E-companion for

Multiple In-Cycle Transshipments with Positive Delivery Times

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Estimation and Convergence of the Demand

Estimation of Multinomial Parameters: Estimation equations are set up using the Method of Moments: $\hat{\mu}_i = Np_i$, for $i \in \{1,2\}$, $\hat{\rho} = -\left[((1-p_1)^{-1}-1)((1-p_2)^{-1}-1)\right]^{0.5}$, where $\hat{\mu}_1, \hat{\mu}_s$ are empirical means, and $\hat{\rho}$ is the empirical correlation. These three statistics can be computed with historical demand data. The three equations in three unknowns yield a quadratic equation in N : $N^2 - (\hat{\mu}_1 + \hat{\mu}_2) - \hat{\mu}_1\hat{\mu}_2(1/\hat{\rho}^2 - 1) = 0$, which has the nonnegative, unique, real root: $N = (\hat{\mu}_1 + \hat{\mu}_2)/2 + [((\hat{\mu}_1 - \hat{\mu}_2)/2)^2 + \hat{\mu}_1\hat{\mu}_2/\hat{\rho}^2]^{0.5}$. N converges to infinity as $\hat{\rho}$ becomes smaller. This is consistent with the convergence of the multinomial distribution to independent Poisson distributions.

Convergence of the Multinomial to Poisson: We rearrange the terms in the multinomial probability and take the limit as $N \to \infty$ and $p_1, p_2 \to 0$ while keeping Np_1 and Np_2 constant:

$$\begin{aligned}
&\mathrm{P}(\xi_1^N = k_1, \xi_2^N = k_2; N, p_1, p_2)\\
&= \frac{(Np_1)^{k_1}}{k_1!}\frac{(Np_2)^{k_2}}{k_2!}\underbrace{\frac{N}{N}\frac{N-1}{N}\cdots\frac{N-k_1-k_2+1}{N}}_{\to 1}\underbrace{\left(1-\frac{Np_1+Np_2}{N}\right)^{N}}_{\to\, e^{-Np_1-Np_2}}\underbrace{\left(1-\frac{Np_1+Np_2}{N}\right)^{-k_1-k_2}}_{\to 1}\\
&\longrightarrow \frac{e^{-Np_1}(Np_1)^{k_1}}{k_1!}\frac{e^{-Np_2}(Np_2)^{k_2}}{k_2!}.
\end{aligned}$$

The last expression is the product of two Poisson probabilities with rates Np_1 and Np_2. Thus our demand model captures independent Poisson demands as a limiting case.

To find hold-back levels for independent Poisson demands with means λ_1 and λ_2, N and the corresponding p_1 and p_2 should be determined. This can be done iteratively with an iteration indicator j such that N_j and $p_{i,0}$ denote, respectively, the number of periods in a cycle and the per period demand probability at retailer i in iteration j, for $i \in \{1,2\}$. For initialization, set $j = 0$. At $j = 0$, set N_0 such that $\lambda_i = N_0 p_{i,0}$ and $p_{i,0} \leq 1$. Next compute hold-back level $\bar{x}_{i,0}^n$ with $N = N_0$ and $p_i = p_{i,0}$ for $i \in \{1,2\}$ and $n \in \{0,1,...,N\}$. Repeatedly, increment j by one and compute hold-back level $\bar{x}_{i,j}^n$ with $N = N_j$ and $p_i = p_{i,j}$, where $N_j = 2^j N_0$ and $p_{i,j} = p_{i,0}/2^j$. As soon as the difference between $\bar{x}_{i,j}^n$ and $\bar{x}_{i,j-1}^n$ is small, stop computations as $\bar{x}_{i,j}^n$ is approximately the hold-back level for the Poisson demands. For example, Figures 1 and 2 illustrate that for the given problem instance, setting $N = 22$ is sufficient as increasing N from 22 to 88 changes hold-back levels only slightly. The complexity of hold-back level computation is linear in N, so problems with large N can be also solved easily.

Proofs of Lemmas/Theorems

Lemma 7 below is a property of the difference of two min functions. It is occasionally used for proving the previously stated lemmas.

Lemma 7. *For any four real numbers $a, b, c,$ and d, we have* $\min\{a-c, b-d\} \leq \min\{a,b\} - \min\{c,d\} \leq \max\{a-c, b-d\}$.

Proof:

$$\begin{aligned}
\min\{a,b\} - \min\{c,d\} &= \min\{a,b\} + \max\{-c,-d\} \\
&= \min\{a + \max\{-c,-d\}, b + \max\{-c,-d\}\} \geq \min\{a-c, b-d\}, \\
\min\{a,b\} - \min\{c,d\} &= \min\{a,b\} + \max\{-c,-d\} \\
&= \max\{-c + \min\{a,b\}, -d + \min\{a,b\}\} \leq \max\{a-c, b-d\}. \quad \square
\end{aligned}$$

Proof of Lemma 1: (i) We prove that $V_n(x_1, x_2 - 1) = V_n(x_1, x_2)$ by induction on n. For $n = 0$, $V_0(x_1, x_2 - 1) = V_0(x_1, x_2) = 0$ for $x_2 \leq 0$. As an induction hypothesis, assume that $V_{n-1}(x_1, x_2 - 1) = V_{n-1}(x_1, x_2)$ for $x_2 \leq 0$. Let us study two cases, $x_1 \leq 0$ and $x_1 \geq 1$.

For $x_1, x_2 \leq 0$, (3) is used to obtain

$$\begin{aligned}
V_n(x_1, x_2 - 1) - V_n(x_1, x_2) &= p_1[V_{n-1}(x_1 - 1, x_2 - 1) - V_{n-1}(x_1 - 1, x_2)] \\
&+ p_2[V_{n-1}(x_1, x_2 - 2) - V_{n-1}(x_1, x_2 - 1)] \\
&+ (1 - p_1 - p_2)[V_{n-1}(x_1, x_2 - 1) - V_{n-1}(x_1, x_2)].
\end{aligned}$$

Each of the terms in the square brackets above is zero by the induction hypothesis. So we obtain $V_n(x_1, x_2 - 1) = V_n(x_1, x_2)$, for $x_1, x_2 \leq 0$.

For $x_1 \geq 1$ and $x_2 \leq 0$, (4) is used to obtain

$$\begin{aligned}
& V_n(x_1, x_2 - 1) - V_n(x_1, x_2) \\
&= p_1[V_{n-1}(x_1 - 1, x_2 - 1) - V_{n-1}(x_1 - 1, x_2)] + (1 - p_1 - p_2)[V_{n-1}(x_1, x_2 - 1) - V_{n-1}(x_1, x_2)] \\
&\quad + p_2[\min\{n\pi + V_{n-1}(x_1, x_2 - 2) + h_1 x_1, T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2 - 1) + h_1(x_1 - 1)\} \\
&\qquad - \min\{n\pi + V_{n-1}(x_1, x_2 - 1) + h_1 x_1, T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2) + h_1(x_1 - 1)\}] \\
&= p_2[\min\{n\pi + V_{n-1}(x_1, x_2 - 2) + h_1 x_1, T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2 - 1) + h_1(x_1 - 1)\} \\
&\qquad - \min\{n\pi + V_{n-1}(x_1, x_2 - 1) + h_1 x_1, T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2) + h_1(x_1 - 1)\}],
\end{aligned}$$

where the second equality follows from the induction hypothesis. Now Lemma 7 can be used with $a := n\pi + V_{n-1}(x_1, x_2 - 2) + h_1 x_1$, $b := T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2 - 1) + h_1(x_1 - 1)$, $c := n\pi + V_{n-1}(x_1, x_2 - 1) + h_1 x_1$, and $d := T(\pi + h_0) + K + V_{n-1}(x_1 - 1, x_2) + h_1(x_1 - 1)$. Since $a - c = 0$ and $b - d = 0$ by the induction hypothesis, we obtain $V_n(x_1, x_2 - 1) = V_n(x_1, x_2)$ for $x_1 \geq 1$, $x_2 \leq 0$. This completes the inductive step and the proof of (i).

(ii) Proof of $V_n(x_1, x_2) = V_n(x'_1, x'_2)$ can also be done with an inductive argument on n while using (3). Then as a particular case, $V_n(x_1, x_2) = V_n(0, 0)$ for all $x_1, x_2 \leq 0$. Plugging this into (3) and using standard algebra, we obtain $V_n(x_1, x_2) = V_n(0, 0) \;=\; (p_1 + p_2)\frac{n(n+1)}{2}\pi$. □

Proof of Lemma 2: (i) Proof is by induction on the remaining number of periods n. Since $\delta_0(\cdot) = 0$, we have $\delta_0(x_1) \leq \delta_0(x_1 - 1)$, for $n = 0$.

To complete the proof, the inductive hypothesis is given that $\delta_{n-1}(x_1)$ is non-increasing in x_1 for $x_1 \geq 3$. By the inductive hypothesis, $\delta_{n-1}(x) \leq \delta_{n-1}(x-1) \leq \delta_{n-1}(x-2)$. Thus, $(n-T)\pi + h_1 - Th_0 - K$ can fall into one of the four intervals defined by these marginal benefits. For ease of notation, we define a function $b(n)$ such that $b(n) = (n-T)\pi + h_1 - Th_0 - K$. To indicate this formally, the following indicator variables are defined.

$$
\begin{aligned}
\mathbb{I}_1 &= \{1, \text{ if } b(n) < \delta_{n-1}(x);\ 0, \text{ otherwise}\}, \\
\mathbb{I}_2 &= \{1, \text{ if } \delta_{n-1}(x) \leq b(n) < \delta_{n-1}(x-1);\ 0, \text{ otherwise}\}, \\
\mathbb{I}_3 &= \{1, \text{ if } \delta_{n-1}(x-1) \leq b(n) < \delta_{n-1}(x-2);\ 0, \text{ otherwise}\}, \\
\mathbb{I}_4 &= \{1, \text{ if } \delta_{n-1}(x-2) \leq b(n);\ 0, \text{ otherwise}\},
\end{aligned}
$$

where $\mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 = 1$. The proof of $\delta_n(x_1) \leq \delta_n(x_1 - 1)$ can be done separately for $x_1 \geq 3$ and $x_1 = 2$.

For $x_1 \geq 3$, (11) can be rewritten for inventory levels x_1 and $x_1 - 1$ by using the indicator variables.

$$
\begin{aligned}
\delta_n(x_1) &= \mathbb{I}_1[-h_1 + (1-p_1)\delta_{n-1}(x_1) + p_1\delta_{n-1}(x_1 - 1)] \\
&\quad + \mathbb{I}_2[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1) + p_1\delta_{n-1}(x_1-1) + p_2 b(n)] \\
&\quad + \mathbb{I}_3[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1) + (p_1+p_2)\delta_{n-1}(x_1-1)] \\
&\quad + \mathbb{I}_4[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1) + (p_1+p_2)\delta_{n-1}(x_1-1)]. \qquad (17) \\
\delta_n(x_1 - 1) &= \mathbb{I}_1[-h_1 + (1-p_1)\delta_{n-1}(x_1-1) + p_1\delta_{n-1}(x_1 - 2)] \\
&\quad + \mathbb{I}_2[-h_1 + (1-p_1)\delta_{n-1}(x_1-1) + p_1\delta_{n-1}(x_1-2)] \\
&\quad + \mathbb{I}_3[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1-1) + p_1\delta_{n-1}(x_1-2) + p_2 b(n)] \\
&\quad + \mathbb{I}_4[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1-1) + (p_1+p_2)\delta_{n-1}(x_1-2)]. \qquad (18)
\end{aligned}
$$

When $\mathbb{I}_2 = 1$, $\delta_{n-1}(x_1)$ is replaced with $b(n)$, in (17). When $\mathbb{I}_3 = 1$, $\delta_{n-1}(x_1 - 1)$ is replaced with $b(n)$, in (17). Thus, (17) can be rewritten as follows

$$
\begin{aligned}
\delta_n(x_1) &\leq \mathbb{I}_1[-h_1 + (1-p_1)\delta_{n-1}(x_1) + p_1\delta_{n-1}(x_1 - 1)] \\
&\quad + \mathbb{I}_2[-h_1 + (1-p_1)b(n) + p_1\delta_{n-1}(x_1-1)] \\
&\quad + \mathbb{I}_3[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1) + (p_1+p_2)b(n)] \\
&\quad + \mathbb{I}_4[-h_1 + (1-p_1-p_2)\delta_{n-1}(x_1) + (p_1+p_2)\delta_{n-1}(x_1-1)] \qquad (19) \\
&\leq \delta_n(x_1 - 1). \qquad (20)
\end{aligned}
$$

Inequality (20) follows from comparing the right hand sides of (18) and (19) while using the induction hypothesis $\delta_{n-1}(x_1) \leq \delta_{n-1}(x_1 - 1)$.

For $x_1 = 2$, $\delta_n(x_1 - 1)$ must be based on (12) so its expression slightly differs from (18). The proof can be done by induction on n. Thus the proof of (i) is completed.

(ii) $\delta_0(\cdot) = 0$ and the induction hypothesis, $\delta_{n-1}(x_1) \leq (n-1)\pi$, is made to prove Lemma 2(i). In particular,

$$\begin{aligned} \delta_n(x_1) &\leq -h_1 + (1-p_1)\delta_{n-1}(x_1) + p_1\delta_{n-1}(x_1 - 1) + p_2[\max\{0, \delta_{n-1}(x_1 - 1) - \delta_{n-1}(x_1)\}] \\ &\leq -h_1 + (1 - p_1 - p_2)\delta_{n-1}(x_1) + (p_1 + p_2) \leq n\pi, \end{aligned}$$

where the first inequality follows from Lemma 7, the second inequality is a result of Lemma 2(i), and the last inequality follows from the induction hypothesis.

(iii) First the existence of the limit should be established. Since $\delta_n(x_1)$ is non-increasing, it suffices to establish a lower bound for $\delta_n(x_1)$. A crude lower bound for $\delta_n(x_1)$ comes from keeping the extra unit in inventory until the end of the cycle without attempting to use this unit to meet any demand. So $\delta_n(x_1) \geq -nh_1$ for any $x_1 \in \mathcal{N}$. Hence, the limit exists and it can be inferred that $\delta_n^\infty := \lim_{x_1\to\infty} \delta_n(x_1)$.

Now when the limit of both sides of (11) is taken and the limits are pushed into the minima by using the continuity of minimum, we get

$$\begin{aligned} \lim_{x_1\to\infty} \delta_n(x_1) &= -h_1 + (1-p_1)\lim_{x_1\to\infty} \delta_{n-1}(x_1) + p_1 \lim_{x_1\to\infty} \delta_{n-1}(x_1 - 1) \\ &\quad + p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \lim_{x_1\to\infty} \delta_{n-1}(x_1 - 1)\} \\ &\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \lim_{x_1\to\infty} \delta_{n-1}(x_1)\}] \\ &= -h_1 + (1-p_1)\delta_{n-1}^\infty + p_1\delta_{n-1}^\infty + p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}^\infty\} \\ &\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}^\infty\}] = -h_1 + \delta_{n-1}^\infty. \end{aligned}$$

Thus we obtain $\delta_n^\infty = -h_1 + \delta_{n-1}^\infty$. This in conjunction with $\delta_0(\cdot) = 0$ implies that $\delta_n^\infty = -nh_1$. $\square$

Proof of Theorem 1: (ii) Note that the hold-back level is infinite if and only if $\lim_{x_1\to\infty} \delta_{n-1}(x) > (n-T)\pi + h_1 - Th_0 - K$. By recalling $\lim_{x_1\to\infty} \delta_{n-1}(x) = -(n-1)h_1$ and using some algebra, $\lim_{x_1\to\infty} \delta_{n-1}(x) > (n-T)\pi + h_1 - Th_0 - K$ is equivalent to $n < \frac{T(\pi+h_0)+K}{\pi+h_1}$. As a result, the hold-back level is infinite if and only if $n < \frac{T(\pi+h_0)+K}{\pi+h_1}$, which is essentially a restatement of (ii). $\square$

Proof of Lemma 3: To provide some justification before the proof, we illustrate the marginal benefit and cost functions in Figure 8. The marginal benefit $\delta_{n-1}(x)$ is a nonlinear function of n and illustrated separately for each inventory level, 1 through 11.

The lemma is proved by induction first on n, second on x_1, and then on both. First we show that Lemma 3 is valid for $x_1 = 1$: $\delta_n(1) \leq \delta_{n-1}(1) + \pi$. To start the induction on n, the inequality is checked for $n = 1$. Note that $\delta_0(\cdot) = 0$. Then from (12), we have the equality below.

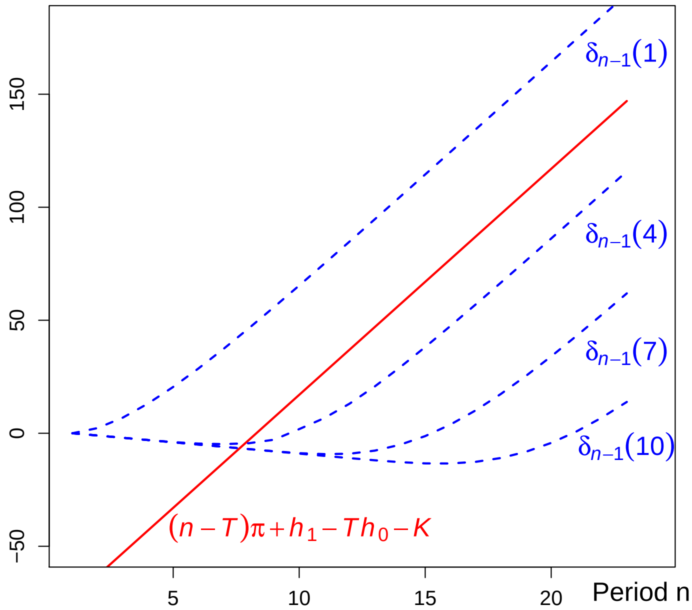

Figure 8: The marginal cost $(n-T)\pi + h_1 - Th_0 - K$ and marginal benefit $\delta_{n-1}(x)$ of rejecting a request for $p_1 = p_2 = 0.3$, $h_0 = h_1 = 1$, $\pi = 10$, $T = 4$, $K = 40$, and $x \in \{1, 4, 7, 10\}$.

$$\begin{aligned}\delta_1(1) - \delta_0(1) &= -h_1 + (p_1 + p_2)(\pi + h_1) - p_2 \min\{\pi + h_1, T(\pi + h_0) + K\}\\ &\le (p_1 + p_2)\pi - (1 - p_1 - p_2)h_1 \le \pi.\end{aligned}$$

The first inequality above is obtained by dropping the term $p_2 \min\{\pi + h_1, T(\pi + h_0) + K\}$. The second inequality follows from $p_1 + p_2 \le 1$.

As an induction hypothesis for $n \ge 2$, we assume that $\delta_{n-1}(1) - \delta_{n-2}(1) \le \pi$. Then

$$\begin{aligned}&\delta_n(1) - \delta_{n-1}(1) = (1 - p_1)[\delta_{n-1}(1) - \delta_{n-2}(1)] + (p_1 + p_2)\pi\\ &\quad + p_2[\min\{(n-1)\pi + h_1, T(\pi + h_0) + K + \delta_{n-2}(1)\} - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(1)\}]\\ &\le (1 - p_1)[\delta_{n-1}(1) - \delta_{n-2}(1)] + (p_1 + p_2)\pi + p_2[\max\{-\pi, \delta_{n-2}(1) - \delta_{n-1}(1)\}] \quad (21)\\ &= (1 - p_1 - p_2)[\delta_{n-1}(1) - \delta_{n-2}(1)] + (p_1 + p_2)\pi \quad (22)\\ &\le (1 - p_1 - p_2)\pi + (p_1 + p_2)\pi = \pi. \quad (23)\end{aligned}$$

Inequality (21) follows from applying Lemma 7 with $a := (n-1)\pi + h_1$, $b := T(\pi + h_0) + K + \delta_{n-2}(1)$, $c := n\pi + h_1$ and $d := T(\pi + h_0) + K + \delta_{n-1}(1)$. Equality (22) and inequality (23) both follow from the induction hypothesis $\delta_{n-1}(1) - \delta_{n-2}(1) \le \pi$, which completes the proof of $\delta_n(1) \le \delta_{n-1}(1) + \pi$ for every $n \in \mathcal{N}$.

From (11), $\delta_1(x_1) = -h_1 \le \pi$. Since $\delta_0(x_1) = 0$, we arrive at $\delta_1(x_1) - \delta_0(x_1) \le \pi$ for $x_1 \ge 2$. Thus, the lemma holds for $n = 1$ and $x_1 \in \mathcal{N}$.

Up to now, it is established that $\delta_n(x_1) \le \delta_{n-1}(x_1) + \pi$ over $\{(x_1, n) : [x_1 = 1 \text{ and } n \in \mathcal{N}] \text{ or } [x_1 \in \mathcal{N} \text{ and } n = 1]\}$. These inequalities are taken to start an induction on both x_1 and n. Formally, for x_1 and $n \ge 2$, the induction hypothesis $\delta_{n-1}(x_1) \le \delta_{n-2}(x_1) + \pi$ is assumed.

In order to ensure that the induction hypothesis holds at all (x_1, n), consider these pairs in the following sequence: First consider all pairs of the form $(x_1, n = 2)$ in an increasing order of x_1. Then increase n by 1 and again consider the pairs in an increasing order of x_1. This sequence ensures that $\delta_{n-1}(x_1) \le \delta_{n-2}(x_1) + \pi$ and $\delta_{n-1}(x_1 - 1) \le \delta_{n-2}(x_1 - 1) + \pi$ while we are proving $\delta_n(x_1) \le \delta_{n-1}(x_1) + \pi$.

For an arbitrary (x_1, n), (11) is used to obtain

$$\begin{aligned}
&\delta_n(x_1) - \delta_{n-1}(x_1) = (1-p_1)[\delta_{n-1}(x_1) - \delta_{n-2}(x_1)] + p_1[\delta_{n-1}(x_1-1) - \delta_{n-2}(x_1-1)] \\
&\quad + p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 - 1)\} \\
&\quad\quad - \min\{(n-1)\pi + h_1, T(\pi + h_0) + K + \delta_{n-2}(x_1 - 1)\}] \\
&\quad + p_2[\min\{(n-1)\pi + h_1, T(\pi + h_0) + K + \delta_{n-2}(x_1)\} \\
&\quad\quad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1)\}] \\
&\le (1-p_1)[\delta_{n-1}(x_1) - \delta_{n-2}(x_1)] + p_1[\delta_{n-1}(x_1-1) - \delta_{n-2}(x_1-1)] \\
&\quad + p_2[\max\{\pi, \delta_{n-1}(x_1-1) - \delta_{n-2}(x_1-1)\}] + p_2[\max\{-\pi, \delta_{n-2}(x_1) - \delta_{n-1}(x_1)\}] \qquad (24) \\
&\le (1-p_1)[\delta_{n-1}(x_1) - \delta_{n-2}(x_1)] + p_1[\delta_{n-1}(x_1-1) - \delta_{n-2}(x_1-1)] \\
&\quad + p_2[\pi] + p_2[\delta_{n-2}(x_1) - \delta_{n-1}(x_1)] \le \pi. \qquad (25)
\end{aligned}$$

Inequality (24) is by Lemma 7. (25) follows from the induction hypothesis. This establishes $\delta_n(x_1) \le \delta_{n-1}(x_1) + \pi$ over $\{(x_1, n) : x_1, n \in \mathcal{N}\}$. □

Proof of Lemma 4: First we show that $V_n(x_1, 0)$ is convex for any $n, x_1 \in \mathcal{N}$. Since $V_n(x_1, 0) - V_n(x_1 - 1, 0) = -\delta_n(x_1)$ and $-\delta_n(x_1)$ is non-decreasing in x_1 by Lemma 2, the cost $V_n(x_1, 0)$ is convex. Similarly, we can argue that $V_n(0, x_2)$ is convex for any $n, x_2 \in \mathcal{N}$.

The rest of the proof is by induction on n while using (2). Since $V_0 = 0$, the convexity of V_n for $n = 0$ trivially holds. Now assume that $V_{n-1}(x_1, x_2)$ is convex for $x_1, x_2 \in \mathcal{N}$. The term $V_{n-1}(x_1 - 1, x_2)$ is convex by the induction hypothesis when $x_1 \ge 2$ and by the convexity of $V_{n-1}(0, x_2)$ when $x_1 = 1$. Adding linear functions of x_1 and x_2 to this, we obtain the convexity of the term in the first square brackets. The convexity of the term in the second square brackets can be argued similarly. The third term $V_{n-1}(x_1, x_2)$ is convex by the induction hypothesis. Summing three convex terms, the proof is finished. □

Proof of Lemma 5: (i) The lemma is proved by induction first on n for $x_1 = 1$ and then for arbitrary x_1 and n. First we show that Lemma 5(i) is valid for $x_1 = 1$: $\delta_n(1; T + \varepsilon) - \delta_n(1; T) \ge -\varepsilon(\pi + h_0)$. Note that $\delta_0(\cdot) = 0$. Thus $\delta_0(1; T + \varepsilon) - \delta_0(1; T) = 0 \ge -\varepsilon(\pi + h_0)$.

As an induction hypothesis for $n \ge 1$, assume that $\delta_{n-1}(1; T + \varepsilon) - \delta_{n-1}(1; T) \ge -\varepsilon(\pi + h_0)$. Then by (12),

$$
\begin{aligned}
&\delta_n(1;T+\varepsilon)-\delta_n(1;T)\\
&=(1-p_1)[\delta_{n-1}(1;T+\varepsilon)-\delta_{n-1}(1;T)]-p_2[\min\{n\pi+h_1,(T+\varepsilon)(\pi+h_0)+K+\delta_{n-1}(1;T+\varepsilon)\}\\
&\quad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;T)\}]\\
&\geq(1-p_1)[\delta_{n-1}(1;T+\varepsilon)-\delta_{n-1}(1;T)]-p_2[\max\{0,\varepsilon(\pi+h_0)+\delta_{n-1}(1;T+\varepsilon)-\delta_{n-1}(1;T)\}] \quad (26)\\
&=(1-p_1-p_2)[\delta_{n-1}(1;T+\varepsilon)-\delta_{n-1}(1;T)]-p_2\varepsilon(\pi+h_0) \quad (27)\\
&\geq-(1-p_1)\varepsilon(\pi+h_0)\geq-\varepsilon(\pi+h_0). \quad (28)
\end{aligned}
$$

Inequality (26) is a result of Lemma 7. (27) and (28) both follow from the induction hypothesis $\delta_{n-1}(1;T+\varepsilon)-\delta_{n-1}(1;T)\geq-\varepsilon(\pi+h_0)$, which completes the proof for $x_1=1$, $n\in\mathcal{N}$.

For $x_1\geq 2$, we also have $\delta_0(x_1;T+\varepsilon)-\delta_0(x_1;T)=0\geq-\varepsilon(\pi+h_0)$. To prove Lemma 5(i) for an arbitrary x_1 and n, the induction hypothesis $\delta_{n-1}(x_1;T+\varepsilon)-\delta_{n-1}(x_1;T)\geq-\varepsilon(\pi+h_0)$ is assumed. For an arbitrary (x_1,n), (11) is used to obtain the first equality below.

$$
\begin{aligned}
&\delta_n(x_1;T+\varepsilon)-\delta_n(x_1;T)\\
&=(1-p_1)[\delta_{n-1}(x_1;T+\varepsilon)-\delta_{n-1}(x_1;T)]+p_1[\delta_{n-1}(x_1-1;T+\varepsilon)-\delta_{n-1}(x_1-1;T)]\\
&\quad+p_2[\min\{n\pi+h_1,(T+\varepsilon)(\pi+h_0)+K+\delta_{n-1}(x_1-1;T+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;T)\}]\\
&\quad-p_2[\min\{n\pi+h_1,(T+\varepsilon)(\pi+h_0)+K+\delta_{n-1}(x_1;T+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;T)\}]\\
&\geq(1-p_1)[\delta_{n-1}(x_1;T+\varepsilon)-\delta_{n-1}(x_1;T)]+p_1[\delta_{n-1}(x_1-1;T+\varepsilon)-\delta_{n-1}(x_1-1;T)]\\
&\quad+p_2[\min\{0,\varepsilon(\pi+h_0)+\delta_{n-1}(x_1-1;T+\varepsilon)-\delta_{n-1}(x_1-1;T)\}]\\
&\quad-p_2[\max\{0,\varepsilon(\pi+h_0)+\delta_{n-1}(x_1;T+\varepsilon)-\delta_{n-1}(x_1;T)\}] \quad (29)\\
&=(1-p_1-p_2)[\delta_{n-1}(x_1;T+\varepsilon)-\delta_{n-1}(x_1;T)]\\
&\quad+p_1[\delta_{n-1}(x_1-1;T+\varepsilon)-\delta_{n-1}(x_1-1;T)]-p_2\varepsilon(\pi+h_0)\geq-\varepsilon(\pi+h_0). \quad (30)
\end{aligned}
$$

Inequality (29) is by Lemma 7. (30) follows from the induction hypothesis and this establishes $\delta_n(x_1;T+\varepsilon)-\delta_n(x_1;T)\geq-\varepsilon(\pi+h_0)$ over $\{(x_1,n):x_1,n\in\mathcal{N}\}$.

(ii) The lemma is proved by induction first on n for $x_1=1$ and then for arbitrary x_1 and n as in (i). We first show that the lemma is valid for $x_1=1$: $\delta_n(1;p_1+\varepsilon)-\delta_n(1;p_1)\geq 0$. Note that $\delta_0(\cdot)=0$. Thus, $\delta_0(1;p_1+\varepsilon)-\delta_0(1;p_1)=0$.

As an induction hypothesis for $n\geq 2$, assume that $\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)\geq 0$. Then

$$
\begin{aligned}
&\delta_n(1;p_1+\varepsilon)-\delta_n(1;p_1)\\
&=(1-p_1)[\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)]+\varepsilon(n\pi+h_1-\delta_{n-1}(1;p_1+\varepsilon))\\
&\quad-p_2[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;p_1+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;p_1)\}]
\end{aligned}
$$

$$\geq (1-p_1)[\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)]-p_2[\max\{0,\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)\}] \quad (31)$$
$$=(1-p_1-p_2)[\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)]\geq 0. \quad (32)$$

Inequality (31) follows from applying Lemma 7 and Lemma 2(i), which induces that $n\pi+h_1-\delta_{n-1}(1;p_1+\varepsilon)>0$. (32) is by the induction hypothesis $\delta_{n-1}(1;p_1+\varepsilon)-\delta_{n-1}(1;p_1)\geq 0$ and this completes the proof for $\delta_n(1;p_1+\varepsilon)-\delta_n(1;p_1)\geq 0$ for every $n\in\mathcal{N}$.

To prove Lemma 5(ii) for an arbitrary x_1 and n, the induction hypothesis $\delta_{n-1}(x_1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1)\geq 0$ is assumed. For an arbitrary (x_1,n), (11) is used to obtain the equality below.

$$\begin{aligned}
&\delta_n(x_1;p_1+\varepsilon)-\delta_n(x_1;p_1)\\
&=(1-p_1)[\delta_{n-1}(x_1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1)]+\varepsilon[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1+\varepsilon)]\\
&\quad+p_1[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1-1;p_1)]\\
&\quad+p_2[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_1+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_1)\}]\\
&\quad-p_2[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_1+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_1)\}]\\
&\geq(1-p_1)[\delta_{n-1}(x_1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1)]+\varepsilon[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1+\varepsilon)]\\
&\quad+p_1[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1-1;p_1)]\\
&\quad+p_2[\min\{0,\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1-1;p_1)\}]\\
&\quad-p_2[\max\{0,\delta_{n-1}(x_1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1)\}] &(33)\\
&=(1-p_1-p_2)[\delta_{n-1}(x_1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1)]+\varepsilon[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1;p_1+\varepsilon)]\\
&\quad+p_1[\delta_{n-1}(x_1-1;p_1+\varepsilon)-\delta_{n-1}(x_1-1;p_1)]\geq 0. &(34)
\end{aligned}$$

Inequality (33) is by Lemma 7. (34) is a result of both the induction hypothesis and Lemma 2(i). This establishes $\delta_n(x_1;p_1+\varepsilon)-\delta_n(x_1;p_1)\geq 0$ over $\{(x_1,n):x_1,n\in\mathcal{N}\}$.

(iii) The proof for p_2 is very similar to (ii). To show that Lemma is valid for $x_1=1$, first note that $\delta_0(1;p_2+\varepsilon)-\delta_0(1;p_2)$ from $\delta_0(.)=0$.

As an induction hypothesis for $n\geq 2$, assume that $\delta_{n-1}(1;p_2+\varepsilon)-\delta_{n-1}(1;p_2)\geq 0$. Then

$$\begin{aligned}
&\delta_n(1;p_2+\varepsilon)-\delta_n(1;p_2)=(1-p_1)[\delta_{n-1}(1;p_2+\varepsilon)-\delta_{n-1}(1;p_2)]\\
&\quad+\varepsilon[n\pi+h_1-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;p_2+\varepsilon)\}]\\
&\quad-p_2[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;p_2+\varepsilon)\}\\
&\qquad-\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(1;p_2)\}]\\
&\geq(1-p_1)[\delta_{n-1}(1;p_2+\varepsilon)-\delta_{n-1}(1;p_2)]-p_2[\max\{0,\delta_{n-1}(1;p_2+\varepsilon)-\delta_{n-1}(1;p_2)\}] &(35)\\
&=(1-p_1-p_2)[\delta_{n-1}(1;p_2+\varepsilon)-\delta_{n-1}(1;p_2)]\geq 0. &(36)
\end{aligned}$$

Inequality (35) follows from applying Lemma 7 and seeing that the second term is always non-negative. Inequality (36) is by the induction hypothesis $\delta_{n-1}(1; p_2 + \varepsilon) - \delta_{n-1}(1; p_2) \geq 0$ and it completes the proof for $\delta_n(1; p_2 + \varepsilon) - \delta_n(1; p_2) \geq 0$ for every $n \in \mathcal{N}$.

The proof for arbitrary x_1 and n is made by following the induction hypothesis $\delta_{n-1}(x_1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2) \geq 0$. For an arbitrary (x_1, n), define the equality below by the definition of $\delta_n(x_1)$.

$$\begin{aligned}
&\delta_n(x_1; p_2 + \varepsilon) - \delta_n(x_1; p_2) \\
&= (1 - p_1)[\delta_{n-1}(x_1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2)] + p_1[\delta_{n-1}(x_1 - 1; p_2 + \varepsilon) - \delta_{n-1}(x_1 - 1; p_2)] \\
&\quad + p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 - 1; p_2 + \varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 - 1; p_2)\}] \\
&\quad - p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1; p_2 + \varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1; p_2)\}] \\
&\quad + \varepsilon[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 - 1; p_2 + \varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1; p_2 + \varepsilon)\}] \\
&\geq (1 - p_1)[\delta_{n-1}(x_1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2)] + p_1[\delta_{n-1}(x_1 - 1; p_2 + \varepsilon) - \delta_{n-1}(x_1 - 1; p_2)] \\
&\quad + p_2[\min\{0, \delta_{n-1}(x_1 - 1; p_2 + \varepsilon) - \delta_{n-1}(x_1 - 1; p_2)\}] \\
&\quad - p_2[\max\{0, \delta_{n-1}(x_1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2)\}] \\
&\quad + \varepsilon[\min\{0, \delta_{n-1}(x_1 - 1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2 + \varepsilon)\}] && (37) \\
&= (1 - p_1 - p_2)[\delta_{n-1}(x_1; p_2 + \varepsilon) - \delta_{n-1}(x_1; p_2)] + p_1[\delta_{n-1}(x_1 - 1; p_2 + \varepsilon) - \delta_{n-1}(x_1 - 1; p_2)] && (38) \\
&\geq 0. && (39)
\end{aligned}$$

Inequality (37) is by Lemma 7. Equality (38) follows from Lemma 2(i) and the induction hypothesis, where (39) follows directly from the induction hypothesis.

(iv) The proof is similar to the proof of (i), where the induction is made first on n for $x_1 = 1$ and then for arbitrary x_1 and n. We first show that Lemma 5(iv) is valid for $x_1 = 1$: $\delta_n(1; h_1 + \varepsilon) - \delta_n(1; h_1) \leq \varepsilon$. Note that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(1; h_1 + \varepsilon) - \delta_0(1; h_1) = 0 \leq \varepsilon$.

As an induction hypothesis for $n \geq 2$, assume that $\delta_{n-1}(1; h_1 + \varepsilon) - \delta_{n-1}(1; h_1) \leq \varepsilon$. Then

$$\begin{aligned}
&\delta_n(1; h_1 + \varepsilon) - \delta_n(1; h_1) = -\varepsilon + (1 - p_1)[\delta_{n-1}(1; h_1 + \varepsilon) - \delta_{n-1}(1; h_1)] + (p_1 + p_2)\varepsilon \\
&\quad - p_2[\min\{n\pi + h_1 + \varepsilon, T(\pi + h_0) + K + \delta_{n-1}(1; h_1 + \varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(1; h_1)\}] \\
&\leq (1 - p_1)[\delta_{n-1}(1; h_1 + \varepsilon) - \delta_{n-1}(1; h_1)] - (1 - p_1 - p_2)\varepsilon \\
&\quad - p_2[\min\{\varepsilon, \delta_{n-1}(1; h_1 + \varepsilon) - \delta_{n-1}(1; h_1)\}] && (40) \\
&= (1 - p_1 - p_2)[\delta_{n-1}(1; h_1 + \varepsilon) - \delta_{n-1}(1; h_1) - \varepsilon] \leq 0. && (41)
\end{aligned}$$

Inequality (40) follows from Lemma 7. (41) is by the induction hypothesis $\delta_{n-1}(1; h_1 + \varepsilon) -$

$\delta_{n-1}(1;h_1) \leq \varepsilon$, which completes the proof for $\delta_n(1;h_1+\varepsilon) - \delta_n(1;h_1) \leq \varepsilon$, for $n \in \mathcal{N}$, where $\varepsilon \geq 0$.

Before proceeding, it can be easily shown that Lemma 5(iv) holds for $n = 0$ and $x_1 \in \mathcal{N}$, by noting that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(x_1;h_1+\varepsilon) - \delta_0(x_1;h_1) = 0 \leq \varepsilon$. To prove Lemma 5(iv) for arbitrary x_1 and n, the induction hypothesis $\delta_{n-1}(x_1;h_1+\varepsilon) - \delta_{n-1}(x_1;h_1) \leq \varepsilon$ is assumed. For an (x_1, n) for which the induction hypothesis holds and $x_1 \geq 2$, $n \geq 1$, (11) is used to obtain

$$
\begin{aligned}
&\delta_n(x_1;h_1+\varepsilon) - \delta_n(x_1;h_1) \\
&= -\varepsilon + (1-p_1)[\delta_{n-1}(x_1;h_1+\varepsilon) - \delta_{n-1}(x_1;h_1)] + p_1[\delta_{n-1}(x_1-1;h_1+\varepsilon) - \delta_{n-1}(x_1-1;h_1)] \\
&\quad + p_2[\min\{n\pi + h_1 + \varepsilon, T(\pi+h_0) + K + \delta_{n-1}(x_1-1;h_1+\varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi+h_0) + K + \delta_{n-1}(x_1-1;h_1)\}] \\
&\quad - p_2[\min\{n\pi + h_1 + \varepsilon, T(\pi+h_0) + K + \delta_{n-1}(x_1;h_1+\varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi+h_0) + K + \delta_{n-1}(x_1;h_1)\}] \\
&\leq -\varepsilon + (1-p_1)[\delta_{n-1}(x_1;h_1+\varepsilon) - \delta_{n-1}(x_1;h_1)] + p_1[\delta_{n-1}(x_1-1;h_1+\varepsilon) - \delta_{n-1}(x_1-1;h_1)] \\
&\quad + p_2[\max\{\varepsilon, \delta_{n-1}(x_1-1;h_1+\varepsilon) - \delta_{n-1}(x_1-1;h_1)\}] \\
&\quad - p_2[\min\{\varepsilon, \delta_{n-1}(x_1;h_1+\varepsilon) - \delta_{n-1}(x_1;h_1)\}] \qquad (42) \\
&= -\varepsilon + (1-p_1-p_2)[\delta_{n-1}(x_1;h_1+\varepsilon) - \delta_{n-1}(x_1;h_1)] \\
&\quad + p_1[\delta_{n-1}(x_1-1;h_1+\varepsilon) - \delta_{n-1}(x_1-1;h_1)] + p_2\varepsilon \leq 0. \qquad (43)
\end{aligned}
$$

Inequality (42) is by Lemma 7. (43) follows from the induction hypothesis. Thus it can be established that $\delta_n(x_1;h_1+\varepsilon) - \delta_n(x_1;h_1) \leq \varepsilon$ over $\{(x_1,n) : x_1, n \in \mathcal{N}\}$.

(v) Similar to the proof of (i), the proof is done by induction first on n for $x_1 = 1$ and then for arbitrary x_1 and n. We first show that Lemma 5(v) is valid for $x_1 = 1$: $\delta_n(1;h_0+\varepsilon) - \delta_n(1;h_0) \geq -T\varepsilon$. Note that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(1;h_0+\varepsilon) - \delta_0(1;h_0) = 0 \geq -T\varepsilon$.

As an induction hypothesis for $n \geq 2$, assume that $\delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0) \geq -t\varepsilon$. Then

$$
\begin{aligned}
\delta_n(1;h_0+\varepsilon) - \delta_n(1;h_0) &= (1-p_1)[\delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0)] \\
&\quad - p_2[\min\{n\pi + h_1, T(\pi+h_0) + K + T\varepsilon + \delta_{n-1}(1;h_0+\varepsilon)\} \\
&\qquad - \min\{n\pi + h_1, T(\pi+h_0) + K + \delta_{n-1}(1;h_0)\}] \\
&\geq (1-p_1)[\delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0)] \\
&\quad - p_2[\max\{0, T\varepsilon + \delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0)\}] \qquad (44) \\
&= (1-p_1-p_2)(\delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0)) - p_2 T\varepsilon \geq -T\varepsilon. \qquad (45)
\end{aligned}
$$

Inequality (44) follows from applying Lemma 7. (45) is by the induction hypothesis $\delta_{n-1}(1;h_0+\varepsilon) - \delta_{n-1}(1;h_0) \geq -T\varepsilon$, which completes the proof for $\delta_n(1;h_0+\varepsilon) - \delta_n(1;h_0) \geq -T\varepsilon$, for every $n \in \mathcal{N}$.

Before proceeding, it can be easily shown that Lemma 5(v) holds for $n = 0$ and $x_1 \in \mathcal{N}$, by noting that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(x_1; h_0 + \varepsilon) - \delta_0(x_1; h_0) = 0 \geq -T\varepsilon$. To prove Lemma 5(v) for arbitrary x_1 and n, the induction hypothesis $\delta_{n-1}(x_1; h_0 + \varepsilon) - \delta_{n-1}(x_1; h_0) \geq -T\varepsilon$ is assumed. For an (x_1, n) for which the induction hypothesis holds and $x_1 \geq 2$, $n \geq 1$, (11) is used to obtain

$$\begin{aligned}
&\delta_n(x_1; h_0 + \varepsilon) - \delta_n(x_1; h_0) \\
&= (1 - p_1)[\delta_{n-1}(x_1; h_0 + \varepsilon) - \delta_{n-1}(x_1; h_0)] + p_1[\delta_{n-1}(x_1 - 1; h_0 + \varepsilon) - \delta_{n-1}(x_1 - 1; h_0)] \\
&\quad + p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + T\varepsilon + \delta_{n-1}(x_1 - 1; h_0 + \varepsilon)\} \\
&\quad - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 - 1; h_0)\}] \\
&\quad - p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + T\varepsilon + \delta_{n-1}(x_1; h_0 + \varepsilon)\} \\
&\quad\ - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1; h_0)\}] && (46) \\
&\geq (1 - p_1)[\delta_{n-1}(x_1; h_0 + \varepsilon) - \delta_{n-1}(x_1; h_0)] + p_1[\delta_{n-1}(x_1 - 1; h_0 + \varepsilon) - \delta_{n-1}(x_1 - 1; h_0)] \\
&\quad + p_2[\min\{0, \varepsilon + \delta_{n-1}(x_1 - 1; h_0 + \varepsilon) - \delta_{n-1}(x_1 - 1; h_0)\}] \\
&\quad - p_2[\max\{0, \varepsilon + \delta_{n-1}(x_1; h_0 + \varepsilon) - \delta_{n-1}(x_1; h_0)\}] && (47) \\
&= (1 - p_1 - p_2)[\delta_{n-1}(x_1; h_0 + \varepsilon) - \delta_{n-1}(x_1; h_0)] \\
&\quad + p_1[\delta_{n-1}(x_1 - 1; h_0 + \varepsilon) - \delta_{n-1}(x_1 - 1; h_0)] - p_2 T\varepsilon \geq -T\varepsilon. && (48)
\end{aligned}$$

Inequality (47) is by Lemma 7. (48) follows from the induction hypothesis. Thus it can be established that $\delta_n(x_1; h_0 + \varepsilon) - \delta_n(x_1; h_0) \geq -T\varepsilon$ over $\{(x_1, n) : x_1, n \in \mathcal{N}\}$.

(vi) Similar to the proof of (v), the proof is done by induction first on n for $x_1 = 1$ and then for arbitrary x_1 and n. We first show that lemma is valid for $x_1 = 1$: $\delta_n(1; K+\varepsilon) - \delta_n(1; K) \geq -\varepsilon$. Note that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(1; K + \varepsilon) - \delta_0(1; K) = 0 \geq -\varepsilon$.

As an induction hypothesis for $n \geq 2$, assume that $\delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K) \geq -\varepsilon$. Then

$$\begin{aligned}
&\delta_n(1; K + \varepsilon) - \delta_n(1; K) = (1 - p_1)[\delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K)] \\
&\quad - p_2[\min\{n\pi + h_1, T(\pi + h_0) + K + \varepsilon + \delta_{n-1}(1; K + \varepsilon)\} \\
&\quad\ - \min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(1; K)\}] \\
&\geq (1 - p_1)[\delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K)] \\
&\quad - p_2[\max\{0, \varepsilon + \delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K)\}] && (49) \\
&= (1 - p_1 - p_2)(\delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K)) - p_2\varepsilon \geq -\varepsilon. && (50)
\end{aligned}$$

Inequality (49) follows from applying Lemma 7. (50) is by the induction hypothesis $\delta_{n-1}(1; K + \varepsilon) - \delta_{n-1}(1; K) \geq -\varepsilon$, which completes the proof for $\delta_n(1; K+\varepsilon) - \delta_n(1; K) \geq -\varepsilon$, for every $n \in \mathcal{N}$.

Before proceeding, it can be easily shown that Lemma 5(vi) holds for $n = 0$ and $x_1 \in \mathcal{N}$, by noting that $\delta_0(\cdot) = 0$. Thus it satisfies $\delta_0(x_1; K + \varepsilon) - \delta_0(x_1; K) = 0 \geq -\varepsilon$. To prove Lemma 5(v) for arbitrary x_1 and n, the induction hypothesis $\delta_{n-1}(x_1; K+\varepsilon) - \delta_{n-1}(x_1; K) \geq -\varepsilon$ is assumed. For an (x_1, n) for which the induction hypothesis holds and $x_1 \geq 2$, $n \geq 1$, (11) is used to obtain

$$\begin{aligned}
&\delta_n(x_1; K+\varepsilon) - \delta_n(x_1; K) \\
&= (1-p_1)[\delta_{n-1}(x_1; K+\varepsilon) - \delta_{n-1}(x_1; K)] + p_1[\delta_{n-1}(x_1-1; K+\varepsilon) - \delta_{n-1}(x_1-1; K)] \\
&\quad + p_2[\min\{n\pi + h_1, T(\pi+h_0) + K + \varepsilon + \delta_{n-1}(x_1-1; K+\varepsilon)\} \\
&\quad\quad - \min\{n\pi + h_1, T(\pi+h_0) + K + \delta_{n-1}(x_1-1; K)\}] \\
&\quad - p_2[\min\{n\pi + h_1, T(\pi+h_0) + K + \varepsilon + \delta_{n-1}(x_1; K+\varepsilon)\} \\
&\quad\quad - \min\{n\pi + h_1, T(\pi+h_0) + K + \delta_{n-1}(x_1; K)\}] \\
&\geq (1-p_1)[\delta_{n-1}(x_1; K+\varepsilon) - \delta_{n-1}(x_1; K)] + p_1[\delta_{n-1}(x_1-1; K+\varepsilon) - \delta_{n-1}(x_1-1; K)] \\
&\quad + p_2[\min\{0, \varepsilon + \delta_{n-1}(x_1-1; K+\varepsilon) - \delta_{n-1}(x_1-1; K)\}] \\
&\quad - p_2[\max\{0, \varepsilon + \delta_{n-1}(x_1; K+\varepsilon) - \delta_{n-1}(x_1; K)\}] \qquad (51) \\
&= (1-p_1-p_2)[\delta_{n-1}(x_1; K+\varepsilon) - \delta_{n-1}(x_1; K)] \\
&\quad + p_1[\delta_{n-1}(x_1-1; K+\varepsilon) - \delta_{n-1}(x_1-1; K)] - p_2\varepsilon \geq -\varepsilon. \qquad (52)
\end{aligned}$$

Inequality (51) is by Lemma 7. (52) follows from the induction hypothesis. Thus it can be established that $\delta_n(x_1; K+\varepsilon) - \delta_n(x_1; K) \geq -\varepsilon$ over $\{(x_1, n) : x_1, n \in \mathcal{N}\}$. □

Proof of Lemma 6: Lemma 6 says that either $\delta_n(x_1; p_2) > (n+1-T)\pi + h_1 - Th_0 - K$ or $\delta_n(x_1; p_2) \leq (n+1-T)\pi + h_1 - Th_0 - K$. More importantly, the inequality that holds with p_2 continues to hold with $p_2 + \varepsilon$. Similarly, the inequality that holds with $p_2 + \varepsilon$ continues to hold with p_2. These later allow us to draw (in)sensitivity conclusions when $\varepsilon < 0$ as well as when $\varepsilon > 0$.

The proof is by induction on n and starts by checking the inequalities with $n = 0$. Note that $\delta_0(x_1; p_2) = \delta_0(x_1; p_2 + \varepsilon) = 0$. Either $\delta_0(x_1; p_2) = \delta_0(x_1; p_2 + \varepsilon) > (n+1-T)\pi + h_1 - Th_0 - K$ or $\delta_0(x_1; p_2) = \delta_0(x_1; p_2 + \varepsilon) \leq (n+1-T)\pi + h_1 - Th_0 - K$, so either (i) or (ii) is satisfied.

In the proof, we consider two mutually exclusive cases: Rejection and Acceptance of a request by retailer 1. In the rejection case, we first prove that $\delta_n(x_1; p_2) = \delta_n(x_1; p_2 + \varepsilon)$. This equality immediately leads to either (i) or (ii), as in the last paragraph. In the acceptance case, we show that statement (ii) holds.

The induction argument first addresses $x_1 = 1$ and then addresses $x_1 \geq 2$. For the induction argument with $x_1 = 1$, we make the induction hypothesis that $\delta_{n-1}(x_1 = 1; p_2)$ and $\delta_{n-1}(x_1 = 1; p_2 + \varepsilon)$ satisfy either inequality (i) or (ii) in period $n-1$. Then we analyze the validity of the lemma in period n.

Rejection: $\delta_{n-1}(x_1 = 1; p_2) > (n-T)\pi + h_1 - Th_0 - K$. The induction hypothesis provides $\delta_{n-1}(x_1 = 1; p_2) = \delta_{n-1}(x_1 = 1; p_2 + \varepsilon) > (n-T)\pi + h_1 - Th_0 - K$, which can be used to deal with the terms in the three brackets below one by one:

$$\begin{aligned}
&\delta_n(x_1 = 1; p_2 + \varepsilon) - \delta_n(x_1 = 1; p_2) \\
= \quad & (1 - p_1)\left[\delta_{n-1}(x_1 = 1; p_2 + \varepsilon) - \delta_{n-1}(x_1 = 1; p_2)\right] + \varepsilon(n\pi + h_1) \\
& -(p_2 + \varepsilon)\left[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 = 1; p_2 + \varepsilon)\}\right] \\
& +p_2\left[\min\{n\pi + h_1, T(\pi + h_0) + K + \delta_{n-1}(x_1 = 1; p_2)\}\right] \\
= \quad & 0 + \varepsilon(n\pi + h_1) - (p_2 + \varepsilon)(n\pi + h_1) + p_2(n\pi + h_1) \quad = \quad 0.
\end{aligned}$$

Since $\delta_n(x_1 = 1; p_2) = \delta_n(x_1 = 1; p_2 + \varepsilon)$, either (i) or (ii) must be true. This completes the inductive argument for $x_1 = 1$ for the rejection case.

Acceptance: $\delta_{n-1}(x_1 = 1; p_2) \leq (n - T)\pi + h_1 - Th_0 - K$. From the induction hypothesis, it follows that $\delta_{n-1}(x_1 = 1; p_2 + \varepsilon) \leq (n - T)\pi + h_1 - Th_0 - K$. From Lemma 3, we know that $\delta_n(x_1 = 1; p_2 + \varepsilon) \leq \delta_{n-1}(x_1 = 1; p_2 + \varepsilon) + \pi$. Thus,

$$\delta_n(x_1 = 1; p_2 + \varepsilon) \leq \delta_{n-1}(x_1 = 1; p_2 + \varepsilon) + \pi \leq (n + 1 - T)\pi + h_1 - Th_0 - K.$$

Combining Lemma 5(iii) with $\delta_n(x_1 = 1; p_2 + \varepsilon) \leq (n + 1 - T)\pi + h_1 - Th_0 - K$, we obtain $\delta_n(x_1 = 1; p_2) \leq \delta_n(x_1 = 1; p_2 + \varepsilon) \leq (n + 1 - T)\pi + h_1 - Th_0 - K$. This establishes (ii) and completes the inductive argument for $x_1 = 1$ for the acceptance case.

Until now, we have proved that either (i) or (ii) holds over the pairs $(n = 0, x_1 \in \mathcal{N})$ and $(n \in \{0, ..., N\}, x_1 = 1)$. What remains is to extend this result to a pair (n, x_1) for $n \geq 1$ and $x_1 \geq 2$. This can be done inductively by moving from $(n \in \{0, ..., N\}, x_1 = 1)$ to $(n \in \{0, ..., N\}, x_1 = 2)$, then to $(n \in \{0, ..., N\}, x_1 = 3)$, and so on. For a fixed x_1, we also traverse the points in $(n \in \{0, ..., N\}, x_1)$ in order of increasing n. Thus, while arguing for (i) or (ii) at a pair (n, x_1), we can assume the following induction hypotheses.

At the pair $(n - 1, x_1)$:

$$\begin{aligned}
&(i) \quad \delta_{n-1}(x_1; p_2) = \delta_{n-1}(x_1; p_2 + \varepsilon) > (n - T)\pi + h_1 - Th_0 - K, \text{ or} \\
&(ii) \quad \delta_{n-1}(x_1; p_2) \leq \delta_{n-1}(x_1; p_2 + \varepsilon) \leq (n - T)\pi + h_1 - Th_0 - K.
\end{aligned}$$

At the pair $(n - 1, x_1 - 1)$:

$$\begin{aligned}
&(i) \quad \delta_{n-1}(x_1 - 1; p_2) = \delta_{n-1}(x_1 - 1; p_2 + \varepsilon) > (n - T)\pi + h_1 - Th_0 - K, \text{ or} \\
&(ii) \quad \delta_{n-1}(x_1 - 1; p_2) \leq \delta_{n-1}(x_1 - 1; p_2 + \varepsilon) \leq (n - T)\pi + h_1 - Th_0 - K.
\end{aligned}$$

We now analyze the validity of the lemma in period n again for two cases.

Rejection: $\delta_{n-1}(x_1; p_2) > (n - T)\pi + h_1 - Th_0 - K$. The induction hypothesis at $(n - 1, x_1)$ then implies

$$\delta_{n-1}(x_1; p_2) = \delta_{n-1}(x_1; p_2 + \varepsilon) > (n - T)\pi + h_1 - Th_0 - K, \tag{53}$$

which gives

$$\begin{aligned}\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_2)\} &= \min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_2+\varepsilon)\} \\ &= n\pi+h_1. \qquad (54)\end{aligned}$$

The statement in (53), when combined with Lemma 2(i), yields $\delta_{n-1}(x_1-1;p_2) > (n-T)\pi + h_1 - Th_0 - K$ and $\delta_{n-1}(x_1-1;p_2+\varepsilon) > (n-T)\pi + h_1 - Th_0 - K$. Because of the induction hypothesis at $(n-1, x_1-1)$, these two last inequalities are possible only if

$$\delta_{n-1}(x_1-1;p_2) = \delta_{n-1}(x_1-1;p_2+\varepsilon) > (n-T)\pi + h_1 - Th_0 - K, \qquad (55)$$

which yields

$$\begin{aligned}&\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_2)\} \\ &= \min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_2+\varepsilon)\} \\ &= n\pi+h_1. \qquad (56)\end{aligned}$$

Then by using (53-56), we have

$$\begin{aligned}&\delta_n(x_1;p_2+\varepsilon)-\delta_n(x_1;p_2) \\ =\ & (1-p_1)(\delta_{n-1}(x_1;p_2+\varepsilon)-\delta_{n-1}(x_1;p_2)) + p_1(\delta_{n-1}(x_1-1;p_2+\varepsilon)-\delta_{n-1}(x_1-1;p_2)) \\ & -p_2\Big[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_2+\varepsilon)\} \\ & \qquad -\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_2)\} \\ & \quad -\Big(\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_2+\varepsilon)\} \\ & \qquad -\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_2)\}\Big)\Big] \qquad (57) \\ & +\varepsilon\Big[\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1-1;p_2+\varepsilon)\} \\ & \quad -\min\{n\pi+h_1,T(\pi+h_0)+K+\delta_{n-1}(x_1;p_2+\varepsilon)\}\Big] \\ =\ & 0.\end{aligned}$$

Since $\delta_n(x_1;p_2) = \delta_n(x_1;p_2+\varepsilon)$, either (i) or (ii) must be true. This completes the inductive argument for $x_1 \geq 2$ for the rejection case.

Acceptance: $\delta_{n-1}(x_1;p_2) \leq (n-T)\pi + h_1 - Th_0 - K$. From the induction hypothesis at $(n-1, x_1)$, we have $\delta_{n-1}(x_1;p_2) \leq \delta_{n-1}(x_1;p_2+\varepsilon) \leq (n-T)\pi + h_1 - Th_0 - K$. Also from Lemma 3, we know that $\delta_n(x_1;p_2) \leq \delta_{n-1}(x_1;p_2) + \pi$. Thus,

$$\delta_n(x_1;p_2+\varepsilon) \leq \delta_{n-1}(x_1;p_2+\varepsilon) + \pi \leq (n+1-T)\pi + h_1 - Th_0 - K.$$

Combining Lemma 5(iii) with $\delta_n(x_1;p_2+\varepsilon) \leq (n+1-T)\pi + h_1 - Th_0 - K$, we obtain $\delta_n(x_1;p_2) \leq \delta_n(x_1;p_2+\varepsilon) \leq (n+1-T)\pi + h_1 - Th_0 - K$. This establishes (ii) and completes the inductive argument for $x_1 \geq 2$ for the acceptance case. □

Proof of Theorem 5: The key idea is that the replenishment amounts q_1^m and q_2^m in (14) and (15) must be nonnegative. However, the current restriction on β^m, i.e., $0 \leq \beta^m \leq \xi_1^L + \xi_2^L$, does not guarantee $q_1^m, q_2^m \geq 0$. This nonnegativity can be achieved by restricting β^m in view of (14) and (15), such as

$$-z_2^m + y_2^m - \hat{\xi}_2^L(y_1^m, y_2^m; \tau) + \xi_1^N + x_2^N \leq \beta^m \leq z_1^m - y_1^m + \hat{\xi}_1^L(y_1^m, y_2^m; \tau). \tag{58}$$

The cost with rebalancing can be obtained by minimizing

$$\min_Z \lim_{M \to \infty} \frac{1}{M} E\left[\sum_{m=1}^{M} \min\{V_N(z_1^m - \beta^m, z_2^m - \xi_1^L - \xi_2^L + \beta^m) : 0 \leq \beta^m \leq \xi_1^L + \xi_2^L\}\right]; \tag{59}$$

subject to (58). By ignoring (58), we obtain a lower bound for the cost with rebalancing. Since the cost with rebalancing is less than the cost without rebalancing, (59) is less than (13).

What remains is to mold (59) into (13). The infinite horizon objective function in (59) states that for each cycle m, L periods before the beginning of the cycle, (z_1^m, z_2^m) are chosen considering expected lead time demand. It can be seen from (59) that the problem to be minimized for each cycle m can be denoted as

$$g(z_1^m, z_2^m) = E\left[\min\{V_N(z_1^m - \beta^m, z_2^m - \xi_1^L - \xi_2^L + \beta^m) : 0 \leq \beta^m \leq \xi_1^L + \xi_2^L\}\right].$$

$g(\cdot, \cdot)$ is a stationary function, i.e., independent of cycles. Therefore, with a similar argument that led to (10), the long-run average cost problem in (59) can be reduced to

$$\min_{z_1, z_2} E\left[\min_\beta\{V_N(z_1 - \beta, z_2 - \xi_1^L - \xi_2^L + \beta) : 0 \leq \beta \leq \xi_1^L + \xi_2^L\}\right]. \tag{60}$$

(60) is equal to (59) and is a *lower bound* on the average cost given in (13). □

Justification for Not Transshipping for Backorders

To model transshipments for backorders, we need to extend our formulation to allow for two stages of decision making in each period: In the first stage, we decide on a transshipment for a current customer and in the second stage, we consider a transshipment for an outstanding backorder. This model requires a dynamic programming formulation with two stages in each period. The costs-to-go in period n are denoted by V_n and Y_n for stages 1 and 2, respectively.

We let $V_n(x_1, x_2)$ denote the minimum expected cost of the system in the remaining n periods with current inventories x_1 and x_2. V_n is the sum of the cost of transshipment for a current demand in period n and Y_n. We also let $Y_n(x_1, x_2)$ be the minimum expected cost of the two-retailer system including the cost of transshipment for a backorder in period n, plus the holding and backorder costs in period n, as well as all costs incurred in periods $n-1, n-2, \ldots, 1$. Since this cost V_n includes Y_n, which accounts for transshipments for backorders, it is an extension of the cost defined by (2-5).

For ease of notation, we let $K' = T(\pi + h_0) + K$. The cost Y_n can be expressed in terms of V_{n-1} after setting $V_0(x_1, x_2) = 0$:

$$\begin{aligned} Y_n(x_1,x_2) &= \min\{V_{n-1}(x_1,x_2) + \pi(-x_2) + h_1 x_1, \\ &\qquad K' + V_{n-1}(x_1-1, x_2+1) + \pi(-x_2-1) + h_1(x_1-1)\}, \\ &\qquad x_1 \geq 1,\ x_2 \leq -1,\ n \geq 1. \end{aligned} \tag{61}$$

$$\begin{aligned} Y_n(x_1,x_2) &= \min\{V_{n-1}(x_1,x_2) + \pi(-x_1) + h_2 x_2, \\ &\qquad K' + V_{n-1}(x_1+1, x_2-1) + \pi(-x_1-1) + h_2(x_2-1)\}, \\ &\qquad x_1 \leq -1,\ x_2 \geq 1,\ n \geq 1. \end{aligned} \tag{62}$$

$$Y_n(x_1,x_2) = V_{n-1}(x_1,x_2) + h_1 x_1 + h_2 x_2, \quad x_1, x_2 \geq 0,\ n \geq 1. \tag{63}$$

$$Y_n(x_1,x_2) = V_{n-1}(x_1,x_2) + \pi(-x_1 - x_2), \quad x_1, x_2 \leq 0,\ n \geq 1. \tag{64}$$

Depending on inventory and backorder levels in period n, we can write V_n in terms of Y_n:

$$\begin{aligned} V_n(x_1,x_2) &= p_1 Y_n(x_1-1, x_2) + (1-p_1-p_2) Y_n(x_1,x_2) \\ &\quad + p_2 \min\{Y_n(x_1, x_2-1), K' + Y_n(x_1-1, x_2)\}, \quad x_1 \geq 1,\ x_2 \leq 0. \end{aligned} \tag{65}$$

$$\begin{aligned} V_n(x_1,x_2) &= p_2 Y_n(x_1, x_2-1) + (1-p_1-p_2) Y_n(x_1,x_2) \\ &\quad + p_1 \min\{Y_n(x_1-1, x_2), K' + Y_n(x_1, x_2-1)\}, \quad x_1 \leq 0,\ x_2 \geq 1. \end{aligned} \tag{66}$$

$$\begin{aligned} V_n(x_1,x_2) &= p_1 Y_n(x_1-1, x_2) + p_2 Y_n(x_1, x_2-1) \\ &\quad + (1-p_1-p_2) Y_n(x_1,x_2), \quad x_1, x_2 \leq 0 \text{ or } x_1, x_2 \geq 1. \end{aligned} \tag{67}$$

$$V_0(x_1,x_2) = 0, \quad \text{for all } x_1, x_2. \tag{68}$$

In short, (61-68) replaces (2-5) to study transshipping to meet backorders.

In the remainder, we show that cost V_n can be minimized without transshipping to meet backorders. Formally, what this means is that $Y_n(x_1, x_2) = V_{n-1}(x_1, x_2) + \pi(-x_2) + h_1 x_1$ in (61) and $Y_n(x_1, x_2) = V_{n-1}(x_1, x_2) + \pi(-x_1) + h_2 x_2$ in (62). To obtain such simplifications in (61-62), we work with marginal costs and obtain their properties. Since we are interested in transshipments, we define the marginal costs only when one of the retailers, say retailer 2, runs out of inventory:

$$\delta_n(x_1,x_2) := Y_n(x_1-1, x_2) - Y_n(x_1, x_2-1) \text{ for } x_2 \leq 0,\ n \geq 1, \tag{69}$$

$$\gamma_n(x_1,x_2) := V_n(x_1-1, x_2) - V_n(x_1, x_2-1) \text{ for } x_2 \leq 0,\ n \geq 0. \tag{70}$$

When $x_1 \geq 2$ and $x_2 \leq -1$, $Y_n(x_1, x_2 - 1)$ and $Y_n(x_1 - 1, x_2)$ are both given by (61). Then by using (69-70), we obtain

$$\begin{aligned} \delta_n(x_1,x_2) &= \gamma_{n-1}(x_1,x_2) - \pi - h_1 + \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2+1)\} \\ &\quad - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1,x_2)\}, \quad x_1 \geq 2, x_2 \leq -1. \end{aligned} \tag{71}$$

Similarly,

$$\delta_n(x_1,x_2) = \gamma_{n-1}(x_1,x_2) - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1,x_2)\},$$
$$x_1 \ge 1, x_2 = 0 \text{ or } x_1 = 1, x_2 \le -1. \quad (72)$$
$$\delta_n(x_1,x_2) = \gamma_{n-1}(x_1,x_2), \quad x_1, x_2 \le 0. \quad (73)$$

Moreover, for $n \ge 2$, γ_n is obtained from δ_n.

$$\gamma_n(x_1,x_2) = p_1\delta_n(x_1-1,x_2) + (1-p_1-p_2)\delta_n(x_1,x_2) + p_2\Big[\delta_n(x_1,x_2-1) + \min\{0, K' + \delta_n(x_1-1,x_2)\} - \min\{0, K' + \delta_n(x_1,x_2-1)\}\Big],$$
$$x_1 \ge 2,\ x_2 \le 0. \quad (74)$$

$$\gamma_n(x_1,x_2) = p_1\delta_n(0,x_2) + (1-p_1-p_2)\delta_n(1,x_2) + p_2[\delta_n(1,x_2-1) - \min\{0, K' + \delta_n(1,x_2-1)\}], \quad x_1 = 1,\ x_2 \le 0. \quad (75)$$
$$\gamma_n(x_1,x_2) = p_1\delta_n(x_1-1,x_2) + p_2\delta_n(x_1,x_2-1) + (1-p_1-p_2)\delta_n(x_1,x_2), \quad x_1, x_2 \le 0. \quad (76)$$

In addition, for $n = 1$, we have $\gamma_{n-1} = \gamma_0$ and $\gamma_0(x_1,x_2) = 0$ for all x_1, x_2 from (68). These along with (71-76) give us a recursive representation of δ_n and γ_n in terms of each other.

Having obtained δ_n and γ_n, we can cast transshipment decisions in terms of these marginal costs. When $x_1 \ge 1$ and $x_2 \le 0$ in period n, a unit is transshipped from retailer 1 to retailer 2 to satisfy a new demand at retailer 2 (the first stage decision) if and only if

$$\delta_n(x_1,x_2) + K' \le 0. \quad (77)$$

When $x_1 \ge 1$ and $x_2 \le -1$ in period n, a unit is transshipped from retailer 1 to retailer 2 to satisfy a backorder at retailer 2 (the second stage decision) if and only if

$$\gamma_{n-1}(x_1,x_2+1) + K' \le \pi + h_1. \quad (78)$$

Note that these marginal costs can potentially depend on inventories at both retailers unlike the case when transshipments for backorders are not allowed.

We already know that there is a stopping time $\mathcal{T}$ that separates a cycle into transshipment acceptance and rejection windows when transshipments for backorders are not allowed. We can have similar acceptance and rejection windows for transshipments for new demands and backorders by showing that $\delta_n(x_1,x_2)$ and $\gamma_n(x_1,x_2)$ are both non-increasing in x_1 and n. These properties can be established by following the footsteps of the proof of Lemma 2. In short, we call these as the non-increasing property of marginal cost functions. These monotone properties imply that there are two stopping times $\mathcal{T}_N$ and $\mathcal{T}_B$. We reject transshipments for new demands in period $n \in \{1,2,\ldots,\mathcal{T}_N\}$ and reject transshipments for backorders in period $n \in \{1,2,\ldots,\mathcal{T}_B\}$.

Otherwise, transshipments are accepted. Optimal transshipment policies remain to be hold-back level based even when we allow transshipments for backorders.

Finding hold-back policies with transshipments for backorders encourages us to check if these policies really depend on the number of outstanding backorders. Statements (I) and (II) will help there.

(I) One and only one of the following two statements holds for $x_1 \geq 1$ and $x_2 \leq 0$.

$$\begin{aligned} (a) \quad \gamma_n(x_1, x_2 - 1) \ &= \ \gamma_n(x_1, x_2) \ > \ \pi + h_1 - K', \text{ or} \\ (b) \quad \gamma_n(x_1, x_2 - 1) \ &\leq \ \gamma_n(x_1, x_2) \ \leq \ \pi + h_1 - K'. \end{aligned}$$

(II) One and only one of the following two statements holds for $x_1 \geq 1$ and $x_2 \leq 0$.

$$\begin{aligned} (c) \quad \delta_n(x_1, x_2 - 1) \ &= \ \delta_n(x_1, x_2) \ > \ -K', \text{ or} \\ (d) \quad \delta_n(x_1, x_2 - 1) \ &\leq \ \delta_n(x_1, x_2) \ \leq \ -K'. \end{aligned}$$

We postpone the validation of these statements and focus on their implication. Suppose that $\delta_n(x_1^0, x_2^0) \leq -K'$ for given $x_1^0 \geq 1, x_2^0 \leq 0$. By (d), we have $\delta_n(x_1^0, x_2^0 - k) \leq -K'$ for $k \geq 0$, so we transship with backorders of x_2^0 units or more. If $\delta_n(x_1^0, x_2^0 + k) > -K'$ for some $0 \leq k \leq -x_2^0$, using (c), we obtain $\delta_n(x_1^0, x_2^0) > -K'$ which is a contradiction. Therefore, we must have $\delta_n(x_1^0, x_2^0 + k) \leq -K'$ for all $0 \leq k \leq -x_2^0$ and we transship with backorders of x_2^0 units or fewer. Similar arguments can be used with conditions (a) and (b) to establish that transshipment decisions do not depend on the number of outstanding backorders.

We now connect the transshipments for new demands to transshipments for backorders via following statements:

(e) $\delta_n(x_1, x_2) > -K'$, if $\gamma_{n-1}(x_1, x_2 + 1) > \pi + h_1 - K'$.

(f) $\delta_n(x_1, x_2) \leq -K'$, if $\gamma_{n-1}(x_1, x_2 + 1) \leq \pi + h_1 - K'$.

To verify (e) and (f) when $[x_1 \geq 1,\ x_2 = 0 \text{ or } x_1 = 1,\ x_2 \leq -1]$, (72) can be rewritten as

$$\delta_n(1, x_2) \ = \ -\min\{\pi + h_1 - \gamma_{n-1}(1, x_2), K'\}.$$

If $\gamma_{n-1}(1, x_2 + 1) > \pi + h_1 - K'$, then $\gamma_{n-1}(1, x_2) = \gamma_{n-1}(1, x_2 + 1) > \pi + h_1 - K'$ by (a). Using $\pi + h_1 - \gamma_{n-1}(1, x_2) < K'$

$$\delta_n(1, x_2) \ = \ -(\pi + h_1 - \gamma_{n-1}(1, x_2)) \ > \ -K',$$

which establishes (e).

On the other hand, if $\gamma_{n-1}(1, x_2 + 1) \leq \pi + h_1 - K'$, then $\gamma_{n-1}(1, x_2) \leq \gamma_{n-1}(1, x_2 + 1) \leq \pi + h_1 - K'$ by (b). Then $\delta_n(1, x_2) = -K'$, which establishes (f). When $[x_1 \geq 2,\ x_2 \leq -1]$, (e) and (f) can be verified similarly.

When (e) and (f) are combined, $\delta_n(x_1, x_2) \leq -K'$ if and only if $\gamma_{n-1}(x_1, x_2 + 1) \leq \pi + h_1 - K'$. Thus, the optimal response to a transshipment for a new demand or a backorder is the same. If a transshipment request for a new demand is accepted (rejected) with inventory (x_1, x_2) in period n, a transshipment request for a backorder is also accepted (rejected) with inventory (x_1, x_2) in period n. That is $\mathcal{T} = \mathcal{T}_N = \mathcal{T}_B$, so we switch from accepting transshipments to rejecting them at the same time for a new demand and a backorder. In particular, when we are considering a transshipment for a backorder, we must already be in the rejection window. Thus, every transshipment request for a backorder must be rejected. As a result, a backordered demand remains backordered until the next replenishment cycle.

We now verify statements (I) and (II) simultaneously by induction on n. First note that $\gamma_0(x_1, x_2 - 1) = \gamma_0(x_1, x_2) = 0$. Either $0 > \pi + h_1 - K'$ or $0 \le \pi + h_1 - K'$. So γ_0 satisfies either (a) or (b).

At $n = 1$, $\delta_1(x_1, x_2) = 0$ for $x_1, x_2 \le 0$, $\delta_1(x_1, x_2) = -\min\{\pi + h_1, K'\}$ for $x_1 \ge 1$, $x_2 = 0$ or $x_1 = 1$, $x_2 \le -1$, and $\delta_1(x_1, x_2) = -(\pi + h_1)$ for $x_1 \ge 2$, $x_2 \le -1$. For $x_1 \le 0$, $\delta_1(x_1, x_2 - 1) = \delta_1(x_1, x_2) = 0$, so either (c) or (d) is satisfied. Similarly, for $x_1 = 1$, $\delta_1(1, x_2 - 1) = \delta_1(1, x_2) = -\min\{\pi + h_1, K'\}$. Thus either (c) or (d) is satisfied. When $x_1 \ge 2$ and $x_2 \le -1$, $\delta_1(x_1, x_2 - 1) = \delta_1(x_1, x_2) = -(\pi + h_1)$, so either (c) or (d) is satisfied. When $x_1 \ge 2$, $x_2 = 0$, and $\pi + h_1 < K'$, $\delta_1(x_1, x_2 - 1) = -(\pi + h_1) = -\min\{\pi + h_1, K'\} = \delta_1(x_1, x_2) > -K'$. So (c) holds. When $x_1 \ge 2$, $x_2 = 0$, and $\pi + h_1 \ge K'$, $\delta_1(x_1, x_2 - 1) = -(\pi + h_1) \le -K' = -\min\{\pi + h_1, K'\} = \delta_1(x_1, x_2)$. So (d) holds. Thus δ_1 satisfies either (c) or (d).

When statement (I) is established for γ_n, the induction hypothesis assumes that statement (II) holds for δ_n. On the other hand, when statement (II) is established for δ_n, the induction hypothesis assumes statement (I) for γ_{n-1}. Since (I) and (II) hold for both γ_0 and δ_1, induction can start with either of these.

Consider two mutually exclusive sets of cases as an induction hypothesis: rejection and acceptance of a request by retailer 1. When an induction hypothesis is made on δ_n, the acceptance case refers to the acceptance of a transshipment request for a new demand. When the induction hypothesis is made on γ_n, the acceptance case refers to the acceptance of transshipment for an outstanding backorder. The induction argument addresses $x_1 \ge 1$. Verification for $x_1 = 1$ can be done similarly.

Statement (II) for $x_1 \ge 2$: To verify statement (II) for period n and inventory level x_1, assume the following induction hypotheses for $x_2 \le -1$, which stem from statement (I).

In period $n - 1$ with x_1, one and only one of the following statements holds.

$$\begin{array}{lllll} (a) & \gamma_{n-1}(x_1, x_2 - 1) = \gamma_{n-1}(x_1, x_2) = \gamma_{n-1}(x_1, x_2 + 1) & > & \pi + h_1 - K' \text{ or} \\ (b) & \gamma_{n-1}(x_1, x_2 - 1) \le \gamma_{n-1}(x_1, x_2) \le \gamma_{n-1}(x_1, x_2 + 1) & \le & \pi + h_1 - K'. \end{array}$$

In period $n - 1$ with $x_1 - 1$, one and only one of the following statements holds.

$$\begin{array}{ll} (a) & \gamma_{n-1}(x_1 - 1, x_2) = \gamma_{n-1}(x_1 - 1, x_2 + 1) > \pi + h_1 - K' \text{ or} \\ (b) & \gamma_{n-1}(x_1 - 1, x_2) \le \gamma_{n-1}(x_1 - 1, x_2 + 1) \le \pi + h_1 - K'. \end{array}$$

Rejection: $\gamma_{n-1}(x_1, x_2+1) > \pi + h_1 - K'$. The induction hypothesis says that (a) holds for $(n-1, x_1)$. Combining this fact with the monotonicity of marginal functions, it also follows that (a) holds for $(n-1, x_1-1)$. Then by using (71) for $x_2 \le -1$,

$$\begin{aligned}
&\delta_n(x_1, x_2) - \delta_n(x_1, x_2-1) \\
&= \gamma_{n-1}(x_1, x_2) - \gamma_{n-1}(x_1, x_2-1) + \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2+1)\} \\
&\quad - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2)\} - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2)\} \\
&\quad + \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2-1)\} = 0.
\end{aligned}$$

Since $\delta_n(x_1, x_2) = \delta_n(x_1, x_2-1)$, either (c) or (d) of statement (II) is satisfied.

Acceptance: $\gamma_{n-1}(x_1, x_2+1) \le \pi + h_1 - K'$. The induction hypothesis says that (b) holds for $(n-1, x_1)$. This fact is used to obtain the below difference equation.

$$\begin{aligned}
&\delta_n(x_1, x_2) - \delta_n(x_1, x_2-1) \\
&= \gamma_{n-1}(x_1, x_2) - \gamma_{n-1}(x_1, x_2-1) + \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2+1)\} \\
&\quad - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2)\} - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2)\} \\
&\quad + \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2-1)\} \\
&= \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2+1)\} - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2)\}.
\end{aligned}$$

For $(n-1, x_1-1)$, either induction hypothesis (a) or (b) holds. If (a) holds, $\delta_n(x_1, x_2) - \delta_n(x_1, x_2-1) = 0$. So either (c) or (d) of statement (II) is satisfied. If (b) holds, $\delta_n(x_1, x_2) - \delta_n(x_1, x_2-1) = \gamma_{n-1}(x_1-1, x_2+1) - \gamma_{n-1}(x_1-1, x_2) \ge 0$.

For the acceptance case, it should be also shown that when the induction hypothesis (b) holds for $(n-1, x_1-1)$, then $\delta_n(x_1, x_2) \le -K'$ for $x_2 \le -1$. From (71), $\delta_n(x_1, x_2)$ is as follows.

$$\begin{aligned}
\delta_n(x_1, x_2) &= \gamma_{n-1}(x_1, x_2) - \pi - h_1 \\
&+ \min\{\pi + h_1, K' + \gamma_{n-1}(x_1-1, x_2+1)\} - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2)\} \\
&= \gamma_{n-1}(x_1-1, x_2+1) - \pi - h_1 \le -K'.
\end{aligned}$$

The two inequalities used above follow from the induction hypotheses. Thus, statement (II) is verified for $x_1 \ge 2$ and $x_2 \le -1$. The case for $x_2 = 0$ can be verified similarly.

Statement (I) for $x_1 \ge 2$: Assume the following induction hypotheses, which stem from statement (II).

In period n with x_1, one and only one of the following statements holds.

$$\begin{array}{ll}
(c) & \delta_n(x_1, x_2-2) = \delta_n(x_1, x_2-1) = \delta_n(x_1, x_2) \;>\; -K' \text{ or} \\
(d) & \delta_n(x_1, x_2-2) \le \delta_n(x_1, x_2-1) \le \delta_n(x_1, x_2) \;\le\; -K'.
\end{array}$$

In period n with $x_1 - 1$, one and only one of the following statements holds.

$$\begin{array}{ll}
(c) & \delta_n(x_1-1, x_2-1) = \delta_n(x_1-1, x_2) > -K' \text{ or} \\
(d) & \delta_n(x_1-1, x_2-1) \le \delta_n(x_1-1, x_2) \le -K'.
\end{array}$$

Rejection: $\delta_n(x_1,x_2) > -K'$. The induction hypothesis states that (c) is satisfied for (n,x_1). Combining $\delta_n(x_1,x_2) > -K'$ with the monotonicity of marginal functions, $\delta_n(x_1-1,x_2) \geq \delta_n(x_1,x_2) > -K'$. So for (n,x_1-1), only (c) can be satisfied. By using the fact that for both (n,x_1) and (n,x_1-1), only statement (c)'s are satisfied, the difference equation of γ_n can be written starting with (74).

$$
\begin{aligned}
&\gamma_n(x_1,x_2) - \gamma_n(x_1,x_2-1)\\
&= p_1(\delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1)) + (1-p_1-p_2)(\delta_n(x_1,x_2) - \delta_n(x_1,x_2-1))\\
&\quad + p_2\Big[\delta_n(x_1,x_2-1) - \delta_n(x_1,x_2-2) + \min\{0, K' + \delta_n(x_1-1,x_2)\}\\
&\qquad - \min\{0, K' + \delta_n(x_1,x_2-1)\} - \min\{0, K' + \delta_n(x_1-1,x_2-1)\} + \min\{0, K' + \delta_n(x_1,x_2-2)\}\Big] = 0.
\end{aligned}
$$

Since $\gamma_n(x_1,x_2) = \gamma_n(x_1,x_2-1)$, either (a) or (b) of statement (I) is satisfied.

Acceptance: $\delta_n(x_1,x_2) \leq -K'$. The induction hypothesis states that (d) holds for (n,x_1). This hypothesis is used in the following difference equation for γ_n.

$$
\begin{aligned}
&\gamma_n(x_1,x_2) - \gamma_n(x_1,x_2-1)\\
&= p_1(\delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1)) + (1-p_1-p_2)(\delta_n(x_1,x_2) - \delta_n(x_1,x_2-1))\\
&\quad + p_2\Big[\delta_n(x_1,x_2-1) - \delta_n(x_1,x_2-2) + \min\{0, K' + \delta_n(x_1-1,x_2)\}\\
&\qquad - \min\{0, K' + \delta_n(x_1,x_2-1)\} - \min\{0, K' + \delta_n(x_1-1,x_2-1)\} + \min\{0, K' + \delta_n(x_1,x_2-2)\}\Big]\\
&= p_1(\delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1)) + (1-p_1-p_2)(\delta_n(x_1,x_2) - \delta_n(x_1,x_2-1))\\
&\quad + p_2\Big[\min\{0, K' + \delta_n(x_1-1,x_2)\} - \min\{0, K' + \delta_n(x_1-1,x_2-1)\}\Big]\\
&\geq p_1(\delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1)) + (1-p_1-p_2)(\delta_n(x_1,x_2) - \delta_n(x_1,x_2-1))\\
&\quad + p_2 \min\{0, \delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1)\} \geq 0.
\end{aligned}
$$

The second equality above is obtained by the induction hypothesis that $\delta_n(x_1,x_2) \leq -K'$. Then Lemma 7 is used to get the first inequality. For (n,x_1-1), whether (c) or (d) is satisfied, it follows that $\delta_n(x_1-1,x_2) - \delta_n(x_1-1,x_2-1) \geq 0$. Then, the result follows from this fact and the induction hypothesis.

For the acceptance case, it is also shown that $\gamma_n(x_1,x_2) \leq \pi + h_1 - K'$. For $x_2 \leq -1$, using (71) and Lemma 7,

$$
\begin{aligned}
&\delta_{n+1}(x_1,x_2)\\
&\geq \gamma_n(x_1,x_2) - \pi - h_1 + \min\{0, \gamma_n(x_1-1,x_2+1) - \gamma_{n-1}(x_1,x_2)\}\\
&= \gamma_n(x_1,x_2) - \pi - h_1 + \min\{0, \gamma_n(x_1-1,x_2+1) - \gamma_n(x_1,x_2+1) + \gamma_n(x_1,x_2+1) - \gamma_{n-1}(x_1,x_2)\}\\
&= \gamma_n(x_1,x_2) - \pi - h_1.
\end{aligned}
$$

$\gamma_n(x_1-1,x_2+1) - \gamma_n(x_1,x_2+1) \geq 0$ is by the monotonicity of marginal functions. $\gamma_n(x_1,x_2+1) - \gamma_{n-1}(x_1,x_2)$ follows from the induction hypothesis.

On the other hand, combining the monotonicity of marginal functions with the acceptance condition $\delta_n(x_1, x_2) \leq -K'$, $\delta_{n+1}(x_1, x_2) \leq \delta_n(x_1, x_2) \leq -K'$ or simply $\delta_{n+1}(x_1, x_2) \leq -K'$. Combining this result with $\delta_{n+1}(x_1, x_2) \geq \gamma_n(x_1, x_2) - \pi - h_1$, we get $\gamma_n(x_1, x_2) \leq \pi + h_1 - K'$.

For $x_2 = 0$, using (72), $\delta_{n+1}(x_1, x_2) = \gamma_n(x_1, x_2) - \min\{\pi + h_1, K' + \gamma_{n-1}(x_1, x_2)\} \geq \gamma_n(x_1, x_2) - \pi - h_1$. Again, by combining this result with the acceptance condition, it follows that $\gamma_n(x_1, x_2) < \pi + h_1 - K'$. Thus this completes the verification of statements (I) and (II).