ESSAYS ON IMPLEMENTABILITY AND MONOTONICITY

A Ph.D. Dissertation

by
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Department of Economics
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Ankara

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To my grandmother, Şaziye

ESSAYS ON IMPLEMENTABILITY AND MONOTONICITY

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of
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by

PELİN PASİN

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ESSAYS ON IMPLEMENTABILITY AND MONOTONICITY

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September, 2009

In this thesis we study the implementation problem with regard to the relation between monotonicity and implementability. Recent work in the field has shown that the implementability of a social choice rule strongly depends upon the compatibility between the monotonicity structures of the social choice rule and of the solution concept according to which implementation takes place. Different degrees of monotonicity of the social choice rules and game theoretic solution concepts can be determined via a generalized monotonicity function, strongest of which is called self-monotonicity. In this study, we determine the unique self-monotonicity of the Nash equilibrium concept and show that the monotonicities of a social choice rule are inherited from the unique self-monotonicity of the Nash equilibrium concept via the mechanisms that implement it. In particular, we show that the essential monotonicity is inherited via the Maskin-Vind type mechanism which is widely used in the characterization results. We also give a new characterization of strong Nash implementable social choice rules via critical profiles. We show that coalitional monotonicity when conjoined with three more conditions is both necessary and sufficient for implementability. Finally we determine a subset of subgame perfect Nash implementable social choice rules that satisfies conditions defined obtained by critical profiles. The results that are obtained in this thesis strongly support the view that implementation theory can be rewritten in terms of monotonicity and that this provides a better understanding of the theory.

Keywords: Implementation, Monotonicity, Self-monotonicity, Critical Profiles.

ÖZET

UYGULANABİLİRLİK VE TEKDÜZELİK ÜZERİNE ÇALIŞMALAR

Pasin, Pelin
Doktora, İktisat Bölümü
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Bu tezde, tekdüzelik ve uygulanabilirlik arasındaki ilişki bağlamında uygulama problemi çalışılmıştır. Güncel çalışmalar, bir sosyal seçim kuralının uygulanabilirliğinin, uygulanacak sosyal seçim kuralının sahip olduğu tekdüzelik yapılarıyla uygulamanın gerçekleşeceği çözüm kavramının tekdüzelik yapıları arasındaki uyumluluğa bağlı olduğunu göstermiştir. Sosyal seçim kurallarının ve oyun kuramsal çözüm kavramlarının, genelleştirilmiş bir tekdüzelik fonksiyonu aracılığıyla, en kuvvetlisinin öz-tekdüzelik olarak adlandırıldığı, değişik tekdüzelik dereceleri tanımlanabilir. Bu çalışmada, Nash denge kavramının öz-tekdüzeliği tek biçimde belirlenmiştir ve bir sosyal seçim kuralının tekdüzeliklerinin, Nash denge kavramının bu öz-tekdüzeliğinden, uygulayan mekanizmalar aracılığıyla taşındığı gösterilmiştir. Özellikle, temel tekdüzeliğin Maskin-Vind tarzı mekanizmalar aracılığıyla taşındığı gösterilmitir. Ayrıca, kuvvetli Nash uygulanabilir seçim kurallarının kritik profiller aracılığıyla yeni bir karakterizasyonu yapılmıştır. Son olarak, üstyetkin denge uygulanabilir seçim kuralları için yeni yeter şartlar tanımlanmıştır. Bu tezde elde edilen sonuçlar, uygulama kuramının tekdüzelik cinsinden yeniden yazılabileceği ve bunun kuramı daha iyi anlamak için önemli olduğu görüşünü güçlü bir şekilde destekler.

Anahtar Kelimeler: Uygulama, Tekdüzelik, Öz-tekdüzelik, Kritik profiller.

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CHAPTER 1

INTRODUCTION

In many economic and social situations, decisions must be made whose outcomes affect the agents in the society. Often it is desirable to make such decisions by taking into account the preferences of each agent who will directly or indirectly be subjected to the consequences of these decisions. Social choice theory is concerned with the various rules that take agents' preferences over a set of alternatives and return a collective decision. Social choice theory studies the merits and flaws of different rules in different situations and, as a result, attempts to determine which rules are best suited to be applied in these situations.

Achieving a collective decision by using a predetermined social choice rule is only possible on the assumption that the agents' preferences are known by the central authority (social planner) who is going to employ the social choice rule. In most real life cases the central authority does not have this information. Sometimes it is not possible to collect all this information because of physical constraints. When it is possible, asking agents their preferences directly may seem to be a natural way to overcome this problem. However, there is no guarantee that the agents will state their true preferences. Especially in situations where they have information about others' preferences, they may have an incentive to change the outcome by stating false preferences.

Economic institutions can be viewed as coordination mechanisms via which information from the agents in a society is communicated and processed to achieve socially desirable outcomes. Taking this view, Hurwicz (1960, 1972), in two highly influential papers, developed an analytic framework for the study of economic institutions. Hurwicz's formalization gave precision to many of the concepts that had been used in a long standing debate on how to organize markets and make social decisions. Furthermore, he incorporated the incentives problem mentioned above into his formalization. In search of a better analysis and understanding of economic institutions, Hurwicz provided a mathematical foundation for mechanism design and introduced incentive compatibility.

A mechanism has two components; a joint message space and an outcome function. Each agent is endowed with a set of messages which identifies the possible actions that the agent can take. The outcome function assigns to each joint message received from all the agents, an outcome from the set of alternatives. When combined with the preferences of the agents over the alternatives, this construction leads to a normal form game. A game theoretic solution concept that reflects the mode of behavior of the agents can then be used to specify the equilibria of the game. In contrast to game theory, in mechanism design, a game is not a given structure but rather something to design to obtain a socially desirable outcome. Hurwicz's formulation of the design problem puts emphasis on two issues; informational efficiency and incentive compatibility. The informational efficiency of a mechanism is determined by the size of its message space and the complexity of computing each message. A mechanism is incentive compatible if it is immune to manipulation by the agents.

Implementation theory studies an abstract generalization of the design problem. A mechanism is said to implement a social choice rule in a solution concept if the equilibrium outcomes of each game at each preference profile coincide with the alternatives that are chosen by the social choice rule at each preference profile. A social choice rule is said to be implementable in a solution concept if there exists a mechanism that implements it. As the true preferences of the agents are not observable to the central authority (social planner) the implementing mechanism identifies the outcome at each possible profile. Implementation theory is concerned with determining the properties that should be satisfied by a social choice rule to be implemented in a solution concept. When each agent has complete information about the preferences of the others, it is natural to consider Nash equilibrium and its refinements for implementation. Maskin was the first to show which social choice rules can be implemented in Nash equilibrium. In his seminal paper, Maskin (1977) introduced a condition called "monotonicity" and showed that every Nash implementable social choice rule satisfies monotonicity. He also showed that when there are three or more agents, a social choice rule is Nash implementable if it satisfies monotonicity and no veto power. A social choice rule is monotonic if any alternative that is chosen at a preference profile is also chosen at all the profiles where the chosen alternative's ranking relative to any other alternative doesn't get worse from the view point of each agent. The lower contour set of an alternative for an agent is the set of alternatives that are preferred less than or equally to this alternative. Then, a social choice rule is monotonic, if any alternative that is chosen at a preference profile is also chosen at all the profiles where the chosen alternative's lower contour set is preserved for each agent. A social choice rule satisfies no veto power if, when all but one agent top ranks the same alternative, then this alternative is chosen.

The full characterization of Nash implementable social choice rules was first given by Moore and Repullo (1990). Their characterization result is based on the existence of a system of sets which satisfies some specific properties. They consider a Nash implementable social choice rule. At each preference profile they look at the set of alternatives that each agent can attain by unilateral deviation. By the definition of Nash equilibrium, at each profile for each agent these sets should be a subset of the lower contour set of the alternative that is chosen. They identified three conditions on these sets which should be satisfied by a Nash implementable social choice rule. The first is a strengthening of monotonicity and the other two are a weakening of no veto power. They then show that the existence of a system of sets satisfying these conditions is also sufficient for Nash implementability.

Danilov (1992) introduced essential monotonicity and showed that it is both necessary and sufficient for Nash implementability when the domain of the social choice rule consists of all possible orderings of the alternatives. Essential monotonicity is stronger than monotonicity. While in monotonicity the lower contour sets should be preserved, in essential monotonicity only the "essential elements" in the lower contour sets should be preserved. A weaker version of no veto power is embedded in the definition of an essential element. The sets of essential elements constitute a system of sets that satisfy Moore and Repullo's condition. Danilov's rather direct approach makes it much easier to check whether a social choice rule is implementable.[1]

The significance of monotonicity for implementability was supported by Danilov's result. A deeper understanding of the relation between monotonicity and implementability was provided by Kaya and Koray (2000). They proved

[1]The conditions that are mentioned for both characterization results are necessary and sufficient when there are at least three agents. When there are only two agents they both introduced additional conditions for implementability.

that the monotonicity properties of the social choice rules implementable in a particular solution concept are inherited from the monotonicity properties inherent to the solution concept itself, and they characterized the solution concepts which only implement monotonic social choice rules.

The notion of self-monotonicity which is introduced by Koray (2002) is based on these observations. As he puts it, regarding monotonicity as the preservation of an order structure on the domain of a social rule, it is natural to introduce different degrees of monotonicity in accordance with the strength of the order structure that is preserved. Roughly, a self-monotonicity for a social choice rule is the strongest monotonicity condition that it satisfies; meaning that the social choice rule fails to satisfy a stronger condition. This generalization easily carries over to the inheritance theorem by Kaya and Koray (2000) in the following way: a self-monotonicity of a social choice rule implementable in a particular solution concept must be inherited from a self-monotonicity inherent to the solution concept itself.

A dual approach to the implementation problem is expressed by the notion of a "critical profile." It was first introduced in a study by Koray, Adali, Erol, and Ordulu (2001) where they provided a simpler proof of the well known Müller-Satterthwaite theorem vial critical profiles. In another study Doğan and Koray (2007) explored more social choice theoretic implications of the notion and also provided a new characterization for two-person Nash implementable social choice rules. Roughly, a critical profile for an alternative a is a preference profile at which a is chosen and which has the following property: at any preference profile that is obtained from the critical profile by a reversal of an ordering between a and any alternative that is less preferred than a from the view point of some agent, a is not chosen by the social choice rule.

In this thesis, we further explore the relation of monotonicity and critical profiles to implementability. First, as an example to the inheritance property mentioned above we identify the self-monotonicity of the Nash equilibrium concept and determine the monotonicities that are inherited by Nash implementable social choice rules which are not necessarily self-monotonicities. The Nash equilibrium concept has a unique self-monotonicity which is carried over to the social choice rule via the mechanism that implements it. As there may be several mechanisms that implement a social choice rule the unique self-monotonicity of the Nash equilibrium concept may induce several monotonicities for the social choice rule. The smaller the size of the message space of the implementing mechanism, the stronger the monotonicity inherited. The self-monotonicity that is carried over by Maskin-Vind type mechanisms, which are commonly used in the characterization results of Nash implementability, turns out to be the essential monotonicity of Danilov.

Second, we consider the strong Nash equilibrium concept which incorporates coalitional deviations. It is more appropriate to use strong Nash equilibrium for implementation in situations where cooperation among agents is likely. Maskin (1979) showed that monotonicity is a necessary condition for strong Nash implementability as well. The first full characterization result is given by Dutta and Sen (1991). They use a similar approach to Moore and Repullo (1990) and their characterization result also depends on the existence of a system of sets satisfying a set of properties. Suh (1996a) showed that one of Dutta and Sen's conditions was not necessary for implementability and gave a characterization result with the remaining conditions. He also provided an algorithm to construct the system of sets used in both results. We give a new characterization of strong Nash implementable social choice rules via critical profiles. We modify the definition of critical profiles so that it applies to coalitions and we determine the critical alternatives for each coalition at each profile. We introduce three new conditions for social choice rules; coalitional monotonicity, preservation of criticals and unique common critical, which together with Pareto optimality characterize social choice rules that are strong Nash implementable. We introduce a new mechanism for the sufficiency part of our result which provides a drastic decrease in the size of the message space relative to the other mechanisms used in the literature. Finally, we identify a subset of subgame perfect Nash implementable social choice rules via critical profiles.

The results obtained in this thesis supports the idea that implementation theory can be rewritten in terms of monotonicity. Many of the results in the literature attempt to explain closely related problems with somewhat different approaches. Expressing them all in terms of monotonicity in different environments provides a better understanding of the relation between various problems and makes it easier to solve those problems that remain open.

CHAPTER 2

SELF-MONOTONICITY FOR THE NASH EQUILIBRIUM CONCEPT

Maskin (1977) showed that monotonicity, which is also referred as Maskin-monotonicity in the literature, is a necessary condition for Nash implementability. He also argued that Maskin-monotonicity conjoined with certain conditions, as is exemplified by no veto power or neutrality, is sufficient for Nash implementability in the presence of at least three agents. Danilov (1992) strengthened Maskin-monotonicity to "essential monotonicity" so that the stronger concept is both necessary and sufficient for Nash implementability on a restricted domain when the number of agents is not less then three. Thus, the Nash implementability of a social choice rule entirely depends upon how monotonic it is. In fact, later it was shown by Kaya and Koray (2000) that implementability strongly depends upon the compatibility between the monotonicity structures of the social choice rule to be implemented and of the solution concept according to which implementation is to take place. In particular, they modified Maskin-monotonicity in a natural way so as to make it fit the content of game theoretical solution concepts. Given a solution concept σ, it turned out that every σ-implementable social choice rule was Maskin-monotonic if and only if σ itself was "Maskin-monotonic" in the modified sense. In other words Maskin-monotonicity of σ-implementable social choice rules was inherited from "Maskin-monotonicity" of σ itself.

The notion of self-monotonicity which is introduced by Koray (2002), is based on these observations. In Maskin-monotonocity preservation of lower contour sets for each agent is required for an alternative to continue to be chosen. In essential monotonicity, only the essential alternatives which are subsets of the lower contour sets of each agent should be preserved. More generally, one can introduce different degrees of monotonicity via "a monotonicity function" which assigns to each point in the graph of a social choice rule a vector each component of which is a subset of the lower contour set at each point for each agent. The strongest of these monotonicities that is satisfied by a social choice rule constitutes a self-monotonicity of this social choice rule. This generalization easily carries over to the inheritance theorem by Kaya and Koray (2000) in the following way; a self-monotonicity of a social choice rule implementable in a certain solution concept must be inherited from a self-monotonicity inherent to the solution concept itself.

In this study we focus on the Nash equilibrium concept. We determine the unique self-monotonicity of the Nash equilibrium concept and in the light of the inheritance theorem mentioned above we investigate the implications of this condition for Nash implementable social choice rules. With a fixed set of agents, a message space and agents' preferences over the joint messages, the Nash equilibrium concept can be viewed as a social choice rule that chooses at each profile the equilibria of the game that is induced at that profile. Then a joint strategy which is an equilibrium at some profile continues to be an equilibrium at some other profile if and only if the joint strategies that can be achieved by each agent by a unilateral deviation, are less preferred to the chosen joint strategy at the initial profile. A Nash implementable social choice rule inherits this self-monotonicity via the mechanisms that implement it. However, the inherited monotonicity doesn't need to be a self-monotonicity for the implemented social choice rule. We define the self-monotonicity of the solution concept at each joint message. There may be several equilibria at some profile that is mapped to the same alternative via the implementing mechanism which implies several monotonicities of different degrees. Moreover, a social choice rule can be implemented via different mechanisms. One monotonicity for all Nash implementable social choice rules which is inherited via the Maskin-Vind type mechanisms is essential monotonicity. However it is not necessarily a self-monotonicity for a Nash implementable social choice rule.

In the next section we will introduce the main notation and definitions. In the second section we will define the self-monotonicity of the Nash equilibrium concept and determine the monotonicities that are inherited by a Nash implementable social choice rule. In the third section we will establish the relation between the self-monotonicity of the Nash equilibrium concept and essential monotonicity.

2.1 Self-Monotonicity

Let N be a nonempty finite set of agents and A be a nonempty finite set of alternatives. The set of all linear orders on A is denoted by $\mathcal{L}(A)$, while $\mathcal{L}(A)^N$ stands for the set of all linear order profiles. Given $R \in \mathcal{L}(A)^N$ and $i \in N$, R_i is the linear order representing agent i's preferences over A. A social choice rule (SCR) is a mapping $F : \mathcal{L}(A)^N \to 2^A$ which assigns to every linear order profile $R \in \mathcal{L}(A)^N$ a subset of A. Given an SCR F, its *graph* is defined as $Gr\,F = \{(a, R) \in A \times \mathcal{L}(A)^N \mid a \in F(R)\}$. For any $R \in \mathcal{L}(A)^N$, $a \in A$ and $i \in N$, the set $L_i(a, R) = \{b \in A \mid aR_ib\}$ is the *lower contour set* of a for i at

R. An SCR F is *Maskin-monotonic* if, for any $(a, R) \in Gr\, F$ and $R' \in \mathcal{L}(A)^N$, one has $(a, R') \in Gr\, F$ whenever $L_i(a, R) \subset L_i(a, R')$ for all $i \in N$.

Maskin (1977) showed that any Nash implementable SCR is Maskin-monotonic and any SCR which is Maskin-monotonic and satisfies no veto power is Nash implementable in the presence of three or more alternatives. We define a strengthening of Maskin-monotonicity which allows one to introduce different degrees of monotonicity of SCRs.

Definition 1. Let $F : \mathcal{L}(A)^N \to 2^A$ be an SCR. A mapping $h : Gr\, F \to (2^A)^N$ is a monotonicity function if $\{a\} \subset h_i(a, R) \subset L_i(a, R)$ for all $(a, R) \in Gr\, F$ and $i \in N$. Given a monotonicity function h, we say that F is *h-monotonic* if, for any $(a, R) \in Gr\, F$ and $R' \in \mathcal{L}(A)^N$, one has $(a, R') \in Gr\, F$, whenever $h_i(a, R) \subset L_i(a, R')$ for all $i \in N$.

h is a *stronger monotonicity* of F than h' if $h_i(a, R) \subset h'_i(a, R)$ for all $(a, R) \in Gr\, F$ and $i \in N$ and it is *strictly stronger* if $h_i(a, R) \subset h'_i(a, R)$ for all $(a, R) \in Gr\, F$ and $i \in N$, and $h_j(b, R') \subsetneq h'_j(b, R')$ for some $(b, R') \in Gr\, F$ and $j \in N$. Note that an h-monotonicity of F is equivalent to F being Maskin-monotonic whenever $L_i(a, R) \subset h_i(a, R) \subset A$ for all $(a, R) \in Gr\, F$ and $i \in N$.

We say that F is *more monotonic* than G if $Gr\, G \subset Gr\, F$ and there exists h-monotonicities h^F, h^G of F and G, respectively, such that h^F is stronger than h^G.

Example 1. Let F^1 be a dictatorial SCR where agent 1 is the dictator and F^1 assigns the top ranked alternative of agent 1 at each profile. An h-monotonicity of F is defined as follows: $h^1(a, R) = (A, \{a\}, ..., \{a\})$.

Example 2. Let F^{IR} be the *individually rational correspondence* defined as follows: $F^{IR}(R) = \{a \in A \mid aR_i a_0 \text{ for all } i \in N\}$. An h-monotonicity of F is defined as follows: $h^{IR}(a, R) = (\{a, a_0\}, ..., \{a, a_0\})$ for all $a \in A \setminus \{a_0\}$ and $h(a_0, R) = (\{a_0\}, ..., \{a_0\})$.

Essential monotonicity which was introduced by Danilov (1992) is an example of an h-monotonicity of Nash implementable SCRs. Essential monotonicity fully characterizes the Nash implementable SCRs when there are three or more agents and it is stronger than Maskin-monotonicity. We will first give the original definition of essential monotonicity and then express it as an h-monotonicity.

Example 3. Let $i \in N$ and $X \subset A$. An alternative $b \in X$ is *essential* for i in set X if $b \in F(R)$ for some $R \in \mathcal{L}(A)^N$ such that $L_i(b, R) \subset X$. The set of all essential elements with respect to $X \subset A$ is denoted as $Ess(F; i, X)$. An SCR F is *essentially-monotonic* if for any $R, R' \in \mathcal{L}(A)^N$ and $(a, R) \in Gr\, F$, we have $(a, R') \in Gr\, F$ whenever $Ess(F; i, L_i(a, R)) \subset L_i(a, R')$ for all $i \in N$. An SCR F is essentially monotonic if and only if it is h^{ess}-monotonic where $h^{ess} : Gr\, F \to (2^A)^N$ is defined as follows:
$h_i^{ess}(a, R) = \{b \in L_i(a, R) \mid$ such that $b \in F(R'')$ for some $R'' \in \mathcal{L}(A)^N$ with $L_i(b, R'') \subset L_i(a, R)\}$ for all $i \in N$.

Next we define one of the primary notions of this paper, *self-monotonicity*. We refer to the strongest monotonicities of F as its self-monotonicities which will be made precise in the following definition.

Definition 2. Let $F : \mathcal{R}^N \to 2^A$ be an SCR and $h : Gr\, F \to (2^A)^N$ be a monotonicity function. We say that h is a self-monotonicity of F if F is h-monotonic and h is minimal, i.e., there does not exist a monotonicity function $h' : Gr\, F \to (2^A)^N$ such that F is h'-monotonic and h' is strictly stronger than h.

A self-monotonicity h of an SCR F specifies a minimal subset of the alternative set for each agent $i \in N$ at any $(a, R) \in Gr\, F$ such that the preservation of these sets in the lower contour set for each agent i at outcome a according to some profile R' ensures $(a, R') \in Gr\, F$.

Note that h^1 and h^{IR} defined in the examples above constitute self-monotonicities for F^1 and F^{IR} respectively.

Example 4. Let F^a be a constant rule where F assigns $a \in A$ at each profile. h^1 and h^{IR} are h-monotonicities of F^a. However, neither of them are self-monotonicities of F^a. The self-monotonicity of F^a is defined as follows: $h^a(a, R) = (\{a\}, ..., \{a\})$. Note that this is the strongest monotonicity of an SCR.

In the above examples the self-monotonicities of the given SCRs are determined uniquely. However the self-monotonicity of an SCR does not have to be unique. In the following example we give the family of the self-monotonicities of the Pareto correspondence.

Example 5. The Pareto correspondence defined by $F^{PC}(R) = \{a \in A \mid \nexists b \in A$ such that bR_ia for all $i \in N\}$ has self-monotonicities characterized as follows: $\mathfrak{h}^{PC} = \{h \in (2^A)^N \mid$ for all $(a, R) \in Gr\, F$ and $i, j \in N$ $h_i(a, R) \subset L_i(a, R), h_i(a, R) \cap h_j(a, R) = \emptyset$ with $i \neq j$ and $\bigcup h_i(a, R) = A\}$.

2.2 Self-Monotonicity of the Nash Equilibrium Concept

When we fix the player set N and the joint strategy space $M = \prod_{i \in N} M_i$ a solution concept for normal form games can be viewed as an SCR so that the notions of h-monotonicity and self-monotonicity will apply to solution concepts as well. Denoting by $\mathcal{R}$ the set of all complete preorders on M, a solution concept for normal form games with a joint strategy space M, now becomes a mapping $\sigma : \mathcal{R}^N \rightarrow 2^M$. In this setting the joint strategy space is considered as the alternative set and the agents' rankings over the joint strategies as the preferences over the alternative set. A solution concept assigns a subset of the joint strategy space to each preference profile in the same way an SCR does.

The notions of h-monotonicity and self-monotonicity now become applicable to solution concepts for normal form games. h-monotonicity of a solution concept will be denoted by H for convenience.

The object of interest of this paper is defining a self-monotonicity of the Nash equilibrium concept for normal form games and establishing its relation to implementability. A joint strategy $m \in M$ constitutes a *Nash equilibrium* at $\succeq \in \mathcal{R}^N$ if $m \succeq_i (m'_i, m_{-i})$ for all $m'_i \in M_i$ and for all $i \in N$. We will denote the set of all Nash equilibria at $\succeq \in \mathcal{R}^N$ by $\sigma^{NE}(\succeq)$. The set (M_i, m_{-i}) is called agent i's *attainable set* at m. Note that a joint strategy m is a Nash equilibrium at $\succeq \in \mathcal{R}^N$ if and only if each agent i's attainable set at m is included in the lower contour set for each agent i at m according to $\succeq$. Based on this observation the unique self-monotonicity of the Nash equilibrium concept is obtained as follows:

Proposition. Let $\sigma^{NE} : \mathcal{R}^N \to 2^M$ be the Nash equilibrium concept. Let $H^{NE} : Gr\,\sigma^{NE} \to (2^M)^N$ be defined as follows:

$H_i^{NE}(m, \succeq) = \{m' \in L_i(m, \succeq) \mid m' = (m'_i, m_{-i}) \text{ for some } m'_i \in M_i\}$ for all $i \in N$.

H^{NE} is the unique self-monotonicity of the Nash equilibrium.

Proof: First we will show that σ^{NE} is H^{NE}-monotonic. Let $(m, \succeq) \in Gr\,\sigma^{NE}$ and $\succeq' \in \mathcal{R}^N$ be such that $H_i^{NE}(m, \succeq) \subset L_i(m, \succeq')$ for all $i \in N$. Then by the definition of H^{NE} we have, for any $i \in N$ and for any $m'_i \in M_i$, $(m_i, m_{-i}) \succeq' (m'_i, m_{-i})$. So, m_i is a best response for m_{-i} at $\succeq'$ and since this is true for any $i \in N$, m is a Nash equilibrium at $\succeq'$, i.e., $(m, \succeq') \in Gr\,\sigma^{NE}$. Hence σ^{NE} is H^{NE}-monotonic.

Next we show that H^{NE} is minimal. Suppose not. Then there exists $H' : Gr\,\sigma^{NE} \to (2^M)^N$ such that σ^{NE} is H'-monotonic with $H'_i(m, \succeq) \subset H_i^{NE}(m, \succeq)$ for all $(m, \succeq) \in Gr\,\sigma^{NE}$ and $i \in N$, and $H'_j(m', \succeq') \subsetneq H_j^{NE}(m', \succeq')$

for some $(m', \succeq') \in Gr\,\sigma^{NE}$ and $j \in N$. Let $(m, \succeq) \in Gr\,\sigma^{NE}$ be such that $H'_i(m, \succeq) \subset H_i^{NE}(m, \succeq)$ for all $i \in N$, and $H'_j(m, \succeq) \subsetneq H_j^{NE}(m, \succeq)$ for some $j \in N$. Then by definition of H^{NE}, there exists $m' = (m'_j, m_{-j}) \in M$ such that $m' \in H_j^{NE}(m, \succeq)$ and $m' \notin H'_j(m, \succeq)$. Define $\succeq'$ such that $\succeq'_i = \succeq_i$ for all $i \in N \backslash \{j\}$ and $L_j(m, \succeq') = L_j(m, \succeq) \setminus \{m'\}$. By the construction we have $H'_i(m, \succeq) \subset L_i(m, \succeq')$ for all $i \in N$. But $(m, \succeq') \notin Gr\,\sigma^{NE}$ as $(m'_j, m_{-j}) \succ'_j (m_j, m_{-j})$. So σ^{NE} is not H'-monotonic. Hence H^{NE} is minimal.

Finally we will show that H^{NE} is unique. Suppose not. Then there exists $H' : Gr\,\sigma^{NE} \rightarrow (2^M)^N$ such that H' is a self-monotonicity for σ^{NE} and for some $(m, \succeq) \in Gr\sigma^{NE}$ and $j \in N$ there exists $m' = (m'_j, m_{-j}) \in M$ such that $m' \in H_j^{NE}(m, \succeq)$ and $m' \notin H'_j(m, \succeq)$. Define $\succeq'$ as above: $\succeq'_i = \succeq_i$ for all $i \in N \backslash \{j\}$ and $L_j(m, \succeq') = L_j(m, \succeq) \setminus \{m'\}$. Then $H'_i(m, \succeq) \subset L_i(m, \succeq')$ for all $i \in N$. As σ^{NE} is H'-monotonic, the above inclusions imply that $m \in \sigma^{NE}(\succeq')$. But $(m, \succeq') \notin Gr\,\sigma^{NE}$ as $(m'_j, m_{-j}) \succ'_j (m_j, m_{-j})$, which is a contradiction. Hence H^{NE} is the unique self-monotonicity of the Nash equilibrium concept. $\square$

Next we will examine the relation between the self-monotonicity of the Nash equilibrium concept and the monotonicity properties of Nash implementable SCRs. First we introduce some more notation and definitions.

An onto function $g : M \rightarrow A$ is called an *outcome function*. A *mechanism* consists of a joint strategy space and an outcome function and is denoted by $\mu = (M, g)$. The following notation and definition was introduced by Kaya and Koray (2000). A complete preorder profile $\succeq \in \mathcal{R}^N$ is called *admissible* if one has $m \sim_j m'$ $\forall j \in N$, whenever $m \sim_i m'$ for some $i \in N$, where $m, m' \in M$. $\succeq$ is admissible if all the agents have exactly the same indifference classes. $\mathcal{A}$ denotes the set of all admissible profiles. Each admissible profile $\succeq \in \mathcal{A}$ induces a partition on M consisting of the common indifference classes which will be denoted by $\rho(\succeq)$. An outcome function $g : M \to A$ also induces a partition $\{g^{-1}(x) \mid x \in A\}$ on M which will be denoted by $p(g)$. Finally set $\mathcal{A}\ (g) = \{\succeq \in \mathcal{A} \mid \rho\ (\succeq) = p(g)\}$.

Let $\succeq \in \mathcal{A}\ (g)$ and $R \in \mathcal{L}(A)^N$ be given. We say that R is induced by $\succeq$ via g if for any $a, b \in A$ with $g(m) = a, g(m') = b$ where $m, m' \in M$, and for any $i \in N$, we have aR_ib if and only if $m \succ_i m'$ and $a = b$ if and only if $m \sim_i m'$. Similarly, we say that $\succeq$ is induced by R via g if and only if for any m, $m' \in M$ and $i \in N$, we have that $m \succ_i m'$ if and only if $g(m)R_ig(m')$ and $m \sim_i m'$ if and only if $g(m) = g(m')$. It is clear that R is induced by $\succeq$ via g if and only if $\succeq$ is induced by R via g. Note that there is a one-to-one correspondence between linear orders on A and the admissible complete preorder profiles on M with $\rho\ (\succeq) = p(g)$.

Given an SCR F, a solution concept σ and a mechanism $\mu = (M, g)$ we say that F is σ-implementable via μ if for every $R \in \mathcal{L}(A)^N$, one has $F(R) = g(\sigma(\succeq))$ where $\succeq$ is the complete preorder profile on M induced by R. F is said to be σ-implementable if there is some $\mu = (M, g)$ via which F is σ-implementable.

Now we define an h-monotonicity of Nash-implementable SCRs as follows:

Theorem. Let F be an SCR which is Nash-implementable via $\mu = (M, g)$. Let $h^{NE,\mu} : Gr\,F \to (2^A)^N$ be defined as follows:

$h_i^{NE,\mu}(a, R) = \{b \in L_i(a, R) \mid b \in g(H^{NE}(m, \succeq))$ where $\succeq \in \mathcal{A}(g)$ is induced by R via g and $m \in \sigma^{NE}(\succeq)\}$ for all $i \in N$.

$h^{NE,\mu}$ is an h-monotonicity for F.

Proof: Let $(a, R) \in Gr\,F$ and $R' \in \mathcal{L}(A)^N$ such that $h_i^{NE,\mu}(a, R) \subset L_i(a, R')$ for all $i \in N$. Consider $\succeq' \in \mathcal{A}\ (g)$ that is induced by R' via g. By the above inclusions and the definition of $h^{NE,\mu}$, for all $(m_i', m_{-i}) \in M_i \times \{m_{-i}\}$ there exists $b \in L_i(a, R')$ such that $g(m_i', m_{-i}) = b$ which implies $(m_i', m_{-i}) \in$

$L_i(m, \succeq')$ for all $(m_i', m_{-i}) \in M_i \times \{m_{-i}\}$. Now we have $(m, \succeq) \in Gr\sigma^{NE}$ and $L_i(m, \succeq) \cap H^{NE}(m, \succeq) \subset L_i(m, \succeq')$ for all $i \in N$. Then by the proposition above $m \in \sigma^{NE}(\succeq')$ and as F is Nash implementable via μ $g(m) = a \in F(R')$. Hence F is $h^{NE,\mu}$-monotonic.□

The theorem establishes the relation between the unique self-monotonicity of the Nash equilibrium concept and the monotonicity properties of a Nash implementable SCR. The h-monotonicities of a Nash implementable SCR are inherited from the unique self-monotonicity of the Nash equilibrium concept via the mechanisms that implements the SCR. However the uniqueness is not inherited due to the following two reasons: One is that an SCR may be Nash implemented via different mechanisms each of which may impose a different h-monotonicity on the SCR implemented. We will explore this point in more detail in the next section. The other reason is that the same mechanism may also induce several different h-monotonicities of F as there may be several Nash equilibria leading to the same outcome at a given preference profile. Every choice of a Nash equilibrium for each outcome in the image of F at a given profile results in a different h-monotonicity of F. Moore and Repullo (1990) gave a necessary and sufficient condition for Nash implementability which is called Condition-μ. Condition-μ requires the existence of systems of sets which satisfies some specific properties for a given SCR. Each h-monotonicity of an SCR induced by a mechanism gives us such system of sets.

In the following example we will consider a situation where different equilibria lead to different h-monotonicities.

Example 6. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Consider all the linear order profiles, $\mathcal{L}(A)^N$, on A. Let $M_1 = \{m, m', m''\}, M_2 = \{m, m'\}$ and $M_3 = \{m\}$ be the strategy spaces of each agent and $M = \prod_{i \in N} M_i$ be the joint strategy space. The outcome function is defined as follows:

		2,3	
		mm	$m'm$
	m	a	c
1	m'	b	a
	m''	c	c

Consider all the admissible profiles on M such that $\rho\ (\succeq) = p(g)$. Let $F(R) = \sigma^{NE}(\succeq)$ for all $\succeq \in \mathcal{A}(g)$ where R is induced by $\succeq$. Let R be the following preference profile: aR_1bR_1c, aR_2cR_2b and bR_3aR_3c. The set of Nash equilibria at $\succeq$ which is induced by R is $\sigma^{NE}(\succeq) = \{mmm, m'm'm\}$ and by definition, $F(R) = a$. Let $\mathbf{m}$, $\mathbf{m}'$ denote $mmm, m'm'm$ respectively. The h-monotonicity of F which is induced by $\mathbf{m}$ at R maps $(a, R) \in Gr\,F$ into the following subsets of A: $h_1^{NE,\mu}(a, R) = \{a, b, c\}$, $h_2^{NE,\mu}(a, R) = \{a, c\}$, $h_3^{NE,\mu}(a, R) = \{a\}$. On the other hand, if we consider $\mathbf{m}'$ we obtain the following mapping at $(a, R) \in Gr\,F$: $h_1^{NE,\mu}(a, R) = \{a, b, c\}$, $h_2^{NE,\mu}(a, R) = \{a, b\}$, $h_3^{NE,\mu}(a, R) = \{a\}$.

2.3 Essential Monotonicity as an h-Monotonicity of a Nash Implementable Social Choice Rule

In this section we will investigate the relation between the *essential monotonicity* and the h-monotonicities of Nash implementable SCRs. In the previous section we showed that the h-monotonicities of Nash implementable SCRs are inherited from the unique self-monotonicity of the Nash solution concept via the mechanisms that implement it. We also know that essential monotonicity, introduced by Danilov (1992), characterizes the Nash implementable SCRs on the full domain of linear orders when there are at least three agents. Essential monotonicity turns out to be an h-monotonicity of Nash implementable SCRs that is inherited via the *Maskin-Vind* mechanism. First we will give the formal definition of essential monotonicity that was mentioned in Example 1.

Definition 3. Let $i \in N$ and $X \subset A$. An alternative $b \in X$ is essential for i in set X if $b \in F(R)$ for some $R \in \mathcal{L}(A)^N$ such that $L_i(b, R) \subset X$. The set of all essential elements is denoted as $Ess(F; i, X)$.

Definition 4. An SCR F is essentially-monotonic if for any $R, R' \in \mathcal{L}(A)^N$ and $(a, R) \in Gr\, F$, we have $(a, R') \in Gr\, F$ whenever

$$Ess(F; i, L_i(a, R)) \subset L_i(a, R') \text{ for all } i \in N.$$

The definition of essential monotonicity as an h-monotonicity is explicitly given by the following monotonicity function:

Let F be an SCR and $h^{ess} : Gr\, F \to (2^A)^N$ be a monotonicity function which is defined as follows:

$h_i^{ess}(a, R) = \{b \in L_i(a, R) \mid b \in F(R'') \text{ for some } R'' \text{ with } L_i(b, R'') \subset L_i(a, R)\}$ for all $i \in N$.

F is essentially-monotonic if and only if it is h^{ess}-monotonic.

The definition above applies to solution concepts for normal form games with a fixed player set N and joint strategy space M, and in particular to the Nash equilibrium concept, as discussed in section 2.

Definition 5. Let σ be a solution concept and $H^{ess} : Gr\, \sigma \to (2^M)^N$ be a monotonicity function which is defined as follows:

$H_i^{ess}(m, \succeq) = \{m' \in L_i(m, \succeq) \mid m' \in \sigma(\succeq'') \text{ for some } \succeq'' \text{ with } L_i(m', \succeq'') \subset L_i(m, \succeq)\}$ for all $i \in N$.

σ is H^{ess}-monotonic if for any $\succeq, \succeq' \in \mathcal{R}^N$ and $(m, \succeq) \in Gr\, \sigma$, we have $(m, \succeq') \in Gr\, \sigma$ whenever $H_i^{ess}(m, \succeq) \subset L_i(m, \succeq')$ for all $i \in N$.

One natural question that arises then is the relation between essential monotonicity and the self-monotonicity of the Nash equilibrium concept. It turns out that the Nash equilibrium concept is H^{ess}-monotonic and the self-monotonicity of the Nash equilibrium concept is strictly stronger than H^{ess}-monotonicity. However when we consider the normal form games induced by

Maskin-Vind type mechanisms, H^{ess} and H^{NE} turn out to be equivalent.

Proposition. The Nash equilibrium concept is H^{ess}-monotonic.

Proof: Let $(m, \succeq) \in Gr\sigma^{NE}$ and $m' = (m'_i, m_{-i}) \in H_i^{NE}(m, \succeq)$ for some $i \in N$. Consider $\succeq'$ such that $\mathrm{L}(m', \succeq'_j) = \mathrm{M}$ for all $j \in N \backslash \{i\}$ and $L_i(m', \succeq') = L_i(m, \succeq)$. Now m'_j is a best response to m'_{-j} for all $j \in N \setminus \{i\}$ at $\succeq'$. For i, m'_i is a best response to m'_{-i} by definition of H^{NE}. So $m' \in \sigma^{NE}(\succeq')$. Hence $m' \in H_i^{ess}(m, \succeq_i)$, since $m' \in L_i(m, \succeq)$ and there exists $\succeq'$ such that $m' \in \sigma^{NE}(\succeq')$ with $L_i(m', \succeq') \subset L_i(m, \succeq)$. $\square$

Next we define a mechanism which is referred as a Maskin-Vind type mechanism in the literature and widely used in the sufficiency results for Nash implementation. We define the version of the Maskin-Vind type mechanism which is used by Danilov (1992) for his sufficiency result.

For each agent $i \in N$ the message space is defined as follows:

$M_i = \{(a, R, n) \in \mathcal{L}(A)^N \times A \times \mathbb{N} \mid (a, R) \in Gr\, F\}$ where $\mathbb{N}$ is the set of nonnegative integers.

The outcome function $g : M \to A$ is defined as follows:

(1) If there exists $m \in M$ such that $m_i = (a, R, n)$ for all $i \in N$ then $g(m) = a$.

(2) If there exists $i \in N$ such that $m_j = (a, R, n)$ for all $j \neq i$ and $m_i = (a', R', n')$ where $a' \neq a$, then $g(m) = a'$ if $a' \in Ess(F; i, L_i(a, R))$ and $g(m) = a$ otherwise.

(3) In all other situations let $g(m) = a_i$ where i is the agent announcing the highest integer. Ties are broken in favor of the agent with the smallest index.

We will denote the Maskin-Vind mechanism by μ_{M-V}.

Proposition. Let F be a Nash implementable SCR. $h^{NE, \mu_{m-v}}$ and h^{ess} monotonicities of F are equivalent.

Proof: By the above proposition and the theorem from the previous section it follows that $h^{NE,\mu_{m-v}} \subset h^{ess}$. We need to show that $h^{ess} \subset h^{NE,\mu_{m-v}}$. Let $(a, R) \in Gr\, F$ and $b \in h_i^{ess}(a, r)$ for some $i \in N$. As F is Nash implementable there exists $m \in M$ such that $m \in \sigma^{NE}(\succeq)$, with $g(m) = a$ where $\succeq$ is induced by R. We want to show that there exists $m_i' \in M_i$ such that $g(m_i', m_{-i}) = b$. We have two cases to consider. First assume condition one of the outcome function applies, i.e., $m_i = (a, R, n)$ for all $i \in N$. Then, as $b \in h_i^{ess}(a, r)$, agent i can make the outcome b by announcing (b, R', n'). Second assume condition two or three applies. Then agent i can make the outcome b by announcing the highest integer. So, in both cases there exists $m_i' \in M_i$ such that $g(m_i', m_{-i}) = b$. As this is true for any $(a, R) \in Gr\, F$, $i \in N$ and $b \in h_i^{ess}(a, R)$ we conclude that $h^{ess} \subset h^{NE,\mu_{m-v}}$. $\square$

Note that if we implement the SCR in Example 6 by the Maskin-Vind mechanism essential monotonicity is inherited as an h-monotonicity of F. However, with the finite mechanism defined in the example we obtain a stronger monotonicity than essential monotonicity of F.

CHAPTER 3

STRONG NASH IMPLEMENTABILITY VIA CRITICAL PROFILES

While the Nash equilibrium concept considers individual deviations, strong Nash equilibrium allows for cooperation among agents and incorporates coalitional deviations. Roughly, a joint message constitutes a strong Nash equilibrium if there is no coalitional deviation which will benefit all the members of the coalition. It is therefore more appropriate to use strong Nash equilibrium for implementation in situations where cooperation among agents is likely. Maskin (1979) showed that monotonicity is a necessary condition for strong Nash implementability as well. The full characterization results are given by Dutta and Sen (1991) and Suh (1996a). They both use a similar approach to Moore and Repullo (1990) and their characterization results also depend on the existence of a system of sets satisfying a complex set of properties.

In this paper we pursue a rather direct approach, like that of Danilov's for Nash implementation, and gives an explicit definition for the system of sets defined by Dutta and Sen. Our main tool in doing so is the notion of a "critical profile" which was first introduced in a study by Koray et al. (2001). They provided a simpler proof of the well known Müller-Satterthwaite theorem via critical profiles.

In another study Doğan and Koray (2007) explored more social choice theoretic implication of the notion and also provided a new characterization for two-person Nash implementable social choice rules. Roughly, a critical profile for an alternative, a, is a preference profile at which a is chosen and which has the following property: at any preference profile that is obtained from the critical profile by a reversal of an ordering between a and any alternative that is less preferred than a from the view point of some individual, a is not chosen by the social choice rule. We modify the general definition of a critical profile for coalitions and we obtain the set of critical profiles for each coalition from a given profile. Then we determine the critical alternatives for each coalition by applying a test to the alternatives that are less preferred than the alternative that is chosen by the individuals in this coalition. We introduce three new conditions for social choice rules; coalitional monotonicity, preservation of criticals and unique common critical, which together with Pareto optimality characterize social choice rules that are strong Nash implementable.

The mechanism that we use for the characterization result is simple. Each individual announces an alternative and a critical profile for that alternative. At each joint message the outcome is the "unique common critical" that is defined by the unique common critical condition. Here, it should be noted that the size of the message space depends on the size of the set of critical profiles. So once the critical profiles are determined for a social choice rule we not only conclude whether it is implementable or not but also implement it very easily. The mechanism that was introduced by Dutta and Sen, and used for the existing characterization results is more complicated. There each individual announces a profile, an alternative that is chosen at this profile, a positive integer, and raises a flag or not. Then they define a suitable outcome function for the implementation to take place.

In the next section we introduce the general environment and basic definitions. In the third section, we define critical profiles and introduce new conditions for social choice rules regarding their critical profiles. Then we show that the conditions are both necessary and sufficient for strong Nash implementable social choice rules.

3.1 Notation and Definitions

Let $N = \{1, \ldots, n\}$ be a nonempty finite set of agents and A be a nonempty finite set of alternatives. A preference profile is an n-tuple, $R = (R_1, \ldots, R_n)$ where each R_i is a linear order [1] on A which represents agent i's preferences on A. The set of all linear order profiles on A is denoted by $\mathcal{L}(A)^N$. A *social choice rule (SCR)* is a mapping $F : \mathcal{L}(A)^N \rightarrow 2^A$ which assigns to every linear order profile $R \in \mathcal{L}(A)^N$ a subset of A.[2] *The graph of an SCR F*, is defined as $Gr\,F = \{(a, R) \in A \times \mathcal{L}(A)^N \mid a \in F(R)\}$. For any $R \in \mathcal{L}(A)^N$, $a \in A$ and $i \in N$, the set $L(i, a, R) = \{b \in A \mid aR_ib\}$ is the *lower contour set of a for i at R*. Let $\mathcal{N}$ denote the set of all nonempty subsets of N. *The collective lower contour set of* $T \in \mathcal{N}$ is defined as the union of the lower contour sets of each agent in T : $L(T, a, R) = \bigcup_{i \in T} L(i, a, R)$. An SCR F is *Pareto optimal* if for all $(a, R) \in GrF$, $L(N, a, R) = A$.

A joint strategy space is the product space of nonempty strategy sets of every agent and is denoted by $M = \prod_{i \in N} M_i$. An onto function $g : M \rightarrow A$ is called an *outcome function. A mechanism* consists of a joint strategy space and an outcome function and is denoted by $\mu = (M, g)$. Note that μ defines

[1] A linear order is a complete, transitive, reflexive, antisymmetric binary relation.

[2] We will assume that F is onto. In general, all the results continue to hold if we restrict F to the image of F.

a normal form game at each $R \in \mathcal{L}(A)^N$ via the outcome function g. Given $T \in \mathcal{N}$, a joint strategy for T is $m_T = (m_i)_{i \in T} \in M_T = \prod_{i \in T} M_i$.

Given a preference profile $R \in \mathcal{L}(A)^N$ and a mechanism $\mu = (M, g)$, a joint strategy $m \in M$ constitutes a *strong Nash equilibrium* of the game (μ, R), if for all $T \in \mathcal{N}$ and $m'_T \in M_T$ there exists $i \in T$ such that $g(m) R_i g(m'_T, m_{N \backslash T})$. The set of all strong Nash equilibria of the game (μ, R) is denoted by $SN(\mu, R)$. A mechanism μ *implements an $SCR\,F$ in strong Nash equilibrium* if $g(SN(\mu, R)) = F(R)$ for all $R \in \mathcal{L}(A)^N$. F is said to be *strong Nash implementable* if there is some mechanism μ that implements F.

3.2 Critical Profiles and the Characterization Result

In this section we will first define critical profiles and then introduce three new conditions for SCRs via critical profiles. These conditions, namely, coalitional monotonicity, condition of preservation of criticals, and condition of unique common critical together with Pareto optimality turn out to be both necessary and sufficient conditions for strong Nash implementability.

Definition 1. Given $a \in A$, $T \in \mathcal{N}$, $R, R' \in \mathcal{L}(A)^N$, R' is said to be a (T, a)-*refinement of* R if $L(T, a, R') \subset L(T, a, R)$ and for all $i \in N \backslash T$, $L(i, a, R') = L(i, a, R)$.[3] R' is a strict (T,a)-refinement of R if the inclusion for T is strict.

For an illustration consider an environment with $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ and let R, R' be defined as follows:

R_1	R_2	R_3		R'_1	R'_2	R'_3
b	b	a		b	c	a
a	a	b		a	a	b
c	c	c		c	b	c

Now, R' is a $(23, a), (12, b), (N, a), (N, b)$-refinement and a strict $(2, b), (23, b)$-refinement of R. R is a $(23, a), (12, b), (N, a), (N, b)$-refinement

[3]We denote set inclusion and strict set inclusion by $\subset$ and $\subsetneq$ respectively.

and a strict $(12,a),(2,c),(12,c),(23,c),(N,c)$-refinement of R'. Note that the lower contour set of an alternative a for each agent $i \in T$ at a (T,a)-refinement does not need to be a subset of the lower contour set of a for each $i \in T$ at the original profile: $L(2,a,R) \not\subset L(2,a,R')$ but $L(12,a,R) \subset L(12,a,R')$ and as $L(3,a,R) = L(3,a,R')$, R is a strict $(12,a)$-refinement of R'.

Let $F : \mathcal{L}(A)^N \to 2^A$ be an SCR which will be kept fixed in the definitions below.

Definition 2. Given $(a,R) \in GrF$ and $T \in \mathcal{N}$, R is said to be a (T,a)-*critical profile relative to* F if for any strict (T,a)-refinement R' of R, one has $a \notin F(R')$.

Notation: $\mathcal{C}(T,a)$ = the set of all (T,a)-critical profiles relative to F.

For example, let F be the SCR that chooses all the Pareto optimal alternatives at each profile: $F^{PO}(R) = \{a \in A | L(N,a,R) = A\}$. Then given $(a,R) \in Gr\,F, T \in \mathcal{N}, R$ is a (T,a)-critical profile if $L(T,a,R)\backslash\{a\} = A\backslash L(N\backslash T,a,R)$. As a second example consider the SCR where agent 1 is the dictator: $F^{D1}(R) = \{a \in A | L(1,a,R) = A\}$. Let $(a,R) \in Gr\,F$ and $T \in \mathcal{N}$. We have two cases: R is a (T,a)-critical profile if ($1 \in T$ and $L(T,a,R) = A$) or ($1 \notin T$ and $L(T,a,R) = \{a\}$).

Definition 3. Given $(a,R) \in GrF$, $T \in \mathcal{N}$ and $R' \in \mathcal{L}(A)^N$, we say that R' is a (T,a,R)-*critical profile* if R' is a (T,a)-critical (T,a)-refinement of R.

Notation: $\mathcal{C}(T,a,R)$ = the set of all (T,a,R)-critical profiles.

In Definition (3) we are considering the set of critical profiles obtained from a given profile. Obviously, $\mathcal{C}(T,a,R) \subset \mathcal{C}(T,a)$ for all $(a,R) \in Gr\,F$.

Definition 4. Given $(a,R) \in GrF$, $T \in \mathcal{N}$, $R' \in \mathcal{C}(T,a,R)$, $b \in L(T,a,R')$ and $R'' \in \mathcal{L}(A)^N$ such that $L(i,a,R'') = L(i,a,R')\backslash\{b\}$ for all $i \in T$ and $L(i,a,R'') = L(i,a,R') \cup \{b\}$ for all $i \in N\backslash T$, b is *a* (T,a,R)-*critical element relative to* R' if $b \notin F(R'')$.

Notation: $Cr(T, a, R, R')$ = the set of all (T, a, R)-critical profiles relative to R'.

Definition 5. Given $(a, R) \in GrF$, $T \in \mathcal{N}$, the set $Cr(T, a, R)$ of all (T, a, R)-*critical elements* is defined as $Cr(T, a, R) = \bigcup_{R' \in C(T,a,R)} Cr(T, a, R, R')$.

Let $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ and R be such that aR_1bR_1c, bR_2aR_2c, and cR_3aR_3b. Consider F^{PO} defined above. Then $a \in F^{PO}(R)$. By the above argument, $R' \in \mathcal{C}(12, a, R)$ if $L(12, a, R')\backslash\{a\} = A\backslash L(3, a, R)$, i.e., $L(12, a, R') = \{a, c\}$. Now, let $R'' \in \mathcal{L}(A)^N$ be such that $L(i, a, R'') = L(i, a, R')\backslash\{c\} = \{a\}$ for all $i \in 1, 2$ and $L(3, a, R'') = L(3, a, R') \cup \{c\} = A$ for some $R' \in \mathcal{C}(12, a, R)$. Then $c \in F(R'')$ and as this is true for all $R' \in \mathcal{C}(12, a, R)$, $c \notin Cr(12, a, R)$, and $Cr(12, a, R) = \{a\}$. Similarly, for all $T \in \mathcal{N}\backslash N$ $Cr(T, a, R) = \{a\}$ and $Cr(N, a, R) = A$.

Remark: If $T' \subset T$ then $Cr(T', a, R) \subset Cr(T, a, R)$.

Definition 6. F is *coalitionally monotonic* if, for all $(a, R) \in GrF, R' \in \mathcal{L}(A)^N$, one has for all $T \in \mathcal{N} : Cr(T, a, R) \subset L(T, a, R')$ implies $a \in F(R')$.

The monotonicity condition introduced by Maskin (1977) is well-known in the literature: F is *Msakin-monotonic* if for all $(a, R) \in Gr\,F$, $R' \in \mathcal{L}(A)^N$ one has $a \in F(R')$ whenever $L(i, a, R) \subset L(i, a, R')$ for all $i \in N$. Maskin showed that monotonicity is a necessary condition for both Nash and strong implementability. Coalitional monotonicity which is both a necessary and sufficient condition for strong Nash implementability is stronger than Maskin-monotonicity:

Proposition. If F is coalitionally monotonic then it is Masking monotonic.

Proof: Suppose not. Then there exists $(a, R) \in Gr\,F$ and $R' \in \mathcal{L}(A)^N$ such that $L(i, a, R) \subset L(i, a, R')$ for all $i \in N$, but $a \notin F(R')$. If $Cr(T, a, R) \subset L(T, a, R')$ for all $T \in 2^N\backslash\{\emptyset\}$ then by coalitional monotonicity one has $a \in F(R')$. So there exists $T \in 2^N\backslash\{\emptyset\}$ such that $Cr(T, a, R) \not\subset L(T, a, R')$. Let

$x \in Cr(T,a,R)$ and $x \notin L(T,a,R')$. Then there exists $\bar{R} \in C(T,a,R)$ such that $x \in L(T,a,\bar{R}) \subset L(T,a,R) = \bigcup_{i\in T} L(i,a,R) \subset \bigcup_{i\in T} L(i,a,R') = L(T,a,R')$, which is a contradiction. □

The converse is not true:

Example 1. Let $A = \{a,b,c,d\}$ and $N = \{1,2,3\}$. $F = \mathcal{L}(A)^N \to 2^A$ is defined as follows:

$$F(R) = \begin{cases} x, \text{ if } x \in \{b,c,d\} \text{ and } x \text{ is top-ranked by at least one agent.} \\ a, \text{ if } aR_1c \text{ and } aR_2d \text{ and } [aR_1b \text{ or } aR_2c]. \end{cases}$$

We will first show that F is not coalitionally monotonic. Let R be a preference profile such that aR_1bR_1c and aR_2cR_2d. Then $a \in F(R)$ and $Cr(1,a,R) = \{a,c\}$, $Cr(2,a,R) = \{a,d\}$ and $Cr(12,a,R) = \{a,b,c,d\}$. Now consider a profile R' such that aR'_1c, $aR'_2dR'_2b$. Note that $Cr(T,a,R) \subset L(T,a,R')$ for all $T \in 2^N \backslash \{\emptyset\}$ but $a \notin F(R')$. So F is not coalitionally monotonic.

Next we will show that F is Maskin monotonic. Let $x \in \{b,c,d\}$ and $x \in F(R)$. Then x is top-ranked by at least one agent at R. If $L(i,x,R) \subset L(i,x,R')$ for all $i \in N$, then x will continue to be top-ranked by at least one agent at R', hence $x \in F(R')$. Let $a \in F(R)$. Then aR_1c, aR_2d, and aR_1b or aR_2c. If the lower contour sets of a for each agent are preserved at R', then $a \in F(R')$. So F is Maskin monotonic.□

Next we will introduce two more conditions via critical profiles and critical elements. The first condition requires the existence of an (N,a^*)-critical profile, R^*, for each $(a,R) \in Gr\,F$ such that anything critical at R^* for a coalition T, will be an (S,a,R)-critical element where S is a superset of T. The second condition requires the existence of a unique critical element and a critical profile for any sequence of coalitions and points in the $Gr\,F$ that satisfies some specific conditions. This second condition implies a crucial simplification in the mechanism that we use for the characterization result.

Definition 7. F satisfies the condition of preservation of criticals if for each $(a, R) \in Gr\, F$, $T \in \mathcal{N}$, and $a^* \in Cr(T, a, R)$ there exists $R^* \in \mathcal{C}(N, a^*)$ such that $Cr(S, a^*, R^*) \subset Cr(T \cup S, a, R)$ for all $S \in \mathcal{N}$.

Definition 8. $\{(T^i, a^i, R^i)\}_{i=1}^k$ is *an adequate sequence* in $\mathcal{N} \times Gr\, F$ if for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$ $T^i \cup T^j = N$, $(a^i, R^i) \neq (a^j, R^j)$ and $\bigcap_{i=1}^{k} T^i = \emptyset$.

Definition 9. $a^* \in A$ is a common critical for an adequate sequence $\{(T^i, a^i, R^i)\}_{i=1}^k$ if $a^* \in \bigcap_{i=1}^{k} Cr(T^i, a^i, R^i)$.

Definition 10. F satisfies the condition of unique common critical if for each adequate sequence $\{(T^i, a^i, R^i)\}_{i=1}^k$ there exists a common critical $a^* \in A$ and a profile $R^* \in \mathcal{C}(N, a^*)$ such that $Cr(S, a^*, R^*) \subset \bigcap_{i=1}^{k} Cr(T^i \cup S, a^i, R^i)$ for all $S \in \mathcal{N}$.

Theorem. An $SCR\, F$ is strong Nash implementable if and only if it is Pareto optimal, coalitionally monotonic, satisfies the condition of preservation of criticals and the condition of unique common critical.

Proof:

Sufficiency:

The following mechanism $\mu = (M, g)$ will be used to establish the result: The strategy space of each agent $i \in N$ is,

$$M_i = \{(a^i, R^i) \in Gr\, F | R^i \in \mathcal{C}(N, a^i)\}$$

Define the outcome function $g : M \to A$ for any $m \in M$ as follows:

1. If $(a^i, R^i) = (a, R)$ for all $i \in N$, then $g(m) = a$.
2. Otherwise, $g(m) = a^*$ where a^* is the unique common critical for the adequate sequence $\{(N \backslash T^i, a^i, R^i)\}_{i=1}^k$ where each T^i consists of agents that announces the same message (a^i, R^i) and $(a^i, R^i) \neq (a^j, R^j)$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

Note that $\{T^i\}_{i=1}^k$ partitions N. So, $N \backslash T^i \cup N \backslash T^j = N$ for all $i, j \in$

$\{1,\ldots,k\}$ and $\bigcap_{i=1}^{k} N\backslash T^i = \emptyset$ which guarantee the existence of a^* in (2) by the unique common critical condition.

Step 1: $F(R) \subseteq g(SN(\mu, R))$ for all $R \in \mathcal{L}(A)^N$.

Let $a^* \in F(R^*)$. Consider a strategy $m^* \in M$ such that $m_i^* = (a^*, R)$ for all $i \in N$ where $R \in \mathcal{C}(N, a^*, R^*) \subset \mathcal{C}(N, a^*)$ with $L(T, a^*, R) \subset L(T, a^*, R^*)$ for all $T \in \mathcal{N}$. We want to show that $m^* \in SN(\mu, R^*)$ and $g(m^*) = a^*$. By (1) $g(m^*) = a^*$. Next, consider a deviation m'_T by $T \in \mathcal{N}$ from m^*. We need to show that $g(m'_T, m^*_{N\backslash T}) \in L(T, a^*, R^*)$. If $T = N$ then by Pareto optimality $g(m'_N) \in L(N, a^*, R^*) = A$. Suppose $T \neq N$. Now we have a partition $T^1, ..., T^k$ of N where all the agents in each T^i announces the same strategy. Let T^k be the coalition where $m_i = (a^*, R)$ for all $i \in T^k$. By (2) $g(m'_T, m^*_{N\backslash T}) = a'$ where a' is the unique common critical for the adequate sequence $\{(N\backslash T^i, a^i, R^i)\}_{i=1}^k$ which implies by definition that $a' \in \bigcap_{i=1}^{k} Cr(N\backslash T^i, a^i, R^i)$ and in particular, $a' \in Cr(N\backslash T^k, a^*, R)$. Note that $N\backslash T^k \subset T$. So $a' = g(m'_T, m^*_{N\backslash T}) \in Cr(T, a^*, R)$. We also have $R \in \mathcal{C}(N, a^*, R^*)$ and $L(T, a^*, R) \subset L(T, a^*, R^*)$ for all $T \in \mathcal{N}$ which implies $L(i, a^*, R) \subset L(i, a^*, R^*)$ for all $i \in N$ and there is no strict N-refinement R' of R such that $a^* \in F(R')$. But that means $R \in C(T, a^*, R^*)$ and $Cr(T, a^*, R) \subset L(T, a^*, R)$. As $L(T, a^*, R) \subset L(T, a^*, R^*)$ we have $a' = g(m'_T, s^*_{N\backslash T}) \in L(T, a^*, R^*)$ as required.

Step 2: $g(SN(\mu, R)) \subset F(R)$ for all $R \in \mathcal{L}(A)^N$.

Let $a^* \in g(SN(\mu, R^*))$. We will show that $a^* \in F(R^*)$. There are two cases to consider. First, assume that (1) applies and $m^* \in SN(\mu, R^*)$ is such that $(a^i, R^i) = (a^*, R)$ for all $i \in N$. As $a^* \in F(R)$, if we show that $Cr(T, a^*, R) \subset L(T, a^*, R^*)$ holds for all $T \in \mathcal{N}$ then by coalitional monotonicity we conclude that $a^* \in F(R^*)$. Let $a' \in Cr(T, a^*, R)$ for some $T \in \mathcal{N}$. By the condition of preservation of criticals there exists $R' \in \mathcal{C}(N, a')$ such that $Cr(S, a', R') \subset$

$Cr(T \cup S, a^*, R)$ for all $S \in \mathcal{N}$. Now suppose T deviates and announces (a', R'). We want to show that $g(m'_T, m^*_{N\setminus T}) = a'$, i.e., a' is the unique common critical for the collection $\{(T, a^*, R), (N\setminus T, a', R')\}$. Obviously $a' \in Cr(T, a^*, R) \cap Cr(N\setminus T, a', R')$. Let $\bar{a} \in Cr(S', a', R')$ for some $S' \in \mathcal{N}$. Then by our choice of R', we have $\bar{a} \in Cr(T \cup S', a^*, R) \cap Cr((N\setminus T) \cup S', a', R')$. As this is true for all $\bar{a} \in Cr(S', a', R')$ and for all $S' \in \mathcal{N}$, then $g(m'_T, m^*_{N\setminus T}) = a'$ as required. Since $m^* \in SN(\mu, R^*)$ we have $a' \in L(T, a^*, R^*)$. As this is true for all $a' \in Cr(T, a^*, R)$ and for all $T \subset N$ by coalitional monotonicity $a^* \in F(R^*)$.

Assume (2) applies and $g(m) = a^*$ where a^* is the unique common critical for the collection $\{(N\setminus T^i, a^i, R^i)\}_{i=1}^k$ with $R \in \mathcal{C}(N, a^*)$, i.e., $a^* \in \bigcap_{i=1}^k Cr(N\setminus T^i, a^i, R^i)$ and $Cr(T, a^*, R) \subset \bigcap_{i=1}^k Cr((N\setminus T^i) \cup T, a^i, R^i)$ for all $T \in \mathcal{N}$. Let $a' \in Cr(S, a^*, R)$ for some $S \in \mathcal{N}$. If we show that a' can be obtained by a deviation of S then we have $a' \in L(S, a^*, R^*)$ as $m^* \in SN(\mu, R^*)$. By the condition of preservation of criticals there exists $R' \in \mathcal{C}(N, a')$ such that $Cr(S', a', R') \subset Cr(S \cup S', a^*, R')$ for all $S' \in \mathcal{N}$. Suppose S deviates and announces (a', R'). We want to show that $g(m'_S, m^*_{N\setminus S}) = a'$. Let $\bar{T}^i$'s be the new coalitions which are formed after the deviation of S such that everybody in each $\bar{T}^i$ announces the same message. Note that $(N\setminus T^i) \cup S \subset (N\setminus \bar{T}^i)$ for all $i \in \{1, \ldots, k\}$. Then we have $Cr((N\setminus T^i) \cup S, a^i, R^i) \subset Cr(N\setminus \bar{T}^i, a^i, R^i)$ for all $i \in \{1, \ldots, k\}$, which implies $a' \in Cr(N\setminus S, a', R') \cap \bigcap_{i=1}^k Cr(N\setminus \bar{T}^i, a^i, R^i)$. Next we will show that $Cr(S', a', R') \subset Cr((N\setminus S) \cup S', a', R') \cap \bigcap_{i=1}^k Cr((N\setminus T^i) \cup S', a^i, R^i)$ for all $S' \in \mathcal{N}$. Let $\bar{a} \in Cr(\bar{S}, a', R')$ for some $\bar{S} \in \mathcal{N}$. Then by our choice of R' we have $\bar{a} \in Cr(S \cup \bar{S}, a^*, R)$ and as $Cr(T, a^*, R) \subset \bigcap_{i=1}^k Cr((N\setminus T^i) \cup T, a^i, R^i)$ for all $T \in \mathcal{N}$, $\bar{a} \in \bigcap_{i=1}^k Cr((N\setminus T^i) \cup S \cup \bar{S}, a^i, R^i) \subset \bigcap_{i=1}^k Cr((N\setminus \bar{T}^i) \cup \bar{S}, a^i, R^i)$. As this is true for all $\bar{a} \in Cr(\bar{S}, a', R')$ and for all $\bar{S} \in \mathcal{N}$, then a' is the unique common critical for the collection $\{(N\setminus T^i, a^i, R^i)\}_{i=1}^k \cup \{(N\setminus S, a', R')\}$ with

$R' \in \mathcal{C}(N, a')$ and $g(m'_S, m^*_{N\backslash S}) = a'$ as required. Then again by coalitional monotonicity we conclude that $a^* \in F(R^*)$.

Necessity:

The following lemma will be used for the necessity of the conditions.

Lemma. Let $(a, R) \in Gr\,F$ and $T \in \mathcal{N}$. Then $Cr(T, a, R) = \bigcup_{m \in m_{a,R}} \bigcup_{m'_T \in M_T} g(m'_T, m_{N\backslash T})$ where $m_{a,R} = \{m \in M : m \in SNE(\mu, R)$, with $g(m) = a\}$.

Proof:

$Cr(T, a, R) \subset \bigcup_{m \in m_{a,R}} \bigcup_{m'_T \in M_T} g(m'_T, m_{N\backslash T})$

Let $\bar{a} \in Cr(T, a, R)$. Suppose $\bar{a} \notin \bigcup_{m \in m_{a,R}} \bigcup_{m'_T \in M_T} g(m'_T, m_{N\backslash T})$, i.e., there does not exist $m'_T \in M_T$ such that $g(m'_T, m_{N\backslash T}) = \bar{a}$ for some $m \in m_{a,R}$. $\bar{a} \in Cr(T, a, R)$ implies that there exists $R' \in \mathcal{C}(T, a, R)$ such that $\bar{a} \in L(T, a, R')$ and at $R'' \in \mathcal{L}(A)^N$ with $L(i, a, R'') = L(i, a, R')\backslash\{\bar{a}\}$ for all $i \in T$ and $L(i, a, R'') = L(i, a, R') \cup \{\bar{a}\}$ for all $i \in N\backslash T$, $\bar{a} \notin F(R'')$. Let $m \in SN(\mu, R')$ with $g(m) = a$ and $m' = (m'_S, m_{N\backslash S})$ where $T \subset S$ and $g(m') = \bar{a}$. We have two cases to consider: First, assume that $\bar{a} \in L(S\backslash T, a, R')$. Then if $\bar{a}$ is in the upper contour set of a for all $i \in T$, all else left the same, m continues to constitute a strong Nash equilibrium according to the new profile $\bar{R}$ which is a strict T-refinement of R'. But then $\bar{a} \in F(\bar{R})$ which contradicts with R' being a (T, a, R)-critical profile. Second, assume that $\bar{a} \notin L(S\backslash T, a, R')$. Then we again have a contradiction as $\bar{a} \in F(R'')$ where R'' is as defined above.

$Cr(T, a, R) \supset \bigcup_{m \in m_{a,R}} \bigcup_{m'_T \in M_T} g(m'_T, m_{N\backslash T})$

Let $m \in m_{a,R}$ and $m' = (m'_T, m_{N\backslash T})$ with $g(m') = \bar{a}$. Let $R' \in \mathcal{C}(T, a, R)$ be such that $m \in m_{a,R'}$, i.e., $m \in SNE(\mu, R')$. Then $\bar{a} \in L(T, a, R')$ and at $R'' \in \mathcal{L}(A)^N$ with $L(i, a, R'') = L(i, a, R')\backslash\{\bar{a}\}$ for all $i \in T$ and $L(i, a, R'') = L(i, a, R') \cup \{\bar{a}\}$ for all $i \in N\backslash T$, $\bar{a} \notin F(R'')$ by definition of strong Nash equilibrium. So $\bar{a} \in Cr(T, a, R)$.$\square$

Pareto optimality directly follows from the definition of strong Nash equilibrium.

Coalitional Monotonicity:

Let $(a, R) \in GrF, R' \in \mathcal{L}(A)^N$, and $Cr(T, a, R) \subset L(T, a, R')$ for all $T \in \mathcal{N}$. Let $m \in m_{a,R}$. By the lemma for all $T \in \mathcal{N}$ we have $g(m'_T, m_{N\backslash T}) \in Cr(T, a, R) \subset L(T, a, R')$. Then by the definition of strong Nash equilibrium we have $m \in SNE(\mu, R')$ and $g(m) = a \in F(R')$.□

Condition of Transitive Criticals:

Suppose F is strong Nash implementable. Let $(a, R) \in Gr\,F$, $\emptyset \neq T \subset N$ and $a^* \in Cr(T, a, R)$. As F is strong Nash implementable there exists $m \in M$ such that $g(m) = a$ and m is a strong Nash equilibrium at R. By the above lemma there exists $m^* \in M$ such that $m^* = (m^*_T, m_N, T)$ and $g(m^*) = a^*$. Let $R^* \in \mathcal{C}(N, a^*)$ be the preference profile at which m^* is a strong Nash equilibrium. Let $a' \in Cr(S, a^*, R^*)$ and consider the following joint strategy: $m' = (m^*_{T\backslash S}, m'_{S\backslash T}, m''_{T\cap S}, m_{N\backslash(T\cup S)})$. The by the above lemma $a' \in Cr(T \cup S, a, R)$.□

Condition of Unique Common Critical:

Suppose F is strong Nash implementable. Let $a^* \in A$ be a common critical for the collection $\{(T^i, a^i, R^i)\}_{i=1}^k$ i.e., $a^* \in \bigcup\limits_{i=1}^{k} Cr(T^{\cdot}a^i, R^i)$. As F is strong Nash implementable for each (a^i, R^i) there exists $m^i \in M$ such that $g(m^i) = a^i$ and m^i is a strong Nash equilibrium at R^i. Let m^* be the following joint strategy with $g(m^*) = a^*$: $m^* = (m^1_{N\backslash T^1}, \ldots, m^k_{N\backslash T^k})$. Note that $N\backslash T^i \cap N\backslash T^j) = N\backslash(T^i \cup T^j) = \emptyset$ as $T^i \cup T^j = N$ for all $i, j \in \{1, \ldots, k\}$. Moreover $\bigcup\limits_{i=1}^{k}(N\backslash T^i) = N\backslash \bigcap\limits_{i=1}^{k} T^i = N$ as $\bigcap\limits_{i=1}^{k} T^i = \emptyset$. So by the above lemma m^* is the unique joint strategy such that $g(m^*) = a^* \in \bigcap\limits_{i=1}^{k} Cr(T^i, a^i, R^i)$. Now let $R^* \in \mathcal{C}(N, a^*)$ be the preference profile at which m^* is a strong Nash equilibrium. Let $a' \in Cr(S, a^*, R^*)$. Then by the above lemma there exists

$m' \in M$ such that $g(m') = a'$ and $m' = (m'_S, m^*_{N\backslash S}) = (m'_S, m^1_{N\backslash \bar{T}^1}, \ldots, m^k_{N\backslash \bar{T}^k})$ where $N\backslash \bar{T}^i = N\backslash(T^i \cup S)$. So again by the lemma $a' \in \bigcap_{i=1}^{k} Cr(T^i \cup S, a^i, R^i)$.

Next suppose there is a collection of joint messages such that $m^{*i} = (m'_{T^i}, m^i_{N\backslash T^i})$ for some $m'_{T^i} \in M_{T^i}$, with $g(m^{*i}) = a^*$ for all $i \in \{1, \ldots, k\}$. Then a^* is a common critical for the collection $\{(T^i, a^i, R^i)\}_{i=1}^k$.

Now suppose there exists $R^* \in \mathcal{C}(N, a^*)$ such that for all $\emptyset \neq S \subset N$, $Cr(S, a^*, R^*) \subset \bigcap_{i=1}^{k} Cr(T^i \cup S, a^i, R^i)$, where m^{*i} is the equilibrium at R^* for some $i \in \{1, \ldots, k\}$. Without loss of generality assume $i = 1$, i.e., $m^{*1} = (m^1_{T^1}, m^1_{N\backslash T^1})$. Consider the strategy $\bar{m}^{*1} = (m^2_{T^2}, m^{*1}_{N\backslash T^2})$. By the lemma $g(\bar{m}^{*1}) \in Cr(T^2, a^*, R^*)$. Then by our assumption $g(\bar{m}^{*1}) \in \bigcap_{i=1}^{k} Cr(T^i \cup T^2, a^i, R^i)$, in particular, $g(\bar{m}^{*1}) \in Cr(T^2, a^2, R^2)$. But again by the lemma $g(\bar{m}^{*1}) = g(m^2_{T^2}, m^{*1}_{N\backslash T^2}) \in Cr(N\backslash T^2, a^2, R^2)$, which is a contradiction.

So, there exists a unique $m^* \in M, m^* = (m^1_{N\backslash T^1}, \ldots, m^k_{N\backslash T^k})$ with $g(m^*) = a^*$ and a unique $R^* \in \mathcal{C}(N, a^*)$ where m^* is the equilibrium at R^*, such that for all $\emptyset \neq S \subset N$, $Cr(S, a^*, R^*) \subset \bigcap_{i=1}^{k} Cr(T^i \cup S, a^i, R^i)$.□

We now give some examples of social choice rules and examine whether they are strong Nash implementable.

Example 2. Let $N = \{1, \ldots, n\}$, A be a finite set of alternatives, and $F^{PO} = \{a \in A | L(N, a, R) = A\}$. F^{PO} is coalitionally monotonic and satisfies preservation of criticals, but F^{PO} does not satisfy the condition of unique common critical. Hence, F^{PO} is not strong Nash implementable: First note that for any $(a, R) \in Gr\, F$ and $T \in \mathcal{N}, R' \in C(T, a, R)$ if and only if $L(T, a, R')\backslash\{a\} = A\backslash L(N\backslash T, a, R')$. Moreover $Cr(T, a, R) = \{a\}$ for all $T \in \mathcal{N}\backslash\{N\}$ and $Cr(N, a, R) = A$.

Coalitional monotonicity: Let $(a, R) \in Gr\, F, R' \in \mathcal{L}(A)^N$ and $Cr(T, a, R) \subset L(T, a, R')$ for all $T \in \mathcal{N}$. As $Cr(N, a, R) = A \subset L(N, a, R')$ we have $a \in F(R')$ as required.

Preservation of criticals: Let $(a, R) \in Gr\, F$ and $a^* \in Cr(T, a, R)$ for some $T \in \mathcal{N}$. There are two possibilities: First, $T \neq N$. Then $a^* = a$ and for all $R^* \in \mathcal{C}(N, a)$ and $S \in \mathcal{N}$ we have $Cr(S, a, R^*) \subset Cr(T \cup S, a, R)$. Second, $T = N$. Let $a^* \in Cr(N, a, R) = A$. Then for all $R^* \in \mathcal{C}(N, a^*)$ and $s \in \mathcal{N}$ we have $Cr(S, a^*, R^*) \subset Cr(N \cup S, a, R) = A$.

Unique common critical: Let $\{(T^i, a^i, R^i)\}_{i=1}^k$ be an adequate sequence. By definition $(a^i, R^i) \neq (a^j, R^j)$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$. Then $\bigcap_{i=1}^{k} Cr(T^i, a^i, R^i) = \bigcap_{i=1}^{k} \{a^i\} = \emptyset$. So there does not exist a common critical for $\{(T^i, a^i, R^i)\}_{i=1}^k$. Hence F^{PO} does not satisfy the condition of the unique common critical. □

Example 3. Let $N = \{1, \ldots, n\}$ be a finite set of agents, A be a finite set of alternatives and $F^{IR}(R) = \{a \in A | aR_ib \text{ for all } i \in n, \text{ with } b \in A\}$. Note that for any $(a, R) \in Gr\, F$ and $T \in \mathcal{N}$, $R' \in C(T, a, R)$ if and only if $L(T, a, R') = \{a, b\}$ and $Cr(T, a, R) = \{a, b\}$ for all $T \in \mathcal{N}$. Obviously, F^{IR} is coalitionally monotonic and satisfies preservation of criticals. F^{IR} also satisfies the condition of unique common critical where b is the unique common critical for any adequate sequence with $R^b \in \mathcal{C}(N, b)$ such that b is bottom ranked by all the agents at R^b. However, F^{IR} is not Pareto optimal and hence not strong Nash implementable.

Example 4. Let $N = \{1, \ldots, n\}$ be a finite set of agents, A be a finite set of alternatives and $F^{IR-PO}(R) = \{a \in A | aR_ib \text{ for all } i \in n, \text{ with } b \in A \text{ and } L(N, a, R) = A\}$. F^{IR-PO} is Pareto optimal, coalitionally monotonic and satisfies the conditions of the preservation of criticals and unique common critical, hence, F^{IR-PO} is strong Nash implementable.

3.3 More Sufficient Conditions

In this section we define modifications of the condition of preservation of criticals and condition of unique common critical which are easier to check whether an SCR satisfies them. Together with Pareto optimality and coalitional monotonicity these conditions are sufficient for strong Nash implementability.

Definition 11. F satisfies the condition of preservation of criticals, if for all $R \in \mathcal{C}(N, a)$, $T \in \mathcal{N}$, $a^* \in Cr(T, a, R)$, and $R^* \in \mathcal{C}(N, a^*)$ one has $Cr(S, a^*, R^*) \subset Cr(T, a, R)$ for all $S \subset T$.

Definition 12. F satisfies the *condition of common criticals* if there is a common critical for any adequate sequence.

Theorem. An $SCRF$ is strong Nash implementable if it is Pareto optimal, coalitionally monotonic, satisfies the condition of common criticals, and the condition of preservation of criticals.

Proof:

The following mechanism $\mu = (M, g)$ will be used to establish the result: The strategy space of each agent $i \in N$ is,

$$M_i = \{(a^i, R^i, d^i, n^i) | R^i \in C(N, a^i), d^i \in \{0, 1\}, n^i \in \mathbb{N}\}$$

where $\mathbb{N}$ denotes the set of positive integers.

Let $w(m_T)$ be the integer game winner which is defined as, $w(m_T) = l \in T$, if $m^l \geq m^i, \forall i \in T \backslash \{l\}$ and if $m^l = m^i$ for some $i \in T \backslash \{l\}$, then $l < i$.

Define the outcome function $g : M \to A$ for any $m \in M$ as follows:

1. If there exists $(a, R, 0) \in A \times \mathcal{L}(A)^N \times \{0, 1\}$ such that $(a^i, R^i, d^i) = (a, R, 0)$ for all $i \in N$, then $g(m) = a$.

2. If there exists $T^1, \ldots, T^r, T^{r+1}, \ldots, T^k \subset N$ such that $\bigcup_{i \in \{1,\ldots,k\}} T^i = N$ with $T^i \neq \emptyset$ for some $i \in \{1, \ldots r\}$ and for each $T^i \in$

$\{T^1,\ldots,T^r\},(a^j,R^j,d^j)=(a^i,R^i,0)$ for all $j\in T^i$, and for each $T^i\in\{T^{r+1},\ldots,T^k\},(a^j,R^j,d^j)=(a^i,R^i,1)$ for all $j\in T^i$, then $g(m)=a^*$ where $a^*\in\bigcap_i^r Cr(N\backslash T^i,a^i,R^i)$ and $\bar{a}R^l_l a$ for all $a\in\bigcap_i^r Cr(N\backslash T^i,a^i,R^i)$ with $l=w(m_T)$ where $T=T^{r+1}\cup\ldots\cup T^k$.

3. If $d=1$ for all $i\in N$ then $g(m)=a^l$ with $l=w(m_N)$.

The nonemptiness of the set $\bigcap_i^k Cr(N\backslash T^i,a^i,R^i)$ in is guaranteed by the condition of common criticals. In (2) the outcome is determined by the dictatorship of the agent who announces the highest integer, on the set $\bigcap_i^k Cr(N\backslash T^i,a^i,R^i)$.

Step 1 $F(R)\subseteq g(SN(\mu,R))$ for all $R\in\mathcal{L}(A)^N$.

Let $a^*\in F(R^*)$. Consider a strategy $m^*\in M$ such that $m^*_i=(a^*,R,0,0)$ for all $i\in N$ where $R\in C(N,a^*,R^*)\subset\mathcal{C}(N,a^*)$ with $L(T,a^*,R)\subset L(T,a^*,R^*)$ for all $T\subset 2^N\backslash\{\emptyset\}$. We want to show that $m^*\in SN(\mu,R^*)$ and $g(m^*)=a^*$. By (1) $g(m^*)=a^*$. Next, consider a deviation m'_T by $T\subset N$ from m^*. We need to show that $g(m'_T,m^*_{N\backslash T})\in L(T,a^*,R^*)$. If $T=N$ then by Pareto optimality $g(m'_N)\in L(N,a^*,R^*)=A$. Suppose $T\neq N$. Now we have a partition $T^1,...,T^k$ of N where all the agents in each partition announces the same strategy. Let T^k be the coalition where $m_i=(a^*,R,0,0)$ for all $i\in T^k$. By (2) we have $g(m'_T,m^*_{N\backslash T})\in Cr(N\backslash T^k,a^*,R)$. Note that $N\backslash T^k\subset T$. So $g(m'_T,m^*_{N\backslash T})\in Cr(T,a^*,R)$. We also have $R\in C(N,a^*,R^*)$ and $L(T,a^*,R)\subset L(T,a^*,R^*)$ for all $T\subset 2^N\backslash\{\emptyset\}$ which implies $L(i,a^*,R)\subset L(i,a^*,R^*)$ for all $i\in N$ and there is no strict N-refinement R' of R such that $a^*\in F(R')$. But that means $R\in C(T,a^*,R^*)$ and $Cr(T,a^*,R)=L(T,a^*,R)$. Then $Cr(T,a^*,R)=L(T,a^*,R)\subset Cr(T,a^*,R^*)=\bigcup_{R'\in C(T,a^*,R^*)}L(T,a^*,R')\subset L(T,a^*,R^*)$. So we have $g(m'_T,s^*_{N\backslash T})\in L(T,a^*,R^*)$ as required.

Step 2 $g(SN(\mu,R))\subseteq F(R)$ for all $R\in\mathcal{L}(A)^N$.

Let $a^*\in g(SN(\mu,R^*))$. We will show that $a^*\in F(R^*)$. There are three possible cases to consider. First, assume that (1) applies and $m^* \in SN(\mu, R^*)$ is such that $(a^i, R^i, d^i) = (a^*, R, 0)$ for all $i \in N$. As $a^* \in F(R)$, if we show that $Cr(T, a^*, R) \subset L(T, a^*, R^*)$ holds for all $T \subset N$ then by coalitional monotonicity we conclude that $a^* \in F(R^*)$. Let $a' \in Cr(T, a^*, R)$ for some $T \subset N$. Consider the strategy $m = (m'_T, m^*_{N\backslash T})$ where $m'_i = (a', R', 1, 0)$ for all $i \in T$. Then by (2) $g(m'_T, m^*_{N\backslash T}) = a'$. Since $m^* \in SN(\mu, R^*)$ we have $a' = g(m'_T, m^*_{N\backslash T}) \in L(T, a^*, R^*)$. As this is true for all $a' \in Cr(T, a^*, R)$ and for all $T \subset N$ by coalitional monotonicity $a^* \in F(R^*)$.

Assume (2) applies and $m^* \in SN(\mu, R^*)$ is such that $(a^j, R^j, d^j) = (a^i, R^i, 0)$ for all $j \in T^i, i \in \{1, \ldots r\}$ and $(a^j, R^j, d^j) = (a^i, R^i, 1)$ for all $j \in T^i, i \in \{r+1, \ldots, k\}$ and $g(m^*) = a^* \in \bigcap_{i=1}^{r} Cr(N\backslash T^i, a^i, R^i)$. Let $R \in C(N, a^*)$. If we show that $Cr(T, a^*, R) \subset L(T, a^*, R^*)$ for all $T \in 2^N \backslash \{\emptyset\}$ we conclude again by coalitional monotonicity that $a^* \in F(R^*)$. Let $a' \in Cr(T, a^*, R)$. If we show that a' can be obtained by a deviation of T then we have $a' \in L(T, a^*, R^*)$ as $m* \in SN(\mu, R^*)$. Suppose T deviates and says $(a', R', 1, n')$ where $R' \in \mathcal{C}(N, a')$ and n' is the highest integer announced. Then the outcome is in the intersection $\bigcap_{i=1}^{r} Cr(N\backslash \bar{T}^i, a^i, R^i)$ where $\bar{T}^i$'s are the new coalitions which are formed after the deviation of T such that everybody in each $\bar{T}^i$ announces the same message with $d_i = 1$. As $a^* \in \bigcap_{i=1}^{r} Cr(N\backslash \bar{T}^i, a^i, R^i)$ and $N\backslash T^i \subset N\backslash T^i$ for all $i \in \{1, \ldots, r\}$, we have $a^* \in \bigcap_{i=1}^{r} Cr(N\backslash \bar{T}^i, a^i, R^i)$. Also note that $T \subset N\backslash \bar{T}^i$ for all $i \in \{1, \ldots, r\}$. So now for each $i \in \{1, \ldots, r\}$ we have $a' \in Cr(T, a^*, R)$ and $a^* \in Cr(N\backslash \bar{T}^i, a^i, R^i)$ where $T \subset N\backslash \bar{T}^i$. Then by the transitive criticals condition we have $\bar{a} \in Cr(N\backslash \bar{T}^i, a^i, R^i)$. As this is true for all $i \in \{1, \ldots, r\}$ we have $a' \in \bigcap_{i=1}^{r} Cr(N\backslash \bar{T}^i, a^i, R^i)$ and $g(m'_T, m^*_{N\backslash T}) = a' \in L(T, a^*, R^*)$ where $m'_i = (a', R', 1, n')$ for all $i \in T$. Hence by coalitional monotonicity we conclude that $a^* \in F(R^*)$.

Assume (3) applies and $m^* \in SN(\mu, R^*)$ is such that $d^i = 1$ for all $i \in N$ and $g(m^*) = a^l$ where $l = w(m^*_N)$. Let $R^k \in \mathcal{C}(N, a^k)$ be the profile announced by agent k. Note that every agent can make the outcome anything in the alternative set by announcing that alternative and an integer higher than n^l. Then we have $Cr(T, a^l, R^l) \subset L(T, a^l, R^*) = A$ for all $\emptyset \neq T \subset N$ and by coalitional monotonicity $a^l \in F(R^*)$.$\square$

Condition of common criticals is obviously weaker than the condition of unique common criticals and hence, also a necessary condition for implemetability. However the modified condition of preservation of criticals is not a necessary condition:

Example 5. Let $A = \{a, a*, a'\}$ and $N = \{1, 2\}$. $F : \mathcal{L}(A)^N \to 2^A$ is defined as follows:

$$F(R) = \begin{cases} a', \text{ if } a' \text{ is top ranked by 1 and 2} \\ a, \text{ if } L(a, R) = A \\ a^*, \text{ if } a^* \text{ is top ranked by } i \text{ and } a^* R_j a \text{ for } j \text{ with } i \neq j. \end{cases}$$

First we will show that F does not satisfy the condition of preservation of criticals. Let R, R^* be profiles such that $a' R_1 a R_1 a^*, a^* R_2 a R_2 a'$ and $a^* R^*_1 a R^*_1 a', a' R^*_2 a^* R^*_2 a$. Note that $F(R^*) = a^*$, $F(R) = a$ and $a' \in Cr(1, a^*, R^*), a^* \in Cr(1, a, R)$. But $a' \notin Cr(1, a, R)$. So F does not satisfy the condition of preservation of criticals.

F is strong Nash implementable: Consider the following mechanism. $M_i = \{m, m', m^*\}$ for all $i \in N$ and g is defined as follows where the rows represent agent 1:

	$\mathbf{m}$	$\mathbf{m'}$	$\mathbf{m^*}$
$\mathbf{m}$	a	a	a
$\mathbf{m'}$	a^*	a	a'
$\mathbf{m^*}$	a	a	a^*

It is left as an exercise that $\mu = (M, g)$ implements F.

CHAPTER 4

SUFFICEINT CONDITIONS FOR SUBGAME PERFECT NASH IMPLEMENTABILITY

In this chapter we consider subgame perfect Nash implementability. We identify a subset of implementable social choice rules via critical profiles. The ideas presented here can be viewed as a first step towards a full characterization and the design of different mechanisms for subgame perfect implementation.

4.1 Notation and Definitions

Let $N = \{1, \ldots, n\}$ be a nonempty finite set of agents and A be a nonempty finite set of alternatives. A preference profile is an n-tuple, $R = (R_1, \ldots, R_n)$ where each R_i is a linear order[1] on A which represents agent i's preferences on A. The set of all linear order profiles on A is denoted by $\mathcal{L}(A)^N$. A *social choice rule (SCR)* is a mapping $F : \mathcal{L}(A)^N \to 2^A$ which assigns to every linear order profile $R \in \mathcal{L}(A)^N$ a subset of A.[2] *The graph of an SCR F*, is defined as $Gr\, F = \{(a, R) \in A \times \mathcal{L}(A)^N \mid a \in F(R)\}$.

An extensive mechanism with simultaneous moves is a 5-tuple, $\mu = (X, >$

[1] A linear order is a complete, transitive, reflexive, antisymmetric binary relation.

[2] We will assume that F is onto. In general, all the results continue to hold if we restrict F to the image of F.

$, D, \delta, g)$, where X is a set of nodes, $>$ is a partial ordering on X that represents precedence, D is a set of possible decisions, δ is a one-to-one function that labels each non-initial node with the last decision taken to reach it and g is a function that associates to each terminal node the outcome that is obtained at this node. At each non-initial node all agents know the entire history of the play and they move simultaneously. $M_i = \prod_{x \in X \setminus Z} M_i(x)$ is the message space for agent i where Z is the set of nonterminal nodes and $M_i(x)$ is the set of messages for i at the node x. The joint message space is $M = \prod M_i$. For each $m \in M$ and $t \in T$, let $g(m, t)$ be the outcome when agents use strategy m starting from node t. An extensive game mechanism μ constitutes an extensive game at each preference profile $R \in \mathcal{L}(A)^N$.

$m \in M$ is a Nash equilibrium for (μ, R), if for all $i \in N$, $g(m; t_0) R_i g((m'_i, m_{-i}); t_0)$ for all $m'_i \in M_i$.

A Nash equilibrium $m \in M$ for (μ, R) is subgame perfect if for all $i \in N$, $g(m; t) R_i g((m'_i, m_{-i}); t)$ for all $m'_i \in M_i$ and for all noninitial nodes x. Let $SPE(\mu, R)$ denote the set of subgame perfect equilibria of the game (μ, R). A social choice rule is implementable in subgame perfect equilibria if there exists an extensive mechansim that implements it, i.e., if there exists μ such that for all $R \in \mathcal{L}(A)^N$, $g(SPE(\mu, R); t_0) = F(R)$.

4.2 The Result

We introduce two new conditions via critical profiles which are related to the conditions of common critical and preservation of criticals from Chapter 3. We show that together with Maskin-monotonicity these conditions are sufficient for subgame perfect implementability. The mechanism used is also based on critical profiles.

Theorem. An SCR F is subgame perfect implementable if it is Maskin monotonic, satisfies the condition of preservation of criticals and the condition of common criticals.

Theorem. A social choice rule F is subgame perfect implementable if it is Maskin-monotonic and satisfies the following conditions:

i. For all $T^1, \ldots T^k \subset N \backslash \emptyset$ such that $\bigcup_i T^i = N$, for all $(a^i, R^i) \in Gr\, F$ such that $R^i \in \mathcal{C}(N, a^i)$ with $L(N \backslash T^i, a^i, R^i) = Cr(N, a^i, R^i) \backslash Cr(T^i, a^i, R^i)$ one has $\bigcap_i L(N \backslash T^i, a^i, R^i) \neq \emptyset$.

ii. For all $b \in L(T, a, R)$ such that $R \in \mathcal{C}(N, a)$ and R is defined as in (i), and for all $R' \in \mathcal{C}(N, b)$ one has $L(i, b, R') \subset L(T \cup \{i\}, a, R)$.

Proof:

Stage 1: Each agent announces a pair $(a^i, R^i) \in Gr\, F$. If there exists $(a, R) \in Gr\, F$ such that $(a^i, R^i) = (a, R)$ for all $i \in N$, then implement a. Otherwise go to Stage 2.

Stage 2: Group the agents that announce the same alternative. Let $T^1, \ldots, T^k \subset N \backslash \emptyset$ be the groups that announce the same alternative, i.e., $a^j = a^i$ for all $j \in T^i$. Each agent announces a critical profile defined as follows:

If $j \in T^i$ then j announces some $R^i \in \mathcal{C}(N, a^i)$ with $L(N \backslash T, a^i, R^i) = Cr(N, a^i, R^i) \backslash Cr(T, a^i, R^i)$.

Each agent $i \in N$ also announces a non-negative integer n^i. Consider the intersection $I = \bigcap_{i=1}^{S} L(N \backslash \bar{T}^i, a^i, R^i)$. Now, if there exists a unique common critical a^* for $\{(N \backslash \bar{T}^i, a^i, R^i)\}_{i=1}^{S}$ then implement a^*.

Otherwise implement $a^* \in I$ where a^* is the most preferred alternative of the agent who announced the highest integer in I. Ties are broken in the favor of the lower indexed agent.

$F(R) \subseteq g(SPE(\mu, R))$ for all $R \in \mathcal{L}(A)^N$. Let $a^* \in F(R^*)$. Consider the following strategy $m^* \in M$: $m_i^*(1) = (a^*, R^*)$ for all $i \in N$ and $m_1^*(2) = (b, R^b)$, $m_i^*(2) = (a, R^a)$ for all $i \in N \backslash \{1\}$. Any unilateral deviation only at Stage 2 will result in outcome a^* as the game will not reach that stage. In other cases, i.e., if an agent i deviates at both Stage 1 and 2 or only at Stage 1, the outcome will be $a^1 \in L(i, a^*, R^*)$ by the construction of the mechanism. Therefore, no agent can be better off by a unilateral deviation and hence $a^* \in g(SPE(\mu, R^*))$.

$g(SPE(\mu, R^*)) \subset F(R)$, for all $R \in \mathcal{L}(A)^N$. Let $a^* \in g(SPE(\mu, R^*))$, i.e., there exists $m^* \in M$ with $g(m^*) = a^*$ and $m^* \in SPE(\mu, R^*)$.

First assume m^* is such that $m_i^*(1) = (a^*, R)$ for all i. Then $g(m^*) = a^*$. Let $x \in L(i, a^*, R)$. If agent i deviates and announces x at Stage 1 and announces the highest integer at Stage 2, then one of the cases in Stage 2 will apply. As $x \in L(i, a^*, R)$, $x \in L(T, a^*, R')$ for all possible announcements of agents other than i. So x will be in the intersection I. Then either x is the unique common critical for the reached collection, or as i announced the highest integer, x will be the outcome. As m^* is equilibrium at $R*$ then $x \in L(i, a^*, R^*)$. This is true for all i, so by Masking-monotonicity $a^* \in F(R^*)$.

In other cases by using a similar argument to the one above and to the argument used in the proof of the sufficiency result of strong Nash equilibrium, we obtain the result. □

CHAPTER 5

CONCLUSION

In the previous chapters we explored the relation of monotonicity and critical profiles to implementability. We began by identifying the self-monotonicity of the Nash equilibrium concept and determined the monotonicities that are inherited by Nash implementable social choice rules which are not necessarily self-monotonicities. We showed that the Nash equilibrium concept has a unique self-monotonicity which is carried over to the social choice rule via the mechanism that implements it. As there may be several mechanisms that implement a social choice rule and several equilibria at each preference profile the unique self-monotonicity of the Nash equilibrium concept may induce several monotonicities for the social choice rule. The self-monotonicity that is carried over by Maskin-Vind type mechanisms, which are commonly used in the characterization results of Nash implementability, turns out to be the essential monotonicity of Danilov.

We then looked at the strong Nash equilibrium. In situations where cooperation among agents is likely, it is more appropriate to use strong the Nash equilibrium for implementation. We gave a new characterization of strong Nash implementable social choice rules via critical profiles. The definition of critical profiles was modified so that it applies to coalitions and we determined the critical alternatives for each coalition at each profile. We introduced three new conditions for social choice rules; coalitional monotonicity, preservation of criticals and unique common critical. When these conditions are combined with Pareto optimality, they characterize social choice rules that are strong Nash implementable. We introduced a new mechanism for the sufficiency part of our result where each agent's message space consists of the alternative set and the critical profiles for each alternative.

Monotonicity and critical profiles are essential concepts for implementability. They are not only important for identifying the rules that are implementable in a solution concept but also very useful to design new mechanisms for implementation. The generality of the applications, supports the idea that implementation theory can be cast in terms of monotonicity. With further research, the identification of monotonicity structures and critical profiles will be sufficient to answer most implementation and mechanism design problems. This unified approach will allow us to have a better understanding of the relation between various problems in the field.

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