where $x = x_A$, i.e. $x^d(t_*) = x_A$. Then according to construction of optimal feedback control $u^d_A$ (it maximizes the integral (6.16)) the following estimate holds

$$\int_{t_*}^{T} g_A(x^d(t), y^d(t))dt \geq v_A(T - t_*), \ \forall T \geq t_* \quad (6.26)$$
Figure 6: Guaranteeing dual-step synthesis $u^d_A$, $v^d_B$ for integral payoffs and the Nash equilibrium trajectory $TR = (x^{ds}(\cdot), y^{ds}(\cdot))$. 
In particular, this inequality remains valid when time $T$ tends to infinity

$$\liminf_{T \to +\infty} \frac{1}{(T - t_s)} \int_{t_s}^{T} g_A(x^{ds}(t), y^{ds}(t)) dt \geq v_A$$

\hfill (6.27)

Inequalities (6.26), (6.27) mean that the averaged value of the integral functional is not worse than the value $v_A$ of the static matrix game.

The analogous result is valid for trajectories generated by optimal control $v_B^{ts}$ (6.25) corresponding to the switching curve $M_B$ (6.23), (6.24).

**Remark 6.2** Consider the acceptable trajectory $(x^{ds}_A(\cdot), y^{ds}_A(\cdot))$ generated by “positive” feedbacks $u_A^{ts}$ (6.22), $v_B^{ts}$ (6.25). Then according to Proposition 6.2 the following inequalities take place

$$\liminf_{T \to +\infty} \frac{1}{(T - t_s)} \int_{t_s}^{T} g_A(x^{ds}_A(t), y^{ds}_A(t)) dt \geq v_A$$

\hfill (6.28)

$$\liminf_{T \to +\infty} \frac{1}{(T - t_s)} \int_{t_s}^{T} g_A(x^{ds}_A(t), y^{ds}_A(t)) dt \geq v_B$$

\hfill (6.29)

and hence the acceptable trajectory $(x^{ds}_A(\cdot), y^{ds}_A(\cdot))$ provides better results for both coalitions than trajectories leading to points of static Nash equilibrium at which corresponding payoffs are equal to values $v_A$ and $v_B$.

On Fig.6 the acceptable trajectory $TR = (x^{ds}_{AB}(\cdot), y^{ds}_{AB}(\cdot))$ is shown for the game with the payoff matrices $A$ (4.13) and $B$ (4.14). It starts from the initial position $IP = (0,8,0,2)$ and moves at first along the straight line to the corner $(0,0)$ of the unit square $[0,1] \times [0,1]$ with control signals $u = 0$, $v = 0$. Then it crosses the switching line $M_B$ and the second coalition switches its control $v$ from 0 to 1. Further the trajectory $TR$ moves to the corner $(0,1)$ till the meeting with the switching line $M_A$. Here the first coalition changes its signal $u$ from 0 to 1. After that the trajectory directs to the corner $(1,1)$. Next it meets the switching line $M_B$ and turns to the corner $(1,0)$. Then it intersects the switching line $M_A$ and again as at the beginning moves to the corner $(0,0)$. In the process of this cycling the trajectory $TR$ converges to the point $IS = M_A \cap M_B$ of intersection of the switching curves $M_A$, $M_B$.

### 7 Quasioptimal Feedback Controls in Games with the Coordinated “Long-Term” and “Short-Term” Interests of Populations and Individuals

#### 7.1 Three-Step Optimal Control Problems

In this section we examine the three-step optimal control problem for the first coalition when control parameter $u$ is coordinated with “short-term” interests of individuals. The coordination means that components $\dot{x}$, $\dot{y}$ of velocity vector of the system (1.1) should have the same sign as components of gradients $\partial g_A/\partial x$ and $\partial g_B/\partial y$.

In Sections 7.1, 7.2 we find solutions for auxiliary three-step optimal control problems. Then we construct switching curves $N_A$ (7.22), (7.23), $L_A$ (7.24), $N_B$ (7.26), (7.27), $L_B$ (7.28) for “positive” triple-step feedbacks $u_A^{0} = u_A^{ts}(x, y)$ (7.25), $v_B^{0} = v_B^{ts}(x, y)$ (7.29) maximizing integral payoff functional (6.1), (6.2) in presence of additional restrictions on control parameters $u$ and $v$. Substituting “positive” feedbacks $u_A^{ts}$ (7.25), $v_B^{ts}$ (7.29) into dynamics (1.1) we obtain the acceptable trajectory $(x^{ts}_{AB}(\cdot), y^{ts}_{AB}(\cdot))$ which forms the basis of a dynamical Nash equilibrium.
Let us turn our attention to the three-step optimal control problem. We assume that there are restrictions $P(x, y)$ on control parameter $u$ which depend on the sign of the partial derivative $\frac{\partial g_A}{\partial x}$ of the payoff function $g_A$ expressing “short-term” interests of individuals

$$P(x, y) = \{ u : \begin{array}{ll}
x \leq u \leq 1 & \text{if } \frac{\partial g_A}{\partial x} = C_A y - \alpha_1 \geq 0, \\
0 \leq u \leq x & \text{if } \frac{\partial g_A}{\partial x} = C_A y - \alpha_1 < 0\end{array} \} \quad (7.1)$$

In other words the following “noncontradictory” relations are valid for restrictions $P(x, y)$ and “short-term” interests $g_A(x, y)$

$$\max_{u \in P(x, y)} \frac{\partial g_A}{\partial x} u \geq 0, \quad (x, y) \in [0, 1] \times [0, 1]$$

Actions $v = v(t)$ of the second coalition satisfy inequalities

$$0 \leq v \leq 1 \quad (7.2)$$

and can be the most unfavorable.

Thus, we analyze the optimal control problem with the payoff functional

$$J^f_A = \int_0^{T_f} g_A(x(t), y(t)) dt \quad (7.3)$$

on trajectories $(x(t), y(t))$ of the dynamical system (1.1) starting from initial position $x(0) = x_0, y(0) = y_0$ and generated by controls $u(\cdot), v(\cdot)$ satisfying restrictions (7.1),(7.2). The terminal instant $T_f = T_f(x_0, y_0)$ in formula (7.3) will be specified below.

We consider the case when initial positions $(x_0, y_0)$ satisfy relations

$$x_0 = x_A, \quad y_0 > y_A \quad (7.4)$$

Let us suppose that actions of the second coalition are the most unfavorable. For trajectories of the system (1.1) starting from initial positions $(x_0, y_0)$ (7.4) these actions $v^0_A = v^d_A(x, y)$ are determined by formula

$$v^d_A(x, y) \equiv 0 \quad (7.5)$$

The optimal actions $u^s_A = u^s_A(x, y)$ of the first coalition which maximize the payoff functional $J^f_A$ are constructed as the three-step bang-bang control

$$u^s_A(t) = u^s_A(x(t), y(t)) = \begin{cases} 
1 & \text{if } t_0 \leq t < s \\
x(s) & \text{if } s \leq t < T_{y_A} \\
0 & \text{if } T_{y_A} \leq t < T_f \end{cases} \quad (7.6)$$

The parameter $s$ here is the parameter of optimization. The switching instant $T_{y_A}$ is the moment when the trajectory $(x(\cdot), y(\cdot))$ crosses the line $y = y_A$, i.e. $y(T_{y_A}) = y_A$, and the final instant $T_f$ is determined by the condition that the trajectory $(x(\cdot), y(\cdot))$ comes back to the line $x = x_A$, i.e. $x(T_f) = x_A$.

So we consider three collections of characteristics. Characteristics of the first collection satisfy the system of differential equations

$$\begin{align*}
\dot{x} &= -x + 1 \\
\dot{y} &= -y \quad (7.7)
\end{align*}$$
and can be represented by the Cauchy formula

\[ x(t) = (x_0 - 1)e^{-t} + 1 \]
\[ y(t) = y_0e^{-t}, \quad 0 \leq t \leq s \]

(7.8)

where initial positions \((x_0, y_0)\) satisfy conditions (7.4) and the first switching instant \(s\) is the parameter of optimization.

The second aggregate of characteristics is determined by the system of differential equations

\[ \dot{x} = 0 \]
\[ \dot{y} = -y \]

(7.9)
solutions of which are determined by the Cauchy formula

\[ x(t) = x \]
\[ y(t) = ye^{-t}, \quad 0 \leq t \leq p(s) \]

(7.10)

Initial positions \((x, y) = (x(s), y(s))\) are determined by relations

\[ x = x(s) = (x_0 - 1)e^{-s} + 1 \]
\[ y = y(s) = y_0e^{-s} \]

(7.11)

The second switching instant \(p = p(s)\) and the position \((x(p), y(p))\) of the characteristic (7.10) are given by formulas

\[ ye^{-p} = y_A = \frac{\alpha_1}{C_A}, \quad p = p(s) = \ln \frac{C_Ay(s)}{\alpha_1} \]
\[ x(p) = x \]
\[ y(p) = y_A \]

(7.12)

The third collection of characteristics is given by the system of differential equations

\[ \dot{x} = -x \]
\[ \dot{y} = -y \]

(7.13)

and are determined by the Cauchy formula

\[ x(t) = xe^{-t} \]
\[ y(t) = y_Ae^{-t}, \quad 0 \leq t \leq r(s) \]

(7.14)

Here the final instant \(r = r(s)\) and the final position \((x(r), y(r))\) of the characteristic trajectory (7.14) are given by formulas

\[ xe^{-r} = x_A = \frac{\alpha_2}{C_A}, \quad r = r(s) = \ln \frac{C_Ax(s)}{\alpha_2} \]
\[ x(r) = x_A \]
\[ y(r) = y_Ae^{-r} \]

(7.15)

The optimal control problem consists in finding such time \(s\) and corresponding switching point \((x, y) = (x(s), y(s))\) on the trajectory \((x(\cdot), y(\cdot))\) that the integral \(I = I(s)\)
Figure 7: Three families of characteristics $TR_1, TR_2, TR_3$ in the triple-step problem.

attains the maximum value

\[
I(s) = I_1(s) + I_2(s) + I_3(s) \quad (7.16)
\]

\[
I_1(s) = \int_0^s (C_A((x_0 - 1)e^{-t} + 1)y_0e^{-t} - \\
\alpha_1((x_0 - 1)e^{-t} + 1) - \alpha_2y_0e^{-t} + a_{22})dt \quad (7.17)
\]

\[
I_2(s) = \int_0^{p(s)} (C_Ax(s)y(s)e^{-t} - \\
\alpha_1x(s) - \alpha_2y(s)e^{-t} + a_{22})dt \quad (7.18)
\]

\[
I_3(s) = \int_0^{r(s)} (C_Ax(s)y_Ae^{-2t} - \\
\alpha_1x(s)e^{-t} - \alpha_2y_Ae^{-t} + a_{22})dt \quad (7.19)
\]

On Fig.7 initial position $IP = (0.6, .09)$, characteristics of three different types $TR_1 \ (7.8)$, $TR_2 \ (7.10)$, $TR_3 \ (7.14)$, switching points $SP_1 = (x(s), y(s))$, $SP_2 = (x(p), y(p))$ and final point $FP = (x(r), y(r))$ are shown.
7.2 Solution of the Three-step Optimal Control Problem

Let us indicate the scheme for solution of the three-step optimal control problem (7.7)-(7.19). We express integrals $I_k, k = 1, 2, 3$ as functions $I_k(x, x_0, y_0)$ depending on the first coordinate $x$ of optimal switching points $(x, y) = (x(s), y(s))$ and initial positions $(x_0, y_0)$, calculate the derivative by the variable $x$ of the integral $I(x, x_0, y_0)$

$$I(x, x_0, y_0) = I_1(x, x_0, y_0) + I_2(x, x_0, y_0) + I_3(x, x_0, y_0)$$
equate this derivative to zero $dI/dx = 0$ and find the equation $F(x, x_0, y_0) = 0$ for optimal switching coordinates $x$. Then we connect initial positions $(x_0, y_0)$ with the switching positions $(x, y)$ by corresponding relations and obtain the equation $F(x, x_0(x, y), y_0(x, y)) = 0$ for the optimal switching curve.

So, we calculate at first the integrals $I_1$, $I_2$, $I_3$ as functions of variables $x$, $x_0$, $y_0$

$$I_1 = I_1(x, x_0, y_0) = \frac{C_A y_0 ((x_0 - 1)^2 - (x - 1)^2)}{2(x_0 - 1)} + \frac{C_A y_0 (x_0 - x)}{(x_0 - 1)} - \alpha_1((x_0 - x) + \ln(1 - x_0) - \ln(1 - x)) - \frac{\alpha_2 y_0 (x_0 - x)}{(x_0 - 1)} + a_{22}(\ln(1 - x_0) - \ln(1 - x))$$

$$I_2 = I_2(x, x_0, y_0) = \frac{(C_A x - \alpha_2)(C_A y_0 (x - 1) - \alpha_1 (x_0 - 1))}{C_A (x_0 - 1)} + (a_{22} - \alpha_1 x)(\ln \frac{C_A y_0}{\alpha_1 (1 - x_0)} + \ln(1 - x))$$

$$I_3 = I_3(x, x_0, y_0) = \frac{\alpha_1 (\alpha_2 - C_A^2 x^2)}{2C_A^2 x^2} + a_{22}(\ln x + \ln C_A)$$

Now we calculate derivatives $dI_1/dx$, $dI_2/dx$, $dI_3/dx$

$$\frac{dI_1}{dx} = -y_0 (C_A x - \alpha_2) (x_0 - 1) + \frac{(\alpha_1 x - a_{22}) (x_0 - 1)}{(x_0 - 1)}$$

$$\frac{dI_2}{dx} = \frac{(C_A y_0 (x - 1) - \alpha_1 x_0 (x_0 - 1) + C_A y_0 x - \alpha_2 y_0)}{(x_0 - 1)} - \alpha_1 \ln \frac{C_A y_0}{\alpha_1 (1 - x_0)} + \ln(1 - x)) + \frac{(a_{22} - \alpha_1 x)}{(x - 1)}$$

$$\frac{dI_3}{dx} = -\frac{\alpha_1 (C_A^2 x^2 + \alpha_2^2)}{2C_A^2 x^2} + \frac{a_{22}}{x}$$

Summarizing derivatives $dI_1/dx$, $dI_2/dx$, $dI_3/dx$ and equating the sum to zero we obtain the following equation

$$\frac{C_A y_0 (1 - x)}{(1 - x_0)} - \alpha_1 - \alpha_1 \ln \frac{C_A y_0 (1 - x)}{\alpha_1 (1 - x_0)} - \frac{(\alpha_1 C_A^2 x^2 - 2C_A^2 a_{22} x + \alpha_1 \alpha_2^2)}{2C_A^2 x^2} = 0$$

Taking into account that $v_A = a_{22} C_A - \alpha_1 \alpha_2 = 0$ in (6.6), and parameters $x$, $y$, $x_0$, $y_0$ are connected by formula

$$y = \frac{y_0 (1 - x)}{(1 - x_0)}$$

we obtain the expression for the switching curve $N^1_A$

$$\frac{C_A y - \alpha_1}{\alpha_1} - \ln(1 + \frac{(C_A y - \alpha_1)}{\alpha_1}) - \frac{(C_A x - \alpha_2)^2}{2C_A^2 x^2} = 0 \quad (7.20)$$
The curve $N^1_A$ with the accuracy of the second order with respect to variable $y$ is a hyperbola which passes through the point $(x_A, y_A)$

$$
\frac{(C_{Ay} - \alpha_1)}{\alpha_1} - \frac{(C_{Ax} - \alpha_2)}{C_{Ax}} = 0
$$

(7.21)

It has the horizontal asymptote

$$
y = \frac{2\alpha_1}{C_A}
$$

and the vertical asymptote

$$
x = 0
$$

Remark 7.1 The derivative of the curve $N^1_A$ at the point $(x_A, y_A)$ is determined by relation

$$
\frac{dy}{dx} = \frac{\alpha_1 \alpha_2}{C^2_A x^2} \bigg|_{x=x_A} = \frac{\alpha_1}{\alpha_2}
$$

It is equal to the derivative $\frac{dy}{dx} = \frac{\alpha_1}{\alpha_2}$ of the switching line $K^2_A$, $y = (\alpha_1/\alpha_2)x$ (see (4.2)) in the game with the multiterminal functional. Hence, the curve $N^1_A$ is situated below the line $K^2_A$.

To construct the complete switching curve $N_A$ for the optimal strategy of the first coalition we need to supplement the curve $N^1_A$ by the analogous curve $N^2_A$ in the domain where $y \leq y_A$

$$
N_A = N^1_A \cup N^2_A
$$

$$
N^1_A = \{(x, y) \in [0, 1] \times [0, 1]: \\
\frac{(C_{Ay} - \alpha_1)}{\alpha_1} - \ln(1 + \frac{(C_{Ay} - \alpha_1)}{\alpha_1}) - \\
\frac{(C_{Ax} - \alpha_2)^2}{2C^2_A x^2} = 0, \quad y \geq \frac{\alpha_1}{C_A}\}
$$

$$
N^2_A = \{(x, y) \in [0, 1] \times [0, 1]: \\
\frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)} - \ln(1 + \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)}) - \\
\frac{(C_A(1-x) - (C_A - \alpha_2)^2}{2C^2_A(1-x)^2} = 0, \quad y \leq \frac{\alpha_1}{C_A}\}
$$

(7.22)

In the case when $C_A < 0$ the curves $N^1_A$ and $N^2_A$ are described by formulas

$$
N^1_A = \{(x, y) \in [0, 1] \times [0, 1]: \\
\frac{(C_{Ay} - \alpha_1)}{\alpha_1} - \ln(1 + \frac{(C_{Ay} - \alpha_1)}{\alpha_1}) - \\
\frac{(C_A(1-x) - (C_A - \alpha_2)^2}{2C^2_A(1-x)^2} = 0, \quad y \geq \frac{\alpha_1}{C_A}\}
$$

$$
N^2_A = \{(x, y) \in [0, 1] \times [0, 1]: \\
\frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)} - \ln(1 + \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)}) - \\
\frac{(C_A x - \alpha_2)^2}{2C^2_A x^2} = 0, \quad y \leq \frac{\alpha_1}{C_A}\}
$$

(7.23)
Let us remind that the line

\[ L_A = \{(x, y) \in [0, 1] \times [0, 1] : \ 0 \leq x \leq 1, \ y = y_A \} \tag{7.24} \]

is also the switching one for the first coalition according to (7.6). The curves \( N_A, L_A \) divide the unit square \([0, 1] \times [0, 1]\) into three parts - the upper part

\[ S_A^u \supset \{(x, y) : \ x = x_A, \ y > y_A\} \]

the lower part

\[ S_A^l \supset \{(x, y) : \ x = x_A, \ y < y_A\} \]

and the middle part

\[ S_A^m = S_A^u \cup S_A^m \]

\[ S_A^{m_1} \supset \{(x, y) : \ x < x_A, \ y = y_A\} \]

\[ S_A^{m_2} \supset \{(x, y) : \ x > x_A, \ y = y_A\} \]

The “positive” feedback \( u^{ts}_A \) has the following structure

\[ u^{ts}_A = u^{ts}_A(x, y) = \begin{cases} \max\{0, -\text{sgn}(C_A)\} & \text{if } (x, y) \in S_A^u \\ \max\{0, \text{sgn}(C_A)\} & \text{if } (x, y) \in S_A^l \\ x & \text{if } (x, y) \in S_A^m \\ & \text{if } (x, y) \in N_A \cup L_A \end{cases} \tag{7.25} \]

For the second coalition one can obtain the similar switching curves \( N_B \). Namely, in the case when \( C_B > 0 \) the switching curve \( N_B \) is given by relations

\[ N_B = N_B^1 \cup N_B^2 \]

\[ N_B^1 = \{(x, y) \in [0, 1] \times [0, 1] : \frac{C_B(1-x) - (C_B - \beta_2)}{(C_B - \beta_2)} - \ln(1 + \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)}) - \frac{(C_B(1-y) - (C_B - \beta_1))^2}{2C_B^2(1-y)^2} = 0, \ x \leq \frac{\beta_2}{C_B} \}, \tag{7.26} \]

In the case when \( C_B < 0 \) the curves \( N_B^1 \) and \( N_B^2 \) are determined by formulas

\[ N_B^1 = \{(x, y) \in [0, 1] \times [0, 1] : \frac{C_Bx - \beta_2}{\beta_2} - \ln(1 + \frac{(C_Bx - \beta_2)}{\beta_2}) - \frac{(C_B(1-y) - (C_B - \beta_1))^2}{2C_B^2(1-y)^2} = 0, \ x \geq \frac{\beta_2}{C_B} \} \]

\[ N_B^2 = \{(x, y) \in [0, 1] \times [0, 1] : \frac{C_B(1-x) - (C_B - \beta_2)}{(C_B - \beta_2)} - \ln(1 + \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)}) - \frac{(C_By - \beta_1)^2}{2C_B^2y^2} = 0, \ x \leq \frac{\beta_2}{C_B} \} \tag{7.27} \]
Remind that the line
\[ L_B = \{(x, y) \in [0, 1] \times [0, 1] : \quad x = x_B, \quad 0 \leq y \leq 1\} \tag{7.28} \]
is also the switching one for the second coalition. The curves \( N_B, L_B \) divide the unit square \([0, 1] \times [0, 1]\) into three parts - the left part
\[ S_B^l \supset \{(x, y) : \quad x < x_B, \quad y = y_B\} \]
the right part
\[ S_B^r \supset \{(x, y) : \quad x > x_B, \quad y = y_B\} \]
and the middle part
\[ S_B^m = S_B^{m1} \cup S_B^{m2} \]
\[ S_B^{m1} \supset \{(x, y) : \quad x = x_B, \quad y < y_B\} \]
\[ S_B^{m2} \supset \{(x, y) : \quad x = x_B, \quad y > y_B\} \]
The “positive” feedback \( v_B^{ts} \) has the following structure
\[ v_B^{ts} = v_B^{ts}(x, y) = \begin{cases} \max\{0, -\text{sgn}(C_B)\} & \text{if } (x, y) \in S_B^l \\ \max\{0, \text{sgn}(C_B)\} & \text{if } (x, y) \in S_B^r \\ y & \text{if } (x, y) \in S_B^{m1} \\ 0 & \text{if } (x, y) \in S_B^{m2} \end{cases} \tag{7.29} \]

On Fig.8 switching curves \( N_A^1, N_A^2, L_A \) and \( N_B^1, N_B^2, L_B \) are shown for matrixes \( A \) (4.13), \( B \) (4.14). Directions of velocities \( \dot{x} \) are depicted by horizontal arrows: left arrows - in the upper domain \( S_A^l \), right arrows - in the lower domain \( S_A^l \), left-right arrows - in the middle domain \( S_A^m \). Directions of velocities \( y \) are indicated by vertical arrows: up arrows - in the left domain \( S_B^l \), down arrows - in the right domain \( S_B^r \), up-down arrows - in the middle domain \( S_B^m \).

### 7.3 The Guaranteed Value in the Three-Step Optimal Control Problem

Generalizing the considered three-step optimal control problem we can formulate the result which arises from the optimization nature of this problem and provides better index values than values of trajectories tending to static Nash equilibria.

**Proposition 7.1** Consider three-step optimal control \( u_A^{ts}(x, y) \) (7.25) with the switching curves \( N_A \) (7.22),(7.23), \( L_A \) (7.24). Then for any initial position \((x_0, y_0) \in [0, 1] \times [0, 1]\) and for any trajectory
\[ (x^{ts}(\cdot), y^{ts}(\cdot)) \in X(x_0, y_0, u_A^{ts}), \quad x^{ts}(t_0) = x_0, \quad y^{ts}(t_0) = y_0, \quad t_0 = 0 \]
generated by the optimal feedback control \( u_A^{ts} = u_A^{ts}(x, y) \) there exists a finite moment \( t_* \in [0, T_A] \) such that at this moment the trajectory \((x^{ts}(\cdot), y^{ts}(\cdot))\) comes to the line where \( x = x_A \). Then according to construction of three-step optimal feedback control \( u_A^{ts} \) (it maximizes the integral (7.16) and conforms to “short-term” interests of individuals) the following estimate holds
\[ \int_{t_*}^{T} g_A(x^{ts}(t), y^{ts}(t))dt \geq v_A(T - t_*), \quad \forall T \geq t_* \tag{7.30} \]
Figure 8: Guaranteeing triple-step design $u^{ts}_A, v^{ts}_B$ for integral payoffs and the Nash equilibrium trajectory $TR = (x^{ts}(.), y^{ts}(.))$. 
In particular, this inequality remains valid when time $T$ tends to infinity

\[ \liminf_{T \to +\infty} \frac{1}{T - t^s} \int_{t^s}^{T} g_A(x(t), y(t)) dt \geq v_A \]  

(7.31)

Inequalities (7.30), (7.31) mean that the averaged value of the integral functional is not worse than the value $v_A$ of the static matrix game.

The analogous result is valid for trajectories generated by three-step optimal control $v_{ts}^A$ (7.29) corresponding to the switching curves $N_B$ (7.26), (7.27), $L_B$ (7.28).

Remark 7.2 Consider the acceptable trajectory $(x_{ts}^A(\cdot), y_{ts}^A(\cdot))$ generated by “positive” three-step feedbacks $u_{ts}^A$ (7.25), $v_{ts}^B$ (7.29). Then according to Proposition 7.1 the following inequalities take place

\[ \liminf_{T \to +\infty} \frac{1}{T - t^s} \int_{t^s}^{T} g_A(x_{ts}^A(t), y_{ts}^A(t)) dt \geq v_A \]  

(7.32)

\[ \liminf_{T \to +\infty} \frac{1}{T - t^s} \int_{t^s}^{T} g_A(x_{ts}^A(t), y_{ts}^B(t)) dt \geq v_B \]  

(7.33)

and hence the acceptable trajectory $(x_{ts}^A(\cdot), y_{ts}^A(\cdot))$ provides better results for both coalitions than trajectories leading to points of static Nash equilibrium at which corresponding payoffs are equal to values $v_A$ and $v_B$.

On Fig.8 the acceptable trajectory $TR = (x_{ts}^A(\cdot), y_{ts}^A(\cdot))$ is shown for the game with the payoff matrices $A$ (4.13) and $B$ (4.14). It starts from the initial position $IP = (0.6, 0.75)$ and moves at first along the straight line to the corner $(1, 0)$ of the unit square $[0, 1] \times [0, 1]$ with control signals $u = 1$, $v = 0$. Then it crosses the switching line $N_A^1$ and the first coalition switches its control $u$ from 1 to $x$. Further the trajectory $TR$ moves down along the straight line parallel to the axis $y$ till the meeting with the switching line $L_A$. Here the first coalition changes its signal $u$ from $x$ to 0. After that the trajectory directs to the corner $(0, 0)$. Next it meets the switching line $N_B^1$ and the second coalition changes its control signal from 0 to $y$. Then the trajectory $TR$ moves to the left along the straight line parallel to the axis $x$ and intersects the switching curve $N_A^2$. On this curve the first coalition changes its control signal $u$ from 0 to $x$. The trajectory $TR$ stops here at point $FP$ because velocities are equal to zero $\dot{x} = 0$, $\dot{y} = 0$.

Conclusion

In this paper we introduce the control generalization of evolutionary dynamics described by Kolmogorov’s differential equations choosing coefficients of flows on the feedback principle. It is shown that in arising evolutionary (dynamical) nonzero sum games “positive” maximizing feedbacks (guaranteeing feedbacks) provide better results than trajectories which tend to (or circulate near) static Nash equilibria. We construct these “positive” feedbacks analytically for different (terminal, integral) types of functionals determined on infinite horizon. Feedbacks are calculated via methods of the theory of generalized solutions (value functions for corresponding differential games) of Hamilton-Jacobi equations and have bang-bang properties on switching curves. We use multiterminal payoffs which depend on future states and so take into account the foreseeing principle. It is proved that equilibrium trajectories generated by guaranteeing feedbacks are located in domains with better index values for both coalitions than at static Nash equilibria. We indicate
the “evolutionary-revolutionary” properties of equilibrium trajectories and show examples when these trajectories converge to points of intersection of switching curves with the strictly better index values. The risk barrier surrounding the equilibrium trajectory is intended for considering the venturous factor. We give an appropriate economical model for justification of dynamics, payoffs, optimal feedbacks, dynamical Nash equilibria and equilibrium trajectories.

References

Krasovskii, N.N., Control of a Dynamical System, Moscow, Nauka, 1985 (in Russian)
Kleimenov, A.F., Nonantagonistic Positional Differential Games, Ekaterinburg, Nauka, 1993 (in Russian)
Young, P., The Evolution of Conventions, Econometrica, 1993, Vol. 61, No. 1, PP.57-84
Vorobyev, N.N., Games Theory, Moscow, Nauka, 1985 (in Russian)
Tarasyev, A.M., A Differential Model for a $2 \times 2$ Evolutionary Game Dynamics, IIASA Working Paper, Laxenburg, Austria, 1994, WP-94-63
Subbotin, A.I., Minimax Inequalities and Hamilton-Jacobi Equations, Moscow, Nauka, 1991 (in Russian)
