Output Regulation for All-Pole and Minimum Phase LTI / LTV Systems

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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June 2010

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ABSTRACT

Output Regulation for All-Pole and Minimum Phase LTI / LTV Systems

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June 2010

In this thesis, the problem of enabling the output of a system to track the reference signals and reject the disturbances created by the same exogenous system is considered. This problem is widely known as *Output Regulation Problem*. Firstly, we propose a method for *all-pole* LTI systems by using relative degree property and then we apply the same method for *minimum phase* LTI systems along with some modifications. In order to obtain controllers for a minimum phase LTI case, the system is converted into an all-pole system by employing the inverse system as the first part of the controller. Then using the method that we used in all-pole cases, we obtain the second part of the controller. Combining these two controllers gives us an overall controller which solves the output regulation problem. This method for LTI systems is then extended to *all-pole* and *minimum phase* LTV systems. However, in order to apply the same methodology we have to make some assumptions on LTV systems. For minimum phase cases, the normal form is obtained by applying certain Lyapunov transformations and then minimum phaseness is defined in accordance with the normal form. Furthermore we show that, similar to minimum phase LTI cases, pole / zero cancelations occur between the inverse system and the original system in minimum phase LTV cases. The method that we develop depends on analytical calculation of the controller and gives a certain degree of freedom to change the transient behavior of the system by only changing some controller parameters.

Keywords: Output Regulation, Tracking, Disturbance Rejection, All-Pole, Minimum Phase, Relative Degree, LTI System, LTV System, Pole / Zero Cancelation, Inverse System, Lyapunov Transformation

ÖZET

TÜM-KUTUPLU VE ENKÜÇÜK EVRELİ DOĞRUSAL ZAMANDA BAĞIMSIZ VE DOĞRUSAL ZAMANLA DEĞİŞEN SİSTEMLERİN ÇIKIŞ REGÜLASYONU

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Bu tezde Tüm Kutuplu ve Enküçük Evreli Doğrusal Zamanda Değişmez Sistemlerin dışsal bir sistem tarafından üretilen referans sinyalini takibi ve aynı dışsal sistem tarafından üretilen bozucuların engellenmesi konusunu incelenmiştir. Bu problem literatürde *Çıkış Regülasyonu* olarak adlandırılır. İlk olarak, Tüm Kutuplu(All-Pole) Sistemlerin göreli derece özelliğini kullanarak *Çıkış Regülasyonu* problemini çözmek için bir yöntem geliştirilmiştir ve aynı yöntem bazı değişiklikler yapılarak Enküçük Evreli Doğrusal Zamanda Bağımsız sistemlere uygulanmıştır. Bu çözümü, Enküçük Evreli(Minimum Phase) sistemlere uygulamak için, sistemin tersi, denetleyicinin ilk parçası olarak kullanılmıştır ve sistem Tüm Kutuplu hale getirilmiştir. Sonra, bu yöntemle denetleyicinin ikinci kısmı oluşturulmuştur. Bu parçaları birleştirerek, regülasyon koşullarını sağlayan toplam denetleyici elde edilmiştir. Bu yöntem daha sonra Tüm Kutuplu ve Enküçük Evreli Doğrusal Zamanla Değişen sistemlere genişletilmiştir. Ancak, aynı yöntemi Doğrusal Zamanla Değişen sistemlere uygulamak için bazı varsayımlarda bulunmak gerekmektedir. Enküçük Evreli durum için belirli Lyapunov dönüşümleri uygulanarak sistem bir normal forma getirilmiştir ve bu normal form üzerinden Enküçük Evreli olmak tanımlanmıştır. Bunun yanısıra, Enküçük Evreli Doğrusal Zamanda Değişmez sistemlere benzer olarak Enküçük Evreli Doğrusal Zamanla Değişen durumda ters sistem ve orjinal sistem arasında kutup / sıfır sadeleşmesinin olduğu gösterilmiştir. Geliştirilen bu yöntem denetleyicinin analitik olarak hesaplanmasına dayanmaktadır ve bu yöntem bazı denetleyici değişkenlerini değiştirerek sistemin geçici davranışını değiştirilebilmesine de olanak vermektedir.

Anahtar Kelimeler: Çıkış Regülasyonu, Takip, Bozulmaların Engellenmesi Tüm Kutuplu, Enküçük Evreli, Göreli Derece, Doğrusal Zamanda Bağımsız Sistemler, Doğrusal Zamanla Değişen Sistemler, Kutup / Sıfır Sadeleşmesi, Ters Sistem, Lyapunov Dönüşümü

ACKNOWLEDGMENTS

Firstly, I would like to thank my supervisor Prof. Dr. Ömer Morgül for his supervision, support and enlightening guidance throughout the development of this thesis.

I am grateful to thank Prof. Dr. A. Bülent Özgüler and Assist. Prof. Uluç Saranlı for reading and commenting on this thesis.

I would also to thank my parents Muzaffer Saldı and Şahinde Saldı and my brother Necdet Saldı.

Finally, I would like to give special thanks to my fiance Rana Keskin whose endless support, encouragement and understanding made this study possible.

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Chapter 1

INTRODUCTION

In control theory, designing controllers that force the system to track a given reference signal $r(t)$ and reduce (if possible, reject) the effect of unwanted signal $\nu(t)$ (disturbance) at the output is among the most important problems [1–18]. This problem, which is generally referred to as the *output regulation problem*, was studied by many researchers and is still under investigation by considering all possible aspects of the problem. For this problem, some of the researchers take into account either disturbance rejection or reference signal tracking only [3–6]. Alternatively, some of the researchers worked on both the disturbance rejection and the reference signal tracking problem simultaneously [7], [8]. Furthermore, in this formulation some of the researchers assumed that the reference signals and the disturbances are considered the signals which are generated by different dynamical systems. In the latter case, these dynamical systems are assumed to be separate as shown below :

$$\begin{aligned} \dot{\nu}(t) &= S_1\nu(t), \\ \dot{r}(t) &= S_2 r(t). \end{aligned} \tag{1.1}$$

where $r(t)$ and $\nu(t)$ are the reference and disturbance signals, respectively, and S_1 and S_2 are matrices which have appropriate dimensions. In this thesis for the formulation of the output regulation problem, we will try to find a controller structure to track the reference signal and to reject the disturbances simultaneously. Instead of the model given by (1.1), we will assume that the reference signal $r(t)$ and the disturbance signal $\nu(t)$ are generated by the same dynamical system, which is called as the exogenous system [9–11, 19–23]. The dynamics of the exogenous system is assumed to be as given below :

$$\begin{aligned} \dot{w}(t) &= Sw(t), \\ r(t) &= Qw(t), \\ \nu(t) &= Pw(t). \end{aligned} \tag{1.2}$$

where $w(t)$ is called the exogenous signal, S, Q and P are matrices which have appropriate dimensions. Note that, even if the reference signal and disturbances have distinct dynamic behaviors as given in (1.1), one can still transform (1.1) into the form given by (1.2) as shown below :

$$\begin{aligned} \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} &= \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \\ r(t) &= \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \\ \nu(t) &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \end{aligned} \tag{1.3}$$

As can be seen, (1.3) has the same form as given by (1.2) where $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, $Q = \begin{pmatrix} I & 0 \end{pmatrix}$, $P = \begin{pmatrix} 0 & I \end{pmatrix}$ [24]. One disadvantage of using (1.3) instead of (1.1) might be the following: the pairs (Q, S) and (P, S) in (1.3) are not observable. In the remaining of the thesis, we assume that the reference signal and disturbances are generated by the same dynamical system as given by (1.2). Note that the state space form of plant for which we will design controllers is given below :

$$\dot{x}(t) = A(t)x + B(t)u + \nu, \tag{1.4}$$

$$y(t) = C(t)x. \tag{1.5}$$

where $x\epsilon\Re^n$, $u\epsilon\Re$, $y\epsilon\Re$, $\nu\epsilon\Re^n$ represent the system state, input, output and disturbances .respectively and $A(t)\epsilon\Re^{n\times n}$, $B(t)\epsilon\Re^{n\times 1}$, $C(t)\epsilon\Re^{1\times n}$ represent system matrices. If any of the system matrices is time-varying, then our system become linear time-varying. Conversely, if all of the system matrices are time-invariant, then our system becomes linear time-invariant.

In the output regulation problem, the objective is to find such a control law that the closed-loop system tracks the reference signal and rejects the disturbances simultaneously. Actually, if the error is to be defined as the difference between the reference signal $r(t)$ and the system output $y(t)$, then output regulation problem can be converted into obtaining such a controller that the overall system satisfies the conditions given below [24]:

(i) For all the initial conditions of the original system and the exogenous system,

$$\lim_{t\longrightarrow\infty} e(t) = \lim_{t\longrightarrow\infty} (y(t) - r(t)) = 0. \tag{1.6}$$

(ii) The closed-loop system is exponentially stable with $w = 0$.

Throughout the thesis, we will refer to the conditions (i) and (ii) given above as **regulation conditions**. Note that if the output regulation problem is defined as the tracking of the reference signal and the rejecting of the disturbances simultaneously, then any controller which solves the output regulation problem also satisfies the regulation conditions given above. Conversely, any controller which satisfies the regulation conditions given above also solves the output regulation problem. This could be easily seen if the exogenous system given by (1.2) is combined with the closed-loop system dynamics. In this case, the state transition matrix will have a block triangular form. By using this structure, the equivalence stated above can be shown easily [24].

The LTI part of this problem was studied by many authors. The first attempt to solve the problem of linear output regulation was made in [1] and [2] where the case of the constant reference signal and the disturbances was considered. In [9], a set of equations, called *the regulator equations*, were introduced and the controller which solved the output regulation problem was related to the solution of the regulator equations. In [10], a new concept *the internal model principle* was introduced which roughly stated that the controller which solved the output regulation problem included a model of exogenous system. In [11], polynomial matrices were used in the output regulation problem, and a primary condition, which was a single polynomial matrix formulation and that the controller should satisfy was given. In addition, *the internal model principle* ,which was first introduced in [10], was clarified by using polynomial matrices. The robust case of the linear output regulation problem was considered in [12–16]. In [12], the controller which solved the output regulation problem was shown to be robust to the perturbations in plant parameters and unmeasurable disturbances if and only if it regulated a system called the *expanded system*. Furthermore, in [15] the case which the disturbances were unmeasurable, arbitrary signals were considered and two conditions for the solvability of the output regulation problem were given. The application of the frequency domain techniques for the output regulation problem can be found in [17, 18].

LTV case of the output regulation problem did not receive as much attention as its LTI counterpart did, possibly due to the mathematical difficulties which may be encountered in the analysis. In [19], the linear periodic time-varying exogenous system and the LTI plant case were studied. Differential regulator matrix equations which were the counterparts of the regulator equations in LTI cases were found. In [20], the case of minimum phase time-varying systems with a time-varying exogenous system were considered and a differential type of regulator equations were found for the solvability of the output regulation problem. Then, in [21] a general time-varying system with a time-varying exogenous system was considered and differential regulator equations were derived as in the previous cases.

For the solution of the output regulation problem for nonlinear systems, the researchers mainly tried to extend the existing approaches for linear systems to nonlinear cases. In [25], the *internal model principle* was extended to the nonlinear systems defined on differentiable manifolds. In [26], a PI controller was employed for the constant disturbances and the reference signal case. In [27], the work in [9] for linear multivariable systems was extended to the nonlinear setting with slowly varying or constant exosystem and nonlinear equations ,which were the nonlinear counterparts of the linear regulator equations in a special case, were obtained. Then, in [22] the results of [9] was extended to a general setting in which the exosystem was a time-varying nonlinear system. In the latter, the nonlinear regulator equations were obtained and their solution guaranteed the solution of the output regulation problem. After the work in [22], solvability conditions for these nonlinear regulator equations were studied by many researchers, see e.g. [22,23,28,29]. In [30,31] the case of nonlinear system with nonhyperbolic zero dynamics was studied. In [32], the results in [22] were extended to the feedback linearizable system. In [33], the global robust output regulation with error feedback was considered. In [34], internal models were used to design output regulators for nonlinear systems. In [35], global output regulation of uncertain nonlinear systems was studied and a novel high gain internal model was developed.

In most of the existing approaches for the output regulation problem, one tries to obtain a control law which satisfies (i) assuming that (ii) is satisfied. Thus, in the classical approach, one assumes that the closed-loop system is already exponentially stable and consequently one tries to find a controller which satisfies the regulation condition (i). Additionally, the existing solutions for the reduced problem do not give the controller explicitly. Instead, in the classical approach one obtains a set of equations, called *the regulator equations* [24], and the solvability of the output regulation problem depends on the solvability of the regulator equations. The regulator equations for LTI cases are shown below :

$$\begin{aligned} X_c S &= A_c X_c + P_c, \\ 0 &= C_c X_c + Q_c \end{aligned} \tag{1.7}$$

where A_c, P_c, C_c and Q_c are the closed-loop system state transition matrix, disturbance matrix, output matrix and reference signal matrix, respectively [24]. Here, X_c is the unknown matrix to be found, and once X_c is found, one can design a controller by using X_c. Note that (1.7) is called as the *Sylvester* equations. For details, see [24]. If one uses the same approach in LTV cases, the regulator equations become as follows :

$$\begin{aligned} \dot{X}_c(t) + X_c(t)S(t) &= A_c(t)X_c(t) + P_c(t), \\ 0 &= C_c(t)X_c(t) + Q_c(t) \end{aligned} \tag{1.8}$$

where A_c, P_c, C_c and Q_c are the closed-loop system state transition matrix, disturbance matrix, output matrix and reference signal matrix, respectively [20]. As in LTI cases, here $X_c(t)$ is the unknown matrix, and if one finds a solution, then by using $X_c(t)$ one can construct a controller. Note that (1.8) is also called *the differential Sylvester equations.* For more details, refer to [20]. The fulfillment of these equations corresponds to the fulfillment of the condition (i).

In this thesis, we restrict ourselves only to all-pole and minimum phase systems. Since we deal with only some portion of the general systems, our proposed method has advantages over the existing approaches. The advantages of our solution to the output regulation problem are as follows;

- Different from existing approaches our solution depends on the analytical calculation of the controller that satisfies regulation conditions (i), (ii).

This analytical calculation particularly is very important for LTV systems because finding controller by using regulator equations, which include a differential matrix equation, is a very difficult task.

- Our approach does not assume the fulfillment of the condition (ii) like most of the existing approaches do. Instead, we propose a controller which satisfies the conditions (i) and (ii). Moreover, in our approach rather than a single controller, a class of controllers which solve the output regulation problem is constructed.

- In LTI case, the controller that solves the output regulation problem may be found easily by using regulator equations. However, in this methodology we do not have enough degree of freedom to alter the transient behavior of the system. On the other hand, our approach allows one to alter the transient behavior of the closed-loop system upto a certain degree only by changing some controller parameters. By this way, the designer can achieve some desired specifications other than the regulation conditions.

The outline of this thesis is as follows;

In chapter 2, we study the output regulation problem for all-pole and minimum phase LTI systems. First of all, the problem formulation will be given. Then, by defining and using the relative degree property of the LTI systems, a static controller for all-pole systems will be obtained. In addition, observers for the original system and the exogenous system will be designed and combined with the overall system. Afterwards, we consider the minimum phase systems and define minimum phaseness. By introducing an inverse system, a dynamic controller for minimum phase case will be achieved. Then, similar with all-pole cases, observers will be designed for both the original system and the exogenous system. Finally, we will show some numerical results.

In chapter 3, we will extend the technique in chapter 2 to the output regulation problem for the all-pole and the minimum phase LTV systems. First the problem formulation will be given. Then, by defining and employing relative degree property, controller for all-pole systems will be constructed. After that, the inverse systems of the minimum phase LTV systems will be found by applying certain transformations on the original systems. Then, a dynamic controller for the minimum phase systems will be obtained by using the inverse systems as a first part of the controller. Then we will show pole/zero cancelations between the inverse system and the original system. Lastly, some numerical results are shown.

In the section which we will deal with controller design for minimum phase LTV systems, we will use pole/zero definitions of LTV systems and we will try to show pole/zero cancelations between the inverse systems and the original systems. However, in literature there are no unique definition of poles and zeros for LTV systems. Thus, to show pole/zero cancelations we will use definitions of poles and zeros in [36] and these definitions are the generalizations of pole/zero definitions for LTV systems in [37]. In [37], definition of poles and zeros for special class of time-varying systems were given. In these definitions, zeros corresponds to the modes that make output zero when we apply this as an input to the system and poles correspond to the modes which determine the stability of the system. However, in [36] definition of poles and zeros for a general class of time-varying systems were given. Additionally, there are two different zero definition in [36] which are *transmission zeros* and *ordinary zeros*. The *transmission zeros* correspond to the transmission zeros of the MIMO systems and the *ordinary zeros* correspond to the zeros of the SISO systems in LTI cases.

In the last chapter, we conclude our remarks by going over some important points of the output regulation problem, and we propose some further research areas, as well as some possible extensions of our results.

Chapter 2

OUTPUT REGULATION for ALL-POLE and MINIMUM PHASE LTI SYSTEMS

Throughout this section, we consider single-input-single-output (SISO) linear-time-invariant (LTI) systems which have the following form :

$$\dot{x(t)} = Ax + Bu + \nu, \tag{2.1}$$

$$y(t) = Cx, \tag{2.2}$$

where $x \epsilon \Re^n$, $y \epsilon \Re$, $u \epsilon \Re$ represent the system state, output and input respectively and $A \epsilon \Re^{n \times n}$, $B \epsilon \Re^{n \times 1}$, $C \epsilon \Re^{1 \times n}$ represent constant system matrices. We assume the exogenous system that we deal with is in the form which is given below :

$$\dot{w(t)} = Sw(t), \tag{2.3}$$

$$r(t) = -Qw(t), \tag{2.4}$$

$$\nu(t) = Pw(t), \tag{2.5}$$

where $w \epsilon \Re^m$, $d \epsilon \Re^n$ and $r \epsilon \Re$ represent exogenous system states, disturbance signals and reference signal respectively and $S \epsilon \Re^{m \times m}$, $P \epsilon \Re^{n \times m}$, $Q \epsilon \Re^{1 \times m}$ represent constant matrices of exogenous system. The sign ”-” in the equation that gives reference signal (2.4) is chosen to ensure compliance with the use in the literature. Then, the tracking error *e(t)* can be defined as shown below :

$$e(t) = y(t) - r(t) = Cx + Qw. \tag{2.6}$$

In order to use the system states *x(t)* and the exogenous system states *w(t)* in the controller, we should make the assumption of the observability for both of the systems.

Assumption 1. *The pairs* (C, A) *and* (Q, S) *are both observable.*

Assumption 2. S *has distinct eigenvalues with zero real parts*

Assumption 2 guarantees that the solutions of the exogenous system are bounded and do not decay to zero as time goes infinity. If the exogenous system has eigenvalues with negative real parts, then the reference signal or/and the disturbances may decay to zero. But, decaying reference signal or disturbances are not considered in the output regulation problem which is investigated here. Conversely, if the exogenous system has eigenvalues with positive real parts, then the reference signal and the disturbances become unbounded, but it is not considered here for simplicity.

Our objective is to design a feedback control law by using both the original and the exogenous system states such that the closed-loop system satisfies the regulation conditions (i) and (ii). In the simplest form (All-Pole case), we will use the relative degree property of the system in order to find the controller which provides regulation conditions. Therefore we need to define this property first.

2.1 Relative Degree Property

If the system satisfies the following conditions :

$$CA = CAB = \ldots = CA^{r-2}B = 0, \tag{2.7}$$

$$CA^{r-1}B = \alpha \neq 0 \ , \ r \leq n, \tag{2.8}$$

then the system has a "relative degree r". If we take the derivative of the output of the system *y(t)*, input *u(t)* appears at the r^{th} derivative because of the relative degree property : i.e.

$$\begin{aligned}
\dot{y} &= C(Ax + Bu) = CAx + \underbrace{CB}\,u, \\
&\vdots \\
y^{(r-1)} &= CA^{r-2}(Ax + Bu) = CA^{r-1}x + \underbrace{CA^{r-2}B}\,u, \\
y^{(r)} &= CA^{r-1}(Ax + Bu) = CA^{r}x + CA^{r-1}Bu = CA^{r}x + \alpha u.
\end{aligned} \tag{2.9}$$

The parts, indicated by underbrace are equal to zero. Therefore, this property of the system can be used to design controllers for All-Pole systems.

2.2 Controller for All-Pole LTI Systems

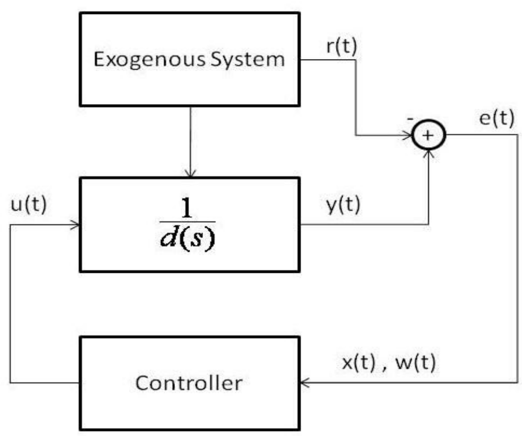

Figure 2.1: Overall System Block Diagram

If we have a system that is *full-relative degree* (i.e. $r = n$ and system dimension is n), then this system is called as an "All-Pole System". Transfer function of this kind of systems are expressed as follows : $G(s) = \frac{1}{d(s)}$ where $d(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0$. A state space model of the system of this type can be given as shown below:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots 0 \\ 0 & 0 & 1 & 0 & \ldots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} x. \tag{2.10}$$

Remark 1. *Using the system model given by (2.10), the proof of full-relative degree property of the system can be done easily.*

If we take the derivative of the error given by (2.6) repeatedly, and if we use the system equations given by (2.1)-(2.5) and the equations given by (2.7)-(2.8) with $r = n$, then we obtain the following equations :

$$\begin{aligned} e &= Cx + Qw, \\ \dot{e} &= C\dot{x} + Q\dot{w}, \\ &= C(Ax + Bu + Pw) + QSw, \\ &= CAx + (CP + QS)w, \\ \ddot{e} &= CA\dot{x} + (CP + QS)\dot{w}, \\ &= CA^2x + CAPw + (CP + QS)Sw, \\ &\vdots \\ e^{(n)} &= CA^n x + \alpha u + CA^{n-1}Pw + S_{n-1}Sw, \end{aligned} \tag{2.11}$$

where in the derivatives we used the relative degree property given by (2.7)-(2.8). Here we have $\alpha = CA^{n-1}B \neq 0$ (see (2.8)), and the matrices S_i are given as below

:

$$S_i = S_{i-1}S + CA^{i-1}P, S_0 = Q, \ \ 1 \leq i \leq n \ \ , \tag{2.12}$$

In this case, we can choose the following control law for *u(t)* :

$$u = \frac{1}{\alpha}\{-CA^n x - S_n w - L_{n-1}e^{n-1} - \ldots - L_1\dot{e} - L_0 e\}. \tag{2.13}$$

By using (2.6) in (2.11) and by separating the multipliers of x and w, we can rewrite (2.13) as follows :

$$u = K_x x + K_w w, \tag{2.14}$$

where K_x and K_w are given as below :

$$K_x = -\frac{1}{\alpha}(CA^n + L_{n-1}CA^{n-1} + \ldots + L_1 CA + L_0 C), \tag{2.15}$$

$$K_w = -\frac{1}{\alpha}(S_n + L_{n-1}S_{n-1} + \ldots + L_1 S_1 + L_0 Q). \tag{2.16}$$

Thus, when equation (2.13) is substituted into equation (2.11), we get the following error dynamics :

$$e^{(n)} + L_{n-1}e^{(n-1)} + \ldots + L_1\dot{e} + L_0 e = 0 \tag{2.17}$$

If we use Laplace transformation, the characteristic polynomial of equation (2.17) will be as follows :

$$ch(s) = s^n + L_{n-1}s^{n-1} + \ldots + L_1 s + L_0. \tag{2.18}$$

Hence, the polynomial given by (2.18) can always be made stable by choosing appropriate controller coefficients L_i. In this case, the solution of the error dynamics given by (2.17) is exponentially stable. Thus, if L_i parameters are selected to make the characteristic equation (2.18) exponentially stable in controller given by equation (2.13), then the regulation condition (i) is satisfied. In order to show regulator problem has been resolved with the controller given by (2.13), the second regulator condition (ii) should be satisfied as well. If the system in the equations (2.1)-(2.2) and the controller in the equations (2.13)-(2.14) are put together, we will obtain the closed-loop system state space form as shown below :

$$\dot{x} = (A + BK_x)x + (P + BK_w)w = A_{cl}x + (P + BK_w)w, \tag{2.19}$$

$$e = y - r = Cx + Qw. \tag{2.20}$$

Lemma 1. *The static controller*

$$u = K_x x, \tag{2.21}$$

where $K_x = -\frac{1}{\alpha}(CA^n + L_{n-1}CA^{n-1} + \ldots + L_1CA + L_0C)$ *makes the closed-loop system (2.19)-(2.20) with* $w = 0$ *exponentially stable and characteristic equation of* A_{cl} *matrix in (2.19) is given by (2.18).*

In order to prove *Lemma 1*, we use the following fact.

Fact 2. *For A, B, C given by system (2.10), the following holds :*

$$BCA^i = \begin{pmatrix} \underline{0} & \ldots & \underline{0} & \underline{e}_{i+1} \end{pmatrix}^T, \tag{2.22}$$

where $0 \leq i \leq n-1$, $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \epsilon \Re^n$ *and* $\underline{e}_{i+1} \epsilon \Re^n$ *is unit vector with* $(i+1)^{th}$ *entry is one.*

Proof. In order to show this fact, we use mathematical induction.

When $i = 0$

$$BC = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} \underline{0} & \ldots & \underline{0} & \underline{e}_1 \end{pmatrix}^T. \tag{2.23}$$

When i=m, $BCA^m = \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{e}_{m+1} \end{pmatrix}^T$ is true. Then,

$$BCA^{m+1} = \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{e}_{m+1} \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix},$$
$$= \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{e}_{m+2} \end{pmatrix}^T. \quad (2.24)$$

Thus, the above statement is true by mathematical induction. □

Then, the proof of *Lemma 1* is given by using the above fact shown below.

Proof. If we combine the controller (2.14) with the system (2.1), the closed-loop system is obtained with $w = 0$. Then, the closed-loop system is given by :

$$\dot{x} = Ax + BK_x x = (A + BK_x)x = A_{cl}x, \quad (2.25)$$

where

$$Acl = A - BCA^n - L_{n-1}BCA_{n-1} - \dots - L_1BCA - L_0BC. \quad (2.26)$$

By using *Fact 1*, we can find A_{cl} as follows :

First we construct BCA^n with the help of the *Fact 1* :

$$BCA^{n} = \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{e}_{n} \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix},$$
$$= \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{a} \end{pmatrix}^T. \quad (2.27)$$

where $\underline{a} = \begin{pmatrix} -\alpha_0 & \dots & -\alpha_{n-1} \end{pmatrix}^T$. Now we know BCA^i for $0 \leq i \leq n-1$ from the *Fact 1*. Hence if we substitute this result and (2.27) into (2.26), the following form is obtained for A_{cl} :

$$Acl = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -L_0 & -L_1 & \dots & -L_{n-2} & -L_{n-1} \end{pmatrix}. \tag{2.28}$$

If we constitute the characteristic equation of A_{cl}, it turns out that it is the same as (2.18) . In addition, in order to make error $e(t)$ exponentially stable L_i coefficients were chosen such that (2.18) became a Hurwitz polynomial. Hence, this shows that Acl matrix is a Hurwitz matrix and the closed-loop system with $w = 0$ is exponentially stable. □

If the results of the equation (2.17) and *Lemma 1* is used, the following theorem can be obtained.

Theorem 3. *The static controller given by equations (2.13)-(2.14) satisfies the regulation conditions (i.e. (i) and (ii)) for the system in the form (2.1)-(2.5) with the system matrices (2.10).*

Proof. (i) From the equations (2.17) and (2.18), it turns out that the error *e(t)* is exponentially stable (i.e. $|e(t)| < k \exp^{-\lambda t}$). Hence,

$$\lim_{t\to\infty} |e(t)| = 0. \tag{2.29}$$

(ii) With $w = 0$ *Lemma 1* showed that the closed-loop system is exponentially stable. i.e.

$$Re\{eig(A_{cl})\} < 0$$

where $eig(A_{cl})$ denotes the eigenvalues of A_{cl} in (2.26) and (2.28).

These two results point out that regulation conditions are satisfied with static controller given by equations (2.13). □

Remark 2. *As we mentioned in Chapter 1, linear output regulation problem has been studied extensively in the past. Most of the existing approaches rely on obtaining set of regulator equations which should be satisfied by the controllers in order to solve the regulator problem. In these approaches, the second part of the regulator conditions (ii) is assumed to be true and the problem is reduced to finding the controller part associated with the exogenous system states, if the controller is a static one. In all-pole case that we deal with above, this corresponds to finding* K_w *assuming that* K_x *is known. In this case, regulator equations become as given below :*

$$X_c S = (A + BK_x)X_c + (P + BK_w), \tag{2.30}$$

$$0 = CX_c + Q. \tag{2.31}$$

If there exists a unique matrix X_c *and* K_w *that satisfies above regulator equations, then the first part of the regulator conditions (i) is satisfied by the controller which is given by below form :*

$$u = K_x x + K_w w \tag{2.32}$$

In our approach, the static controller $u = K_x x$ *makes the closed-loop system exponentially stable when the exogenous signal is not present, i.e. when* $w(t) = 0$*. Actually, we can assign poles of the closed-loop system with this controller anything that we desire because the coefficients of the characteristic polynomial of the closed-loop system given by (2.18) depend only on the controller parameters* L_i*. Thus, this shows that the static controller class given by the equation (2.13) covers all the static controllers that can be designed to make the closed-loop system (2.19)-(2.20) stable without the exogenous system. In addition to this,* K_w *part of the controller in equation (2.13) can be achieved from the regulator equations (2.30)-(2.31) as will be shown below.*

Lemma 4. *If we take the controller part* K_x *associated with the system states* $x(t)$ *as in (2.21), the controller part* K_w *associated with the exogenous system states*

$w(t)$ in (2.13)-(2.14) which is given by (2.16) can be obtained from regulator equations (2.30)-(2.31).

Proof. We know that K_x is in the following form : $K_x = -\frac{1}{\alpha}(CA^n + L_{n-1}CA^{n-1} + \ldots + L_1CA + L_0C)$. Let us try to find K_w from the equations (2.30)-(2.31) by using the relative-degree property. For simplicity, we take α in (2.8) as 1.

The first thing that we observe from (2.30)-(2.31) is the following :

$$e^{(i)} = CA^i x - CA^i X_c w, \tag{2.33}$$

where $0 \leq i \leq n-1$. We can prove (2.33) by mathematical induction. We first show that (2.33) is true for $i = 0$. From (2.20) we have

$$e = Cx + Qw. \tag{2.34}$$

On the other hand, from (2.31) we obtain $Q = -CX_c$. By using this in (2.34) we obtain the following :

$$e = Cx - CX_c w, \tag{2.35}$$

which shows that (2.33) holds for $i = 0$. Now, assume that (2.33) holds for $i = m$, i.e. assume that the following holds :

$$e^{(m)} = CA^m x - CA^m X_c w. \tag{2.36}$$

Then by differentiating (2.36) once more, we obtain :

$$\begin{aligned} e^{(m+1)} &= CA^m \dot{x} - CA^m X_c \dot{w} = CA^{m+1}x + CA^m Bu + CA^m Pw - CA^m X_c Sw, \\ &= CA^{m+1}x + CA^m Pw - CA^m X_c Sw, \end{aligned} \tag{2.37}$$

from the relative degree property. If we multiply (2.30) with CA^m, then the following equation is obtained :

$$CA^m X_c S = CA^m(A + BK_x)X_c + CA^m(P + BK_w) = CA^{m+1}X_c + CA^m P \tag{2.38}$$

where to obtain the last equality we used the relative degree property, see (2.7) and (2.8). Hence, we have the following :

$$CA^m X_c S = CA^{m+1} X_c + CA^m P. \tag{2.39}$$

If we substitute (2.39) into (2.37), then we obtain :

$$\begin{aligned} e^{(m+1)} &= CA^{m+1}x + CA^m Pw - (CA^{m+1}X_c + CA^m P)w, \\ &= CA^{m+1}x - CA^{m+1}X_c w. \end{aligned} \tag{2.40}$$

Hence, by mathematical induction the statement (2.33) is true.

Secondly, we observe that :

$$S_i = -CA^i X_c, \tag{2.41}$$

where S_i are given by (2.12) and $0 \leq i \leq n$. We can again prove this observation by mathematical induction. We first show that (2.41) holds for $r = 0$. Indeed, from (2.12) we see that $S_0 = Q$. On the other hand, from (2.31) we obtain :

$$S_0 = Q = -CX_c, \tag{2.42}$$

which shows that (2.41) holds for $i = 0$. Assume that (2.41) holds for $i = m$, i.e. we have :

$$S^m = -CA^m X_c. \tag{2.43}$$

Then, by using (2.43) in (2.12) we obtain :

$$\begin{aligned} S_{m+1} &= -CA^m X_c S + CA^m P = (-CA^{m+1}X_c - CA^m P) + CA^m P, \\ &= -CA^{m+1}X_c, \end{aligned} \tag{2.44}$$

where we used (2.39) to obtain the final equality. Hence, by mathematical induction we show that (2.41) holds.

Let us use (2.33) for $i = n - 1$, i.e.

$$e^{(n-1)} = CA^{n-1}x - CA^{n-1}X_c w. \tag{2.45}$$

If we differentiate (2.45) with respect to time, and use (2.3) and (2.19), we obtain :

$$e^{(n)} = CA^{n-1}[(A - BK_x)x + (P - BK_w)w] - CA^{n-1}X_cSw. \quad (2.46)$$

Assuming $\alpha = CA^{n-1}B = 1$ (without loss of generality), we obtain :

$$e^{(n)} = CA^n x - K_x x + CA^{n-1}Pw - K_w w - CA^{n-1}X_cSw. \quad (2.47)$$

By using (2.41) for $i = n - 1$ in (2.47), we obtain :

$$e^{(n)} = CA^n x - K_x x + CA^{n-1}Pw - K_w w + S_{n-1}Sw. \quad (2.48)$$

Finally, by using (2.12) for $i = n$ in (2.48), we obtain :

$$e^{(n)} = CA^n x - K_x x - K_w w + S_n w. \quad (2.49)$$

Hence, when K_x is given by (2.15), then K_w can be obtained from the regulator equations (2.30)-(2.31) as given below :

$$-K_w w = e^{(n)} - CA^n x + K_x x - S_n w. \quad (2.50)$$

Now let us consider the control law obtained by our approach, which is given by (2.13)-(2.14). If we use the latter in (2.11), we obtain :

$$e^{(n)} = CA^n x - K_x x - K_w w + CA^{n-1}Pw + S_{n-1}Sw. \quad (2.51)$$

By using (2.12) for $i = n$ in (2.51), we obtain :

$$-K_w w = e^{(n)} - CA^n x + K_x x - S_n w. \quad (2.52)$$

By comparing (2.52) and (2.50), we see that the term $K_w w$ obtained both by our approach and by the regulator equations are the same. Hence, we conclude that if K_x is as given by (2.15), then K_w , which is obtained by our approach, is the same as the one ,which is obtained from the regulator equations (2.30)-(2.31). $\square$

2.3 Observer Based Controller for All-Pole LTI Systems

In order to implement the controller given by equations (2.13)-(2.14), in addition to error term *e(t)* and various derivatives of it, the signals *w(t)*, *x(t)* should also be measurable. If only the system output *y(t)* and the reference signal *r(t)* are known, we can design observers for *x(t)* and *w(t)* through *Assumption 1* and the outputs of these observers can be used in the controller equations (2.13)-(2.14). In the following, we will use the standard full order observer, also known as Luenberger observer [38], for our observer based controller design.

The observer structure for *x(t)* is in this below form :

$$\hat{x} = A\hat{x} + Bu + L_x(y - C\hat{x} + P\hat{w}), \tag{2.53}$$

and the observer structure for *w(t)* is given by the below equations :

$$\hat{w} = S\hat{w} + L_w(r + Q\hat{w}), \tag{2.54}$$

Let us define new state variables e_x and e_w as shown below :

$$e_x = x - \hat{x} \ , \ e_w = w - \hat{w}. \tag{2.55}$$

Then, the dynamic equations of the above states can be found as follows :

$$\begin{aligned} \dot{e_x} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu + Pw - A\hat{x} - Bu - P\hat{w} - L_xCx + L_xC\hat{x} \\ \dot{e_x} &= (A - L_xC)e_x + Pe_w \end{aligned} \tag{2.56}$$

and

$$\begin{aligned} \dot{e_w} &= \dot{w} - \dot{\hat{w}} = Sw - S\hat{w} + L_wQw - L_wQ\hat{w} \\ &= (S + L_wQ)e_w \end{aligned} \tag{2.57}$$

Since both (A, C) and (Q, S) pairs are observable, we can find L_x and L_w such that matrices in (2.56)-(2.57) become Hurwitz . Thus, estimated states

$\hat{x}$, $\hat{w}$ converge true states x, w exponentially. If we combine the system in the equations (2.1)-(2.5) and the controller (2.13)-(2.14) with the observer equations (2.56)-(2.57), overall system can be obtained. The overall system (i.e. observer-controller-plant) state space model with states e_x, e_w become in the following form :

$$\begin{pmatrix} \dot{x} \\ \dot{e_x} \\ \dot{e_w} \end{pmatrix} = \begin{pmatrix} (A+BK_x) & -BK_x & -BK_w \\ 0 & (A-L_xC) & P \\ 0 & 0 & S+L_wQ \end{pmatrix} \begin{pmatrix} x \\ e_x \\ e_w \end{pmatrix} + \begin{pmatrix} P+BK_w \\ 0 \\ 0 \end{pmatrix} w, \tag{2.58}$$

$$e = \begin{pmatrix} C & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ e_x \\ e_w \end{pmatrix} + Qw. \tag{2.59}$$

Lemma 5. *The system in equations (2.58)-(2.59) satisfies regulation conditions (i) and (ii).*

Proof. (i) Since the error *e(t)* is exponentially stable with controller (2.13), the controller-plant system (2.19)-(2.20) satisfies the regulator equations (2.30)-(2.31) and its inverse is also true. Thus, if the system in (2.58)-(2.59) satisfies regulator equation formed by its system matrices, then the error term *e(t)* is exponentially stable. The regulator equations formed by the matrices in (2.58)-(2.59) as follows :

$$\begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} S = \begin{pmatrix} (A+BK_x) & -BK_x & -BK_w \\ 0 & (A-L_xC) & P \\ 0 & 0 & S+L_wQ \end{pmatrix} \begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} + \begin{pmatrix} P+BK_w \\ 0 \\ 0 \end{pmatrix}, \tag{2.60}$$

$$0 = \begin{pmatrix} C & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} + Q. \tag{2.61}$$

If X_{c2} and X_{c3} are chosen as zero vectors, then the regulator equations are reduced to this below form :

$$X_{c1}S = (A + BK_x)X_{c1} + (P + BK_w), \tag{2.62}$$

$$0 = CX_{c1} + Q. \tag{2.63}$$

The equations (2.62)-(2.63) are the same with regulator equations given by (2.30)-(2.31). Also in *Lemma4*, we proved that the regulator equations given by (2.30)-(2.31) are satisfied with the controller given by (2.13)-(2.14). This implies that the equations (2.62)-(2.63) have a solution. Thus, the regulator equations given by (2.60)-(2.61) are satisfied. This indicates that the error *e(t)* exponentially decays to zero for the system given by the state space representation (2.58)-(2.59).

(ii) The closed-loop state transition matrix with $w = 0$ is in block triangular form as can be seen in equation (2.58). In addition, the matrices $(A+BK_x)$, $(A - L_xC)$ and $(S + L_wQ)$ are known to be Hurwitz. We know that the state transition matrix eigenvalues are composed of these three matrices eigenvalues because of the block triangular structure as shown below: i.e

$$eig(A_{cl}) = eig(A + BK_x)\bigcup eig(A - L_xC)\bigcup eig(S + L_wQ), \tag{2.64}$$

It easily follows that the closed-loop matrix in (2.58) is Hurwitz, which means that the closed-loop system with $w = 0$ is exponentially stable.

□

2.4 Controller for Minimum Phase LTI Systems

In general, the transfer function of LTI systems are as follows : $G(s) = \frac{n(s)}{d(s)}$. If the zeros of the system, i.e. the roots of $n(s)$, are on the left-half-plane (LHP), this system is called as a "Minimum Phase System". Besides, these systems have

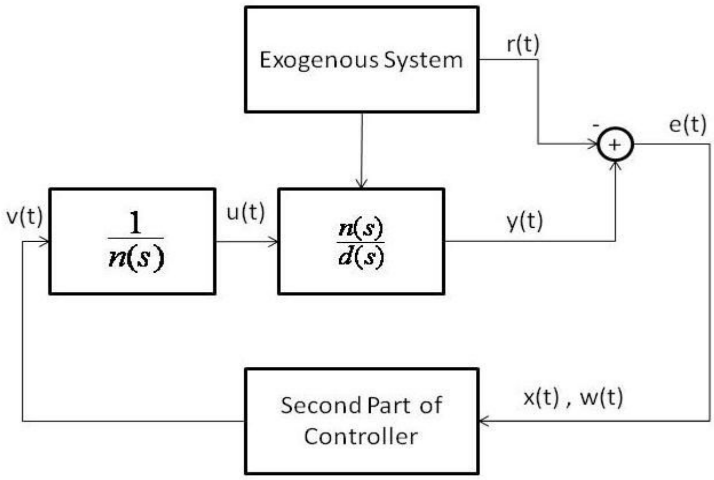

Figure 2.2: Overall System Block Diagram

relative degree r and r is given by $r =$ (number of poles) $-$ (number of zeros) . In order to obtain a state space model for such systems, suppose that $n(s)$ and $d(s)$ are given as follows :

$$n(s) = s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0, \tag{2.65}$$

$$d(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0. \tag{2.66}$$

In this case, the system to be controlled can be given as shown below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots 0 \\ 0 & 0 & 1 & 0 & \ldots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} b_0 & b_1 & \ldots & 1 & 0\ldots & 0 \end{pmatrix} x. \tag{2.67}$$

We know that the zeros of the minimum phase system are stable. Then to make the system in (2.67) equivalent to an all-pole system, we can employ $C_1(s) = \frac{1}{n(s)}$ as the first part of the controller. Since there is no unstable pole/zero cancelations between $C_1(s)$ and the system in (2.67), the first part of the controller does not cause any instability problem. The overall system will become equivalent to all-pole with this first part of the controller and a state space model for $C_1(s)$ can be given as follows :

$$\dot{\xi} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -b_0 & -b_1 & \dots & -b_{m-2} & -b_{m-1} \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} v = G\xi + Hv, \tag{2.68}$$

$$u = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \xi = K\xi, \tag{2.69}$$

where $\xi \epsilon \Re^m$, $v \epsilon \Re$ and $u \epsilon \Re$ denote the controller states, input and output respectively. Then $G \epsilon \Re^{m \times m}$, $H \epsilon \Re^{m \times 1}$, $K \epsilon \Re^{1 \times m}$, given by (2.68)-(2.69), represent constant system matrices . If we form the overall system by combining the original system in the equations (2.1)-(2.5) and the first part of the controller given by (2.68)-(2.69), a state space model of augmented system becomes in the form given below :

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} 0 \\ H \end{pmatrix} v + \begin{pmatrix} P \\ 0 \end{pmatrix} w, \tag{2.70}$$

$$e = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + Sw. \tag{2.71}$$

This overall system has dimension $\tilde{n} = n + m$ where m is the dimension of the inverse system $C_1(s)$. Since the poles of $C_1(s)$ and the zeros of the system in (2.67) are canceled each other, the number of m unobservable states arise in the overall system given by (2.70)-(2.71).

Fact 6. *The system given by (2.70)-(2.71) has relative degree n.*

Proof. The first part of controller has transfer function $C_1(s) = \frac{1}{n(s)}$ and we know from section (1.2) that this kind of transfer functions represent all-pole systems. Since the first part has dimension m, the system in (2.68)-(2.69) has relative degree m : i.e.

$$KH = KGH = \dots = KG^{m-2}H = 0, \tag{2.72}$$

$$KG^{m-1}H = \beta \neq 0. \tag{2.73}$$

In addition, the original system in (2.67) has relative degree r, thus the following equations must hold :

$$CB = CAB = \ldots = CA^{r-2}B = 0, \tag{2.74}$$

$$CA^{r-1}B = \gamma \neq 0. \tag{2.75}$$

Let us denote $z = \begin{pmatrix} x \\ \xi \end{pmatrix}$, $A_c = \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix}$, $C_c = \begin{pmatrix} C & 0 \end{pmatrix}$ and $B_c = \begin{pmatrix} 0 \\ H \end{pmatrix}$. Then,

$$\begin{aligned}
C_c &= \begin{pmatrix} C & 0 \end{pmatrix}, \\
C_cA_c &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix} = \begin{pmatrix} CA & CBK \end{pmatrix}, \\
C_cA_c^2 &= \begin{pmatrix} CA & CBK \end{pmatrix} \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix} = \begin{pmatrix} CA^2 & CABK + CBKG \end{pmatrix}, \\
&\vdots \\
C_cA_c^r &= \begin{pmatrix} CA^r & CA^{r-1}BK + CA^{r-2}BKG + \ldots + CBKG^{r-1} \end{pmatrix} \\
&= \begin{pmatrix} CA^r & CA^{r-1}BK \end{pmatrix}, \\
&\vdots \\
C_cA_c^{n-2} &= \begin{pmatrix} CA^{n-2} & CA^{n-3}BK + \ldots + CA^{n-m-2}BKG^{m-1} + \ldots + CBKG^{n-3} \end{pmatrix} \\
&= \begin{pmatrix} CA^{n-2} & CA^{n-3}BK + \ldots + CA^{r-1}BKG^{m-2} \end{pmatrix}, \\
C_cA_c^{n-1} &= \begin{pmatrix} CA^{n-1} & CA^{n-2}BK + \ldots + CA^{n-m-1}BKG^{m-1} + \ldots + CBKG^{n-2} \end{pmatrix} \\
&= \begin{pmatrix} CA^{n-1} & CA^{n-2}BK + \ldots + CA^{r-1}BKG^{m-1} \end{pmatrix}.
\end{aligned} \tag{2.76}$$

where we used (2.74) and the fact that $r = n - m$. If we multiply these equations from right with B_c, we obtain the following result :

$$\begin{aligned}
C_c A_c B_c &= 0, \\
C_c A_c^2 B_c &= 0, \\
&\vdots \\
C_c A_c^{n-2} B_c &= 0, \\
C_c A_c^{n-1} B_c &= \beta\gamma = \rho \neq 0,
\end{aligned}$$

where we used (2.72), (2.73) and (2.75). This proves that the overall system in (2.70)-(2.71) has relative degree n. □

Thus, as in all-pole systems the input appears at the n^{th} derivative of the error $e(t)$ which is shown below :

$$\begin{aligned}
\dot{e} &= C_c \dot{\tilde{x}} + Qw, \\
&= C_c A_c \tilde{x} + \underbrace{C_c B_c}\, v + C_c P_c + QSw, \\
&\vdots \\
e^{(n)} &= C_c A_c^n \tilde{x} + \beta v + C_c A_c^{n-1} P_c w + \tilde{S}_{n-1} w, \qquad (2.77)
\end{aligned}$$

where $\tilde{S}_i = \tilde{S}_{i-1} S + C_c A_c^{i-1} P_c$, $1 \leq i \leq n$, $\tilde{S}_0 = Q$, and the part, indicated by underbrace is equal to zero. In order to find the second part of the controller that guarantees the regulation conditions, we will use the same methodology, that we applied in all-pole systems. Therefore, we will choose control input $u(t)$ as follows :

$$v = \frac{1}{\gamma}\{-C_c A_c^n \tilde{x} - \tilde{S}_n w - \tilde{L}_{n-1} e^{n-1} - \ldots - \tilde{L}_1 \dot{e} - \tilde{L}_0 e\}. \qquad (2.78)$$

If the equation in (2.78) is substituted into equation in (2.77), we obtain the error dynamics $e(t)$ as shown below :

$$e^{(n)} + \tilde{L}_{n-1} e^{(n-1)} + \ldots + \tilde{L}_1 \dot{e} + \tilde{L}_0 e = 0. \qquad (2.79)$$

The latter is the same with (2.17) that was obtained for all-pole systems. If we again use Laplace transformation, the characteristic polynomial of the equation (2.79) will be as given below :

$$ch(s) = s^n + \tilde{L}_{n-1}s^{n-1} + \ldots + \tilde{L}_1 s + \tilde{L}_0 = 0. \tag{2.80}$$

If the controller parameters $\{\tilde{L}_{n-1}, \ldots, \tilde{L}_1, \tilde{L}_0\}$ are chosen properly, we can make the error $e(t)$ exponentially stable as we discussed in *Section 1.2.* Actually, the second part of the controller is like a static controller and its state space model can be shown as follows :

$$v = K_\xi \xi + K_x x + K_w w. \tag{2.81}$$

In this case, if we combine the first and the second part of the controller given by (2.68)-(2.69), (2.81) respectively, the overall controller becomes as shown below :

$$\begin{aligned} \dot{\xi} &= (G + HK_\xi)\xi + HK_x x + HK_w w, \\ u &= K\xi. \end{aligned} \tag{2.82}$$

Thus, the closed-loop system state space model is in the following form :

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= \begin{pmatrix} A & BK \\ HK_x & G + HK_\xi \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} P \\ HK_w \end{pmatrix} w, \\ e &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + Qw. \end{aligned} \tag{2.83}$$

We denote

$$A_{cl} = \begin{pmatrix} A & BK \\ HK_x & G + HK_\xi \end{pmatrix}. \tag{2.84}$$

In order to satisfy the second regulation condition (ii), A_{cl} should be a Hurwitz matrix. In the system given by (2.70)-(2.71), there are number of m unobservable states as a result of the pole/zero cancelations between the original system and the first part of the controller. This point is proven in the following fact.

Fact 7. *System in (2.70)-(2.71) has m unobservable states.*

Proof. First, let us compute the controllability matrix of the system;

$$R_c = \begin{pmatrix} B_c & A_cB_c & \dots & A_c^{n+m-1}B_c \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \iddots & * \\ 0 & 1 & * & \dots & * \\ 1 & * & \dots & * & * \end{pmatrix}.$$

Clearly $rank(R_c) = n + m$. Hence the system is completely controllable. 0n the other hand, the minimal realization of this system has transfer function $G_m(s) = \frac{1}{d(s)}$ which has dimension n. Thus, there should be n states which is both controllable and observable. Since, we proved that all states are controllable, there has to be m unobservable states in system (2.70)-(2.71) by Kalman decomposition. □

Since the system in (2.70)-(2.71) has m unobservable states as we showed in *Fact 7*, there has to be a similarity transformation T which transforms the state transition matrix of the system in (2.70)-(2.71) into the canonical form shown below :

$$T \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix} T^{-1} = \begin{pmatrix} A_t & 0 \\ \diamond & G_t \end{pmatrix}, \tag{2.85}$$

where the eigenvalues of A are same with the eigenvalues of A_t and the eigenvalues of G is same with the eigenvalues of G_t.

Lemma 8. *The closed-loop state transition matrix A_{cl} in (2.83) is a Hurwitz matrix and its eigenvalues are the combination of the roots of (2.80) and the eigenvalues of the state transition matrix G of the inverse system given by (2.68).*

Proof. From (2.78) we can obtain $\begin{pmatrix} K_x & K_\xi \end{pmatrix}$ as follows :

$$\begin{pmatrix} K_x & K_\xi \end{pmatrix} = C_cA_c^n - \tilde{L}_{n-1}C_cA_c^{n-1} - \dots - \tilde{L}_1C_cA_c - \tilde{L}_0C_c. \tag{2.86}$$

Let us denote

$$T_0 = 0, \;\; T_k = \sum_{i=0}^{k-1} CA^i BKG^{k-1-i}, \;\; 1 \le k \le n. \tag{2.87}$$

Then it easily follows that $C_c A_c^k = \begin{pmatrix} CA^k & T_k \end{pmatrix}$ for $1 \le k \le n$. From this relation, we can obtain A_{cl} as follows :

$$A_{cl} = \begin{pmatrix} A & BK \\ X & G+Y \end{pmatrix}, \tag{2.88}$$

where

$$Y = -H[T_n + \tilde{L}_{n-1} T_{n-1} + \ldots + \tilde{L}_1 T_1], \tag{2.89}$$

and

$$X = -H[CA^n + \tilde{L}_{n-1} CA^{n-1} + \ldots + \tilde{L}_1 CA + \tilde{L}_0 C]. \tag{2.90}$$

Let us write the transformation matrix T in (2.85) as :

$$T = \begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix}. \tag{2.91}$$

Then, from (2.85) we obtain the following :

$$\begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix} \begin{pmatrix} A & BK \\ 0 & G \end{pmatrix} = \begin{pmatrix} A_t & 0 \\ A_1 & G_t \end{pmatrix} \begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix}. \tag{2.92}$$

If we carry out the above matrix multiplications, we obtain the following :

$$T_{11} A = A_t T_{11}, \tag{2.93}$$

$$T_{11} BK + T_{21} G = A_t T_{21}, \tag{2.94}$$

$$T_{12} A = A_1 T_{11} + G_t T_{12}, \tag{2.95}$$

$$T_{12} BK + T_{22} G = A_1 T_{21} + G_t T_{22}. \tag{2.96}$$

Let us apply the same transformation T to A_{cl} :

$$\begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix} \begin{pmatrix} A & BK \\ X & G+Y \end{pmatrix} = \begin{pmatrix} T_{11}A + T_{21}X & T_{11}BK + T_{21}G + T_{21}Y \\ T_{12}A + T_{22}X & T_{12}BK + T_{22}G + T_{22}Y \end{pmatrix}, \tag{2.97}$$

and let us substitute (2.93), (2.94), (2.95) and (2.96) into (2.97) :

$$TA_{cl} = \begin{pmatrix} A_tT_{11} + T_{21}X & A_tT_{21} + T_{21}Y \\ A_1T_{11} + G_tT_{12} + T_{22}X & A_1T_{21} + G_tT_{22} + T_{22}Y \end{pmatrix}. \tag{2.98}$$

In addition, we know from Kalman decomposition of (2.70)-(2.71) that the following holds :

$$\begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix} \begin{pmatrix} 0 \\ H \end{pmatrix} = \begin{pmatrix} B_t \\ B_{uo} \end{pmatrix}, \tag{2.99}$$

$$\begin{pmatrix} C & 0 \end{pmatrix} = \begin{pmatrix} C_t & 0 \end{pmatrix} \begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix}. \tag{2.100}$$

By using (2.99) and (2.100), we obtain the following :

$$T_{21}H = B_t, \tag{2.101}$$

$$T_{22}H = B_{uo}, \tag{2.102}$$

$$C_tT_{11} = C, \tag{2.103}$$

$$C_tT_{21} = 0. \tag{2.104}$$

First we find $T_{21}X$ as follows :

$$\begin{aligned} T_{21}X &= -T_{21}H[CA^n + \tilde{L}_{n-1}CA^{n-1} + \ldots + \tilde{L}_1CA + \tilde{L}_0C], \\ &= -B_tC_tT_{11}[A^n + \tilde{L}_{n-1}A^{n-1} + \ldots + \tilde{L}_1A + \tilde{L}_0], \\ &= -B_tC_t[A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1A_t + \tilde{L}_0]T_{11}, \end{aligned} \tag{2.105}$$

where we used (2.101), (2.103) and (2.93). Then, we obtain the following :

$$A_tT_{11} + T_{21}X = \{A_t - B_tC_t[A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1A_t + \tilde{L}_0]\}T_{11} = A_lT_{11} \tag{2.106}$$

where

$$A_l = A_t - B_tC_t[A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1A_t + \tilde{L}_0].$$

Actually, the triple (A_t, B_t, C_t) describes minimal realization of the system in the form (2.70)-(2.71) with $w = 0$. Additionally, this state space model is all-pole and has transfer function $G_m = \frac{1}{d(s)}$. Since the form of A_l is the same with A_{cl} in (2.26), A_l has the characteristic equation as given by (2.80).

By using (2.87), (2.103) and (2.93), we obtain :

$$\begin{aligned} T_k &= CA^{k-1}BK + CA^{k-2}BKG + \ldots + CBKG^{k-1}, \\ &= (C_tA_t^{k-1}T_{11}BK + C_tA_t^{k-2}T_{11}BKG + \ldots + C_tT_{11}BKG^{k-1}). \end{aligned} \tag{2.107}$$

By using (2.107) and (2.101), we obtain:

$$\begin{aligned} T_{21}Y &= -T_{21}H[T^n + \tilde{L}_{n-1}T^{n-1} + \ldots + \tilde{L}_1T + \tilde{L}_0T_0] \\ &= -B_t[T^n + \tilde{L}_{n-1}T^{n-1} + \ldots + \tilde{L}_1T + \tilde{L}_0T_0]. \end{aligned} \tag{2.108}$$

We know from (2.94) that the following holds :

$$T_{11}BK = -T_{21}G + A_tT_{21}. \tag{2.109}$$

If we substitute the latter into (2.107), we obtain :

$$\begin{aligned} T_k &= (C_tA_t^kT_{21} + C_tA_t^{k-1}T_{21}G + \ldots + C_tA_tT_{21}G^{k-1}) \\ &- (C_tA_t^{k-1}T_{21}G + \ldots + C_tA_tT_{21}G^{k-1} + C_tT_{21}G^k) = C_tA_t^kT_{21} - C_tT_{21}G^k, \\ &= C_tA_t^kT_{21}, \end{aligned} \tag{2.110}$$

where we used (2.104). By substituting (2.110) into (2.108) we can obtain the following :

$$T_{21}Y = -B_t(C_tA_t^n + \tilde{L}_{n-1}C_tA_t^{n-1} + \ldots + \tilde{L}_1C_tA_t + \tilde{L}_0C_t). \tag{2.111}$$

Thus, if we put (2.111) into $A_tT_{21} + T_{21}Y$, then below we obtain :

$$A_tT_{21} + T_{21}Y = A_lT_{21}. \tag{2.112}$$

Finally, we form $T_{22}X$ and $T_{22}Y$ shown below :

$$T_{22}X = -B_{uo}C_t(A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1A_t + \tilde{L}_0)T_{11}, \tag{2.113}$$

$$T_{22}Y = -B_{uo}C_t(A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1A_t + \tilde{L}_0)T_{21}, \tag{2.114}$$

where we used (2.102), (2.103), (2.93) and (2.110). If we substitute (2.106), (2.112), (2.113) and (2.114) into (2.97), we obtain the following form :

$$TA_{cl} = \begin{pmatrix} A_l & 0 \\ \acute{A}_1 & A_g \end{pmatrix} \begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix}, \tag{2.115}$$

where $\acute{A}_1 = A_1 - B_{uo}C_t(A_t^n + \tilde{L}_{n-1}A_t^{n-1} + \ldots + \tilde{L}_1 A_t + \tilde{L}_0)$ and $A_g = G_t$. Thus, transformed closed-loop matrix is in this following form :

$$\tilde{A_{cl}} = TA_{cl}T^{-1} = \begin{pmatrix} A_l & 0 \\ \acute{A}_1 & A_g \end{pmatrix}. \tag{2.116}$$

The eigenvalues of $\tilde{A_{cl}}$ are the same as the eigenvalues of A_{cl}. Hence, from (2.116) we obtain the following :

$$eig(\tilde{A_{cl}}) = eig(A_{cl}) = eig(A_l) \bigcup eig(A_g) \tag{2.117}$$

Since we know that the eigenvalues of A_l are given by the (2.80) and the eigenvalues of A_g are the same with G, which are stable by minimum phase property, the closed-loop system state transition matrix A_{cl} in (2.83) is a Hurwitz matrix. In addition, its eigenvalues are combination of the roots of (2.80) and the eigenvalues of the state transition matrix G of the inverse system in (2.68) □

The *Lemma 8* proves that the closed-loop system with $w = 0$ in (2.83) is exponentially stable.

Theorem 9. *The dynamic controller given by (2.82) satisfies regulation conditions (i), (ii) for the system in the form (2.1)-(2.5) with system matrices as given by (2.67).*

Proof. (i) Equations (2.79)-(2.80) indicates that the error term $e(t)$ is exponentially stable (i.e. $|e(t)| < k \exp^{-\lambda t}$ for some $k > 0$, $\lambda > 0$). Hence, we have

$$\lim_{t \to \infty} |e(t)| = 0. \tag{2.118}$$

(ii) *Lemma 8* proves that the closed-loop system with $w = 0$ is exponentially stable. i.e.

$$Re\{eig(A_{cl})\} < 0$$

where $eig(A_{cl})$ denotes the eigenvalues of A_{cl} in (2.88).

These two results prove that the dynamic controller in the form (2.82) satisfies the regulation conditions for the system in the form (2.1)-(2.5) with system matrices as given by (2.67). □

2.5 Observer Based Controller for Minimum Phase LTI Systems

In order to implement the controller in (2.82), we need to know the system states $x(t)$ and the exogenous system states $w(t)$. If only the system output $y(t)$ and the reference signal $r(t)$ are known, observers for $x(t)$ and $w(t)$ can be designed through *Assumption 1*, see *section 2.3*. The observer structure for $x(t)$ and $w(t)$ is the same with (2.53),(2.54) respectively. Then the observers error terms $e_x = x - \hat{x}$ and $e_w = w - \hat{w}$ are defined as a new state variables for the overall system. The observers error dynamics are the same as the ones found in *section 2.3* which are given by (2.56)-(2.57) and are again given below :

$$\dot{e_x} = (A - L_x C)e_x + Pe_w, \tag{2.119}$$

$$\dot{e_w} = (S + L_w Q)e_w. \tag{2.120}$$

Since both (A, C) and (Q, S) pair are observable, we can find L_x, L_w such that the matrices in (2.119) and (2.120) become Hurwitz. Thus estimated states $\hat{x}$, $\hat{w}$ converge true states x, w asymptotically.

If we combine the system in (2.1)-(2.5) and the controller in (2.82) with the observer error dynamics given by (2.119)-(2.120), the overall controller-observer system can be obtained. The overall system state space model with new states e_x, e_w turns into the following form :

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \\ \dot{e_x} \\ \dot{e_w} \end{pmatrix} = \begin{pmatrix} A & BK & 0 & 0 \\ HK_x & G + HK_\xi & -HK_x & -HK_w \\ 0 & 0 & A - L_xC & P \\ 0 & 0 & 0 & S + L_wQ \end{pmatrix} \begin{pmatrix} x \\ \xi \\ e_x \\ e_w \end{pmatrix} + \begin{pmatrix} P \\ HK_w \\ 0 \\ 0 \end{pmatrix} w, \tag{2.121}$$

$$e = y - r = \begin{pmatrix} C & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \\ e_x \\ e_w \end{pmatrix} + Qw. \tag{2.122}$$

Lemma 10. *The system in equations (2.121)-(2.122) satisfies regulation conditions (i) and (ii).*

Proof. (i) Since the error $e(t)$ is exponentially stable with the controller in (2.82), the regulator equations given below are satisfied by the controller-plant system :

$$X_cS = A_{cl}X_c + \begin{pmatrix} P \\ HK_w \end{pmatrix}. \tag{2.123}$$

$$0 = C_cX_c + Q, \tag{2.124}$$

If the controller-observer-plant system given by (2.121)-(2.122) satisfies regulator equations formed by its system matrices, the error term $e(t)$ also becomes exponentially stable for the observer-controller-plant system. The regulator equations formed by the matrices in (2.121)-(2.122) as follows :

$$\begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} S = \begin{pmatrix} A_{cl} & B_1 & B_2 \\ 0 & (A - L_x C) & P \\ 0 & 0 & S + L_w Q \end{pmatrix} \begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} + \begin{pmatrix} P_c \\ 0 \\ 0 \end{pmatrix}, \tag{2.125}$$

$$0 = \begin{pmatrix} C_c & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{c1} \\ X_{c2} \\ X_{c3} \end{pmatrix} + Q. \tag{2.126}$$

where $B_1 = \begin{pmatrix} 0 \\ -HK_x \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ -HK_w \end{pmatrix}$ and $P_c = \begin{pmatrix} P \\ HK_w \end{pmatrix}$. If X_{c2} and X_{c3} are chosen as zero, then the regulator equations (2.125)-(2.126) are reduced to the form given below :

$$X_{c1} S = A_{cl} X_{c1} + P_c, \tag{2.127}$$

$$0 = C_c X_{c1} + Q. \tag{2.128}$$

The equations (2.127)-(2.128) are the same with the regulator equations (2.123)-(2.124). Hence, there exists an X_{c1} such that (2.127)-(2.128) are satisfied. This implies that the regulator equations given by (2.125)-(2.126) are satisfied. This proves that the error $e(t)$ is exponentially stable in the system given by equations (2.121)-(2.122).

(ii) The closed-loop state transition matrix of the system in (2.121)-(2.122) with $w = 0$ is in block triangular form. Additionally, the matrices $A_{cl} = \begin{pmatrix} A & BK \\ HK_x & G + HK_\xi \end{pmatrix}$, $(A - L_x C)$ and $(S + L_w Q)$ are Hurwitz matrices. Since the eigenvalues of the overall state transition matrix are composed of the eigenvalues of these three matrices, we have :

$$eig(A_{ocl}) = eig(A_{cl}) \bigcup eig(A - L_x C) \bigcup eig(S + L_w Q) \tag{2.129}$$

This proves that the closed-loop system in (2.121)-(2.122) with $w = 0$ is exponentially stable.

□

2.6 Numerical Results

In this section, some simulation results for both All-pole and Minimum Phase LTI Systems are given. Initially, in the figures we will give graph that shows the error signal $e(t)$ between system output $y(t)$ and reference signal $r(t)$. Then, we will put the graph that shows the stability of closed-loop system without exogenous system. Finally, we will give graph of the errors e_x, e_w.

2.6.1 Example 1

In the first simulation, we consider the following system (see (2.1)-(2.5)) :

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -1 & 3 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u + \nu, \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x. \end{aligned} \tag{2.130}$$

The exogenous system is given as follows :

$$\begin{aligned} \dot{w} &= \begin{pmatrix} 0 & 0 & 0 & -\pi \\ 0 & 0 & 1 & 0 \\ 0 & -(\frac{\pi}{2})^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} w, \\ r(t) &= -\begin{pmatrix} 1 & -0.5 & 2 & 0 \end{pmatrix} w, \\ \nu(t) &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0.5 \\ 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 \end{pmatrix} w. \end{aligned} \tag{2.131}$$

Hence according to (2.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -0.5 & 2 & 0 \end{pmatrix} w. \tag{2.132}$$

Note that, when $w(t) = 0$, the transfer function of this system is given as :

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^4 + 2s^3 - 3s^2 + s + 5}. \tag{2.133}$$

Hence, when $w(t) = 0$, the uncontrolled system is all-pole and unstable. By using (2.13)-(2.14), we find the controller which satisfies the regulation conditions as follows :

$$u = \begin{pmatrix} 3 & -4 & -7 & -1 \end{pmatrix} x + \begin{pmatrix} -1.0953 & -10.1245 & -1.5728 & 4.3579 \end{pmatrix} w. \tag{2.134}$$

With K_x as given above, the characteristic polynomial of the closed-loop system becomes as follows :

$$ch(s) = s^4 + 3s^3 + 4s^2 + 5s + 2 \tag{2.135}$$

and roots of (2.135) can be given as follows: $\{-2, -.2151 + 1.3071\imath, -.2151 - 1.3071\imath, -.5698\}$. If we assign the eigenvalues of the state observer matrix $(A - L_xC)$ as $\{-1, -2 + \imath, -2 - \imath, -3\}$ and the exogenous system observer matrix $(S + L_wQ)$ as $\{-0.064 + 1.67\imath, -0.064 - 1.67\imath, -0.26 + 1.34\imath, -0.26 - 1.34\imath\}$, we obtain L_x and L_w as follows :

$$L_x = \begin{pmatrix} 6 & 15 & 19 & 11 \end{pmatrix}^T \tag{2.136}$$

$$L_w = \begin{pmatrix} -0.5 & -0.1 & -0.1 & -0.1 \end{pmatrix}^T \tag{2.137}$$

Simulation results are obtained for these initial conditions :
$x(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$.

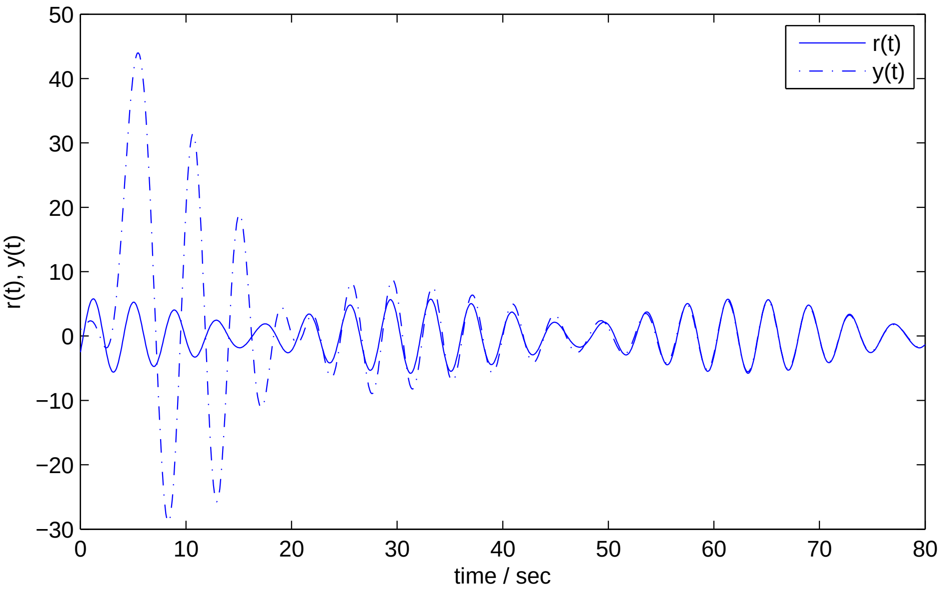

Figure 2.3: Tracking of Reference Signal

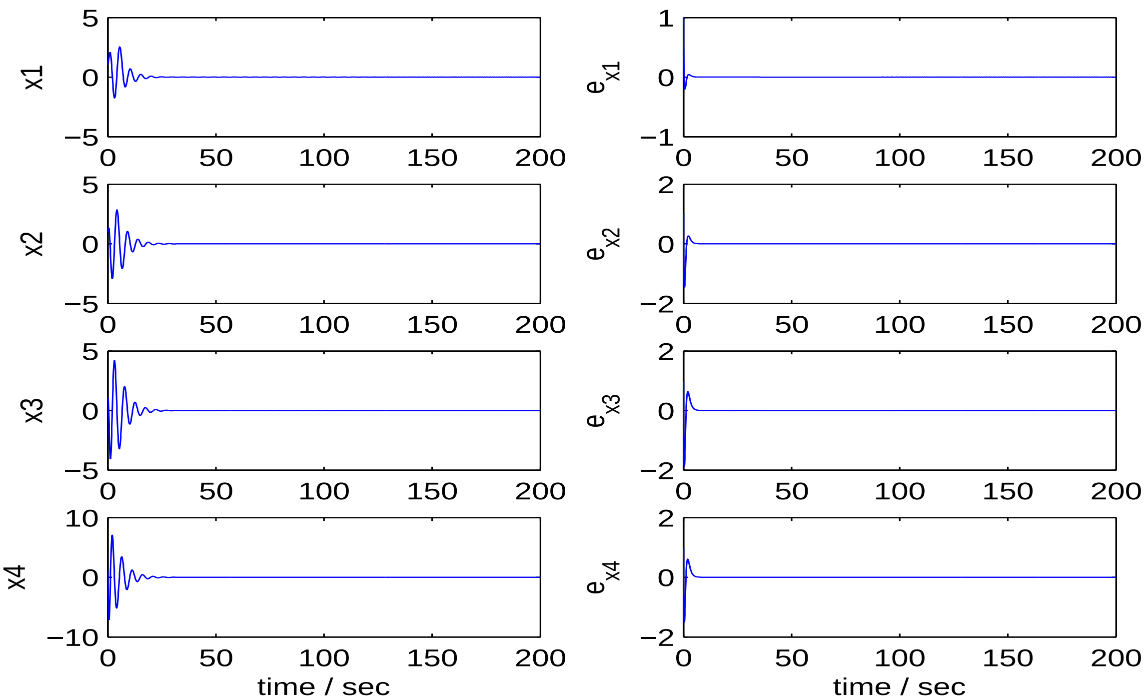

Figure 2.4: Stability of Closed-Loop System

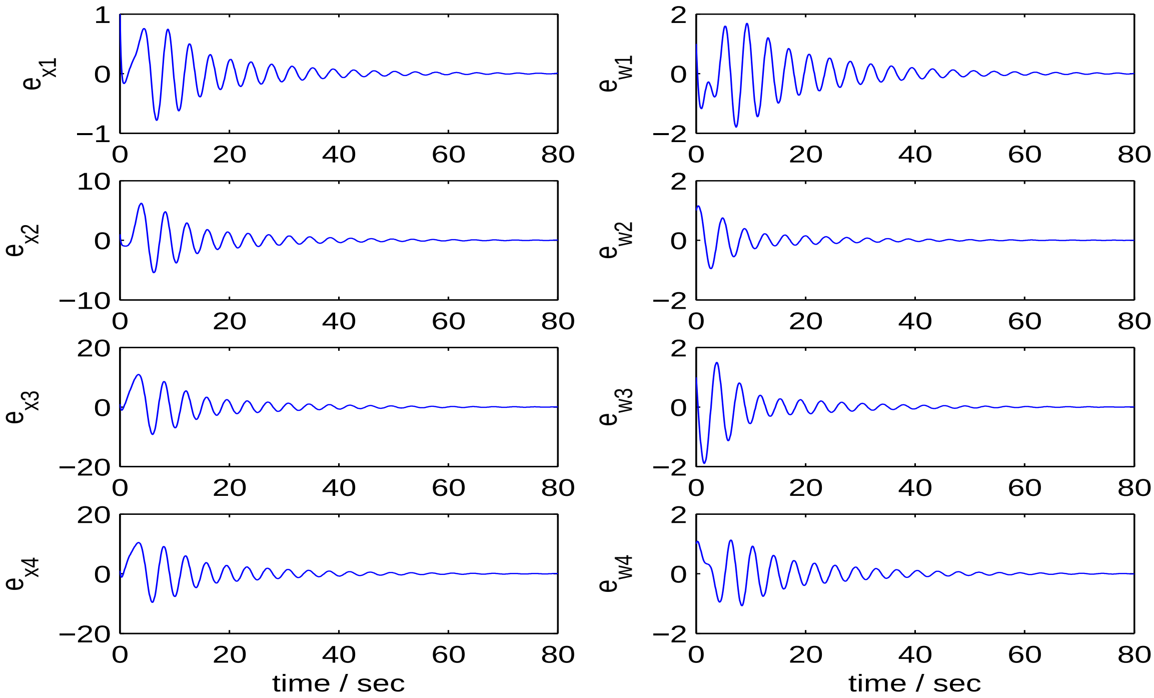

Figure 2.5: Error between x, w and $\hat{x}$, $\hat{w}$

In figure 2.3, figure 2.4 and figure 2.5, we can see the simulation results for system given by (2.130)-(2.131). In figure 2.3, we can observe that the output of the system tracks the reference signal exponentially when the disturbances are effective on the system. In figure 2.4, we observe that the closed-loop system is stable with $w = 0$. In figure 2.5, we see that the estimated states $\hat{x}$, $\hat{w}$ are converge to the true states x, w.

2.6.2 Example 2

The system that we will deal with as a second example is given below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u + \nu,$$
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x. \tag{2.138}$$

The exogenous system is given as follows :

$$\dot{w} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} w,$$
$$r(t) = -\begin{pmatrix} 1 & 2 \end{pmatrix} w,$$
$$\nu(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} w \tag{2.139}$$

Hence according to (2.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 2 \end{pmatrix} w. \tag{2.140}$$

Note that, when $w(t) = 0$, the transfer function of this system is given as :

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^3 + 3s^2 + s + 2}. \tag{2.141}$$

Hence, when $w(t) = 0$, the uncontrolled system is all-pole and stable. By using (2.13)-(2.14), we find the controller which satisfies the regulation conditions as follows :

$$u = \begin{pmatrix} 0.284 & -3.31 & -0.63 \end{pmatrix} x + \begin{pmatrix} 7.414 & 14.258 \end{pmatrix} w. \tag{2.142}$$

With K_x as given above, the characteristic polynomial of the closed-loop system becomes as follows :

$$ch(s) = s^3 + 3.6s^2 + 4.31s + 1.716, \tag{2.143}$$

and roots of this polynomial can be given as follows: $\{-1.1, -1.2, -1.3\}$. If we assign the eigenvalues of the state observer matrix $(A - L_x C)$ as $\{-1, -2, -3\}$ and the exogenous system observer matrix $(S + L_w Q)$ as $\{-2, -1\}$, we obtain L_x and L_w as given below :

$$L_x = \begin{pmatrix} 3 & 1 & -2 \end{pmatrix}^T \tag{2.144}$$

$$L_w = \begin{pmatrix} -1 & -1 \end{pmatrix}^T \tag{2.145}$$

Simulation results are obtained for the below initial conditions :
$x(0) = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 0.2 & 0.5 \end{pmatrix}^T$.

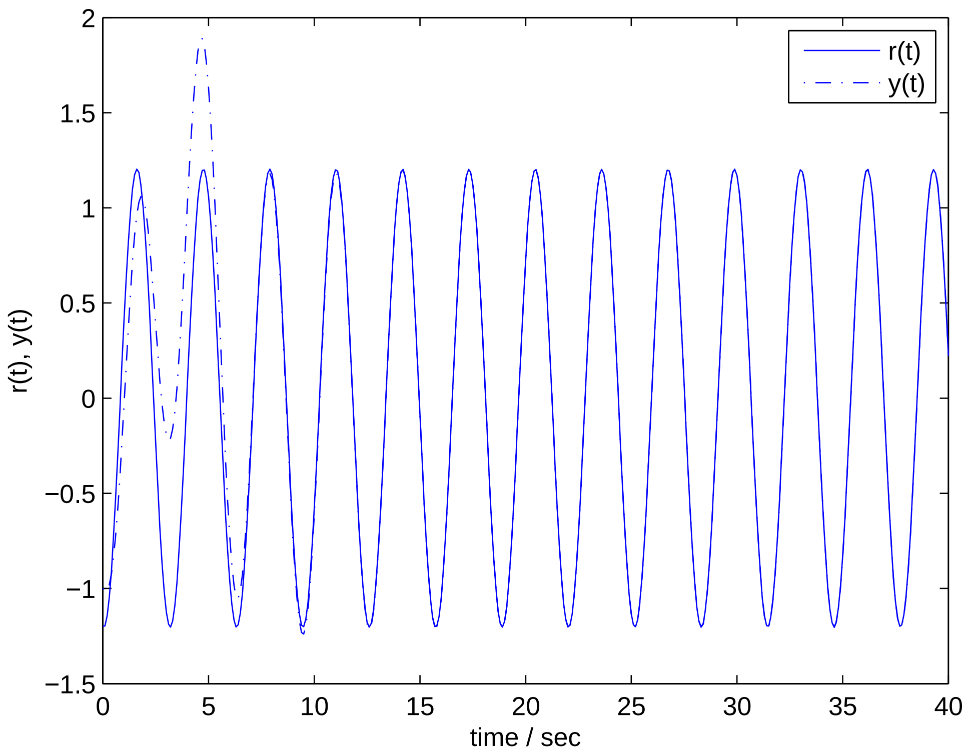

Figure 2.6: Tracking of Reference Signal

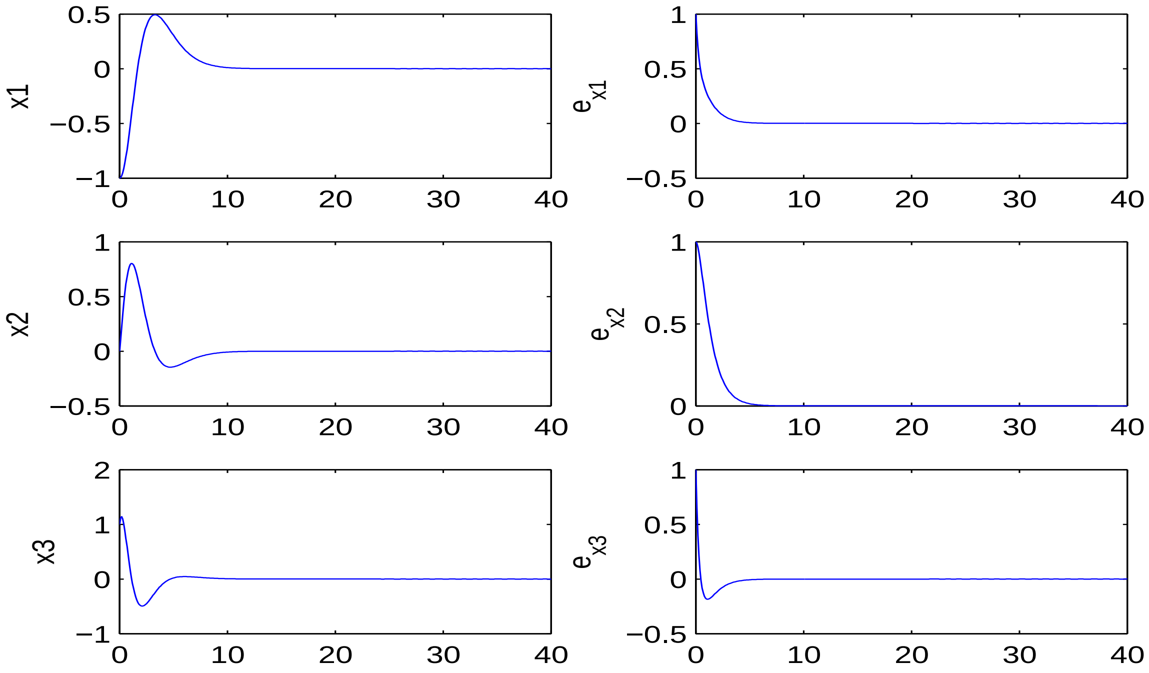

Figure 2.7: Stability of Closed-Loop System

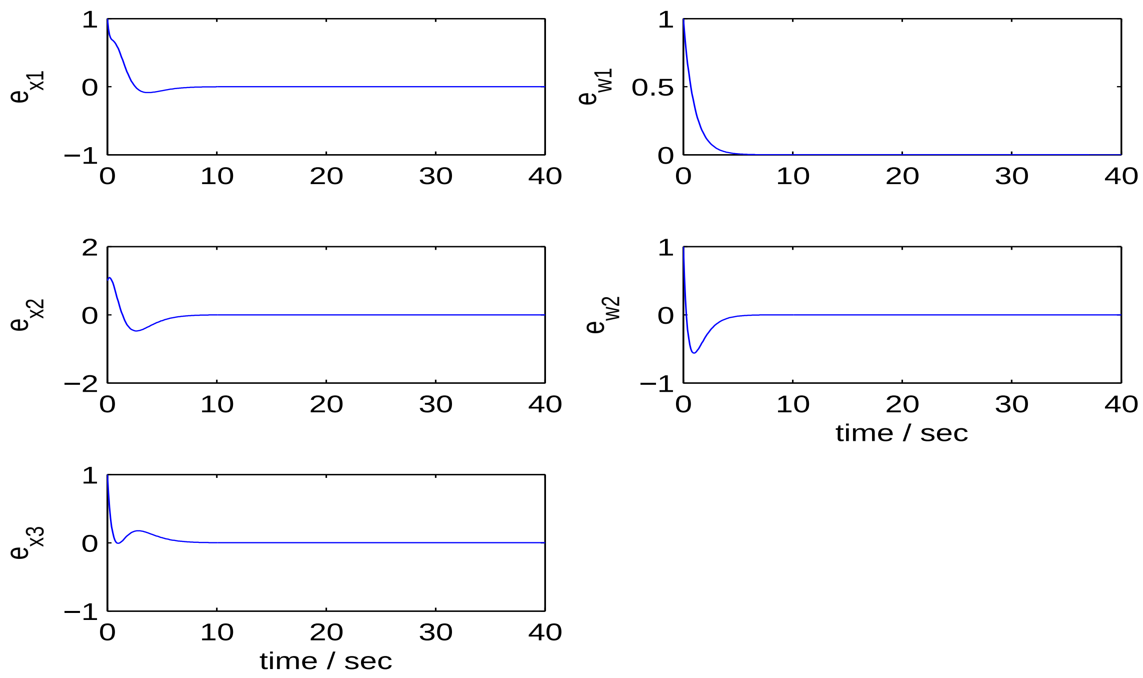

Figure 2.8: Error between x, w and $\hat{x}$, $\hat{w}$

In figure 2.6, figure 2.7 and figure 2.8, we can see the simulation results for the system in (2.138)-(2.139). The graphs are ordered same with *Example 1*.

2.6.3 Example 3

In *Example 3*, we will examine a minimum phase system. State space model of the system is shown below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 1 & 3 & -5 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u + \nu,$$

$$y = \begin{pmatrix} 2 & 2 & 1 & 0 \end{pmatrix} x. \tag{2.146}$$

The exogenous system is given as follows :

$$\dot{w} = \begin{pmatrix} 0 & 0 & 0 & -\pi \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\pi}{2}^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} w,$$

$$r(t) = -\begin{pmatrix} 1 & -0.5 & 2 & 0 \end{pmatrix} w,$$

$$\nu(t) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0.5 \\ 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 \end{pmatrix} w. \tag{2.147}$$

Hence according to (2.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 2 & 2 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -0.5 & 2 & 0 \end{pmatrix} w. \tag{2.148}$$

When $w(t) = 0$, the transfer function of this system is given as :

$$G(s) = C(sI - A)^{-1}B = \frac{s^2 + 2s + 2}{s^4 + 5s^3 - 3s^2 - s + 4}. \tag{2.149}$$

Hence, when $w(t) = 0$, the uncontrolled system is minimum phase and unstable. By using (2.68), (2.69), (2.78), (2.81) and (2.82), we find the controller which satisfies the regulation conditions as follows :

$$\dot{\xi} = \begin{pmatrix} 0 & 1 \\ -9 & -3 \end{pmatrix} \xi + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 41 & 10 & -39 & -9 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 28.16 & 5.39 & -10.7 & 21.37 \end{pmatrix} w,$$
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \xi. \tag{2.150}$$

With the dynamic controller as given above, the characteristic polynomial of the closed-loop system becomes as given below :

$$ch(s) = s^6 + 5s^5 + 12s^4 + 19s^3 + 20s^2 + 14s + 4, \tag{2.151}$$

and the roots of (2.159) can be given as follows: $\{-0.215 + 1.307\imath, -0.215 - 1.307\imath, -2, -1 + \imath, -1 - \imath, -0.569\}$. If we assign the eigenvalues of the state observer matrix $(A - L_x C)$ as $\{-4.22, -1.02 + 0.48\imath, -1.02 - 0.48\imath, -1.13$ and the exogenous system observer matrix $(S + L_w Q)$ as $\{-0.064 + 1.67\imath, -0.064 - 1.67\imath, -0.26 + 1.34\imath, -0.26 - 1.34\imath\}$, we obtain L_x and L_w as follows :

$$L_x = \begin{pmatrix} 0.1 & 1.1 & 3 & 1 \end{pmatrix}^T, \tag{2.152}$$
$$L_w = \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1 \end{pmatrix}^T. \tag{2.153}$$

The initial conditions for this simulation are taken as follows :
$x(0) = \begin{pmatrix} 0.2 & 0 & -0.4 & 0 \end{pmatrix}^T$, $\xi(0) = \begin{pmatrix} 0.9 & 0 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$.

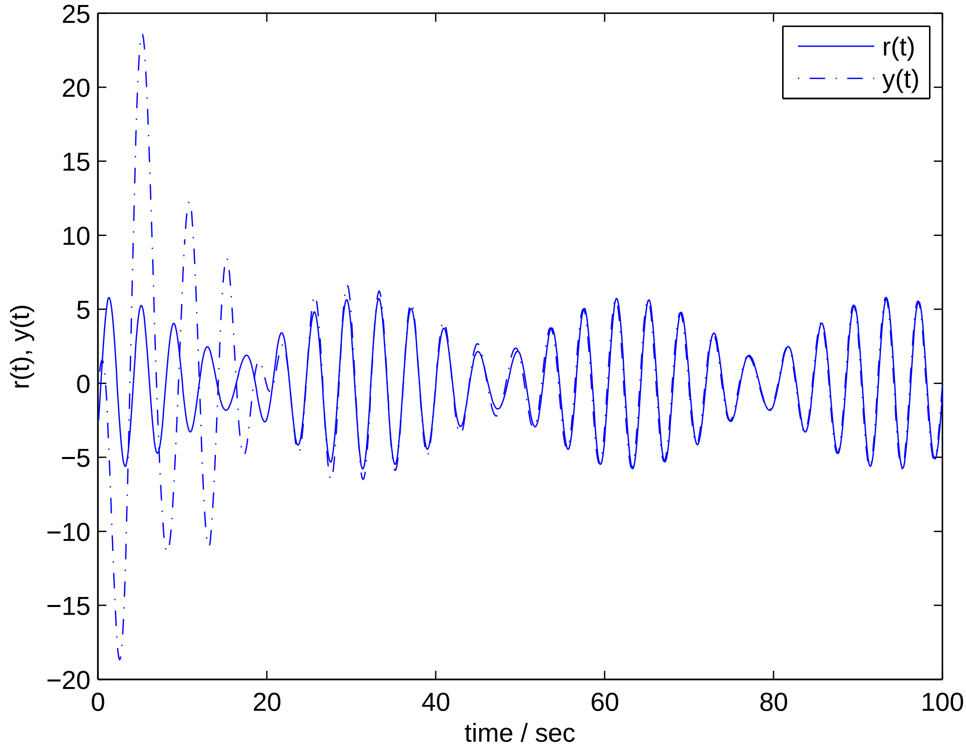

Figure 2.9: Tracking of Reference Signal

The simulation results for the system in (2.146)-(2.147) can be seen in figure 2.9, figure 2.10 and figure 2.11 respectively.

2.6.4 Example 4

Finally, we consider below minimum phase system as an *Example 4* :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u + \nu,$$

$$y = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} x. \tag{2.154}$$

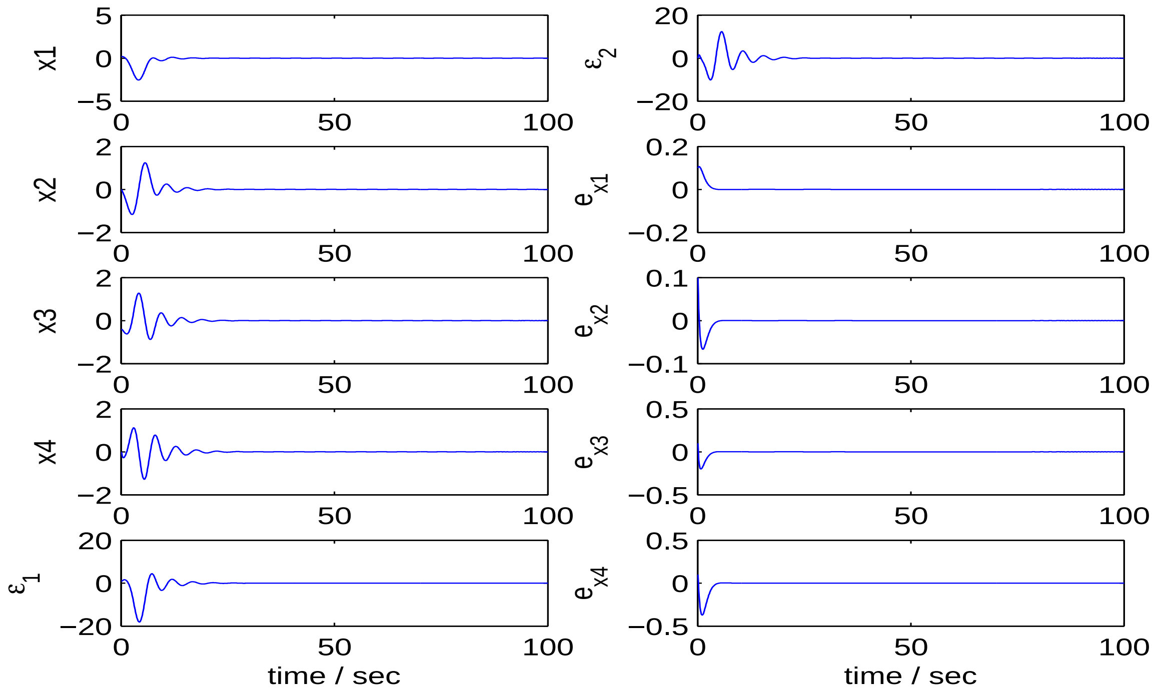

Figure 2.10: Stability of Closed-Loop System

The exogenous system is given as follows :

$$\dot{w} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} w,$$

$$r(t) = -\begin{pmatrix} 1 & -3 \end{pmatrix} w,$$

$$\nu(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} w. \tag{2.155}$$

Hence according to (2.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & -3 \end{pmatrix} w. \tag{2.156}$$

Note that, when $w(t) = 0$, the transfer function of this system is given as :

$$G(s) = C(sI - A)^{-1}B = \frac{s^2 + 3s + 2}{s^3 + 3s^2 + 5s + 2} \tag{2.157}$$

Hence, when $w(t) = 0$, the uncontrolled system is minimum phase and stable. By using (2.68), (2.69), (2.78), (2.81) and (2.82), we find the controller which

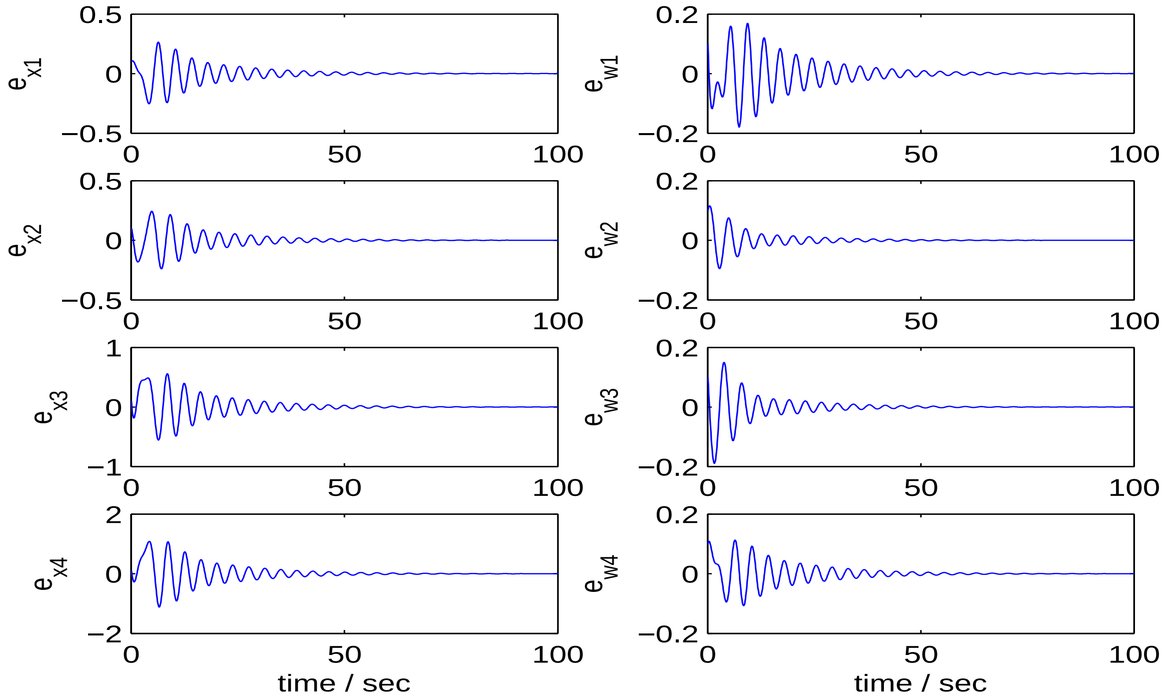

Figure 2.11: Error between x, w and $\hat{x}$, $\hat{w}$

satisfies the regulation conditions as follows :

$$\dot{\xi} = \begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix} \xi + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 20 & 2 \end{pmatrix} w$$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \xi \tag{2.158}$$

With dynamic controller as given above, the characteristic polynomial of the closed-loop system becomes as given below :

$$ch(s) = s^5 + 7s^4 + 19s^3 + 25s^2 + 16s + 4, \tag{2.159}$$

and roots of (2.159) can be given as follows: $\{-2, -2, -1, -1, -1\}$. If we assign the eigenvalues of the state observer matrix $(A - L_xC)$ as $\{-6.61, -1.06, -1.8$ and the exogenous system observer matrix $(S + L_wQ)$ as $\{-1, -2\}$, we obtain L_x and L_w as follows :

$$L_x = \begin{pmatrix} 0.1 & 1.1 & 3 \end{pmatrix}^T \tag{2.160}$$

$$L_w = \begin{pmatrix} \frac{3}{19} & \frac{-18}{19} \end{pmatrix}^T \tag{2.161}$$

Simulations are done for the below initial conditions :
$x(0) = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}^T$, $\xi(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

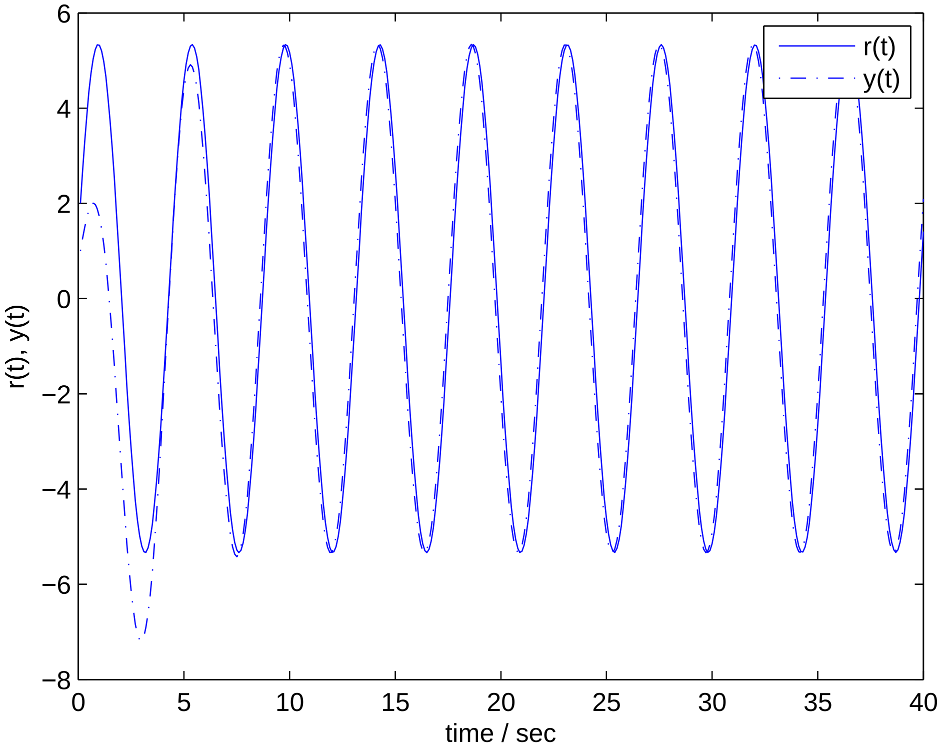

Figure 2.12: Tracking of Reference Signal

The simulation results for the system in (2.154)-(2.155) with the controller (2.158) and observers gain matrices (2.161) can be seen in figure 2.12, figure 2.13 and figure 2.14 respectively.

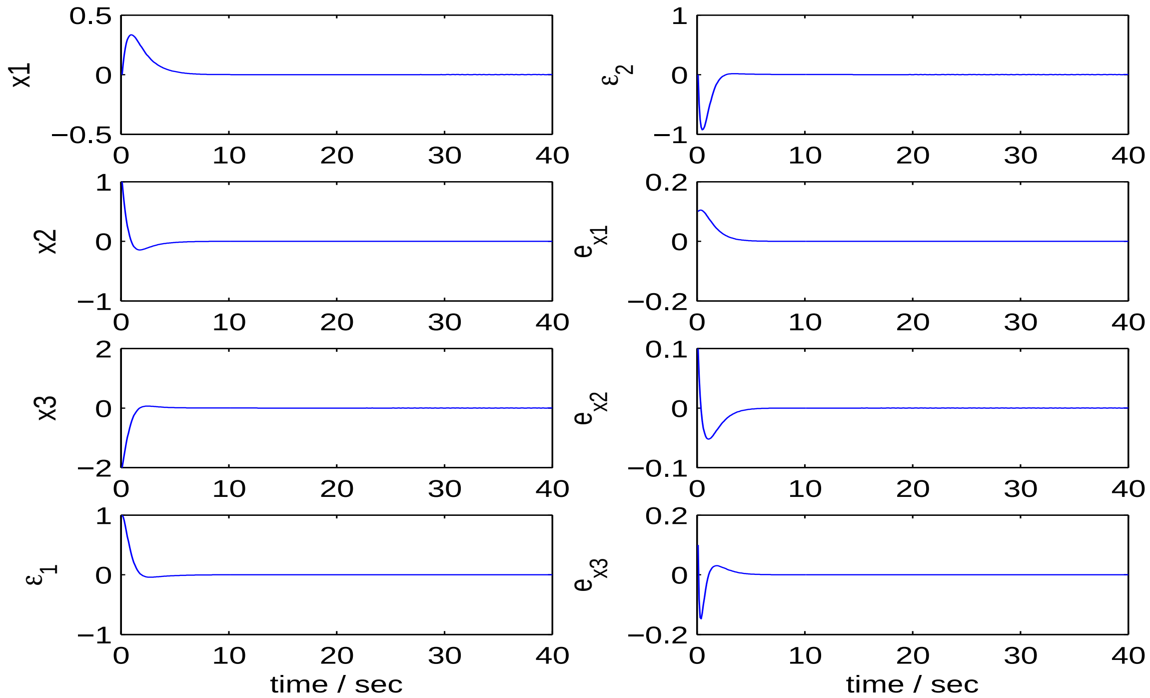

Figure 2.13: Stability of Closed-Loop System

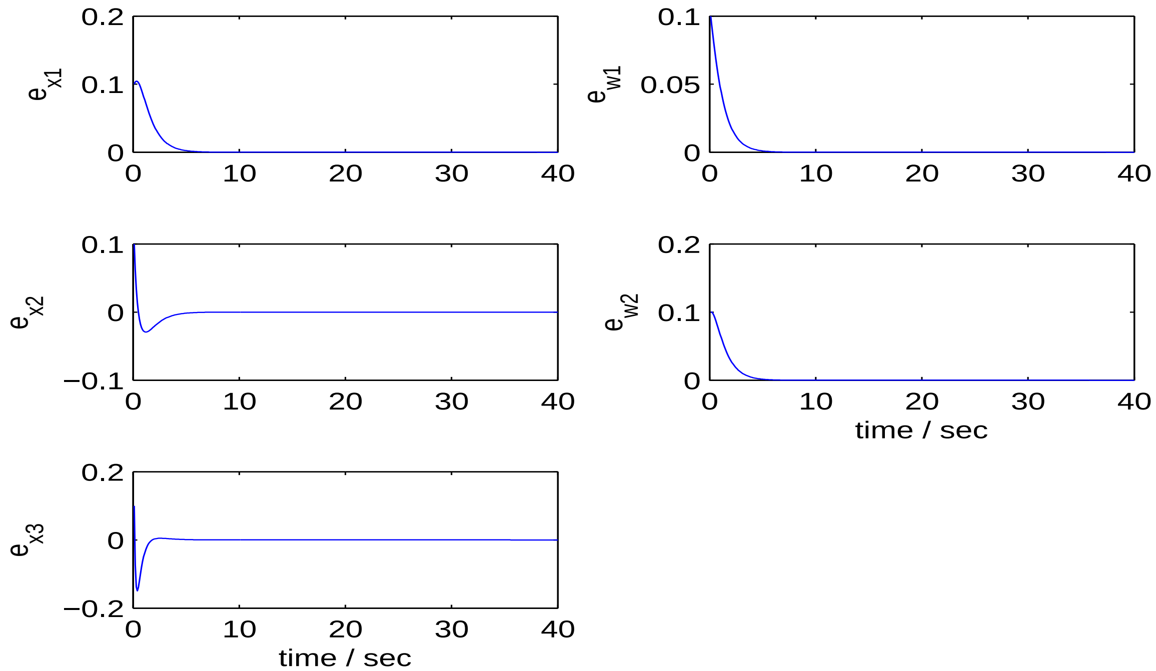

Figure 2.14: Error between x, w and $\hat{x}$, $\hat{w}$

Chapter 3

OUTPUT REGULATION for ALL-POLE and MINIMUM PHASE LTV SYSTEMS

In this chapter, we will consider single-input-single-output (SISO) all-pole and minimum phase Linear-Time Varying systems. Since the definitions of *all-pole* and *minimum phase* are not standard for time varying systems, they will be defined in the sequel. The general state space representation of these systems are given below :

$$\dot{x}(t) = A(t)x + B(t)u + \nu, \tag{3.1}$$

$$y(t) = C(t)x. \tag{3.2}$$

where $x \epsilon \Re^n$, $u \epsilon \Re$, $y \epsilon \Re$ represent the system state, input and output respectively and $A(t) \epsilon \Re^{n \times n}$, $B(t) \epsilon \Re^{n \times 1}$, $C(t) \epsilon \Re^{1 \times n}$ represent the time-varying system matrices. The exogenous system model is to be used in this chapter is in the below form :

$$\dot{w}(t) = S(t)w(t), \tag{3.3}$$

$$r(t) = -Q(t)w(t), \tag{3.4}$$

$$\nu(t) = P(t)w(t), \tag{3.5}$$

where $w\epsilon\Re^m$, $\nu\epsilon\Re^n$,$r\epsilon\Re$ represent exogenous system state, disturbance signals and reference signal respectively and $S(t)\epsilon\Re^{m\times m}$, $P(t)\epsilon\Re^{n\times m}$, $Q(t)\epsilon\Re^{1\times m}$ represent time-varying matrices of the exogenous system. The matrices $A(t)$, $B(t)$, $C(t)$, $S(t)$, $Q(t)$ and $P(t)$ are continuous and bounded functions of time. The negative sign in the equation that gives reference signal (3.4) is again used to ensure compliance with the use in the literature. Thus, with this reference signal definition the tracking error $e(t) = y(t) - r(t)$ becomes as given below :

$$e(t) = y(t) - r(t) = C(t)x + Q(t)w. \tag{3.6}$$

In the time-varying linear systems, designing an observer for the system states is not an easy task, even if we will assume that the system states are observable. Thus, we make an assumption of the observability of the system states and the exogenous system states but we actually will not design observers for these states in this chapter.

Assumption 3. *The pairs* $(C(t), A(t))$ *and* $(Q(t), S(t))$ *are both observable.*

The cases in which the reference signal $r(t)$ and/or the disturbance signals $\nu(t)$ converge to infinity, although may be meaningful for some applications, are not considered in this thesis for simplicity. Conversely, the cases ,in which the reference signal $r(t)$ and/or the disturbances $\nu(t)$ converge to zero, are not considered in the output regulation problem which is investigated here. Thus, this implies that $r(t)$ and $\nu(t)$ should be bounded below and above (i.e. $c_1 \leq r(t) \leq c_2$ and $a_1 \leq r(t) \leq a_2$ for any $t \geq t_0$). Therefore, in order to prevent above cases we should make below assumption on the exogenous system :

Assumption 4. *$d_1\|w(t_0)\| \le \|\Phi_s(t,t_0)w(t_0)\| \le d_2\|w(t_0)\|$ for any $t \ge t_0$, where $\Phi_s(t,t_0)$ is the transition matrix of the exogenous system, where d_1 and d_2 are real constants.*

Our objective is to find a control law such that with this control law our closed-loop system satisfies regulation conditions (i), (ii). Similar to LTI systems, in the simplest case (All-pole case), we will use the relative degree property of the time-varying systems. Additionally, we will define all-pole systems by using the relative degree property, because in linear time-varying systems there is no transfer function representation in the Laplace domain that will help us to define the all-pole systems unlike LTI case. Thus, we should first define the relative degree property of LTV systems.

3.1 Relative Degree Property

If the system in (3.1)-(3.2) satisfies the conditions given below :

$$T_1(t) = C(t)$$
$$T_i(t) = T_{i-1}(t)A(t) + \dot{T}_{i-1}(t)\ ,\ 2 \le i \le n \tag{3.7}$$
$$T_i(t)B(t) = 0\ ,\ 1 \le i \le r-1$$
$$T_r(t)B(t) = b(t) \ne 0\ ,\ \forall t \ge t_0 \tag{3.8}$$

then, the system has a "relative degree r system" [20]. If the derivative of the system output $y(t)$ is taken with $w = 0$, input appears at the r^{th} derivative because of the relative degree property as shown below :

$$
\begin{aligned}
\dot{y} &= \dot{C}(t)x + C(t)(A(t)x + B(t)u) = (\dot{C}(t) + C(t)A(t))x + \underbrace{C(t)B(t)}\, u \\
&= T_2(t)x \\
&\vdots \\
y^{(r-1)} &= \dot{T}_{r-1}(t)x + T_{r-1}(t)(A(t)x + B(t)u) = (\dot{T}_{r-1}(t) + T_{r-1}(t)A(t))x + \underbrace{T_{r-1}(t)B(t)}\, u \\
&= T_r(t) \\
y^{(r)} &= \dot{T}_r(t)x + T_r(t)(A(t)x + B(t)u) = (\dot{T}_r(t) + T_r(t)A(t))x + b(t)u
\end{aligned} \tag{3.9}
$$

The parts, indicated by underbrace, are equal to zero as a result of the relative degree property. Therefore, above property will be used to design the controller for the all-pole systems in the following section.

Remark 3. *If the system in question is actually LTI, by using constant matrices A, B, C instead of $A(t)$, $B(t)$, $C(t)$, it is straightforward to show that the conditions given by (3.7)-(3.8) reduces to (2.7)-(2.8).*

3.2 Controller for All-Pole LTV Systems

If the system has *full-relative degree* (i.e. $r = n$ and n is system dimension), then this system is called an "All-Pole LTV System". Actually, this all-pole definition is the same with the LTI case. In order to obtain controller for all-pole LTV systems, we need to make an assumption on the observability matrices of these systems. To be able to make this assumption, we first need to define a new kind of transformation, which is called *Lyapunov transformation*, and it is defined below.

Definition 11. *A matrix $T(t)$ is called a Lyapunov transformation if $T(t)$ is nonsingular, $T(t)$ and $\dot{T}(t)$ are continuous, and $T(t)$ and $T(t)^{-1}$ are bounded for all t [39].*

The similarity transformation in LTI systems is special case of the Lyapunov transformation. Additionally, Lyapunov transformation is a stability preserving transformation, which is obvious from Definition 11. Note that the observability matrix of the system (3.1)-(3.2) can be given as follows :

$$T(t) = \begin{pmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{pmatrix}, \tag{3.10}$$

see e.g. [36]. Note that if the system is LTI, then by using (3.7), it is easy to see that $T(t)$ becomes the well known observability matrix for LTI systems.

Assumption 5. *The observability matrix given by (3.10) is a Lyapunov transformation.*

Remark 4. *In the LTI SISO case, assumption 5 is automatically satisfied if the system is observable, since in this case the observability matrix is nonsingular. However, in LTV SISO case, one can easily construct examples in which the system is observable but the observability grammian given by (3.10) is not a Lyapunov transformation.*

If $T(t)$ is a Lyapunov transformation, then we can apply this transformation to the system in (3.1)-(3.2) to get a canonical form similar with (2.10). Let us define the new state variables $\tilde{x}$ as $\tilde{x} = T(t)x$. To guarantee that the stability properties of x and $\tilde{x}$ are the same, we need to assume that $T(t)$ is a Lyapunov transformation. In the new state variables $\tilde{x}$, the state equations become:

$$\begin{aligned} \dot{\tilde{x}} &= \dot{T}(t)x + T(t)\dot{x} = \dot{T}(t)x + T(t)(A(t)x + B(t)u), \\ &= (\dot{T}(t) + T(t)A(t))x + T(t)B(t)u, \\ &= (\dot{T}(t) + T(t)A(t))T(t)^{-1}\tilde{x} + T(t)B(t)u, \qquad (3.11) \\ y &= C(t)x = C(t)T(t)^{-1}\tilde{x}. \qquad (3.12) \end{aligned}$$

Let us define the new system matrices $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}(t)$ as :

$$\tilde{A}(t) = (\dot{T}(t) + T(t)A(t))T(t)^{-1} \tag{3.13}$$

$$\tilde{B}(t) = T(t)B(t) \tag{3.14}$$

$$\tilde{C}(t) = C(t)T(t)^{-1} \tag{3.15}$$

Next we will obtain the structure of $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}(t)$ by using the relative degree property. By using (3.14), (3.10) and (3.8), it can be easily shown that $\tilde{B}(t)$ has the following form :

$$\begin{aligned}\tilde{B}(t) &= \begin{pmatrix} T_1(t) & T_2(t) & \dots & T_n(t) \end{pmatrix}^T B(t), \\ &= \begin{pmatrix} 0 & \dots & 0 & b(t) \end{pmatrix}^T.\end{aligned} \tag{3.16}$$

Similarly, from (3.15) we obtain :

$$\tilde{C}(t) \begin{pmatrix} T_1(t) & T_2(t) & \dots & T_n(t) \end{pmatrix}^T = C(t). \tag{3.17}$$

Since by (3.7) $T_1(t) = C(t)$, from (3.17) we obtain :

$$\tilde{C}(t) = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}. \tag{3.18}$$

Finally, to find the form of $A(t)$, let us define :

$$\tilde{A}(t) = \begin{pmatrix} \tilde{a}_1(t) \\ \vdots \\ \tilde{a}_{n-1}(t) \\ \tilde{a}_n(t) \end{pmatrix}. \tag{3.19}$$

Note that (3.13) can be written as :

$$\begin{pmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{pmatrix} A(t) + \begin{pmatrix} \dot{T}_1(t) \\ \vdots \\ \dot{T}_n(t) \end{pmatrix} = \tilde{A}(t) \begin{pmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{pmatrix}. \tag{3.20}$$

Note that i^{th} row of (3.20) can be written as :

$$T_i(t)A(t) + \dot{T}_i(t) = a_i(t)T(t) \ , \ 1 \leq i \leq n. \tag{3.21}$$

From (3.7), it easily follows that :

$$a_i(t) = \underline{e}_{i+1}^T \ , \ 1 \leq i \leq n-1, \tag{3.22}$$

where e_i denotes i^{th} unit vector, i.e. a vector whose only i^{th} entry is 1 and the rest are zero. For n, we have

$$a_n(t) = (T_n(t)A(t) + \dot{T}_n(t))T(t)^{-1} = \begin{pmatrix} \lambda_1(t) & \dots & \lambda_n(t) \end{pmatrix}. \tag{3.23}$$

Thus, if we put together the equations (3.18),(3.20),(3.22) and (3.23), we obtain the following state-space representation for the transformed system :

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_1(t) & \lambda_2(t) & \dots & \lambda_{n-1}(t) & \lambda_n(t) \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix} u, \\ y &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x. \end{aligned} \tag{3.24}$$

This form is similar with (2.10) in all-pole LTI case except for time-varying functions. Let us define the observability matrix $\tilde{T}(t)$ of the transformed system as follows :

$$\tilde{T}(t) = \begin{pmatrix} \tilde{T}_1(t) & \tilde{T}_2(t) & \dots & \tilde{T}_n(t) \end{pmatrix}^T. \tag{3.25}$$

Fact 12. *The rows $\tilde{T}_i(t)$ of $\tilde{T}(t)$ defined similar with (3.7) has the following form :*

$$\tilde{T}_i(t) = T_i(t)T(t)^{-1} \ ,1 \leq i \leq n. \tag{3.26}$$

Proof. We use mathematical induction to prove this argument.

When i=1, we have :

$$\tilde{T}_1(t) = \tilde{C}(t) = C(t)T(t)^{-1} = T_1(t)T(t)^{-1}$$

Hence, (3.26) holds for $i = 1$. Now, assume that (3.26) holds for $i = k \geq 1$, i.e.$\tilde{T}_k(t) = T_k(t)T(t)^{-1}$. Then, we try to find $\tilde{T}_{k+1}(t)$ as follows :

$$\begin{aligned}\tilde{T}_{k+1}(t) &= \tilde{T}_k(t)\tilde{A}(t) + \dot{\tilde{T}}_k(t)\\ &= T_k(t)T(t)^{-1}(T(t)A(t)T(t)^{-1} + \dot{T}(t)T(t)^{-1})\\ &+ \dot{T}_k(t)T(t)^{-1} + T_k(t)\dot{T}(t)^{-1}. \end{aligned} \tag{3.27}$$

Since $T(t)T(t)^{-1} = I$, by differentiating this equation we obtain $\dot{T}(t)T(t)^{-1} + T(t)\dot{T}(t)^{-1} = 0$. Hence $\dot{T}(t)^{-1} = T(t)^{-1}\dot{T}(t)T(t)^{-1}$. By using the latter in (3.27), we obtain :

$$\begin{aligned}\tilde{T}_{k+1}(t) &= T_k(t)A(t)T(t)^{-1} + T_k(t)T(t)^{-1}\dot{T}(t)T(t)^{-1}\\ &- T_k(t)T(t)^{-1}\dot{T}(t)T(t)^{-1} + \dot{T}_k(t)T(t)^{-1},\\ &= (T_k(t)A(t) + \dot{T}_k(t))T(t)^{-1},\\ &= T_{k+1}(t)T(t)^{-1}, \end{aligned} \tag{3.28}$$

where in the last equation we used (3.7). Hence, by mathematical induction the *Fact 12* is true. □

By using (3.14) and the *Fact 12*, one can easily show that the transformed system given by (3.24) has relative degree n. Furthermore, from the *Fact 12*, it easily follows that :

$$\tilde{T}(t) = \begin{pmatrix}\tilde{T}_1(t) & \tilde{T}_2(t) & \dots & \tilde{T}_n(t)\end{pmatrix}^T = \begin{pmatrix}T_1(t) & T_2(t) & \dots & T_n(t)\end{pmatrix}^T T(t)^{-1} = I_{n\times n}, \tag{3.29}$$

hence we have :

$$\tilde{T}_i(t) = \underline{e}_i^T \;\; 1 \leq i \leq n, \tag{3.30}$$

where $\underline{e}_i \epsilon \Re^n$ is i^{th} unit vector as defined before. After we apply the transformation $T(t)$, the state space representation of the transformed system with exogenous signal is obtained as shown below :

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{A}(t)\tilde{x} + \tilde{B}(t)u + \tilde{P}(t)w\\ e &= \tilde{C}(t)x + Q(t)w \end{aligned} \tag{3.31}$$

where $\tilde{P}(t) = T(t)P(t)$ and the exogenous system is same with (3.3). If the derivatives of the error $e(t) = y(t) - r(t)$ are taken successively, and if we use the system equations in (3.31) and the relative degree property, then the input appears at the n^{th} derivative. This fact is shown below :

$$\begin{aligned}
e &= \tilde{C}(t)x + Q(t)w \\
\dot{e} &= \dot{\tilde{C}}(t)x + \tilde{C}(t)(\tilde{A}(t)\tilde{x} + \tilde{B}(t)u + \tilde{P}(t)w) + \dot{Q}(t)w + Q(t)S(t)w \\
&= \tilde{T}_2(t)x + \underbrace{\tilde{C(t)}\tilde{B}(t)}\, u + (\tilde{C}(t)\tilde{P}(t) + \dot{Q}(t) + Q(t)S(t))w \\
&= \tilde{T}_2(t)x + S_1(t)w \\
&\vdots \\
e^n &= (\dot{\tilde{T}}_n(t) + \tilde{T}_n(t)A(t))(x) + \tilde{T}_n(t)\tilde{B}(t)u + \tilde{T}_n(t)\tilde{P}(t)w + S_{n-1}(t)w \\
&= \tilde{T}_{n+1}(t)(x) + b(t)u + S_n(t)w
\end{aligned} \tag{3.32}$$

The parts, indicated by underbraces are zero. In (3.32), we denote $\tilde{T}_{n+1}(t) = \dot{\tilde{T}}_n(t) + \tilde{T}_n(t)A(t)$ and $S_i(t)$ terms are given as below :

$$S_i(t) = \tilde{T}_i(t)\tilde{P}(t) + \dot{Q}(t) + Q(t)S(t) \ , \ \ S_0(t) = Q(t) \ , \ \ 1 \leq i \leq n. \tag{3.33}$$

Similar to the LTI all-pole case, see (2.13), the control input $u(t)$ is chosen as given below :

$$u = -\frac{1}{b(t)}\{\tilde{T}_{n+1}(t)\tilde{x} + S_n(t)w + L_{n-1}e^{n-1} + \ldots + L_1\dot{e} + L_0 e\} \tag{3.34}$$

As in the LTI case, see (2.14), we can express $u(t)$ given by (3.34) :

$$u = K_{\tilde{x}}(t)\tilde{x} + K_w(t)w, \tag{3.35}$$

where

$$K_{\tilde{x}}(t) = -\frac{1}{b(t)}\{\tilde{T}_{n+1}(t) + L_{n-1}\tilde{T}_n(t) + \ldots + L_1\tilde{T}_2(t) + L_0\tilde{T}_1(t)\}, \tag{3.36}$$

and

$$K_w(t) = -\frac{1}{b(t)}\{S_n(t) + L_{n-1}S_{n-1}(t) + \ldots + L_1 S_1(t) + L_0 S_0\}. \tag{3.37}$$

Since $b(t) \neq 0$ for all $t \geq t_0$, division of $b(t)$ in the controller does not cause any instability problem. Then, if we substitute the control input $u(t)$ given by (3.34) into (3.32), the error dynamics becomes as follows :

$$e^n + L_{n-1}e^{n-1} + \ldots + L_1\dot{e} + L_0 e = 0 \tag{3.38}$$

If we use Laplace transform for this differential equation, the characteristic polynomial of (3.38) can be given as follows :

$$ch(s) = s^n + L_{n-1}s^{n-1} + \ldots + L_1 s + L_0. \tag{3.39}$$

If we will chose coefficients L_i in (3.39) properly, the characteristic polynomial given by (3.39) can always be made exponentially stable. This implies that error $e(t)$ is exponentially stable with the controller in (3.34). Thus, the regulation condition (i) is achieved with the controller in (3.34), if controller coefficients are chosen such that the roots of the polynomial in (3.39) are in the LHP. In order to claim that the output regulation problem is solved with the controller structure given by (3.34), we need to prove that the regulation condition (ii) is also satisfied with this structure. If the system equations in (3.31) and the controller in (3.35) are put together, the following closed-loop system is obtained :

$$\begin{aligned}\dot{\tilde{x}} &= (\tilde{A}(t) + \tilde{B}(t)K_{\tilde{x}}(t))\tilde{x} + (\tilde{B}(t)K_w(t) + \tilde{P}(t))w \\ &= \tilde{A}_{cl}(t)\tilde{x} + (\tilde{B}(t)K_w(t) + \tilde{P}(t))w \end{aligned} \tag{3.40}$$

$$e = y - r = \tilde{C}(t)x + Q(t)w \tag{3.41}$$

Lemma 13. *Consider the time varying controller given by :*

$$u = K_{\tilde{x}}(t)x \tag{3.42}$$

where $K_{\tilde{x}}(t)$ is as given by (3.36). Then the closed-loop system given by (3.40)-(3.41) with $w = 0$ is exponentially stable. Additionally, the closed-loop state transition matrix $\tilde{A}_{cl}(t)$ turns out to be constant and the characteristic equation of $\tilde{A}_{cl}(t)$ is given by the equation (3.39).

Proof. We know from the equations in (3.30) that $\tilde{T}_i(t) = \underline{e}_i^T$ for $1 \le i \le n$. Thus, from there we can find $\tilde{B}(t)\tilde{T}_i(t)$ as follows :

$$\tilde{B}(t)\tilde{T}_i(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix} \underline{e}_i^T = b(t) \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{e}_i \end{pmatrix}^T \quad , \; 1 \le i \le n. \tag{3.43}$$

In addition to this, we can find $\tilde{T}_{n+1}(t)$ by using $\tilde{T}_n(t)$ as shown below :

$$\begin{aligned} \tilde{T}_{n+1}(t) &= \dot{\tilde{T}}_n(t) + \tilde{T}_n(t)\tilde{A}(t) \\ &= \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_1(t) & \lambda_2(t) & \dots & \lambda_{n-1}(t) & \lambda_n(t) \end{pmatrix} \\ \tilde{T}_{n+1}(t) &= \begin{pmatrix} \lambda_1(t) & \lambda_2(t) & \dots & \lambda_{n-1}(t) & \lambda_n(t) \end{pmatrix}. \end{aligned} \tag{3.44}$$

Then, we obtain :

$$\tilde{B}(t)\tilde{T}_{n+1}(t) = b(t) \begin{pmatrix} \underline{0} & \dots & \underline{0} & \underline{\lambda}(t) \end{pmatrix}^T \tag{3.45}$$

where $\underline{\lambda}(t) = \begin{pmatrix} \lambda_1(t) & \lambda_2(t) & \dots & \lambda_{n-1}(t) & \lambda_n(t) \end{pmatrix}^T$. If we substitute (3.44)-(3.45) into closed-loop state transition matrix $\tilde{A}_{cl}(t) = (\tilde{A}(t) + \tilde{B}(t)K_{\tilde{x}}(t))$ with $K_x(t)$

in (3.36), we obtain the following form :

$$\tilde{A}(t) + \tilde{B}(t)K_{\tilde{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_1(t) & \lambda_2(t) & \dots & \lambda_{n-1}(t) & \lambda_n(t) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\lambda_1(t) - L_0 & -\lambda_2(t) - L_1 & \dots & -\lambda_{n-1}(t) - L_{n-2} & -\lambda_n(t) - L_{n-1} \end{pmatrix}$$

$$\tilde{A}_{cl}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -L_0 & -L_1 & \dots & -L_{n-2} & -L_{n-1} \end{pmatrix} \tag{3.46}$$

Thus, the closed-loop state transition matrix $\tilde{A}_{cl}(t)$ is constant and the characteristic equation of $\tilde{A}_{cl}(t)$ is given by the polynomial in (3.39) as we can easily compute. Additionally, L_i coefficients were chosen such that the polynomial in (3.39) becomes a Hurwitz polynomial. This implies that all the eigenvalues of $\tilde{A}_{cl}(t)$ are in the LHP and the closed-loop system is exponentially stable with $w = 0$. □

By using *Lemma 13* and (3.38), we obtain the following result.

Theorem 14. *The time-varying controller given by (3.34)-(3.35) satisfies the regulation conditions (i) and (ii) for the system in the form (3.1)-(3.5).*

Proof. (i) From (3.38) and (3.39) it turns out that $e(t)$ is exponentially stable (i.e. we have $\|e(t)\| < k \exp^{-\beta t}$ for some $k > 0$, $\beta > 0$). Hence,

$$\lim_{t \longrightarrow \infty} \|e(t)\| = 0 \tag{3.47}$$

(ii) In *Lemma 13*, we proved that the transformed closed-loop system in (3.40)-(3.41) is exponentially stable with $w = 0$. By *Assumption 5*, $T(t)$ is a Lyapunov transformation, hence it preserves the stability properties. Thus, if the transformed system in (3.24) is exponentially stable with controller (3.34)-(3.35) and $w = 0$, then the original system given by (3.1)-(3.5) is also exponentially stable with controller (3.34)-(3.35) and $w = 0$. Hence, we have :

$$\|\tilde{x}(t)\| \leq \alpha \exp^{-\mu t} \tag{3.48}$$

for some $\alpha > 0$ and $\mu > 0$. Hence we have

$$\|x(t)\| = \|T(t)^{-1}\tilde{x}(t)\| \leq \|T(t)^{-1}\|\|\tilde{x}(t)\| < \gamma\alpha \exp^{-\mu t} \tag{3.49}$$

where $\|T(t)^{-1}\| \leq \gamma$.

Therefore, the results given above prove that the controller in (3.34)-(3.35) with $K_x(t) = K_{\tilde{x}}(t)T(t)$ satisfies the regulation conditions (i)-(ii) for the system given by (3.1)-(3.5). □

3.3 Controller for Minimum Phase LTV Systems

In LTV systems, we can not find Laplace representations of the systems as we do in LTI cases. For this reason, the inverse systems of the minimum phase LTV systems can not be specified easily as we did in the minimum phase LTI cases in *Section 2.4*. In LTI systems, we can find a similarity transformation such that this transformation puts minimum phase LTI systems in a normal form. The state transition matrix of this normal form contains a submatrix whose eigenvalues correspond to the zeros of the original system [40]. Hence, instead of using Laplace representations in order to find the inverse systems in minimum phase

LTI cases, we can use state space representations. Then similar to this method we can find Lyapunov transformations such that these transformations put the LTV systems in a normal form which includes the inverse systems dynamics in the state transition matrices of the original systems.

Firstly, we will show that the minimum phase LTI systems can be put in a normal form by applying certain transformations. Actually, obtaining of the normal form and the transformation matrix were carried out in [40] and here we perform the same methodology to obtain the normal form. To illustrate the methodology mentioned above, let us first consider an LTI, SISO, and minimum phase plant model as given below :

$$G(s) = \frac{n(s)}{d(s)} = \frac{s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0} \tag{3.50}$$

where $n(s)$ is a stable polynomial and $m < n$. This minimum phase system has relative degree r where $r = n - m$. A state space representation of $G(s)$ in (3.50) can be obtained as shown below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots 0 \\ 0 & 0 & 1 & 0 & \ldots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_0 & b_1 & \ldots & 1 & 0 \ldots & 0 \end{pmatrix} x \tag{3.51}$$

Let us define the new state variables $\tilde{x} = \begin{pmatrix} z_1 & \ldots & z_r & \varepsilon_1 & \ldots & \varepsilon_m \end{pmatrix} \epsilon \Re^n$, which are given below :

$$z_i = x_i \text{ for } 1 \leq i \leq r \tag{3.52}$$

$$\varepsilon_i = CA^{i-1}x \text{ for } 1 \leq i \leq m \tag{3.53}$$

where A is the state transition matrix and C is the output vector given in (3.51). If we define the transformation matrix T as $\tilde{x} = Tx$, $T \epsilon \Re^{n \times n}$ can be easily obtained as follows :

$$T = \begin{pmatrix} \underline{e}_1^T \\ \vdots \\ \underline{e}_m^T \\ C \\ \vdots \\ CA^{r-1} \end{pmatrix} \tag{3.54}$$

where $\underline{e}_i$ is unit vector. If we perform this transformation, we obtain the representation given below, see e.g. [40]:

$$\begin{aligned} \dot{z} &= Gz + P\varepsilon_1 \\ \dot{\varepsilon}_1 &= \varepsilon_2 \\ &\vdots \\ \dot{\varepsilon}_{r-1} &= \varepsilon_r \\ \dot{\varepsilon}_r &= \tilde{K}z + S\varepsilon + u \end{aligned} \tag{3.55}$$

$$y = \tilde{H} \begin{pmatrix} z \\ \varepsilon \end{pmatrix} = \varepsilon_1 \tag{3.56}$$

where P, $\tilde{H}$, $\tilde{K}$ are matrices with appropriate dimensions [40]. The form of G is given below :

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -b_0 & -b_1 & \dots & -b_{m-2} & -b_{m-1} \end{pmatrix} \tag{3.57}$$

The characteristic equation of the G matrix is the same as the $n(s)$ polynomial, so the eigenvalues of the G matrix correspond to the zeros of the original system in (3.50)-(3.51). Therefore, we can view the first equation in (3.55) as *the inverse system dynamics*, if it is without ε_1. Because $(sI-G)^{-1}$ contains $\frac{1}{n(s)}$. In addition, we can choose the input vector H and the output vector K used in (2.68)-(2.69), which makes the inverse system complete.

As shown above, the inverse of the minimum phase LTI system can be achieved only by using the state space forms and transformations. Therefore, since we do not use Laplace representation above, we can apply similar methodology to the LTV systems to obtain the inverse dynamics .Then, this inverse dynamics can be used as the first part of the controller. Additionally, after we obtain the inverse dynamics of the system, we will give the definition of the minimum phaseness in LTV systems. Her, we only assume that our LTV system has relative degree r.

In order to apply the similar methodology that we used in the LTI case, the LTV system should be converted into a form similar to (3.51). If we make appropriate assumptions on the observability and the controllability matrices of our LTV system, this state space form can be obtained. Let us first define the controllability matrix of the LTV system.

Definition 15. *The controllability matrix of the system in the form (3.1)-(3.2) is* $W(t) = \begin{pmatrix} W_1(t) & \dots & W_n(t) \end{pmatrix}$ *where*

$$\begin{aligned} W_1(t) &= B(t) \\ W_i(t) &= A(t)W_{i-1}(t) - \dot{W}_{i-1}(t) \ , \ 2 \le i \le n \end{aligned} \tag{3.58}$$

see [36].

Assumption 6. *The observability matrix* $T(t) = \begin{pmatrix} T_1(t) \\ T_2(t) \\ \vdots \\ T_n(t) \end{pmatrix}$ *is a Lyapunov transformation (see (3.7), (3.10)).*

Assumption 7. *The controllability matrix $W(t)$ given by (3.58) of the system in (3.1)-(3.2) is a Lyapunov transformation.*

Remark 5. *The above assumptions indicate that the minimum and maximum singular values for the controllability and the observability matrices are bounded* *below and above for all t, respectively. Actually, the boundedness of the minimum singular values of the controllability and the observability matrices correspond to the instantaneous controllability and observability, see [36].*

Lemma 16. *If LTV system in (3.1)-(3.2) has relative degree r and satisfy (3.8), then the following holds :*

$$T_i(t)W_j(t) = 0 \ , \ i + j - 1 \leq r - 1 \tag{3.59}$$

Proof.

- First, for $j = 1$, from (3.59) we obtain $i \leq r - 1$. For this case, by using (3.7) we obtain :

$$T_i(t)W_1(t) = 0. \tag{3.60}$$

- For j=2, from (3.59) we obtain $i \leq r - 2$. For this case, by using (3.58) we obtain :

$$\begin{aligned} T_i(t)W_2(t) &= T_i(t)(A(t)W_1(t) - \dot{W}_1(t)) \\ &= T_i(t)A(t)W_1(t) + \dot{T}_i(t)W_1(t) \end{aligned} \tag{3.61}$$

Since in this case we have $T_i(t)W_1(t) = 0$, hence $\dot{T}_i(t)W_1(t)+T_i(t)\dot{W}_1(t) = 0$, by using the latter and (3.58) in (3.61), we obtain :

$$T_i(t)W_2(t) = T_{i+1}W_1(t) = 0 \tag{3.62}$$

Note that since $i \leq r - 2$, we have $i + 1 \leq r - 1$, hence the (3.62) follows from (3.60).

- For $j = 3$, from (3.59) we obtain $i \leq r - 3$. For this case, by using (3.58) we obtain :

$$\begin{aligned} T_i(t)W_3(t) &= T_i(t)(A(t)W_2(t) - \dot{W}_2(t)) \\ &= T_i(t)A(t)W_2(t) + \dot{T}_i(t)W_2(t) \end{aligned} \tag{3.63}$$

Since in this case we have $T_i(t)W_2(t) = 0$, hence $\dot{T}_i(t)W_2(t)+T_i(i)\dot{W}_2(t) = 0$, by using the latter and (3.58) in (3.63), we obtain :

$$T_i(t)W_3(t) = T_{i+1}(t)W_2(t) = 0. \tag{3.64}$$

Note that since $i \leq r-3$, we have $i+1 \leq r-2$, hence (3.64) follows from (3.62). Following recursively, by increasing j, and following exactly the same analysis, one can show that (3.59) holds. For example, assume that for $j = r-2$, (3.59) holds. Then for $j = r-1$, from (3.59) we have $i \leq 1$. For this case, by using (3.58) we obtain :

$$\begin{aligned} T_i(t)W_{r-1}(t) &= T_i(t)(A(t)W_{r-2}(t) - \dot{W}_{r-2}(t)) \\ &= T_i(t)A(t)W_{r-2}(t) + \dot{T}_i(t)W_{r-2}(t) \end{aligned} \tag{3.65}$$

Since in this case we have $T_i(t)W_{r-2}(t) = 0$, hence $\dot{T}_i(t)W_{r-2}(t) + T_i(i)\dot{W}_{r-2}(t) = 0$, by using the latter and (3.58) in (3.65) we obtain :

$$T_i(t)W_{r-1}(t) = T_{i+1}(t)W_{r-2}(t) = 0 \tag{3.66}$$

Note that the latter equality holds since we assume that (3.59) holds for $j = r-2$. The equations (3.60)-(3.66) show that (3.59) holds (Note that alternatively, we could prove this lemma by using the mathematical induction, which would utilize essentially the same calculations given above).

$\square$

Remark 6. *If the system is an LTI, SISO system with relative degree r, then by using constant system matrices A, B, C, we obtain $CA^{i-1}B$ for $i = 1, \dots, r-1$. In this case, we have $T_i = CA^{i-1}$ and $W_j = A^{j-1}B$ for $i = 1, \dots, n$, $j = 1, \dots, n$. Hence we have $T_iW_j = CA^{i+j-1}B$. Obviously, for relative degree r case, we have $T_iW_j = 0$ for $i+j-1 \leq r-1$. This argument shows that Lemma 16 holds for LTI systems, hence it could be considered as a generalization of this result to LTV case.*

Remark 7. *If we take $i = 1$ in* Lemma 16*, then $j \leq r - 1$. Since $T_1(t) = C(t)$, this implies the following :*

$$C(t)W_i(t) = 0 \text{ for } 1 \leq i \leq r - 1 \tag{3.67}$$

$$C(t)W_r(t) = \beta(t) \tag{3.68}$$

Fact 17. *$\beta(t)$ in equation (3.68) and $b(t)$ in equation (3.8) are the same.*

Proof.

$$C(t)W_r(t) = C(t)A(t)W_{r-1}(t) - C(t)\dot{W}_{r-1}(t) = C(t)A(t)W_{r-1}(t) + \dot{C}(t)W_{r-1}(t)$$
$$= T_2(t)W_{r-1}(t) \tag{3.69}$$

where we used the facts $C(t)W_{r-1}(t) = 0$ and hence $-C(t)\dot{W}_{r-1}(t) = \dot{C}(t)W_{r-1}(t)$. Then, if we apply similar steps, we obtain $T_2(t)W_{r-1}(t) = T_3(t)W_{r-2}(t)$. Thus, if we repeat this procedure recursively, we obtain the following :

$$C(t)W_r(t) = T_r(t)W_1(t) = T_r(t)B(t) = b(t) \tag{3.70}$$

□

Remark 8. *Again, in LTI SISO case, by using constant system matrices A, B, C, assuming that the system has relative degree r, and by using the fact that $T_i = CA^{i-1}$ and $W_j = A^{j-1}B$, we obtain*

$$\beta = CW_r = CA^{r-1}B = T_rB = b \tag{3.71}$$

Hence fact 17 holds for LTI case as well.

In order to obtain an appropriate normal form for the LTV system and the inverse dynamics, we will apply some Lyapunov transformations to the system. Thus, we should first show that the relative degree property, Lyapunov transformation property of the controllability and the observability matrices should be preserved under Lyapunov transformations.

Fact 18. *If we apply Lyapunov transformation $P(t)$ to the LTV system in the form (3.1)-(3.2), the new observability matrix $\tilde{T}(t)$ and the new controllability matrix $\tilde{W}(t)$ are still Lyapunov transformations.*

Proof. After transformation $P(t)$ is applied to (3.1)-(3.2), the following system matrices are obtained : $\tilde{A}(t) = (P(t)A(t) + \dot{P}(t))P(t)^{-1}$, $\tilde{B}(t) = P(t)B(t)$ and $\tilde{C}(t) = C(t)P(t)^{-1}$. We first prove the observability part and then we prove the controllability part.

(a)

We use mathematical induction to prove the following equation :

$$\tilde{T}_i(t) = T_i(t)P(t)^{-1} \quad 1 \leq i \leq n. \tag{3.72}$$

where $\tilde{T}_i(t)$ are the rows of the observability matrix of the transformed system. For i=1, we have $\tilde{T}_1(t) = C(t)P(t)^{-1} = T_1 P(t)^{-1}$. But since $T_1(t) = C(t)$, it follows that (3.72) holds for $i = 1$. Now assume that (3.72) holds for $i = m > 1$. Then, we have :

$$\begin{aligned} \tilde{T}_{m+1}(t) &= \tilde{T}_m(t)\tilde{A}(t) + \dot{\tilde{T}}_m(t) \\ &= T_m P(t)^{-1}((P(t)A(t) + \dot{P}(t))P(t)^{-1} + \dot{T}_m P(t)^{-1} + T_m \dot{P}(t)^{-1} \end{aligned} \tag{3.73}$$

Since $P(t)P(t)^{-1} = I$, by differentiating we obtain $\dot{P}(t)P(t)^{-1} = -P(t)\dot{P}(t)^{-1}$, by using latter in (3.73) we obtain :

$$\begin{aligned} \tilde{T}_{m+1}(t) &= T_m A(t)P(t)^{-1} - T_m P(t)^{-1}P(t)\dot{P}(t)^{-1} + \dot{T}_m P(t)^{-1} + T_m \dot{P}(t)^{-1} \\ &= (T_m A(t) + \dot{T}_m)P(t)^{-1} \\ &= T_{m+1}(t)P(t)^{-1} \end{aligned}$$

By mathematical induction $\tilde{T}_i(t) = T_i(t)P(t)^{-1}$ for $1 \leq i \leq n$. Thus, the new observability matrix is given as follows : $\tilde{T}(t) = T(t)P(t)^{-1}$. Since $T(t)$ and $P(t)$ are Lyapunov transformations, the new observability matrix $\tilde{T}(t)$ is also a Lyapunov transformation.

(b)

We use mathematical induction to prove the following equation :

$$\tilde{W}_i(t) = P(t)W_i(t) \;\; 1 \le i \le n \tag{3.74}$$

where $\tilde{W}_i(t)$ are the columns of the controllability matrices of the transformed system. For $i = 1$, we have $\tilde{W}_1(t) = \tilde{B}(t) = P(t)B(t)$. But since $W_1(t) = B(t)$, it follows that (3.74) holds for $i = 1$. Now assume that (3.74) holds for $i = m > 1$. Then we have :

$$\begin{aligned}\tilde{W}_{m+1}(t) &= \tilde{A}(t)\tilde{W}_m(t) - \dot{\tilde{W}}_m(t)\\ &= (P(t)A(t) + \dot{P}(t))P(t)^{-1}P(t)W_m(t) - \dot{P}(t)W_m(t) - P(t)\dot{W}_m(t)\\ &= P(t)A(t)W_m(t) - P(t)\dot{W}_m(t)\\ &= P(t)\dot{W}_{m+1}(t)\end{aligned}$$

By mathematical induction $\tilde{W}_i(t) = P(t)W_i(t)$ for $1 \le i \le n$. Thus, the new controllability matrix is given as follows : $\tilde{W}(t) = P(t)W(t)$. Since both $W(t)$ and $P(t)$ are Lyapunov transformation, the new controllability matrix $\tilde{W}(t)$ is also Lyapunov transformation.

□

Fact 19. *If we apply Lyapunov transformation $P(t)$ to the LTV system in the form (3.1)-(3.2), the transformed system has also relative degree r.*

Proof. We know from *Fact (18)* that $\tilde{T}_i(t) = T_i(t)P(t)^{-1}$ and also $\tilde{B}(t) = P(t)B(t)$. Then,

$$\tilde{T}_i(t)\tilde{B}(t) = T_i(t)P(t)^{-1}P(t)B(t) = T_i(t)B(t) = 0, \;\; 1 \le i \le r-1$$

and

$$\tilde{T}_r(t)\tilde{B}(t) = T_r(t)P(t)^{-1}P(t)B(t) = T_r(t)B(t) = b(t) \ne 0$$

Thus, transformed system has also relative degree r. □

In order to obtain the inverse dynamics, the system given by (3.1)-(3.2) should be firstly transformed to the similar form given in (3.51), but instead of constant coefficients in the system matrices, we will have time-varying coefficients. Let us apply first the inverse controllability matrix $W(t)^{-1}$ as the first transformation in order to get a certain form that will help us to transform the system into a form similar to the one in (3.51). In that case, the transformed system matrices become as shown below :

$$\begin{aligned} A_n(t) &= W(t)^{-1}(A(t)W(t) - \dot{W}(t)) \\ B_n(t) &= W(t)^{-1}B(t) \\ C_n(t) &= C(t)W(t) \end{aligned} \tag{3.75}$$

Now, let us try to find out the form of $A_n(t)$, $B_n(t)$ and $C_n(t)$. First, note that from (3.75) we obtain :

$$B(t) = W(t)B_n(t) = \begin{pmatrix} B(t) & W_2(t) & \dots & W_n(t) \end{pmatrix} B_n(t) \tag{3.76}$$

Since we assume that the system is controllable, $W(t)$ has a full rank, therefore $B(t)$ has the following form :

$$B_n(t) = \begin{pmatrix} 1 & 0 & \dots 0 \end{pmatrix}^T \tag{3.77}$$

The form of $C_n(t)$ can be obtained as follows :

$$C_n(t) = C(t)\begin{pmatrix} W_1(t) & W_2(t) & \dots & W_n(t) \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & b(t) & c_1(t) & \dots & c_{n-r}(t) \end{pmatrix} \tag{3.78}$$

where we used the relative degree property and *Lemma 16.* Finally, we can find the $A_n(t)$ as shown below :

$$A(t)W(t) - \dot{W}(t) = W(t)A_n(t) \tag{3.79}$$

Let us denote $A_n(t) = \begin{pmatrix} a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}$. Then :

$$A(t)W_i(t) - \dot{W}_i(t) = W_{i+1}(t) = W(t)a_{ni}(t) \;\; 1 \leq i \leq n \tag{3.80}$$

Since we assume that the system is controllable, $W(t)$ has a full rank. Therefore, from (3.80) we obtain :

$$a_{ni}(t) = \underline{e}_{i+1} \;\;,\;\; 1 \leq i \leq n-1 \tag{3.81}$$

$$a_{nn}(t) = (A(t)W_n(t) - \dot{W}_n(t))W(t)^{-1} = \begin{pmatrix} \beta_1(t) & \dots & \beta_n(t) \end{pmatrix} \tag{3.82}$$

where $\underline{e}_i$ is the i^{th} unit vector. If we combine (3.77),(3.78) and (3.82), we obtain the following form for the transformed system :

$$\dot{x}_n = \begin{pmatrix} 0 & 0 & \dots & \beta_1(t) \\ 1 & 0 & \dots & \beta_2(t) \\ 0 & 1 & \dots & \beta_3(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_2(t) \end{pmatrix} x_n + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & \dots & 0 & b(t) & c_1(t) & \dots & c_{n-r}(t) \end{pmatrix} x_n \tag{3.83}$$

In order to obtain a form similar to (3.51), we will apply another transformation to the system in (3.83). However, to guarantee that the transformation we apply is a Lyapunov transformation, we need to make the assumption given below.

Assumption 8. *The terms $\beta_i(t)$ and their derivatives are continuous and bounded.*

We want to transform system given by (3.83) into the form given below;

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_1(t) & \alpha_2(t) & \dots & \alpha_{n-1}(t) & \alpha_n(t) \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} m_1(t) & m_2(t) & \dots & m_{n-r}(t) & k(t) & 0 \dots & 0 \end{pmatrix} x \tag{3.84}$$

The required transformation matrix is the controllability matrix of the system (3.84), but we know neither the transformation matrix nor the transformed system yet. First we will try to identify the transformation matrix which is the controllability matrix of the system (3.84) by using the equations obtained from the transformation. Then, with this information we will form the system matrices in (3.84). Let us denote system matrices in (3.84) as $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$. The controllability matrix of this transformed system is given below :

$$\begin{aligned} \tilde{W}_1(t) &= \tilde{B}(t) \\ \tilde{W}_i(t) &= A_n(t)\tilde{W}_{i-1}(t) - \dot{\tilde{W}}_{i-1}(t) \text{ for } 1 \leq i \leq n \\ \tilde{W}(t) &= \begin{pmatrix} \tilde{W}_1(t) & \dots & \tilde{W}_n(t) \end{pmatrix} \end{aligned} \tag{3.85}$$

and the form of controllability matrix is lower triangular matrix as shown below :

$$\tilde{W}(t) = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & \alpha_n(t) \\ \vdots & \iddots & \iddots & \diamond \\ 1 & \alpha_n(t) & \diamond & \diamond \end{pmatrix} \tag{3.86}$$

If we apply the above controllability matrix to the system (3.83), we get the following system matrices :

$$\tilde{B}(t) = \begin{pmatrix} \tilde{B}(t) & \dots & \tilde{W}_n(t) \end{pmatrix} B_n(t)$$

Since $B_n(t)$ is in the form (3.77), the above equation is satisfied. This implies that if we can find the transformation matrix $\tilde{W}(t)$, the transformed system input vector $\tilde{B}(t)$ is in the form (3.84). Secondly, we will show output vector $\tilde{C}(t)$ is in the form (3.84) as shown below :

$$\begin{aligned} C_n(t) &= \begin{pmatrix} 0 & \dots & 0 & b(t) & c_1(t) & \dots & c_{n-r}(t) \end{pmatrix} \\ &= \tilde{C}(t)\tilde{W}(t) = \begin{pmatrix} m_1 & m_2 & \dots & m_{n-r}(t) & k(t) & 0\dots & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & 1 \\ 0 & \dots & \alpha_n(t) \\ \vdots & \iddots & \vdots \\ 1 & \alpha_n(t) & \dots \end{pmatrix} \end{aligned}$$

Since $\tilde{W}(t)$ is lower triangular matrix , above equation is also satisfied. Additionally, because of the diagonal elements of $\tilde{W}(t)$ is one, $b(t) = k(t) \neq 0$.

Finally, we will form $\tilde{A}(t)$ matrix. Actually, when the transformation equations are written for $\tilde{A}(t)$, we will obtain the time varying coefficients of $\tilde{A}(t)$ and the transformation matrix.

Remark 9. *The columns of the transformation matrix $\tilde{W}(t)$ have the structure given below :*

$$\tilde{W}_1(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \tilde{W}_2(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \alpha_n(t) \end{pmatrix}, \tilde{W}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_n(t) \\ r_{21}(t) \end{pmatrix}$$

$$\ldots \ \tilde{W}_i(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_n(t) \\ r_{i1}(t) \\ \vdots \\ r_{i(i-2)}(t) \end{pmatrix} \ \ldots \ \tilde{W}_n(t) = \begin{pmatrix} 1 \\ \alpha_n(t) \\ r_{n1}(t) \\ r_{n2}(t) \\ r_{n3}(t) \\ \vdots \\ \vdots \\ r_{n(n-2)}(t) \end{pmatrix} \tag{3.87}$$

where $r_{im}(t) = (\alpha_{n-m}(t) + \ldots)$ and the term which is shown by dots only contains $(\alpha_{n-m+1}(t), \ldots, \alpha_n(t))$, their derivatives and multiplications, and it does not contain $(\alpha_{n-m-1}(t), \alpha_{n-m-2}(t), \ldots)$ terms.

Next, we will show that the transformation between $A_n(t)$ and $\tilde{A}(t)$ is consistent. By consistency, we mean the following transformation equation holds for some $\alpha_i(t)$, where $\tilde{W}(t)$, $A_n(t)$ and $\tilde{A}(t)$ are given by (3.87), (3.83) and (3.84) respectively. Let us write the transformation equations between $A_n(t)$ and $\tilde{A}(t)$

as given below :

$$\begin{aligned}\tilde{W}(t)A_n(t) &= \tilde{A}(t)\tilde{W}(t) - \dot{\tilde{W}}(t) \\ &= \tilde{A}(t)\begin{pmatrix}\tilde{W}_1(t) & \ldots & \tilde{W}_n(t)\end{pmatrix} - \begin{pmatrix}\dot{\tilde{W}}_1(t) & \ldots & \dot{\tilde{W}}_n(t)\end{pmatrix}\end{aligned} \tag{3.88}$$

By using (3.88), we obtain:

$$\begin{aligned}\tilde{W}(t)a_{ni}(t) &= \tilde{W}(t)\underline{e}_{i+1} \\ &= \tilde{A}(t)\tilde{W}_i(t) - \dot{\tilde{W}}_i(t) = \tilde{A}(t)\tilde{W}_{i+1}(t) \ , \ 1 \le i \le n-1\end{aligned}$$

This implies that the transformation between $A_n(t)$ and $\tilde{A}(t)$ is consistent up to the first $n-1$ column of $A_n(t)$. If we do the procedure given above for the last column of $A_n(t)$, the time-varying coefficients of $\tilde{A}(t)$ can be found as indicated below:

$$\begin{pmatrix} 0 & \ldots & 1 \\ 0 & \ldots & \alpha_n(t) \\ \vdots & \iddots & \vdots \\ 1 & \alpha_n(t) & \ldots \end{pmatrix}\begin{pmatrix}\beta_1(t) \\ \beta_2(t) \\ \vdots \\ \beta_n(t)\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots 0 \\ 0 & 0 & 1 & 0 & \ldots 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_1(t) & \alpha_2(t) & \ldots & \alpha_{n-1}(t) & \alpha_n(t)\end{pmatrix}\begin{pmatrix} 1 \\ \alpha_n(t) \\ r_{n1}(t) \\ \vdots \\ r_{n(n-2)}\end{pmatrix} - \begin{pmatrix} 0 \\ \dot{\alpha}_n(t) \\ \dot{r}_{n1}(t) \\ \vdots \\ \dot{r}_{n(n-2)}\end{pmatrix} \tag{3.89}$$

From the property of $\tilde{W}(t)$ matrix which is mentioned in *Remark 9*, we can find the time varying coefficients $\{\alpha_n(t), \ldots, \alpha_1(t)\}$ by using the substitution of the coefficients found in each step. If we specify these time-varying coefficients, then we can easily form the transformation matrix $\tilde{W}(t)$ and from that transformation matrix, the output vector $\tilde{C}(t)$ and the state transition matrix $\tilde{A}(t)$ can be constructed. Therefore, we can always transform the system given by (3.83) into (3.84) with the transformation matrix given by (3.85)-(3.86). The procedure that we applied to transform the system given by (3.83) into (3.84) is applied to the following example for further clarification

Let us consider the system given below :

$$\dot{x} = \begin{pmatrix} 0 & 0 & \beta_1(t) \\ 1 & 0 & \beta_2(t) \\ 0 & 1 & \beta_3(t) \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & b(t) & c_2(t) \end{pmatrix} x \tag{3.90}$$

Note that (3.90) is in the form given by (3.83). We want to transform (3.90) into the form given below :

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} m_1(t) & b(t) & 0 \end{pmatrix} \tag{3.91}$$

Note that (3.91) is in the form given by (3.84). The transformation matrix that converts (3.90) into (3.91) is the controllability matrix of the system given by (3.91). By using (3.87), we find the columns of $\tilde{W}(t)$ as follows :

$$\tilde{W}_1(t) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$$

$$\tilde{W}_2(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \alpha_3(t) \end{pmatrix}^T$$

$$\tilde{W}_3(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \alpha_3(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \dot{\alpha}_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha_3(t) \\ \alpha_2(t) + \alpha_3(t)^2 - \dot{\alpha}_3(t) \end{pmatrix}^T$$

Hence we have :

$$\tilde{W}(t) = \begin{pmatrix} \tilde{W}_1(t) & \tilde{W}_2(t) & \tilde{W}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \alpha_3(t) \\ 1 & \alpha_3(t) & \alpha_2(t) + \alpha_3(t)^2 - \dot{\alpha}_3(t) \end{pmatrix} \tag{3.92}$$

If we perform the multiplication similar to (3.89) so as to find the coefficients of $\tilde{W}(t)$, the following matrix equation is obtained :

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \alpha_3(t) \\ 1 & \alpha_3(t) & \alpha_2(t)+\alpha_3(t)^2-\dot{\alpha}_3(t) \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_3(t) \\ \alpha_2(t)+\alpha_3(t)^2-\dot{\alpha}_3(t) \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_3(t) \\ \dot{\alpha}_2(t)+2\alpha_3(t)\dot{\alpha}_3(t)-\ddot{\alpha}_3(t) \end{pmatrix}$$

Then, from above matrix multiplications we obtain the following :

$$\begin{aligned} &\beta_3(t) = \alpha_3(t), \\ &\beta_2(t)+\alpha_3(t)\beta_3(t) = \alpha_2(t)+\alpha_3(t)^2-\dot{\alpha}_3(t)-\alpha_3(t), \\ &\beta_1(t)+\alpha_3(t)\beta_2(t)+(\alpha_2(t)+\alpha_3(t)^2-\dot{\alpha}_3(t))\beta_3(t) = \alpha_1(t)+\alpha_2(t)\alpha_3(t), \\ &\ +\alpha_3(t)(\alpha_2(t)+\alpha_3(t)^2-\dot{\alpha}_3(t))-(\dot{\alpha}_2(t)+2\alpha_3(t)\dot{\alpha}_3(t)-\ddot{\alpha}_3(t)). \end{aligned} \tag{3.93}$$

Thus, we can see that since the coefficients $\{\beta_1(t), \beta_2(t), \beta_1(t)\}$ are already known, the coefficients $\{\alpha_1(t), \alpha_2(t), \alpha_1(t)\}$ is calculated by using the equations in (3.93) recursively. Indeed, from the first equation in (3.93) we obtain $\alpha_3(t)$, by using $\alpha_3(t)$ in the second equation in (3.93), we obtain $\alpha_2(t)$ and finally by using $\alpha_3(t)$ and $\alpha_2(t)$ in the last equation of (3.93), we obtain $\alpha_1(t)$ recursively. Consequently, the transformation matrix $\tilde{W}(t)$ that converts the system given by (3.83) into the system in (3.84) can be calculated by using the procedure outlined above and shown in the preceding example for illustrative purposes. However, in order to preserve stability, the transformation matrix $\tilde{W}(t)$ should be a Lyapunov transformation.

Fact 20. *The transformation matrix $\tilde{W}(t)$ in (3.85) is a Lyapunov transformation.*

Proof. If we look at the form of $\tilde{W}(t)$ in (3.86), it is a lower triangular matrix with diagonals 1. In the lower part of diagonals, there are time-varying functions and these functions are composed of the coefficients $\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$, their derivatives and multiplications. From *Assumption 8*, coefficients $\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$, their derivatives and multiplications are bounded and continuous. This implies that $\tilde{W}(t)$ matrix is bounded and continuous. Additionally, if we write the inverse of $\tilde{W}(t)$, we obtain the form given below:

$$\tilde{W}(t)^{-1} = \frac{adj(\tilde{W}_n(t))}{det(\tilde{W}_n(t))} \tag{3.94}$$

Since $det(\tilde{W}_n(t)) = (-1)^n$ and the elements of the $adj(\tilde{W}_n(t))$ contain the coefficients $\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$, their derivatives and their multiplications which are bounded and continuous, it follows that $\tilde{W}(t)^{-1}$ is bounded and continuous. Thus, the transformation matrix $\tilde{W}(t)$ is a Lyapunov transformation. □

Therefore, the system given by (3.83) can be transformed the system given by (3.84) by employing the Lyapunov transformation $\tilde{W}(t)$ in (3.85). After obtaining the transformed system given by (3.84) which is similar with system given by (3.51) in LTI case, another transformation should be applied to this system in order to get inverse dynamics. The transformation, that will be applied, is time-varying counterpart of the transformation in (3.54) and is shown below :

$$\tilde{T}(t) = \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \underline{e}_{n-r} \\ \tilde{T}_1(t) \\ \vdots \\ \tilde{T}_r(t) \end{pmatrix} \tag{3.95}$$

where $\tilde{T}_1(t), \ldots, \tilde{T}_r(t)$ are first r rows of the observability matrix of the system in (3.84) and is formed as follows :

$$\begin{aligned} \tilde{T}_1(t) &= \tilde{C}(t) \\ \tilde{T}_i(t) &= \tilde{T}_{i-1}(t)\tilde{A}(t) + \dot{\tilde{T}}_{i-1}(t) \ , \ 1 < i \leq r \end{aligned} \tag{3.96}$$

We know that $\tilde{T}_1(t), \ldots, \tilde{T}_r(t)$ are bounded by the *Assumption 6* and the *Fact 18*. Additionally, by using the *Fact 19* we can say that the transformed system given by (3.84) has relative degree r as shown below :

$$\begin{aligned} \tilde{T}_i(t)\tilde{B}(t) &= 0 \ , \ 1 \leq i \leq r-1 \\ \tilde{T}_r(t)\tilde{B}(t) &= b(t) \neq 0 \ \ \forall t \geq t_0. \end{aligned} \tag{3.97}$$

If the transformation given by (3.95) is applied to the system given by (3.84), the following system matrices are obtained :

$$\begin{aligned} \hat{A}(t) &= (\tilde{T}(t)\tilde{A}(t) + \dot{\tilde{T}}(t))\tilde{T}(t)^{-1} \\ \hat{B}(t) &= \tilde{T}(t)\tilde{B}(t) \\ \hat{C}(t) &= \tilde{C}(t)\tilde{T}(t) \end{aligned} \tag{3.98}$$

First, we will find the input vector $\hat{B}(t)$ of the transformed system as follows :

$$\hat{B}(t) = \begin{pmatrix} \underline{e}_1 \\ \vdots \\ \underline{e}_{n-r} \\ \tilde{T}_1(t) \\ \vdots \\ \tilde{T}_r(t) \end{pmatrix} \tilde{B}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ b(t) \end{pmatrix} \tag{3.99}$$

where we used the relative degree property and the form of $\tilde{B}(t)$ is given by (3.84).

Secondly, we will construct the output vector $\hat{C}(t)$ as follows :

$$\tilde{C}(t) = \hat{C}(t) \begin{pmatrix} \underline{e}_1^T \\ \vdots \\ \underline{e}_{n-r}^T \\ \tilde{T}_1(t) \\ \vdots \\ \tilde{T}_r(t) \end{pmatrix}.$$

Since $\tilde{T}_1(t) = \tilde{C}(t)$, the following form will be obtained for $\hat{C}(t)$:

$$\hat{C}(t) = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} = \underline{e}_{n-r+1}^T. \tag{3.100}$$

Finally, we will form the new state transition matrix $\hat{A}(t)$. The states of the transformed system is as follows :

$$\hat{x} = \tilde{T}(t)\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{n-r} \\ \tilde{T}_1(t)\tilde{x} \\ \vdots \\ \tilde{T}_r(t)\tilde{x} \end{pmatrix} \tag{3.101}$$

where $\hat{x}$ are the states of the transformed system and $\tilde{x}$ are the states of the system given by (3.84). Since the first $n-r$ states do not change, the first $n-r-1$ rows of $\hat{A}(t)$ are same as the first $n-r-1$ rows of $\tilde{A}(t)$, which is shown below :

$$\hat{a}_i(t) = \underline{e}_{i+1} \;\;,\;\; 1 \leq i \leq n-r-1 \tag{3.102}$$

where $\{\hat{a}_1(t), \dots, \hat{a}_{n-r-1}(t)\}$ denotes first $n-r-1$ rows of $\hat{A}(t)$. Besides, from the system given by (3.84) we see that $\dot{\tilde{x}}_{n-r} = \tilde{x}_{n-r+1}$ and also the state $\tilde{x}_{n-r}$ does not change with this transformation as can be seen in (3.101). If the dynamics of the state $\tilde{x}_{n-r} = \hat{x}_{n-r}$ is obtained by using the transformed system state transition matrix, we obtain the following :

$$\dot{\hat{x}}_{n-r} = \hat{a}_{n-r}(t)\hat{x} = \hat{a}_{n-r}(t)\begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{n-r} \\ \tilde{T}_1(t)\tilde{x} \\ \vdots \\ \tilde{T}_r(t)\tilde{x} \end{pmatrix} = \dot{\tilde{x}}_{n-r} = \tilde{x}_{n-r+1} \tag{3.103}$$

Since $\tilde{T}_1(t) = \tilde{C}(t) = \begin{pmatrix} m_1(t) & \ldots & m_{n-r}(t) & b(t) & 0\ldots & 0 \end{pmatrix}$, we obtain :

$$\tilde{T}_1(t)\tilde{x} = m_1(t)\tilde{x}_1 + \ldots + m_{n-r}(t)\tilde{x}_{n-r} + b(t)\tilde{x}_{n-r+1} \tag{3.104}$$

If we combine (3.103) and (3.104), we obtain $\hat{a}_{n-r}(t)$ as follows :

$$\hat{a}_{n-r}(t) = \frac{1}{b(t)}\begin{pmatrix} -m_1(t) & \ldots & -m_{n-r}(t) & 1 & 0 & \ldots & 0 \end{pmatrix} \tag{3.105}$$

If we use (3.105) in (3.103), we obtain the following :

$$\dot{\hat{x}}_{n-r} = \dot{\tilde{x}}_{n-r} = \frac{1}{b(t)}(-m_1(t)\tilde{x}_1 - \ldots - m_{n-r}\tilde{x}_{n-r}(t) + \tilde{T}_1(t)\tilde{x}) = \tilde{x}_{n-r+1} \tag{3.106}$$

Thus, (3.106) proves that $\hat{a}_{n-r}(t)$ in (3.105) is the $(n-r)^{th}$ row of $\hat{A}(t)$. The remaining rows of the matrix $\hat{A}(t)$ can be found by using (3.98) as shown below :

$$\tilde{T}_i(t)\tilde{A}(t) + \dot{\tilde{T}}_i(t) = \tilde{T}_{i+1} = \hat{a}_{n-r+i}\tilde{T}(t) \ \ , \ 1 \le i \le r-1 \tag{3.107}$$

which implies:

$$\hat{a}_{n-r+i} = \underline{e}^T_{n-r+i+1} \tag{3.108}$$

and the last row of the matrix $\hat{A}(t)$ is as follows :

$$\hat{a}_n = (\tilde{T}_r(t)\tilde{A}(t) + \dot{\tilde{T}}_r(t))\tilde{T}(t)^{-1} \tag{3.109}$$

Therefore, if we put together (3.99), (3.100), (3.102), (3.105), (3.109) and if we divide the states of the transformed system into two parts $\tilde{x} = \begin{pmatrix} z \\ \varepsilon \end{pmatrix}$ where $z \epsilon \Re^{n-r}$ and $\varepsilon \epsilon \Re^{r}$, we obtain the following system :

$$\begin{aligned} \dot{z} &= G(t)z + P(t)\varepsilon_1 \\ \dot{\varepsilon}_1 &= \varepsilon_2 \\ &\vdots \\ \dot{\varepsilon}_{r-1} &= \varepsilon_r \\ \dot{\varepsilon}_r &= R(t)z + F(t)\varepsilon + b(t)u \\ y &= \varepsilon_1 \end{aligned} \tag{3.110}$$

where

$$G(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 1 \\ -\frac{m_1(t)}{b(t)} & -\frac{m_2(t)}{b(t)} & \dots & \dots & -\frac{m_{n-r}(t)}{b(t)} \end{pmatrix}, \tag{3.111}$$

$$P(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}. \tag{3.112}$$

and

$$\begin{pmatrix} R(t) & F(t) \end{pmatrix} = (\tilde{T}_n(t)\tilde{A}(t) + \dot{\tilde{T}}(t))T(t)^{-1} \tag{3.113}$$

Note that $G(t)$ in (3.110)-(3.111) is similar with G given by (3.57) except for the time-varying coefficients. Thus, the inverse system state transition matrix for LTV system, which is given by (3.110), is characterized by $G(t)$. Additionally, let us denote the input vector as $H(t)$ and the output vector as $K(t)$ for the inverse system which is shown below :

$$\dot{\eta} = G(t)\eta + H(t)v,$$
$$u = K(t)\eta. \tag{3.114}$$

Note that at this point $H(t)$ and $K(t)$ are not defined yet. In fact, there are many possible selections for $H(t)$ and $K(t)$, and one particular choice will be given in the sequel, see *Remark 10.* The observability matrix of the inverse system given by (3.114) can be obtained as follows :

$$M_1(t) = K(t)$$
$$M_i(t) = M_{i-1}(t)G(t) + \dot{M}_{i-1}(t) \text{ for } 2 \leq i \leq n-r$$
$$M(t) = \begin{pmatrix} M_1(t) \\ \vdots \\ M_{n-r}(t) \end{pmatrix} \tag{3.115}$$

In order to determine the inverse system completely, the vectors $H(t), K(t)$ should be chosen such that :

(1) The inverse system becomes all-pole (i.e. full relative degree)

(2) the observability matrix $M(t)$ should be a Lyapunov transformation.

Hence, from the full relative degree condition (1), the inverse system should satisfy the equations given below :

$$M_i(t)H(t) = 0 \;,\; 1 \leq i \leq n-r-1$$
$$M_i(t)H(t) = d(t) \neq 0 \;\; \forall t \tag{3.116}$$

If we simply choose :

$$K(t) = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}, \tag{3.117}$$

and

$$H(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{3.118}$$

then the inverse system satisfies equations in (3.116) and the inverse system observability matrix ,which is $M(t) = I_{(n-r)\times(n-r)}$, is a Lyapunov transformation. These facts can be shown easily. Hence, the inverse system satisfies conditions (1) and (2), which is given above, with $K(t)$ in (3.117) and $H(t)$ in (3.118).

Remark 10. *The $H(t)$ and $K(t)$ that satisfy conditions (1) and (2) on the inverse system are not unique. We can find other $(H(t), K(t))$ pairs. Thus the inverse system can be given as a class which consist of LTV systems with the triple $\{G(t), H(t), K(t)\}$, where $G(t)$ has the form given by (3.111), and where $H(t)$ and $K(t)$ satisfy the conditions (1) and (2) given above.*

As a result, we can choose the following system as an inverse system :

$$\dot{\eta} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 1 \\ -\frac{m_1(t)}{b(t)} & -\frac{m_2(t)}{b(t)} & \dots & \dots & -\frac{m_{n-r}(t)}{b(t)} \end{pmatrix} \eta + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v$$

$$u = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \eta \tag{3.119}$$

where $\eta \epsilon \Re^{n-r}$, $v \epsilon \Re$ and $u \epsilon \Re$ represent system state, input and output, respectively. However, to preserve the stability results from transformed system to the original system, the transformation matrix $\tilde{T}(t)$ in (3.95) should be a Lyapunov transformation. The following result resolves this question.

Fact 21. *The transformation matrix $\tilde{T}(t)$ in (3.95) is a Lyapunov transformation.*

Proof. Let us write the transformation matrix $\tilde{T}(t)$ in the form given below :

$$\tilde{T}(t) = \begin{pmatrix} I_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ K_1(t) & K_2(t) \end{pmatrix}, \tag{3.120}$$

where

$$\begin{pmatrix} \tilde{T}_1(t) \\ \vdots \\ \tilde{T}_r(t) \end{pmatrix} = \begin{pmatrix} K_1(t) & K_2(t) \end{pmatrix}, \tag{3.121}$$

and $K_1(t)\epsilon\Re^{r\times(n-r)}$, $K_2(t)\epsilon\Re^{r\times r}$ where r is the relative degree of the original system. Also, the elements of $K_1(t)$ and $K_2(t)$ are continuous and bounded as a result of the *Assumption 6* and the *Fact 18*. Thus, the transformation matrix $\tilde{T}(t)$ is also continuous and bounded. Additionally, $K_2(t)$ has the form which is shown below :

$$K_2(t) = \begin{pmatrix} b(t) & 0 & 0 & \dots & 0 \\ \diamond & b(t) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \diamond & \diamond & \dots & \diamond & b(t) \end{pmatrix}, \tag{3.122}$$

where the terms below the diagonal, indicated by diamond, are bounded and $b(t) \neq 0 \; \forall t$, (see (3.8)). Since the transformation matrix is in the block triangular form, its inverse is also in the block triangular form which is shown below :

$$\tilde{T}(t)^{-1} = \begin{pmatrix} I_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ L_1(t) & L_2(t) \end{pmatrix}, \tag{3.123}$$

where $L_1(t)\epsilon\Re^{r\times(n-r)}$, $L_2(t)\epsilon\Re^{r\times r}$. If the transformation matrix in (3.120) is multiplied with its inverse in (3.123) , we will obtain the following :

$$\begin{pmatrix} I_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ K_1(t) & K_2(t) \end{pmatrix} \begin{pmatrix} I_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ L_1(t) & L_2(t) \end{pmatrix} = \begin{pmatrix} I_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ 0_{r\times(n-r)} & I_{r\times r} \end{pmatrix}. \tag{3.124}$$

The above multiplication gives us set of equations which is given below :

$$K_1(t) + K_2(t)L_1(t) = 0, \quad (3.125)$$

$$K_2(t)L_2(t) = I_{r\times r} \quad (3.126)$$

In order to find $L_1(t)$ and $L_2(t)$ from (3.125) and (3.126), $K_2(t)$ should be invertible. The invertibility of $K_2(t)$ easily follows from the lower triangular form of $K_2(t)$ given by (3.122), where $b(t) \neq 0$. Obviously, $K_2(t)^{-1}$ can be calculated as follows :

$$K_2(t)^{-1} = \frac{adj(K_2(t))}{det(K_2(t))}, \quad (3.127)$$

We know that the elements of $adj(K_2(t))$ are the multiplication of theelements of $K_2(t)$ which are continuous and bounded. This implies that the elements of $adj(K_2(t))$ are also continuous and bounded. Since $det(K_2(t)) = b(t)^r \neq 0\ \forall t$, $K_2(t)$ has an inverse and its inverse matrix is continuous and bounded for all t. By using the inverse of $K_2(t)$, we can find $L_1(t)$ and $L_2(t)$ from (3.125)-(3.126) which is shown below :

$$\begin{aligned} L_1(t) &= -K_2(t)^{-1}K_1(t), \\ L_2(t) &= K_2(t)^{-1}. \end{aligned} \quad (3.128)$$

Since the elements of $K_1(t)$ are continuous and bounded, and $K_2(t)^{-1}$ is also a continuous and bounded matrix. Then both $L_1(t)$ and $L_2(t)$ become continuous and bounded matrices for all t as a result of (3.128). Thus, we prove that the transformation matrix $\tilde{T}(t)$ in (3.95) is continuous, invertible and bounded. Additionally, its inverse matrix is also continuous and bounded for all t. Consequently, our transformation matrix which is given by (3.95) is a Lyapunov transformation. □

Until now, the definition of the minimum phaseness for the LTV systems is not given, because we could not identify the inverse of the LTV systems yet. Since the inverse of the LTV system is identified by above calculations, we can make the definition of minimum phaseness in LTV systems.

Definition 22. *The system which is given by (3.110) is a minimum phase, if the inverse of this system is exponentially stable : i.e.*

$$\|\Phi_G(t, t_0)\| \le ke^{-\lambda t} \tag{3.129}$$

where $\|\Phi_G(t, t_0)\|$ *is transition matrix of* $G(t)$*.*

Remark 11. *The systems, which are Lyapunov equivalent to a minimum phase systems, are also called as a minimum phase systems, because there always exist Lyapunov transformations which can be used to convert minimum phase systems into other systems. Thus, the systems which are given by (3.83), (3.84) and original system in (3.1)-(3.2) are also called as a minimum phase, because we proved that the transformation matrices between these systems and minimum phase system in (3.110) are Lyapunov.*

Similar with LTI minimum phase case, we will employ the inverse system given by triple $(G(t), H(t), K(t))$ as a first part of the controller. Then, the combination of the inverse system given by (3.119) and the original system given by (3.110) which is called as an overall system is given as follows :

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K(t) \\ 0 & G(t) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ H(t) \end{pmatrix} v,$$

$$y = \begin{pmatrix} \hat{C}(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix}. \tag{3.130}$$

Fact 23. *where* $\hat{A}(t)$*,* $\hat{B}(t)$ *and* $\hat{C}(t)$ *are the system matrices of the system in (3.110). The overall system which is given by (3.130) has relative degree* n*.*

Proof. Let us compute observability matrix of the system in (3.110) as shown below :

$$\begin{aligned}
\hat{T}_1(t) &= \hat{C}(t),\\
\hat{T}_i(t) &= \dot{\hat{T}}_{i-1}(t) + \hat{T}_{i-1}(t)\hat{A}(t),\ \ 2 \le i \le n,\\
\hat{T}(t) &= \begin{pmatrix} \hat{T}_1(t) \\ \vdots \\ \hat{T}_n(t) \end{pmatrix}.
\end{aligned} \tag{3.131}$$

In addition, the system given by (3.110) has relative degree r and satisfies the following :

$$\begin{aligned}
\hat{T}_i(t)\hat{B}(t) &= 0,\ \ 1 \le i \le r-1,\\
\hat{T}_r(t)\hat{B}(t) &= b(t) \ne 0\ \ \forall t \ge t_0
\end{aligned} \tag{3.132}$$

Then let us compute the first n rows of the observability matrix of the overall system given by (3.130) as shown below :

$$
\begin{aligned}
T_{o1}(t) &= \begin{pmatrix} \hat{T}_1(t) & 0 \end{pmatrix}, \\
T_{o2}(t) &= \begin{pmatrix} \dot{\hat{T}}_1(t) & 0 \end{pmatrix} + \begin{pmatrix} \hat{T}_1(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K \\ 0 & G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \dot{\hat{T}}_1(t) + \hat{T}_1(t)\hat{A}(t) & \hat{T}_1(t)\hat{B}(t)K(t) \end{pmatrix}, \\
&= \begin{pmatrix} \hat{T}_2(t) & 0 \end{pmatrix}, \\
T_{o3}(t) &= \begin{pmatrix} \dot{\hat{T}}_2(t) & 0 \end{pmatrix} + \begin{pmatrix} \hat{T}_2(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K \\ 0 & G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \dot{\hat{T}}_2(t) + \hat{T}_2(t)\hat{A}(t) & \hat{T}_2(t)\hat{B}(t)K(t) \end{pmatrix}, \\
&= \begin{pmatrix} \hat{T}_3(t) & 0 \end{pmatrix}, \\
&\vdots \\
T_{or}(t) &= \begin{pmatrix} \dot{\hat{T}}_{r-1}(t) & 0 \end{pmatrix} + \begin{pmatrix} \hat{T}_{r-1}(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K \\ 0 & G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \dot{\hat{T}}_{r-1}(t) + \hat{T}_{r-1}(t)\hat{A}(t) & \hat{T}_{r-1}(t)\hat{B}(t)K(t) \end{pmatrix}, \\
&= \begin{pmatrix} \hat{T}_r(t) & 0 \end{pmatrix}, \\
T_{o(r+1)}(t) &= \begin{pmatrix} \dot{\hat{T}}_r(t) & 0 \end{pmatrix} + \begin{pmatrix} \hat{T}_r(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K \\ 0 & G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \dot{\hat{T}}_r(t) + \hat{T}_r(t)\hat{A}(t) & \hat{T}_r(t)\hat{B}(t)K(t) \end{pmatrix}, \\
&= \begin{pmatrix} \hat{T}_{r+1}(t) & b(t)M_1(t) \end{pmatrix}, \\
T_{o(r+2)}(t) &= \begin{pmatrix} \dot{\hat{T}}_{r+1}(t) & \dot{b}(t)M_1(t), \\ & +b(t)\dot{M}_1(t) \end{pmatrix} \\
&+ \begin{pmatrix} \hat{T}_{r+1}(t) b(t)M_1(t) \end{pmatrix} \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K(t) \\ 0 & G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \dot{\hat{T}}_{r+1}(t) + \hat{T}_{r+1}(t)\hat{A}(t) & (\dot{b}(t) + \hat{T}_{r+1}(t)\hat{B}(t))M_1(t) + b(t)\dot{M}_1(t) + b(t)M_1(t)G(t) \end{pmatrix}, \\
&= \begin{pmatrix} \hat{T}_{r+2}(t) & \diamond M_1(t) + b(t)M_2(t) \end{pmatrix}, \\
&\vdots
\end{aligned}
$$

$$
T_{on}(t) = \begin{pmatrix} \hat{T}_n(t) & \diamond M_1(t) + \ldots + \diamond M_{n-r-1}(t) + b(t)M_{n-r}(t) \end{pmatrix}, \tag{3.133}
$$

where we used (3.132) and where $\diamond$ refers to arbitrary functions which result in from the development given above and are not important for the development given in the sequel . By using (3.116), we will obtain the following :

$$T_{oi}(t)\begin{pmatrix}0\\ H(t)\end{pmatrix} = 0,\ \ 1 \leq i \leq n-1,$$

$$T_{on}(t)\begin{pmatrix}0\\ H(t)\end{pmatrix} = d(t)b(t) \neq 0\ \ \forall t \geq t_0. \tag{3.134}$$

Thus, we proved that the overall system given by (3.130) has relative degree n. □

Since we applied transformations to the system given by (3.1)-(3.2) in order to obtain an inverse system, the matrix $P(t)$ associated with disturbance ν in the equation (3.5) is also affected by these transformations. Thus, let us denote transformed disturbance matrix as $\hat{P}(t)$ and let us add the exogenous system given by (3.3)-(3.5) to the overall system given by (3.130). Then the state space representation of the overall system and the exogenous system takes the form given below :

$$\begin{pmatrix}\dot{\hat{x}}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}\hat{A}(t) & \hat{B}(t)K(t)\\ 0 & G(t)\end{pmatrix}\begin{pmatrix}\hat{x}\\ \eta\end{pmatrix} + \begin{pmatrix}0\\ H(t)\end{pmatrix}v + \begin{pmatrix}\hat{P}(t)\\ 0\end{pmatrix}w,$$

$$e = \begin{pmatrix}\hat{C}(t) & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \eta\end{pmatrix} + Qw, \tag{3.135}$$

$$\dot{w} = S(t)w. \tag{3.136}$$

Let us denote $x_o = \begin{pmatrix}\hat{x}\\ \eta\end{pmatrix}$, $A_o(t) = \begin{pmatrix}\hat{A}(t) & \hat{B}(t)K(t)\\ 0 & G(t)\end{pmatrix}$, $B_o(t) = \begin{pmatrix}0\\ H(t)\end{pmatrix}$ and $P_o(t) = \begin{pmatrix}\hat{P}(t)\\ 0\end{pmatrix}$. Since the overall system given by (3.130) has relative degree n, if we take derivative of the error in (3.135), the input v appears at the n^{th}

derivative of the error which is shown below :

$$
\begin{aligned}
e &= T_{o1}(t)x_o + Q(t)w, \\
\dot{e} &= \dot{T}_{o1}(t)x_o + T_{o1}(A_o(t)x_o + B_o(t)v + P_o(t)w) + Q(t)\dot{w} + \dot{Q}(t)w, \\
&= (\dot{T}_{o1}(t) + T_{o1}(t)A_o(t))x_o + \underbrace{T_{o1}(t)B_o(t)}\, v + (T_{o1}(t)P_o(t) + Q(t)S(t) + \dot{Q}(t))w, \\
&= T_{o2}(t)x_o + S_{o1}(t)w, \\
\ddot{e} &= \dot{T}_{o2}(t)x_o + T_{o2}(t)(A_o(t)x_o + B_o(t)v + P_o(t)w) + S_{o1}(t)\dot{w} + \dot{S}_{o1}(t)w, \\
&= (\dot{T}_{o2}(t) + T_{o2}(t)A_o(t))x_o + \underbrace{T_{o2}(t)B_o(t)}\, v + (T_{o2}(t)P_o(t) + S_{o1}(t)S(t) + \dot{S}_{o1}(t))w, \\
&= T_{o3}(t)x_o + S_{o2}(t)w, \\
&\vdots \\
e^{(n)} &= T_{on}(t)x_o + b(t)d(t)v + (T_{o(n-1)}(t)P_o(t) + S_{o(n-1)}S + \dot{S}_{o(n-1)})w, \\
&= T_{on}(t)x_o + b(t)d(t)v + S_{on}(t)w,
\end{aligned} \tag{3.137}
$$

where $S_{oi}(t) = S_{o(i-1)}(t)S + \dot{S}_{o(i-1)}(t) + T_{oi}(t)P_o(t)$, $S_{o0}(t) = Q(t)4$, $1 \leq i \leq n$ and the parts, indicated by underbrace are equal to zero. In order to find the second part of the controller, we will use the same methodology that we applied for the all-pole LTV systems. Therefore, we will choose the control input v as follows :

$$
v = \frac{1}{b(t)d(t)}\{-T_{on}(t)x_o - S_{on}(t)w - L_{o(n-1)}e^{(n-1)} - \ldots - L_{o1}\dot{e} - L_{o0}e\}. \tag{3.138}
$$

If (3.138) is substituted into (3.137), we obtain the error dynamics of $e(t)$ as given below :

$$
e^{(n)} + L_{o(n-1)}e^{(n-1)} + \ldots + L_{o1}\dot{e} + L_{o0}e = 0. \tag{3.139}
$$

The latter is the same with (3.39) that has obtained for all-pole LTV systems. If we again use Laplace transformation, the characteristic polynomial of the equation (3.139) will be as follows :

$$
ch(s) = s^n + L_{o(n-1)}s^{n-1} + \ldots + L_{o1}s + L_{o0} = 0. \tag{3.140}
$$

If the controller parameters $\{L_{o(n-1)}, \ldots, L_{o1}, L_{o0}\}$ are chosen properly, we can make the error $e(t)$ exponentially stable as we did for all-pole LTV systems. Actually, the second part of the controller given by (3.138) has the form which is shown below :

$$v = K_\eta(t)\eta + K_{\hat{x}}(t)\hat{x} + K_w(t)w. \tag{3.141}$$

If the first and the second part of the controller given by (3.119), (3.141)-(3.138) are combined, then the overall controller becomes as follows :

$$\begin{aligned} \dot{\eta} &= (G(t) + H(t)K_\eta(t))\eta + H(t)K_{\hat{x}}(t)\hat{x} + H(t)K_w(t)w, \\ u &= K(t)\eta. \end{aligned} \tag{3.142}$$

Therefore, the closed-loop system state space model with controller in (3.142) becomes as shown below :

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} \hat{A}(t) & \hat{B}(t)K(t) \\ H(t)K_{\hat{x}}(t) & G(t) + H(t)K_\eta(t) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} + \begin{pmatrix} \hat{P}(t) \\ H(t)K_w(t) \end{pmatrix} w, \\ e &= \begin{pmatrix} \hat{C}(t) & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} + Qw. \end{aligned} \tag{3.143}$$

In order to complete the solution of the output regulation problem for minimum phase LTV systems, the system given by (3.143) should satisfy the second regulation condition (ii). Thus, this implies that the system in (3.143) should be exponentially stable with $w = 0$. This is indicated by the following lemma.

Lemma 24. *The system given by (3.143) is exponentially stable with $w = 0$. i.e.*

$$\| \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} \| \leq a e^{-\vartheta t} \tag{3.144}$$

Proof. If we take $w = 0$, then the error $e(t)$ becomes equal to the system output $y(t)$. By applying the second part of the controller in (3.138) to the overall system in (3.135), the dynamics of the error $e(t)$ becomes the equation given by

(3.139). This indicates that the dynamics of the output y takes the form which is given below with $w = 0$:

$$y^{(n)} + L_{o(n-1)}y^{(n-1)} + \ldots + L_{o1}\dot{y} + L_{o0}y = 0. \tag{3.145}$$

Since we chose the controller parameters $\{L_{o(n-1)}, \ldots, L_{o1}, L_{o0}\}$ properly, the system output $y(t)$ is exponentially stable. If the $y(t)$ is exponentially stable, then its derivatives are also exponentially stable. This fact will be used to prove the exponential stability of the closed-loop system states. Let us denote $K_{x_o}(t) = \begin{pmatrix} K_\eta(t) & K_{\hat{x}}(t) \end{pmatrix}$, then :

$$\begin{aligned}
y &= T_{o1}(t)x_o, \\
\dot{y} &= \dot{T}_{o1}(t)x_o + T_{o1}(A_o(t) + B_o(t)K_{x_o}(t))x_o \\
&= T_{o2}(t)x_o, \\
\ddot{y} &= \dot{T}_{o2}(t)x_o + T_{o2}(A_o(t) + B_o(t)K_{x_o}(t))x_o \\
&= T_{o2}(t)x_o, \\
&\vdots \\
y^{(n-1)} &= \dot{T}_{o(n-1)}(t)x_o + T_{o(n-1)}(A_o(t) + B_o(t)K_{x_o}(t))x_o \\
&= T_{on}(t)x_o,
\end{aligned} \tag{3.146}$$

where we used (3.134). Thus we obtain the following from (3.146) :

$$\| \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \| = \| \begin{pmatrix} T_{o1} \\ \vdots \\ T_{on} \end{pmatrix} x_o \| \leq ke^{-\lambda t} \tag{3.147}$$

Actually, the matrix in (3.147) is the first n rows of the observability matrix of the closed-loop system given by (3.130). This matrix was already computed in the proof of the *Fact 23* and is given below in a more detailed form :

$$
\begin{aligned}
T_{o1}(t) &= \begin{pmatrix} 0 & \dots & 0 & \underline{e}_1 & 0 & \dots & 0 \end{pmatrix}, \\
T_{o2}(t) &= \begin{pmatrix} 0 & \dots & 0 & \underline{e}_2 & 0 & \dots & 0 \end{pmatrix}, \\
&\vdots \\
T_{or}(t) &= \begin{pmatrix} 0 & \dots & 0 & \underline{e}_r & 0 & \dots & 0 \end{pmatrix}, \\
T_{o(r+1)} &= \begin{pmatrix} Q_1(t) & P_1(t) & b(t)M_1(t) \end{pmatrix}, \\
T_{o(r+2)} &= \begin{pmatrix} Q_2(t) & P_2(t) & \diamond M_1(t) + b(t)M_2(t) \end{pmatrix}, \\
&\vdots \\
T_{on} &= \begin{pmatrix} Q_{n-r}(t) & P_{n-r}(t) & \diamond M_1(t) + \ldots + \diamond M_{n-r-1}(t) + b(t)M_{n-r}(t) \end{pmatrix},
\end{aligned}
\tag{3.148}
$$

where $\underline{e}_i \epsilon \Re^r$ is unit vector, $Q_i(t)\epsilon\Re^{n-r}$ and $P_i(t)\epsilon\Re^r$. If we take the first r row vectors in (3.148) and use (3.147), we will obtain the form given below :

$$
\| \begin{pmatrix} 0_{r\times(n-r)} & I_{r\times r} & 0_{r\times(n-r)} \end{pmatrix} \begin{pmatrix} z \\ \varepsilon \\ \eta \end{pmatrix} \| = \|\varepsilon\| \le m e^{-\gamma t},
\tag{3.149}
$$

This implies that the states ε are exponentially stable.

The dynamics of the states z in the closed-loop system is given below (see (3.110) and (3.143)) :

$$
\dot{z} = G(t)z + P(t)\varepsilon_1. \tag{3.150}
$$

We know that the state transition matrix of the $G(t)$ is exponentially stable because of the minimum phase property and $P(t)$ matrix is bounded and the states ε are also exponentially stable. This implies that the states z are also exponentially stable which can be shown easily. In other words, we have :

$$
\|z\| \le r e^{-\varrho t},
\tag{3.151}
$$

Finally, if we take the last $n-r$ row vectors in (3.148), we will obtain :

$$\left\| \begin{pmatrix} Q_1(t) & P_1(t) & b(t)M_1(t) \\ \vdots & \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) & \diamond M_1(t) + \ldots + \diamond M_{n-r-1}(t) + b(t)M_{n-r}(t) \end{pmatrix} x_o \right\| = \left\| \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \right\| \leq ce^{-\rho t} \tag{3.152}$$

We know that the rows $\begin{pmatrix} Q_i(t) & P_i(t) \end{pmatrix}$ are bounded and the states z, ε are exponentially stable, then from (3.152) we get the below equation :

$$\begin{pmatrix} b(t) & 0 & \ldots & 0 \\ \diamond & b(t) & \ldots & 0 \\ \diamond & \diamond & \ddots & \vdots \\ \diamond & \diamond & \diamond & b(t) \end{pmatrix} \begin{pmatrix} M_1(t) \\ \vdots \\ M_{n-r-1}(t) \\ M_{n-r}(t) \end{pmatrix} \eta = \Gamma(t)M(t)\eta$$

$$= - \begin{pmatrix} Q_1(t) & P_1(t) \\ \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) \end{pmatrix} \begin{pmatrix} z \\ \varepsilon \end{pmatrix} + \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \tag{3.153}$$

We know that the lower triangular elements of the matrix $\Gamma(t)$ are multiplications of the bounded functions and also $b(t) \neq 0 \ \forall t \geq t_0$. Thus, $\Gamma(t)$ is a bounded matrix. Then, if we write the inverse of $\Gamma(t)$, we get the following :

$$\Gamma(t)^{-1} = \frac{adj(\Gamma(t))}{det(\Gamma(t))}. \tag{3.154}$$

Since $det(\Gamma(t)) = b(t)^{n-r} \neq 0 \ \forall t \geq t_0$, $\Gamma(t)^{-1}$ exists and is bounded. Additionally, the matrix $M(t)$ is the observability matrix of the inverse system in (3.115) which is a Lyapunov transformation. Thus, both $\Gamma(t)$ and $M(t)$ are bounded and invertible matrices, and their inverses are also bounded. By using (3.153), we can conclude as follows :

$$
\begin{aligned}
\|\Gamma(t)M(t)\eta\| &= \| -\begin{pmatrix} Q_1(t) & P_1(t) \\ \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) \end{pmatrix} \begin{pmatrix} z \\ \varepsilon \end{pmatrix} + \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \| \\
\|\eta\| &= \|(\Gamma(t)M(t))^{-1}(-\begin{pmatrix} Q_1(t) & P_1(t) \\ \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) \end{pmatrix} \begin{pmatrix} z \\ \varepsilon \end{pmatrix} + \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix})\| \\
&\le \|(\Gamma(t)M(t))^{-1}\|(\| \begin{pmatrix} Q_1(t) & P_1(t) \\ \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) \end{pmatrix} \|\| \begin{pmatrix} z \\ \varepsilon \end{pmatrix} \| + \| \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \|) \\
&\le d(sme^{-\delta t} + ce^{-\rho t}) = ce^{-\tau t}
\end{aligned} \tag{3.155}
$$

where

$$
\begin{aligned}
\|(\Gamma(t)M(t))^{-1}\| &\le d \\
\| \begin{pmatrix} Q_1(t) & P_1(t) \\ \vdots & \vdots \\ Q_{n-r}(t) & P_{n-r}(t) \end{pmatrix} \| &\le s \\
\| \begin{pmatrix} z \\ \varepsilon \end{pmatrix} \| &\le me^{-\delta t} \\
\| \begin{pmatrix} y^{(r)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \| &\le ke^{-\rho t}.
\end{aligned} \tag{3.156}
$$

Therefore, the results in (3.149), (3.151) and (3.155) proves that the closed-loop system given by (3.143) is exponentially stable with $w = 0$. □

Theorem 25. *The dynamic controller given by (3.142) satisfies the regulation conditions (i), (ii) for the minimum phase system given by (3.110) with the exogenous system in (3.3)-(3.5).*

Proof. (i) Equations (3.139)-(3.140) indicates that the error term $e(t)$ is exponentially stable. i.e.

$$|e(t)| < k \exp^{-\lambda t}$$

for some $k > 0$, $\lambda > 0$. Hence, we have :

$$\lim_{t \to \infty} |e(t)| = 0. \tag{3.157}$$

(ii) *Lemma 24* proves that the closed-loop system given by (3.143) with $w = 0$ is exponentially stable. i.e.

$$\| \begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} \| \leq h e^{-\zeta t}, \tag{3.158}$$

These two results prove that the dynamic controller in the form (3.142) satisfies the regulation conditions for the minimum phase system given by (3.110) with the exogenous system in (3.3)-(3.5). □

In *Theorem 25*, we showed that the controller in (3.142) satisfies the regulation conditions (i) and (ii) for the transformed system in (3.110). We know that there is a Lyapunov transformation between the transformed system in (3.110) and the original system in (3.1)-(3.2) as shown below :

$$\hat{x} = T_s(t) x, \tag{3.159}$$

where $T_s(t) = \tilde{T}(t)\tilde{W}(t)W(t)^{-1}$ (see (3.58), (3.85) and (3.95)). And also we have :

$$\begin{pmatrix} \hat{x} \\ \eta \end{pmatrix} = \begin{pmatrix} T_s(t) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \tag{3.160}$$

Since $T_s(t)$ and I are Lyapunov transformations and Lyapunov transformation preserve the stability between the original system and the transformed system, the states of the original system in (3.1)-(3.2) are also exponentially stable. Additionally, the error $e(t)$ dynamics is not affected by the transformations, hence it remains exponentially stable for the original system in (3.1)-(3.2). Therefore, the minimum phase system in (3.1)-(3.5) with transformed system matrices in (3.110) satisfies regulation conditions (i), (ii) with the following dynamic controller :

$$\dot{\eta} = (G(t) + H(t)K_\eta(t))\eta + H(t)K_x(t)x + H(t)K_w(t)w,$$
$$u = K(t)\eta. \tag{3.161}$$

where $K_x(t) = K_{\hat{x}}(t)T_s^{-1}(t)$.

3.4 Pole/Zero Cancelation in LTV Systems

In the minimum phase LTI case, the original system zeros which are the roots of the polynomial given by (2.65) are canceled by the poles of the inverse system given by (2.68)-(2.69). It is the way we obtained a system which is equivalent to an all-pole system and we designed the second part of the controller accordingly. Additionally, these cancelations result in m unobservable states and m denotes the number of pole/zero cancelations.

In this section, we will try to show pole/zero cancelations in the design of the controller for the minimum phase LTV system given in the *Section 3.3*. That's how we can obtain an analogy between the minimum phase LTI and LTV cases and justify the reasoning behind defining the system given by (3.119) as an inverse system.

In order to show pole/zero cancelations in LTV systems, we will use the LTV system pole/zero definition of O'Brien in [36]. Before making definitions we should show some facts which are used later in this section.

Fact 26. *(G(t),R(t)) pair (see (3.110)) is observable.*

Proof. We know that the observability matrix of the system in (3.110) is a Lyapunov transformation. If we form the observability matrix of this system, we will get the rows of the observability matrix as shown below :

$$\begin{aligned}
T_1(t) &= \begin{pmatrix} \underline{0} & \vdots & \underline{e}_1 \end{pmatrix} \\
&\vdots \\
T_r(t) &= \begin{pmatrix} \underline{0} & \vdots & \underline{e}_r \end{pmatrix} \\
T_{r+1}(t) &= \begin{pmatrix} L_1(t) & \vdots & \diamond \end{pmatrix} \\
&\vdots \\
T_n(t) &= \begin{pmatrix} L_{n-r}(t) + \diamond L_{n-r-1}(t) + \ldots + \diamond L_1(t) & \vdots & \diamond \end{pmatrix} \qquad (3.162)
\end{aligned}$$

where $L_i(t)$ are the rows of the observability matrix which is formed by the pair $(G(t), R(t))$ as given below :

$$\begin{aligned}
L_1(t) &= R(t) \\
L_i(t) &= L_{i-1}(t)G(t) + \dot{L}_{i-1}(t), \;\; 2 \le i \le n-r \\
L(t) &= \begin{pmatrix} L_1(t) & \ldots & L_{n-r}(t) \end{pmatrix} \qquad (3.163)
\end{aligned}$$

As we can see, the observability matrix in (3.162) is in the block triangular form and this matrix is invertible. Then, from block triangular property of the matrix in (3.162), we see that the matrix ,which is the left bottom block of the observability matrix in (3.162) is also invertible and it is given below :

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ \diamond & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \diamond & \diamond & \ldots & 1 \end{pmatrix} L(t) = O(t) \qquad (3.164)$$

Since $O(t)$ is invertible, the matrix $L(t)$ is also invertible which implies that the pair $(G(t), R(t))$ is observable. □

Fact 27. *The pair* $(G(t), \frac{-R(t)}{b(t)})$ *is observable.*

Proof. If we form the observability matrix of the pair $(G(t), \frac{-R(t)}{b(t)})$, we will obtain

$$\bar{L}(t) = \begin{pmatrix} \frac{1}{b(t)} & 0 & \dots & 0 \\ \diamond & \frac{1}{b(t)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \diamond & \diamond & \dots & \frac{1}{b(t)} \end{pmatrix} L(t) \tag{3.165}$$

Since $b(t)$ is bounded above and below for all t and $L(t)$ is invertible, the observability matrix $\bar{L}(t)$ is also invertible which implies that the pair $(G(t), \frac{-R(t)}{b(t)})$ is observable. □

In [36], O'Brien gives two types of zero definitions which are called "Transmission Zero" and "Ordinary Zero". The definitions of these zeros are given as follows.

Definition 28. *Suppose the system that we have has a minimal realization. A function* $q(t)$ *is a transmission zero for this system if there exists an initial state and a function* $r(t)$ *,which is bounded above and do not converge to zero as* $t \longrightarrow \infty$*, such that the output of the system is zero for all* $t \geq t_0$ *when the input is* $r(t)\phi_q(t)$ *where* $\phi_q(t)$ *is the transition function of the scalar equation given below*

$$\dot{x} = q(t)x.$$

Definition 29. *Suppose the system that we have has a minimal realization. A function* $q(t)$ *is an ordinary zero for this system if there exists an initial state such that the output of the system is zero for all* $t \geq t_0$ *when the input is the transition function* $\phi_q(t)$ *given in* Definition 28.

The definition of a pole for LTV systems is given below.

Definition 30. *The functions* $\{p_1(t), \ldots, p_n(t)\}$ *are poles of the LTV system if there exists an invertible matrix* S_0 *such that*

$$S(t) = \Phi_A(t,0) S_0 \Phi_P(t,0)^{-1}, \tag{3.166}$$

is a Lyapunov transformation and $\Phi_A(t,0)$ *is the transition matrix of* $A(t)$ *which is system matrix,* $\Phi_P(t,0)$ *is the transition matrix of the* $P(t)$ *which is given below*

$$P(t) = diag\{p_i(t)\} \tag{3.167}$$

Actually, $S(t)$ diagonalizes the system matrix $A(t)$ and preserves the stability property of the system. Then, the diagonal entries are called as the poles of the system.

First we will show that the poles of the inverse system in (3.119) correspond to the transmission zeros of the system in (3.110) and vice versa.

Fact 31. *The poles of the system given by (3.119) and the transmission zeros of the system given by (3.110) are cancel out eachother.*

Proof. If the output of the original system in (3.110) is set to zero, then we obtain :

$$\begin{aligned} y &= \varepsilon_1 = 0 \\ \dot{\varepsilon}_1 &= 0 = \varepsilon_2 \\ &\vdots \\ \varepsilon_r &= 0 \end{aligned} \tag{3.168}$$

Then, by using (3.168) in (3.110), we obtain :

$$\begin{aligned} \dot{z} &= G(t)z \\ u &= \frac{-1}{b(t)} R(t) z \end{aligned} \tag{3.169}$$

Therefore, the output zeroing input $u(t) = r(t)\phi_q(t,0)$ should satisfy (3.169), see *Definition 28*. If we consider (3.169) to be a system with states z and output $u(t)$, then the zero input response of this system is equal to $u(t) = r(t)\phi_q(t,0)$. Since $(G(t), \frac{-R(t)}{b(t)})$ has a minimal realization as we proved, we can use *Lemma 21* in [36]. This lemma claims that a function $p(t)$ is a pole of a system if and only if the zero input response of this system can be written as $r(t)\phi_p(t,0)$ where $r(t)$ is bounded above and does not converge to zero as $t \longrightarrow \infty$. It implies that, if the system in (3.169) has a zero input response $r(t)\phi_\gamma(t,0)$, then this $\gamma(t)$ function is a pole for the system in (3.169) and additionally a transmission zero for the system in (3.110) (see Definition 28). Thus, the poles of the system in (3.169) and the transmission zeros of the system in (3.110) correspond to each other, if they exist. Additionally, we know that the poles of the system in (3.169) actually are the poles of the system matrix $G(t)$. The inverse system in (3.119) also contains $G(t)$ as a system matrix and the inverse system has a minimal realization. It shows that the poles of the system in (3.169) are equivalent to the poles of the inverse system in (3.119). This proves that the poles of the inverse system in (3.119) and the transmission zeros of the system in (3.110) cancel out each other. □

In the following fact we will use equivalence between functions of time, hence we first give the definition of the equivalence below.

Definition 32. *The functions associated with f_1, f_2 ,which are continuous and bounded, are equivalent if there exists a scalar Lyapunov transformation s_{eq} such that*

$$s_{eq}(t)\phi_{f_1}(t,0) = \phi_{f_2}(t,0) \ , \ \forall t \tag{3.170}$$

where $\phi_{f_1}(t,0)$, $\phi_{f_2}(t,0)$ are transition functions [36].

Fact 33. *If $G(t)$ has a pole set $\{p_1(t), \ldots, p_n(t)\}$ in which no two poles are equivalent and additionally if $G(t)^{(k)}$ and $(\frac{-R(t)}{b(t)})^{(k)}$ are continuous, bounded and* *do not decay to zero for $k = 1, \ldots, n-r-1$, then the poles of the inverse system in (3.119) and the ordinary zeros of the system in (3.110) cancel out each other.*

Proof. We know that;

- The pair $(G(t), \frac{-R(t)}{b(t)})$ is observable.
- $G(t), \frac{-R(t)}{b(t)}$ are the analytic functions of t.
- $G(t)^{(k)}$ and $(\frac{-R(t)}{b(t)})^{(k)}$ are continuous, bounded and do not decay to zero for $k = 1, \ldots, n-r-1$.
- $G(t)$ has a pole set $\{p_1(t), \ldots, p_n(t)\}$ in which no two poles are equivalent.

Then we can use *Lemma 22* in [36] which states that $\psi(t)$ is a pole for the system in (3.169) if and only if the zero input response of this system can be written as $u(t) = \phi_\psi(t, 0)$. It implies that if the zero input response of the system in (3.169) can be written as $\phi_\psi(t, 0)$, then $\psi(t)$ is a pole for the system in (3.169) and an ordinary zero for the system in (3.110)(see Definition 29). Thus, poles of the system in (3.169) correspond to the ordinary zeros of the system in (3.110). Additionally, the poles of the system in (3.169) and the poles of the inverse system in (3.119) are equivalent. This proves that the poles of the inverse system in (3.119) and the ordinary zeros of the system in (3.110) cancel out each other. □

In *Fact 31* and *Fact 33*, we showed that there occur transmission zero/pole or/and ordinary zero/pole cancelations between the system in (3.110) and the inverse system in (3.119) similar to the LTI case. If some certain conditions, which are given in the below fact, are satisfied, then the ordinary zero/pole cancelations make the overall system unobservable as a result of the *Theorem 49* in [36] similar to what happens in LTI cases.

Fact 34. *If the inverse system in (3.119) and the system in (3.110) satisfy the hypothesis of* Lemma 48 *in [36] ,and additionally if $H(t)^k$ is continuous, bounded and does not decay to zero for $k = 1, \ldots, n - r - 1$, then ordinary zero/pole cancelations between the inverse system in (3.119) and the system in (3.110) make the overall augmented system unobservable.*

Proof. The proof of above fact follows directly from *Theorem 49* in [36]. □

Therefore, the ordinary zero/pole cancelations in LTV systems and the ones in LTI systems have similar effects on the overall augmented systems. Actually, we can think the ordinary zero/pole cancelations as the generalization of the pole/zero cancelations in LTI cases, if the poles and the ordinary zeros exist for the LTV systems. However, the transmission zero/pole cancelations may not result in unobservable states unlike the LTI cases because the functions $r(t)$ associated with transmission zeros may prevent the formation of the unobservable states. Even if, the transmission zero/pole cancelations do not affect the overall LTV system like they do in LTI systems, we may still show the transmission zero/pole cancelations between the system given by (3.110) and the inverse system given by (3.119). Because by this way we can indicate the analogy between the method that we used in the minimum phase LTV systems and the method that we used in the minimum phase LTI systems.

3.5 Numerical Results

In this section, some simulation results for All-pole and Minimum Phase LTV systems are given. In these simulations, we will use the transformed system matrices in order to prevent complicated matrix computations. Actually, we can go back to the original system matrices by applying transformations ,which we applied, in the reverse direction. In the figures of the examples, we will first give the figure which shows the error signal $e(t)$ between the system output $y(t)$ and the reference signal $r(t)$. Then we give the figure which shows the stability of the closed-loop system when $w = 0$.

3.5.1 Example 1

In the first simulation, we consider the following system (see (3.1)-(3.2)) :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin(wt) & -\sin(wt) & \cos(wt) \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 2+\sin(wt) \end{pmatrix} u + \nu,$$
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \tag{3.171}$$

where $w = 0.2\pi$. The exogenous system is given as follows :

$$\dot{w} = \begin{pmatrix} 0 & 1 \\ -(1.6+1.2\cos(2t)) & 0 \end{pmatrix} w,$$
$$r(t) = -\begin{pmatrix} 1 & 0 \end{pmatrix} w,$$
$$\nu(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} w. \tag{3.172}$$

Hence according to (3.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \end{pmatrix} w. \tag{3.173}$$

By using (3.34)-(3.35), we find the controller which satisfies the regulation conditions as follows :

$$u = \begin{pmatrix} \frac{-\sin(wt)-5}{2+\sin(wt)} & \frac{\sin(wt)-20}{2+\sin(wt)} & \frac{-\cos(wt)-2}{2+\sin(wt)} \end{pmatrix} x$$
$$+ \begin{pmatrix} -2.4\frac{1+2\sin(2t)-4\cos(t)^2}{2+\sin(wt)} & 0.8\frac{-46+3\cos(2t)}{2+\sin(wt)} \end{pmatrix} w \tag{3.174}$$

With $K_{\tilde{x}}(t)$ as given above, the characteristic polynomial of the closed-loop system becomes as follows :

$$ch(s) = s^3 + 2s^2 + 20s + 5, \tag{3.175}$$

and the roots of (3.175) can be given as follows: $\{-0.87 + 4.33\imath, -0.87 - 4.33\imath, -0.25\}$. Simulation results are obtained for these initial conditions : $x(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 0.5 & 0.2 \end{pmatrix}^T$.

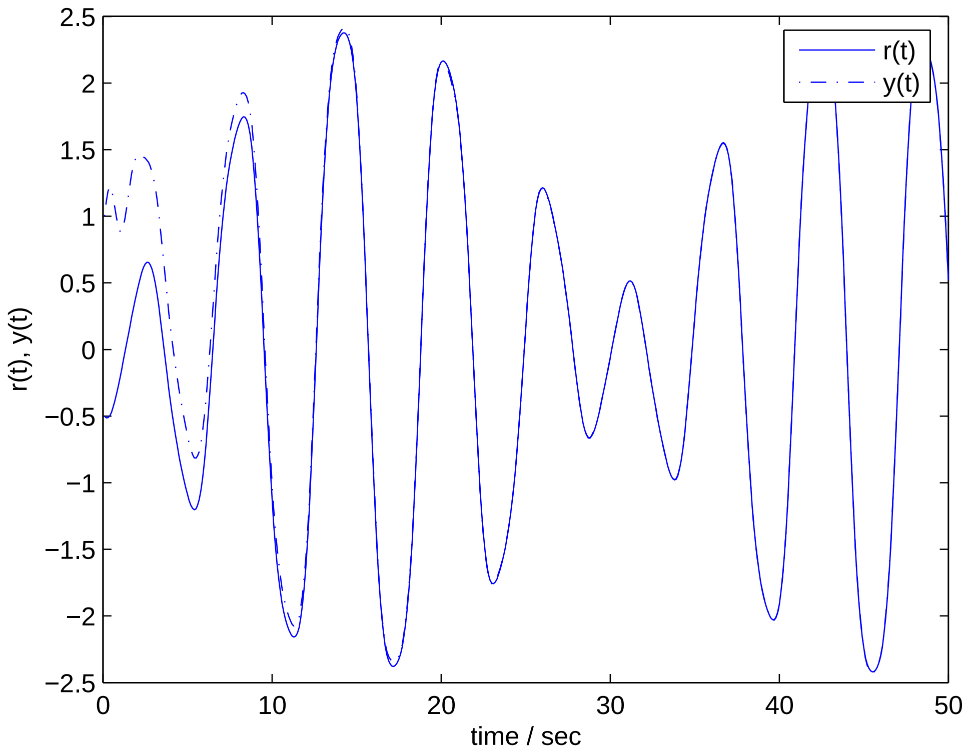

Figure 3.1: Tracking of Reference Signal

In figure 3.1 and 3.2, we can see the simulation results for the system given by (3.171)-(3.172). As can be seen in Figure 3.1, the tracking error decays to zero, in fact exponentially fast. Also Figure 3.2 indicates that the closed-loop system is stable when $w = 0$.

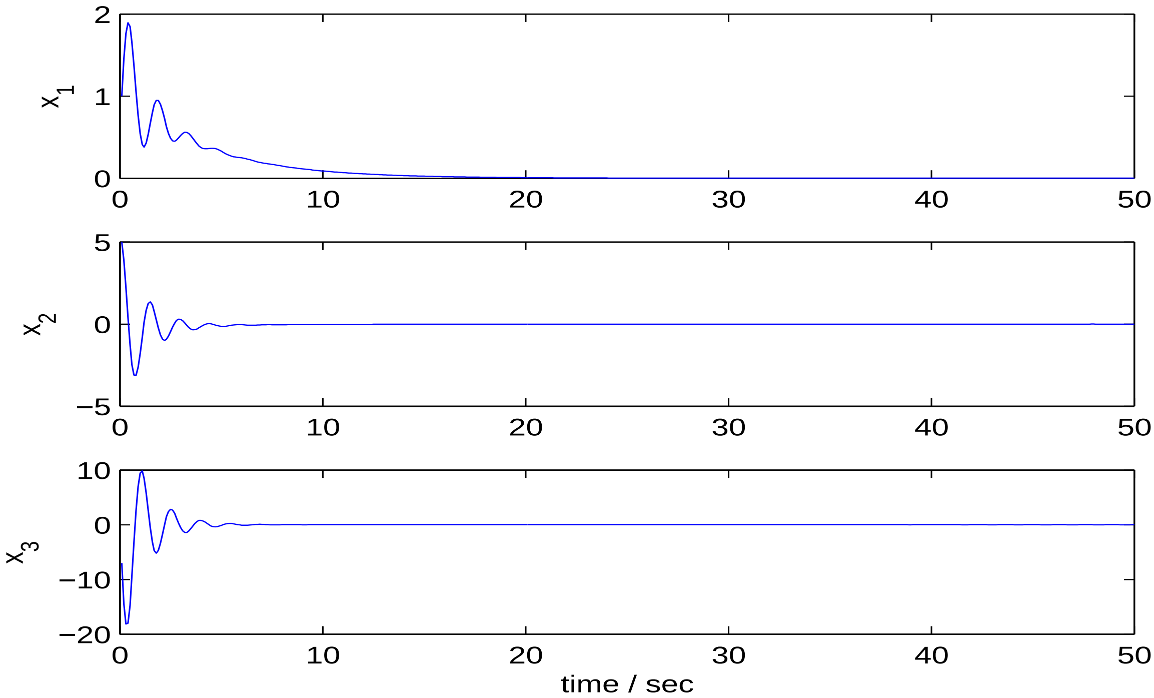

Figure 3.2: Stability of Closed-Loop System

3.5.2 Example 2

In the second simulation, the system that we consider is shown below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin(2wt) & -\cos(wt)^2 & \sin(wt) \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 2+\cos(2wt) \end{pmatrix} u + \nu,$$
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x. \tag{3.176}$$

where $w = 0.2\pi$. The exogenous system is given as follows :

$$\dot{w} = \sin(t) \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -\frac{\pi}{4}^2 & 0 & 0 \end{pmatrix} w,$$

$$r(t) = -\begin{pmatrix} \cos(t) & 0 & -\sin(t) \end{pmatrix} w,$$

$$\nu(t) = \begin{pmatrix} 0 & 0 & \sin(t) \\ 0 & 0 & \cos(t) \\ -\sin(t) & 0 & -\cos(t) \end{pmatrix} w. \tag{3.177}$$

Hence according to (3.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} \cos(t) & 0 & -\sin(t) \end{pmatrix} w. \tag{3.178}$$

By using (3.34)-(3.35), we find the controller which satisfies the regulation conditions as follows :

$$\begin{aligned} u = &\begin{pmatrix} \frac{-\sin(2wt)-60}{2+\cos(2wt)} & \frac{\cos(wt)^2-47}{2+\cos(2wt)} & \frac{-\sin(wt)-12}{2+\cos(2wt)} \end{pmatrix} x \\ &+ \begin{pmatrix} \frac{\alpha(t)}{2+\cos(2wt)} & 0 & -\frac{\beta(t)}{2+\cos(2wt)} \end{pmatrix} w \end{aligned} \tag{3.179}$$

where

$$\begin{aligned} \alpha(t) = &16.5\cos^2(t) - 19.73\cos(t)\sin(t) + 3.7\sin(t)\cos^2(t) + 45.1\sin(t) \\ &- 40.5\cos(t) - 18.74 - 7.4\cos(t) + 0.38\cos^4(t) \end{aligned} \tag{3.180}$$

and

$$\begin{aligned} \beta(t) = &3.78\sin(t) + 39.3\cos(t)\sin(t) - 24 - 19.2\cos(t) - 6.7\sin(t)\cos^2(t) \\ &+ 0.61\sin(t)\cos^3(t) - 3.7\cos(t) + 36\cos^2(t). \end{aligned} \tag{3.181}$$

With $K_{\tilde{x}}(t)$ as given above, the characteristic polynomial of the closed-loop system becomes as follows :

$$ch(s) = s^3 + 12s^2 + 47s + 60, \tag{3.182}$$

and the roots of (3.182) can be given as follows: $\{-3, -4, -5\}$. Simulation results are obtained for these initial conditions :
$x(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 0.7 & 0.9 & 0.8 \end{pmatrix}^T$.

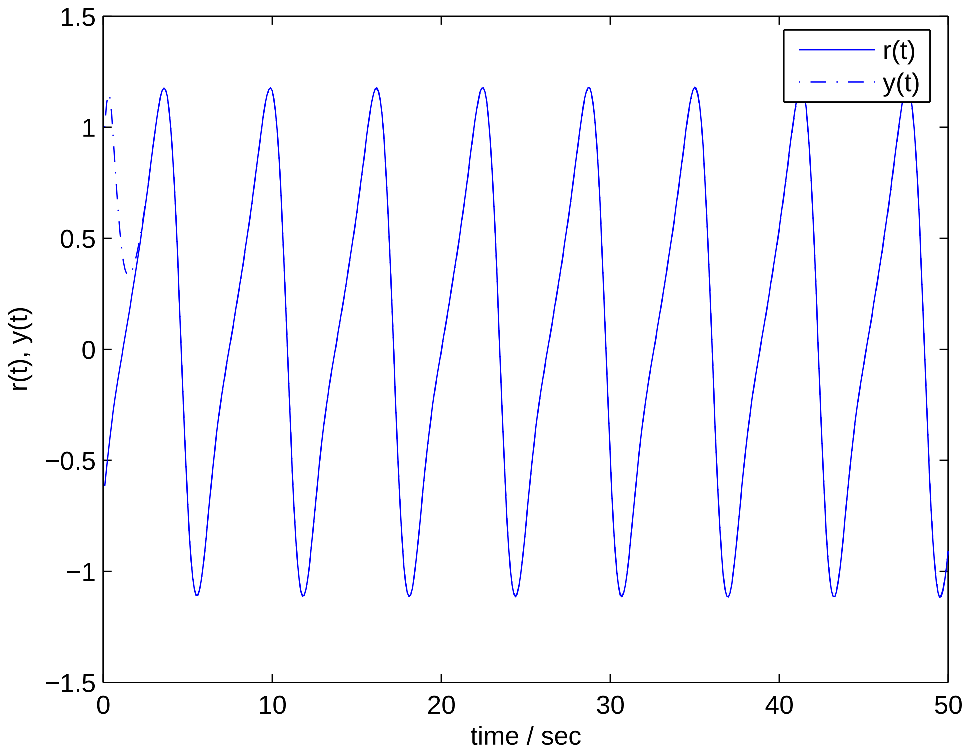

Figure 3.3: Tracking of Reference Signal

In figure 3.3 and 3.4, we can see the simulation results for the system given by (3.176)-(3.177). As can be seen in Figure 3.3, the tracking error decays to zero, in fact exponentially fast. Also Figure 3.4 indicates that the closed-loop system is stable when $w = 0$.

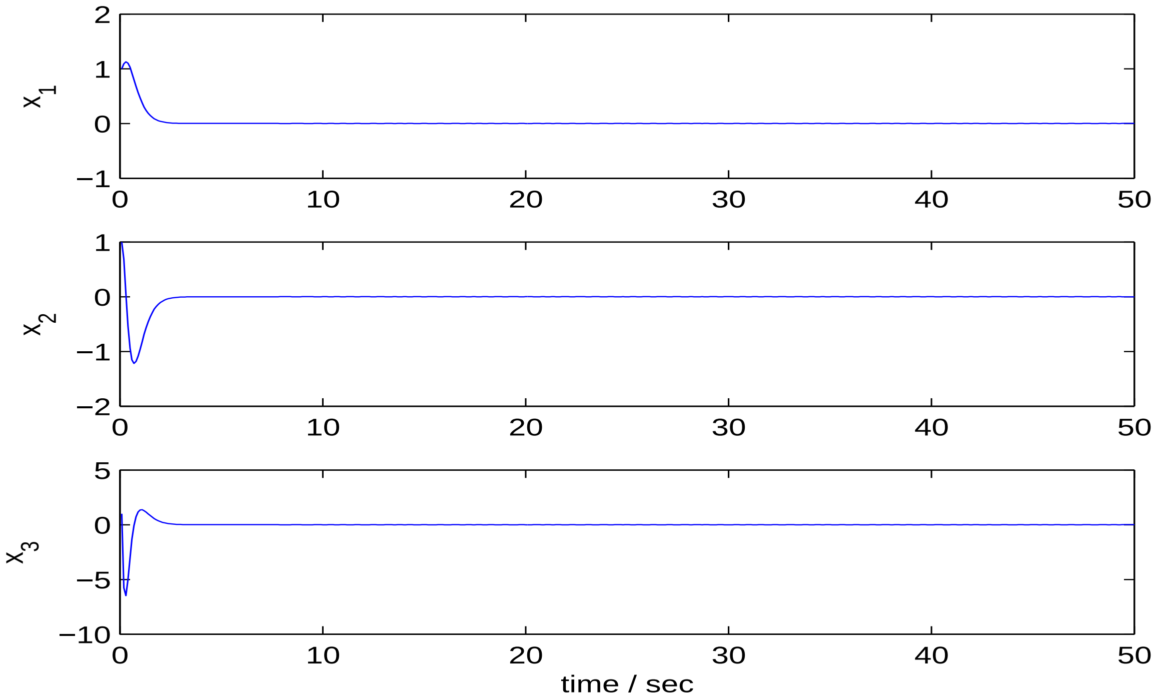

Figure 3.4: Stability of Closed-Loop System

3.5.3 Example 3

In *example 3*, we will consider a minimum phase system. State space model of the system is given below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{6}{t^2+2} & \frac{9}{t+1} & \frac{t+4}{2t+2} \\ \frac{t^2\cos(wt)+1}{t^2+2} & \frac{(t+5)\sin(wt)}{t+2} & 2\frac{t^3\sin(wt)-cos(wt)}{t^3+15} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{2t+2}{t+4} \end{pmatrix} u + \nu,$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x. \tag{3.183}$$

where $w = 0.4\pi$. The exogenous system is shown below :

$$\dot{w} = \begin{pmatrix} 0 & 1 \\ -(1.6 + 1.2\cos(2t)) & 0 \end{pmatrix} w,$$
$$r(t) = -\begin{pmatrix} 1 & 0 \end{pmatrix} w,$$
$$\nu(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} w. \quad (3.184)$$

Hence according to (3.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \end{pmatrix} w. \quad (3.185)$$

By using (3.119), (3.138), (3.141) and (3.142), we find the controller which satisfies the regulation conditions as follows :

$$\dot{\eta} = \begin{pmatrix} 0 & 1 \\ g_1(t) & g_2(t) \end{pmatrix} \eta + \begin{pmatrix} 0 & 0 & 0 \\ h_1(t) & h_2(t) & h_3(t) \end{pmatrix} \hat{x} + \begin{pmatrix} 0 & 0 \\ p_1(t) & p_2(t) \end{pmatrix} w \quad (3.186)$$

Since the functions in the matrices are extremely long and complicated, we cannot give these functions in detail here (see Appendix). Note that these functions are obtained from Symbolic Toolbox of MATLAB. The initial conditions for this simulation are taken as follows :
$x(0) = \begin{pmatrix} 0.1 & 0.2 & 0.1 & -0.1 & 0.3 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}^T$.

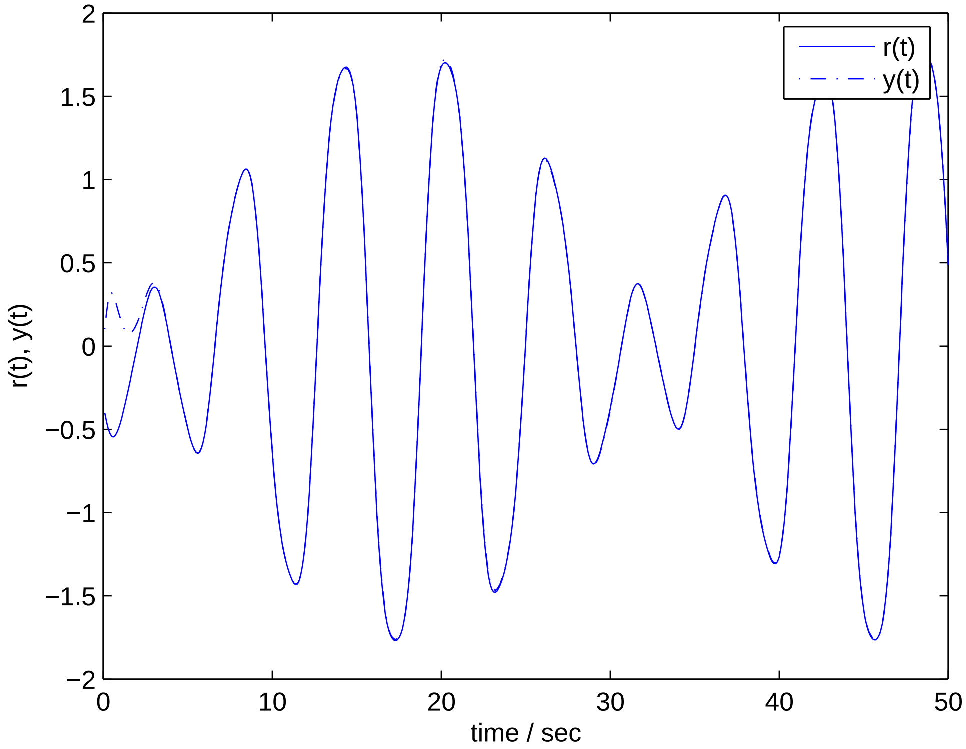

Figure 3.5: Tracking of Reference Signal

In figure 3.5 and 3.6, we can see the simulation results for the system given by (3.183)-(3.184). As can be seen in Figure 3.5, the tracking error decays to zero, in fact exponentially fast. Also Figure 3.6 indicates that the closed-loop system is stable when $w = 0$.

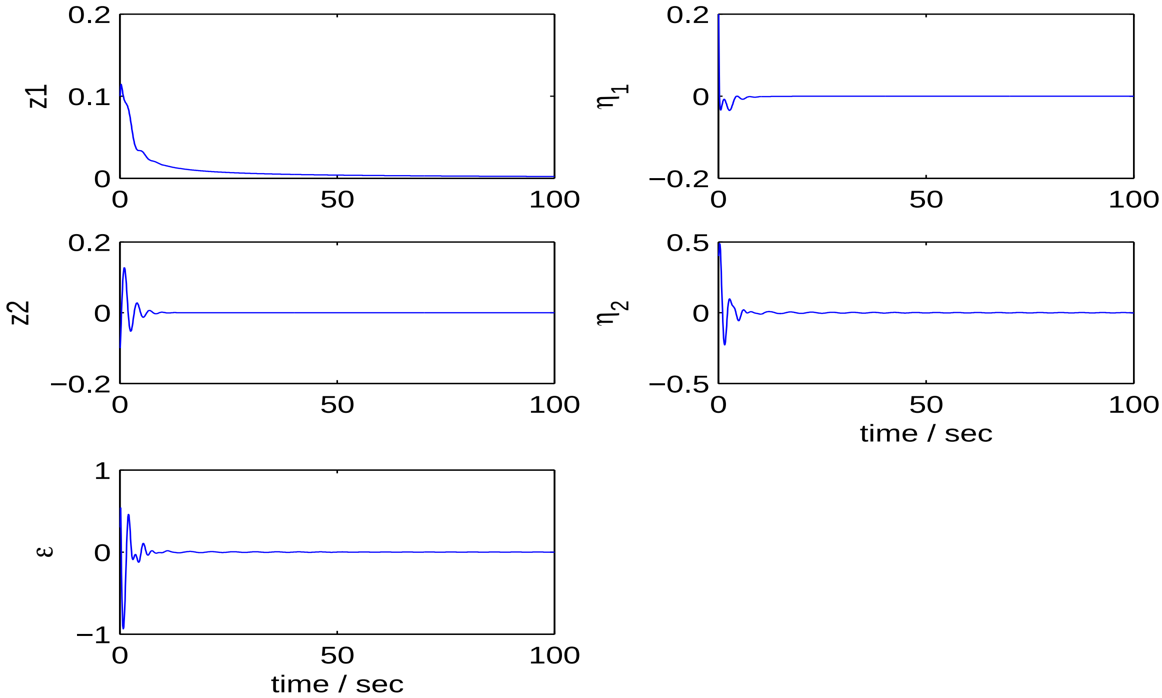

Figure 3.6: Stability of Closed-Loop System

3.5.4 Example 4

In the last example, we will examine minimum phase system which is shown below :

$$\dot{x} = \begin{pmatrix} 0 & 1 & 1 \\ \sin(wt)\cos(wt) & -\sin(wt)(\cos(wt)+2\sin(wt)) & \frac{1}{8+\cos(wt)} \\ \sin(wt) & 2\cos(wt) & \sin(wt)-\cos(wt) \end{pmatrix} x$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 8+\cos(wt) \end{pmatrix} u + \nu,$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x. \tag{3.187}$$

where $w = 0.4\pi$. The exogenous system is shown below :

$$\dot{w} = \sin(t)\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -(\frac{\pi}{4})^2 & 0 & 0 \end{pmatrix} w,$$

$$r(t) = -\begin{pmatrix} \cos(t) & 0 & -\sin(t) \end{pmatrix} w,$$

$$\nu(t) = \begin{pmatrix} 0 & 0 & \sin(t) \\ 0 & 0 & \cos(t) \\ -\sin(t) & 0 & -\cos(t) \end{pmatrix} w. \tag{3.188}$$

The vectors $K(t)$, $H(t)$ of the inverse system can be chosen from a class as *Remark 10* indicated. Hence in this example, instead of using $K(t) = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $H(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the inverse system, we use $K(t)$ and $H(t)$ as given below :

$$K(t) = \begin{pmatrix} -\frac{\sin\left(\frac{2\pi t}{5}\right)}{\cos\left(\frac{2\pi t}{5}\right)+8} & -\frac{2\cos\left(\frac{2\pi t}{5}\right)}{\cos\left(\frac{2\pi t}{5}\right)+8} \end{pmatrix}$$

$$H(t) = \begin{pmatrix} -\frac{320\cos\left(\frac{2\pi t}{5}\right)+20\cos\left(\frac{4\pi t}{5}\right)+20}{16\pi+10\cos\left(\frac{4\pi t}{5}\right)-5\cos\left(\frac{8\pi t}{5}\right)+40\sin\left(\frac{4\pi t}{5}\right)-5} \\ \frac{160\sin\left(\frac{2\pi t}{5}\right)+10\sin\left(\frac{4\pi t}{5}\right)}{16\pi+10\cos\left(\frac{4\pi t}{5}\right)-5\cos\left(\frac{8\pi t}{5}\right)+40\sin\left(\frac{4\pi t}{5}\right)-5} \end{pmatrix} \tag{3.189}$$

Hence according to (3.6), the error $e(t)$ becomes :

$$e = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} \cos(t) & 0 & -\sin(t) \end{pmatrix} w. \tag{3.190}$$

By using (3.119), (3.138), (3.141) and (3.142), we find the controller which satisfies the regulation conditions as follows :

$$\dot{\eta} = \begin{pmatrix} g_3(t) & g_4(t) \\ g_5(t) & g_6(t) \end{pmatrix} \eta + \begin{pmatrix} h_3(t) & h_4(t) & h_5(t) \\ h_6(t) & h_7(t) & h_8(t) \end{pmatrix} \hat{x} + \begin{pmatrix} p_3(t) & p_4(t) & p_5(t) \\ p_6(t) & p_7(t) & p_8(t) \end{pmatrix} w \tag{3.191}$$

Since the functions in the matrices are extremely long and complicated, we cannot give these functions in detail here (see Appendix). Note that these functions are obtained from Symbolic Toolbox of MATLAB. The initial conditions for this simulation are taken as follows;

$x(0) = \begin{pmatrix} -1 & 0.2 & 0 & 0.2 & 2 \end{pmatrix}^T$ and $w(0) = \begin{pmatrix} 0 & 1 & \frac{\pi}{8} \end{pmatrix}^T$.

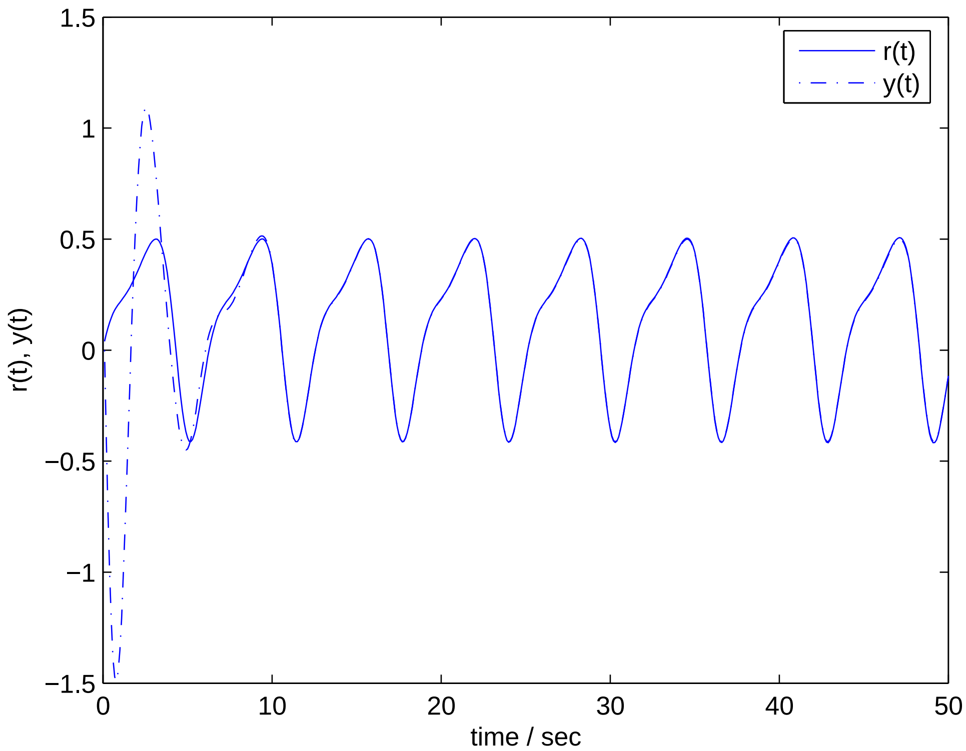

Figure 3.7: Tracking of Reference Signal

In figure 3.7 and 3.8, we can see the simulation results for the system given by (3.187)-(3.188). As can be seen in Figure 3.7, the tracking error decays to zero, in fact exponentially fast. Also Figure 3.8 indicates that the closed-loop system is stable when $w = 0$.

3.5.5 Appendix

In *Example 3* and *Example 4*, we obtain extremely long and complicated functions from Symbolic Toolbox of MATLAB for the dynamic controllers given by (3.186), (3.191). Hence, in below we only give $g_1(t)$,which is in (3.186), in order to indicate how complicated and long these functions are. Since the space is not

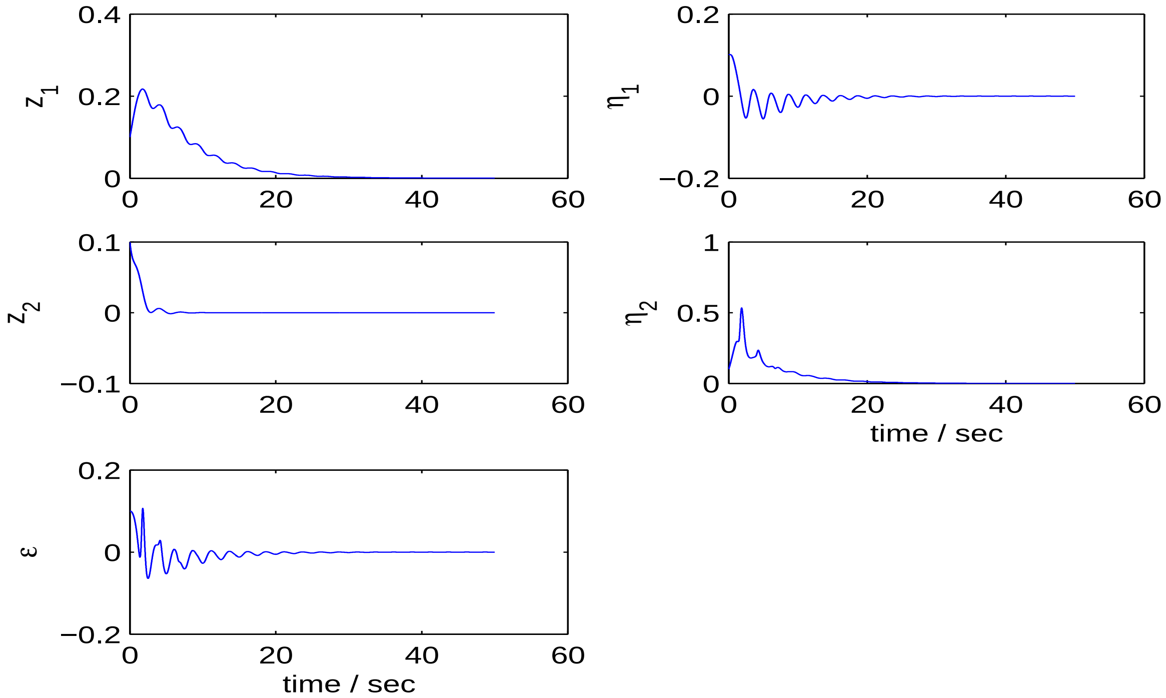

Figure 3.8: Stability of Closed-Loop System

enough to write $g_1(t)$ in a one line, we write $g_1(t)$ term by term as given below :

$$a_1(t) = \frac{4\,t^6\left(\cos\left(\frac{2\,\pi\,t}{5}\right) - \sin\left(\frac{2\,\pi\,t}{5}\right)\right)^2}{\left(t^3+15\right)^2}$$

$$a_2(t) = \frac{2\,t^3\left(\frac{2\,\pi\,\cos\left(\frac{2\,\pi\,t}{5}\right)}{5} + \frac{2\,\pi\,\sin\left(\frac{2\,\pi\,t}{5}\right)}{5}\right)}{t^3+15}$$

$$a_3(t) = \frac{6\,t^2\,\left(\cos\left(\frac{2\,\pi\,t}{5}\right) - \sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{t^3+15}$$

$$a_4(t) = \frac{6\,t^5\,\left(\cos\left(\frac{2\,\pi\,t}{5}\right) - \sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{\left(t^3+15\right)^2}$$

$$a_5(t) = \frac{\sin\left(\frac{2\,\pi\,t}{5}\right)\,(t+4)\,(t+5)}{(2\,t+2)\,(t+2)}$$

$$d_0(t) = \frac{18}{t+4} - \frac{4}{(t+4)^2} + \frac{4\,t+4}{(t+4)^3} - \frac{18\,t+18}{(t+4)^2} + \frac{52\,t+52}{t+4}\frac{12\,t+12}{(t^2+2)\;(t+4)}$$

$$d_1(t) = \frac{(2\,t+2)\;(a_1(t)+a_2(t)-a_3(t)+a_4(t)+a_5(t))}{t+4}$$

$$d_2(t) = \frac{4\,t^3\;\left(\cos\left(\frac{2\,\pi\,t}{5}\right)-\sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{(t^3+15)\;(t+4)}$$

$$d_3(t) = \frac{2\,t^3\;(2\,t+2)\;\left(\frac{2\,\pi\,\cos\left(\frac{2\,\pi\,t}{5}\right)}{5}+\frac{2\,\pi\,\sin\left(\frac{2\,\pi\,t}{5}\right)}{5}\right)}{(t^3+15)\;(t+4)}$$

$$d_4(t) = \frac{6\,t^2\;(2\,t+2)\;\left(\cos\left(\frac{2\,\pi\,t}{5}\right)-\sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{(t^3+15)\;(t+4)}$$

$$d_5(t) = \frac{18\,t^3\;(2\,t+2)\;\left(\cos\left(\frac{2\,\pi\,t}{5}\right)-\sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{(t^3+15)\;(t+4)}$$

$$d_6(t) = \frac{2\,t^3\;(2\,t+2)\;\left(\cos\left(\frac{2\,\pi\,t}{5}\right)-\sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{(t^3+15)\;(t+4)^2}$$

$$d_7(t) = \frac{6\,t^5\;(2\,t+2)\;\left(\cos\left(\frac{2\,\pi\,t}{5}\right)-\sin\left(\frac{2\,\pi\,t}{5}\right)\right)}{(t^3+15)^2\;(t+4)}$$

$$g_1(t) = -\frac{6}{t^2+2} - \frac{(t+4)\;(d_0(t)+d_1(t)-d_2(t)+d_3(t)-d_4(t)-d_5(t)+-d_6(t)+d_7(t))}{2\,t+2} \tag{3.192}$$

As we can see from the expression given above, even we write one function, this is extremely long and complicated. For this reason, we give only $g_1(t)$ as an example and other functions in (3.186), (3.191) can easily be obtained by using Symbolic Toolbox of MATLAB.

Chapter 4

CONCLUSION

In this thesis, we dealt with the output regulation problem for all-pole and minimum phase LTI / LTV systems. Our main approach is to find a controller, which solves the output regulation problem analytically. Since obtaining controllers analytically for the output regulation problem was difficult, we restricted the systems that we dealt with to a certain class of LTI/LTV systems. We developed a solution for the output regulation problem for all-pole and minimum phase systems. First, we found a design procedure for all-pole and minimum phase LTI cases. Then, the same methodology was applied to the LTV cases. However, in the LTV part of the problem we first needed to obtain canonical forms for all-pole and minimum phase systems in order to apply the same methodology as we used in LTI part.

In the first part of the thesis, we considered the output regulation problem for all-pole and minimum phase LTI systems. The relative degree property of LTI systems was first introduced. Then, the relative degree property was used to obtain a controller for all-pole cases by taking the derivative of the error up to a system degree. Since the original system and the exogenous system states are generally not available for measurement and since the designed controller uses these states, we designed observers both for the original system states and the exogenous system states. In the minimum phase part, an inverse system was employed as the first part of the controller. By using this inverse system as the first part, the overall system become equivalent to an all-pole system. Then, we used the same procedure with all-pole cases for this overall system and obtained the second part of the controller. Combining these two parts gave us the total controller which solved the output regulation problem for minimum phase cases.

Secondly, the LTV part of the problem was studied. First we defined the relative degree property for LTV systems. Since Laplace transform techniques are not applicable to time-varying systems, we tried to transform the LTV systems into some certain state space forms similar to the LTI state space forms in order to apply the same methodology with LTI cases. Observability matrix was used as a transformation matrix to obtain canonical form for all-pole case. Actually, while obtaining this canonical form, we used the relative degree property of the all-pole systems. After obtaining the canonical form, we designed a controller with the same method as we used in LTI all-pole cases. In the minimum phase part of the LTV systems, obtaining certain state space form, which is called the *normal form*, was carried out by applying three transformations on the system. However, in order to preserve stability property between the original system and the transformed system, we made some assumptions on controllability and observability matrices of the minimum phase system. Actually, the minimum phaseness definition was given after we obtained the normal form. Then, the same procedure was applied as in the LTI case. The inverse system was employed as the first part of the controller and then by taking the derivative of the error, the second part of the controller was obtained. As a final step, we showed pole/zero cancelations between the inverse system and the minimum phase system like we did in the LTI minimum phase case. In order to show these cancelations we used pole/zero definitions which are given in [36].

Since we restrict ourselves to a certain class of LTI / LTV systems, our approach has some advantages over the previous ones. The advantages of our method for the output regulation problem are as follows:

- Different from existing approaches, our solution depends on the analytical calculation of the controller that satisfies the regulation conditions (i), (ii). This analytical calculation is particularly very important for LTV systems because finding a controller by using regulator equations, which include differential matrix equation, is a very difficult task.
- Our approach does not assume the fulfillment of the condition (ii) like most of the existing approaches. Instead, we proved that the controller which we proposed also satisfies the condition (ii).
- In the LTI cases, the controller that solves the output regulation problem may be found relatively easily by using the regulator equations . However, in this methodology we have no degree of freedom to alter the transient behavior of the system. On the other hand, our approach allows one to alter the transient behavior of the closed-loop system up to a certain degree by only changing some controller parameters. By this way, the designer can achieve some desired specification with no difficulty.

In addition to above advantages, we find a normal form for minimum phase LTV systems under some assumptions on the original systems controllability and observability matrices. Then we define minimum phaseness in LTV systems in accordance with this normal form. Furthermore, we show pole/zero cancelations between the inverse systems and the original systems in minimum phase LTV cases like we do in LTI cases. When we show pole/zero cancelations in LTV systems, we use the definitions for poles and zeros given in [36].

Our contributions in this thesis can be listed as follows :

(I) In an analytical manner, we can find a controller, which solves the output regulation problem.

(II) We do not give only one controller structure. Instead we give a class of controllers and this whole class of controllers can be obtained by only changing some scalar parameters.

(III) In the LTV case, we obtain a normal form for the minimum phase part and define minimum phaseness for time-varying systems.

(IV) We show under which conditions the normal form and the original system are Lyapunov equivalent.

(V) By using the pole/zero definitions given in [36], we show pole/zero cancelations between the inverse system and the original system in LTV minimum phase case. These cancelations are presented in order to point out the analogy between the controller design methodologies that we used both for the LTI and the LTV minimum phase systems.

In the future, we may try to extend our approach to the continuous nonlinear time-invariant and time-varying cases. In the nonlinear part of this problem, we will need to define the relative degree property and the minimum phaness again. In order to obtain an inverse system for minimum phase cases, we will need to find a normal form. Additionally the transformation, which we will apply to the original system in order to find the normal form, should preserve stability. Afterwards, we will try to find the extension of our method to the LTI/LTV/Nonlinear discrete time systems. In discrete time cases, first we will develop methodology for all-pole and minimum phase LTI case and extend this for LTV and Nonlinear discrete systems. However, since stability conditions are different between continuous and discrete systems, this may cause additional assumptions on discrete time systems. In addition to this, pole / zero definition for discrete LTV systems may be defined or existing definitions may be used in order to show cancelations in LTV discrete time minimum phase case. Also, since there is no unique pole/zero definition for the continuous time LTV systems in literature, definitions of the poles and the zeros for the continuous time LTV systems may be modified in order to make the analogy between minimum phase continuous time LTI cases and LTV cases more precise.

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