DISTORTION RISK MEASURES AND ALLOCATION METHODOLOGIES

A Master's Thesis

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September 2009

To my parents Nigar & Ahmet Kurtulan

DISTORTION RISK MEASURES AND ALLOCATION METHODOLOGIES

The Institute of Economics and Social Sciences
of
Bilkent University

by

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In Partial Fulfillment of the Requirements for the Degree of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
BİLKENT UNIVERSITY
ANKARA

September 2009

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ABSTRACT

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September 2009

This study reviews the commonly used risk measures and allocation methodologies for risk capital. The method proposed by Tsanakas (2004) about dynamic capital allocation with distortion risk measures analyzed and for the cases when the events on which the liability processes are conditioned have zero probability, a new *k*-number approach is proposed which helps to behave risk-averse when correlations among liabilities are not accurate.

Keywords: Risk Capital, Capital Allocation, Distortion Risk Measures

ÖZET

DİSTORSİYON RİSK ÖLÇÜMLERİ VE DAĞITIM YÖNTEMLERİ

Kurtulan, Ali Burak
Yüksek Lisans, Ekonomi Bölümü
Tez Yöneticisi: Yrd. Doç. Dr. Taner Yiğit

Eylül 2009

Bu çalışmada, yaygın olarak kullanılan risk ölçümleri ve risk sermayesinin dağıtım metotları gözden geçirilmektedir. Tsanakas (2004) tarafından sunulan distorsiyon ölçümleriyle dinamik sermaye dağıtımı yöntemi incelendi ve yükümlülük süreçlerinin şartlı olduğu olayların sıfır ihtimali olduğundaki durumlar için, yükümlülükler arası korelasyonunun kesin olmadığı zamanda riskten kaçınmaya yardımcı olan yeni bir *k*-sayısı yaklaşımı önerildi.

Anahtar Kelimeler: Risk Sermayesi, Sermaye Dağıtımı, Distorsiyon Risk Ölçümleri

ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisor Asst. Prof. Dr. Taner Yiğit for his instructive comments in the supervision of the thesis.

I would like to express my special thanks and gratitude to Drs. Fatih Cemil Özbuğday, Drs. Hakan Er and Drs. Erdal Yılmaz for showing keen interest to the subject matter and accepting to read and review the thesis.

I would like to thank my friends Abdurrahman Bulut, Rasim Yurtcan, Yunus Yılmaz, Aykut Şahin, Ümit Aslan, Cüneyt Öztürk, Ömer Hamdi Kaya, and Hüseyin Şener Şen for their invaluable friendship and support.

I would like to thank to my beloved friend Henin M. Emin for everything she did for me.

Finally, I would like to thank to my parents Ahmet-Nigar Kurtulan and sisters Betül-Banu Kurtulan for their great efforts to help me overcome any difficulties throughout my life.

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CHAPTER I

INTRODUCTION

Financial institutions are obligated to hold a buffer capital in order to prevent insolvency by regulator. This buffer capital is called as *risk capital* or *economic capital*, which generally exceeds the minimum set by the regulator. This risk capital has to be determined optimally by taking two conflicting effects into account: holding risk capital incurs costs for the company and the more risk capital reduces the probability of ruining of the company (Hancock et al. 2001). Furthermore, the risk capital held by the company is an indicator for rating agencies as a measure of the company's capacity to bear risks.

The determination of risk capital depends on choosing a proper risk measure. Tsanakas (2007) emphasizes that a risk measure is a function that specifies real numbers to random variables representing uncertain pay-offs, e.g. insurance loss. However, as Dowd and Blake (2006) mentioned, it is hard to give a good assessment of financial risk except the cases in which we specify what a measure of financial risk actually means. They elucidate by this example, "the notion of temperature is difficult to conceptualize without a clear notion of a thermometer, which tells us how temperature should be measured (Dowd and Blake, 2006).” Hence, in order to clarify the notion of risk itself, Artzner et al. (1999) proposed several axioms that will be argued later on in chapter II.

Tsanakas (2007) divided the main applications of risk measures into three main areas; as demonstration of risk aversion in asset pricing models, as tools for the calculation of the insurance premium corresponding to a risk and as quantifiers of the risk capital that the holder of a specific portfolio or risks should safely invest in e.g. Artzner et al. (1999).

In this study, the issue of being able to fairly divide the total capital requirement of a diversified (insurance) company across its various business units is examined. Dhaene et al. (2005) defined *capital allocation* as a term referring to the subdivision of the aggregate capital held by the firm across its various constituents, e.g. business lines, types of exposure, territories or even individual products in a portfolio of insurance policies. There is a *diversification benefit* at risk capital, which means that the amount of reserve capital that should be held for a pool of liabilities is less than the sum of the corresponding amounts for the liabilities taken separately. The *capital allocation problem* is how to allocate the benefits of diversification across the business lines.

Given that most financial firms write several lines of business, most of the time it is necessary to allocate risk capital across these lines. Firstly, there is a cost associated with holding capital and the financial firm may want to exactly reveal this cost by line of business so as to redistribute this cost equitably across the lines

(Valdez and Chernih, 2003). Secondly, the allocation of costs across lines of business is a compulsory activity for financial reporting purposes (Dhaene, 2009). Lastly, capital allocation formulas provide a useful device for fair assessment of performance of the different lines of business by determining the return on allocated capital for each line (Valdez and Chernih, 2003). Valdez and Chernih (2003) mentioned that capital allocation is supposed to be helpful in accomplishing the goals of competitive pricing of insurance contracts and making optimal capital budgeting decisions.

The interest on capital allocation has been increasing during last decade. There are many capital allocation algorithms proposed. Dhaene et al. (2009) listed some studies from the literature in their paper. The mostly used tools are RAROC, marginal contribution, game theory, solvency exchange options. Tsanakas (2004) studies allocations where the relevant risk measure belongs to the class of distortion risk measures, while Tsanakas (2008) extends these allocation principles to the more general class of convex risk measures including the exponential risk measures. In chapter III, the study of Tsanakas (2004) will be examined in more detail.

Capital allocation is a very complicated issue, and closed-form solutions for component risks are known only for special cases (Dowd and Blake, 2006). The determination of capital allocation with distortion risk measures is very difficult except using simulation methods.

The capital allocation problem is a developing new concept. However, most of the studies are not applicable into practice. As Valde and Chernih (2003) mentioned, there are more useful studies that are required in reaching the insurer's goals: "prioritizing new capital budgeting projects," "deciding which lines of business to expand or to contract," and "fair assessment of performance of managers of various business units" (Gründl and Schmeiser, 2005).

The thesis provides a wide literature search about risk measures and gives clear path about how to use dynamic capital allocations by using distortion risk measures. The rest of the paper is organized as follows: in Chapter II, risk measures and their desired properties are discussed. Chapter III explains how to allocate the risk capital with distortion risk measures. In Chapter IV, a numerical example is given and Chapter V summarizes the key results in this paper and concludes.

CHAPTER II

RISK MEASURES

Generally, a risk $X \in \chi$ is defined as a real-valued random variable representing losses at a fixed time horizon T. If under a particular state of the world ω the variable $X(\omega) > 0$ then this will be a loss, while negative outcomes will be considered as gains. Artzner et al. (1999) defines the risk measure as follows: a risk measure is a real-valued functional, ρ, defined on a set of random variables χ, standing for risky portfolios of assets and/or liabilities. For a portfolio with risk X, its risk measure, $\rho(X)$, represents the amount of safely invested capital that a regulator would oblige the owner of X to hold. In particular, $\rho(X)$ is interpreted as "the minimum extra cash that the agent has to add to the risky position X, and to invest 'prudently', to be allowed to proceed with his plans" (Artzner et al., 1999).

If a financial firm has X aggregated net risk exposure and risk capital corresponds to $\rho(X)$, then the ruin occurs when $X > \rho(X)$. Here, $\rho(X)$ is known as risk capital and regulatory authorities demand strong economic capital. Therefore, the measure of risk is an essential issue that one might be careful in order to prevent insolvency. In the next chapter, some of the proposed and commonly used risk measures are explained in more detail.

2.1 Examples of Risk Measures

Some of the risk measures proposed in the financial literatures are as follows.

2.1.1 Expected Value Principle

The first use of risk measures in actuarial science and insurance was the development of premium principles. The risk measures such as expected value principle and standard deviation principle were applied to a loss distribution to determine a proper premium to charge for the risk. As a consequence a premium calculation principle can be directly interpreted as a risk measure. First, these traditional premium principles will be explained.

For the expected premium principle, we have:

$$\rho(X) = \lambda E[X], \qquad \lambda \geq 1.$$

$$\lambda = \theta + 1 \;\Rightarrow\; \rho(X) = E[X] + \theta E[X]$$

It represents a proportional *safety loading,* $\theta E[X]$, which means the amount left in addition to expected losses. Moreover, this risk measure in fact underlies simple regulatory minimum requirements, such as the current EU Solvency rules, which determine risk capital as a proportion of an exposure measure such as premium (Tsanakas, 2007).

Bühlmann (1970) used *expected value principle* as one of the frequently encountered principle of premium calculation for insurance. Expected value principle is widely used risk measure especially in life insurance. Nevertheless, it is only seldom used in property and casualty insurance. Bühlmann (1970) explained this reason as probably the heterogeneity of the claims which occur in non-life insurance. For example, extreme events such as earthquake, fire, flood has a high volatility with a high claim does not permit an "average calculation".

Daniel Bernoulli figured out that using the expected value leads to the so called "St.Petersburg paradox"; in other words, the expected value principle corresponds to a risk-neutral individual, and works poorly for the more common case of risk-averse individuals (Novosyolov, 2003). Thus the expected value turns out to not be a proper risk measure.

2.1.2 Standard Deviation Principle

$$\rho(X) = E[X] + \kappa\sigma[X], \qquad \kappa \geq 0.$$

In this case the safety loading, $\kappa\sigma[X]$, is risk-sensitive, as it is a proportion of the standard deviation. This principle is mostly used by reinsurance pricing and also related to Markowitz portfolio theory.

Standard deviation principle is most likely the most commonly used approach in property and casualty insurance (Bülhmann, 1970). It is linear due to a proportional change in the claims experience, and this is most likely the reason for its popularity. In addition, if the probability distribution of X is normal, then all premiums stand an equal chance of being exceeded by related claims experience:

$$\sigma[X] = 1 \Rightarrow \rho(X) = E[X] + \kappa$$

Since the individual premium which differs widely from the normal distribution, it is not that important to take into account that argument.

The more volatile portfolios require more capital and so it makes this measure more realistic in comparison to expected value principle, which does not distinguish the volatilities among assets. However, a risk-averse individual for sure would take into account of volatility in her investments.

In the literature there is a discussion from Denneberg (1990) who proposed that standard deviation should be replaced by absolute deviation. Laeven and Goovaerts (2007) mentioned in their paper that *dynamic versions of the standard deviation principles in an economic environment are studied by Schweizer (2001) and Moller (2001).*

For both expected value and standard deviation principles, these measures have some things in common; each requires a premium which is bigger than the expected loss (Hardy, 2006). The difference between expected loss and premium, which is called premium loading, acts as a buffer against adverse experience.

2.1.3 Exponential Premium Principle

$$\rho(X) = \frac{1}{a} \ln E[e^{aX}], \qquad a > 0.$$

Exponential premium principle, which is also known as the entropic risk measure (see e.g. Föllmer and Schied (2002)) is termed by Gerber (1974).

This principle is widely accepted in the actuarial and insurance literature in order to determine the ruin probabilities, see for example Bühlmann (1985). The ruin is defined as S_t becoming negative at some time $t > 0$ where S_t is the surplus between the total premiums and the total claims. Hence, the ruin probability, ψ, is

$$\psi = P(S_t\colon\ S_t < 0 \text{ and } t > 0)$$

Assuming that Xhas exponentially bounded tails, the probability of ruin, ψ, is bounded by:

$$\psi \le e^{-\lambda S_0}$$

where λ is called the "adjustment coefficient" and is the solution of the following equation:

$$e^{\lambda c} = E\left[e^{\lambda X}\right] \quad \Rightarrow \quad c = \frac{1}{\lambda} \ln E[e^{\lambda X}]$$

where c is the premium required. So, calculating premium, c, by the exponential principle introduces an upper bound of $e^{-\lambda S_0}$ on the probability of ruin. It can be observed that the higher λ is, the lower the probability of ruin.

In comparison to exponential premium principle, the expected value and standard deviation principles can be handled easily.

2.1.4 Value-at-Risk

$$\rho(X) = VaR_p(X) = F_X^{-1}(p), \qquad p \in (0,1),$$

where F_X is the cumulative probability distribution of X. $\mathrm{VaR_p(X)}$ is easily understood as the amount of capital that, when added to the risk X, limits the probability of default to $1-p$. VaR has become more and more popular methodology for the measurement and reporting of risk since the early 1990s, especially among banks. The Market Risk Amendment of the Basel Accord, represented in 1995, permitted the use of VaR to set regulatory capital for market risk.

More practically, it can be expressed as:

$$\mathrm{VaR_i} = \mathrm{v_i} \times \alpha \times \sigma_\mathrm{i} \times \sqrt{(\mathrm{t}/365)}$$

where $\mathrm{v_i}$ is the market value of the ith asset, σ_i is the annualised volatility of that asset, t is the number of days in the chosen holding period, and α represents the desired level of confidence. This structure uses the market value of the position denominated in local currency, and as a result the standard deviation parameter is a dimensionless, annualized volatility.

2.1.5 Expected Shortfall

$$\rho(X) = ES_p[X] = \int_p^1 F_X^{-1}(q)dq, \qquad p \in (0,1).$$

Expected Shortfall is also called Conditional Value-at-Risk (CVaR) and Expected Tail Loss (ETL). This measure has been proposed in the literature as a risk measure to correct some of the theoretical defects of Value-at-Risk (Wirch and Hardy 1999). Expected shortfall is better to obtain extreme events for fat tailed distributions. It has a difficult interpretation and does not provide a clear link to companies.

VaR assesses the *worst case* loss, where worst case is defined as the event with a $1 - p$ probability. However, it does not take into consideration what the loss will be if that $1 - p$ worst case event actually realized. Expected Shortfall addresses these problems by measuring the loss in tails. In other words, Expected Shortfall is the expected loss given that the loss falls in the worst $1 - p$ part of the loss distribution.

Hardy (2006) mentions that Expected shortfall has become very important risk measure in actuarial practice. It is intuitive, easy to understand and to apply with simulation output. As a mean, it is more robust with respect to sampling error than the VaR.

Example: (VaR vs Expected Shortfall)

A simple discrete example is provided in order to show how to calculate VaR and Expected Shortfall. Assume X is a loss random variable with probability function:

$$X = \begin{cases} 0 & \text{with probability } 0.9 \\ 100 & \text{with probability } 0.06 \\ 1000 & \text{with probability } 0.04 \end{cases}$$

For 90% confidence interval:

$$VaR_{90\%}(X) = 0$$

$$ES_{\%90}(X) = E[X|X > 0] = \frac{0.06 \times 100 + 0.04 \times 1000}{0.10} = 460$$

For 95% confidence interval:

$$VaR_{95\%}(X) = 100$$

$$ES_{\%95}(X) = \frac{0.01 \times 100 + 0.04 \times 1000}{0.05} = 820$$

For 99% confidence interval:

$$VaR_{99\%}(X) = ES_{\%99}(X) = 1000$$

The Expected Shortfall increases as confidence interval increases. For a given portfolio the Expected Shortfall ES_p is worse than (or equal) to the Value-at-Risk VaR_p at the same confidence level.

2.1.6 Distortion Risk Measure

$$\rho(X) = -\int_{-\infty}^{0} \Big(1 - g\big(1 - F_X(x)\big)\Big)\, dx + \int_{0}^{\infty} g\big(1 - F_X(x)\big) dx,$$

where g is called as a *distortion function* such that $g\colon [0,1] \longmapsto [0,1]$ is increasing and concave (Wang, 1996). Distortion risk measure can be interpreted as an expectation under a distortion of the probability distribution affected by the function g (Tsanakas, 2007). Distortion risk measures can be seen as Choquet integrals (Denneberg, 1990), which are broadly used in the economics of uncertainty, e.g. Schmeidler (2003). An equivalent class of risk measures defined in the finance literature is known as spectral risk measures, which will not be discussed in this paper (Acerbi 2002).

Hürlimann (2004) argued that, “despite of being coherent, a lot of distortion risk measures, do not always provide incentive for risk management because they lack of giving a capital relief in some simple two scenarios situations of reduced risk”. Additionally, Darkiewicz et al. (2003) also mentioned that distortion risk measures do not always preserve the correlation order.

2.2 Coherent Risk Measures

Artzner et al. (1999) postulates a set of axioms in order to classify how a good risk measure should be. A coherent risk measure is defined by Artzner et al. (1999) as a functional $\rho(X)$ on a collection of random cash flows that satisfies the following properties:

(Axiom M) Monotonicity: If $X \leq Y$ then $\rho(X) \leq \rho(Y)$.

(Axiom S) Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

(Axiom PH) Positive Homogeneity: If $a \in \mathbb{R}_+$ then $\rho(\lambda X) = \lambda \rho(X)$.

(Axiom T) Translation Invariance: If $a \in \mathbb{R}$ then $\rho(a+X) = a + \rho(X)$.

$\rho(X)$ is interpreted as "the minimum extra cash that the agent has to add to the risky position X, and to invest 'prudently' (with zero interest), to be allowed to proceed with his plans" (Artzner et al., 1999).

Axiom M indicates the losses that are always higher should also attract a higher capital requirement. Axiom S states that the merging of risks should yield a decrease in risk capital due to diversification effect. Axiom PH claims that the risk of a portfolio consisting of λ risky asset X should be same as λ portfolios with each has X risky assets, and finally Axiom T postulates that adding a constant loss to a portfolio raises the necessary risk capital by the same amount. Table 1 classifies the risk measures with respect to axioms of coherence. Also Tsanakas (2007) mentioned that all risk measures in Table 1 are "law invariant", which means $\rho(X)$ only depends on the distribution function of X (Wang et al., 1997).

Coherent risk measures are criticized widely in the financial literature since the axioms are too strict. For example, Axiom PH does not take illiquidity risk into account. Also, the most commonly used risk measure in financial sector, Value-at-Risk, generally fails the Axiom S, due to its disregard for the extreme tails of distribution.

For example, Dhaene et al. (2003) gives the following example about subadditivity axiom. In earthquake risk insurance, it is better, in the sense that a lower total price is possible, to insure two independent risks than two positively dependent risks, like two buildings in the same area. For insuring both buildings, the premium should be more than twice the premium for insuring only a single building because these buildings are highly dependent to each other and in case of an earthquake, the weaker one may cause the other building to collapse. So, $\rho(X+Y) \geq \rho(X) + \rho(Y)$ should be.

An additional property for risk measures is additivity for comonotonic risks:

Comonotonic Additivity: If X, Y comonotonic then $\rho(X+Y) = \rho(X) + \rho(Y)$.[1]

Comonotonicity indicates the strongest form of positive dependence between random variables among which there is not a diversification benefit. Table 1 classifies the risk measures with respect to properties provided above.

[1] Two random variables X, Y are called comonotonic if there is a random variable U and non-decreasing real functions e, d such that $X = e(U)$, $Y = d(U)$ (Denneberg, 1994).

Table 1: The properties of risk measures (Tsanakas, 2007)

	Axiom M	Axiom S	Axiom PH	Axiom T	Coherent
Expected value p.	√	√	√		
Standard deviation p.		√	√	√	
Exponential premium p.	√	√		√	
Value-at-Risk	√		√	√	
Expected shortfall	√	√	√	√	√
Distortion risk m.	√	√	√	√	√

CHAPTER III

CAPITAL ALLOCATION WITH DISTORTION RISK MEASURES

Consider a portfolio of n individual losses X_1, X_2,..., X_n at the end of a single period. Let (X_1, X_2,..., X_n) be a random vector on a well-defined probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X_i has a finite mean. Then the total company loss is the random variable:

$$Z = \sum_{i=1}^{n} X_i$$

where, Dhaene et al. (2009) proposed several interpretations for this aggregate loss Z as follows:

i. the total loss of a corporate, e.g. an insurance company, with the individual losses corresponding to the losses of the related business units;
ii. the loss from an insurance portfolio, with the individual losses being those coming up from different policies;
iii. the loss suffered by a financial conglomerate, while the different individual losses stand for the losses suffered by its subsidiaries; or

iv. the total loss across an insurance/financial market, with the individual losses being the ones from different firms in this market.

The risk capital required for the aggregate loss Z can be determined by using a risk measure $\rho: Z \mapsto \mathbb{R}$ as $\rho(Z) = K$ where K is the risk capital.

3.0.1 Diversification Benefit

Pooling of different risk types acquires diversification. The success of the diversification benefits depend on the degree of dependence between the pooled risks. Risk capital should reflect the diversification benefit.

Value-at-Risk (VaR) is a widely used risk measure in order to determine the risk capital required by banks, insurance and pension companies. Below in detail the diversification benefits will be explained by using VaR as a practical example. From section 2.1.4 recall that the basic formula for VaR is:

$$\mathrm{VaR_i} = \mathrm{v_i} \times \alpha \times \sigma_\mathrm{i} \times \sqrt{(t/365)}$$

where:

- $\mathrm{v_i}$ is the market value of the ith asset
- σ_i is the annualised volatility of the ith asset
- t is the number of days in the chosen holding period

- α represents the desired level of confidence

Examining the formula for the variance of the portfolio returns is essential because it reveals how the correlations of the returns of the assets in the portfolio affect volatility. The variance formula is:

$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j$$

where:

- σ_P^2 = the variance of the portfolio returns
- w_i = the portfolio weight invested in position i
- σ_i = the standard deviation of the return in position i
- $\rho_{i,j}$ = the correlation between the returns of asset i and asset i

So, the standard deviation, denoted by σ_P, is:

$$\sigma_P = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j}$$

and the total VaR of the portfolio becomes:

$$\begin{aligned} VaR_P &= v_P \times \alpha \times \sigma_P \times \sqrt{(t/365)} \\ &= v_P \times \alpha \times \sqrt{(t/365)} \times \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j} \\ &= \alpha \sqrt{V^T \Sigma V} \sqrt{t/365} \end{aligned}$$

where V is the vector of N current market values of each assets and Σ is their covariance matrix. For example, in case of there are two assets:

$$VaR_P = v_P \times \alpha \times \sqrt{(t/365)} \times \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2}$$
$$= \sqrt{VaR_1^2 + VaR_2^2 + 2\rho_{1,2}VaR_1VaR_2}$$

VaR for **uncorrelated** assets i.e. when $\sigma_{1,2} = 0$ is:

$$VaR_P = \sqrt{VaR_1^2 + VaR_2^2}$$

VaR for **perfectly correlated** assets i.e. when $\sigma_{1,2} = 1$ is:

$$\sqrt{VaR_1^2 + VaR_2^2 + 2VaR_1VaR_2} = VaR_1 + VaR_2$$

Perfectly correlated assets can also be called as undiversified VaR since the aggregate sum of total risks is equal to sum of individual risks. In this situation, instead of investing in a single asset, using two uncorrelated or less than perfectly correlated assets achieves a diversification benefit which is the difference between summation of individual risks and aggregated risks:

$$(VaR_1 + VaR_2) - \sqrt{VaR_1^2 + VaR_2^2}$$

Diversification arises because not all risks realize in the same period. For example an insurance company insuring cars and ships would not expect claims from accidents in cars and ships to be interlinked. Similarly, a major motor accident (insurance risk) would not necessarily coincide with turbulence in the financial markets (financial risk). As a result, since it is unlikely that different types of events occur at same time, the company may not need to hold capital for all events going wrong at the same time.

3.1 Marginal cost approaches

As the correlation between two random variables decreases, it achieves more diversification benefit because volatility of total portfolio lessens. Therefore, it generally is

$$\rho(Z) \leq \sum_{i=1}^{n} \rho(X_i).$$

After determining the risk capital K, the company aims to allocate K across its various business units as d_1, d_2,..., d_n satisfying the full allocation requirement:

$$\sum_{i=1}^{n} d_i = K. \tag{1}$$

Given the risk capital K, there are countless number of ways to allocate. This allocation must be done in a reasonable framework such that the allocated capital amounts K_i to be 'close' to their corresponding losses X_i. In actuarial literature, early papers dealt with cost allocation problems in insurance. Bühlmann (1996) used a risk theoretical view and Lemaire (1984) used the perspective of cooperative game theory. Valdez and Chenih (2003) proposed a "fair allocation" methodology by using the three properties: no undercut, symmetry and consistency.

Marginal cost approaches are used to check the marginal effect of sub-portfolios on aggregate capital. Let $w \in [0,1]^n$,

$$Z^w = \sum_{i=1}^{n} w_i X_i$$

Then the marginal cost of each sub-portfolio is given by (Tsanakas, 2007):

$$\mathrm{MC}(X_i; Z) = \frac{\partial \rho(Z^w)}{\partial w_i}\bigg|_{w = 1}$$

subject to related differentiability assumptions. Given that the risk measure satisfies Axiom PH, then by Euler's theorem it is derived that

$$\sum_{i=1}^{n} \mathrm{MC}(X_i; Z) = \rho(Z)$$

consequently, the marginal costs $d_i = \mathrm{MC}(X_i; Z)$ can be used as the capital allocation. Furthermore, if the risk measure is subadditive then it follows (Aubin, 1981):

$$d_i = \mathrm{MC}(X_i; Z) \leq \rho(X_i),$$

which indicates that the stand-alone risks of each subportfolio are greater than their risks in the pooled portfolio.

If no such strong assumptions such as Axiom PH and Axiom S are made with respect to the risk measure, then marginal costs will in general not satisfy that their sum is equal to aggregate risk. Hence, Tsanakas (2004) also proposed to use "Aumann-Shapley value" (Aumann and Shapley, 1974) as a generalization of marginal costs

$$\mathrm{AC}(X_i; Z) = \int_0^1 \mathrm{MC}(X_i; \gamma Z)\, d\gamma.$$

So, by assigning $d_i = \mathrm{AC}(X_i; Z)$, $\sum_{i=1}^{n} d_i = \rho(Z)$ is attained and for the ones which satisfies Axiom PH, Aumann-Shapley allocation reduces to marginal costs. Table 2 reveals some examples of capital allocations.

Table 2: Capital allocations of the risk measures used in Table 1 by using marginal costs / Aumann-Shapley:

	Allocated capital amount	Notes
Expected value p.	$d_i = \lambda E[X_i]$	
Standard deviation p.	$d_i = E[X_i] + \kappa \frac{Cov(X_i, Z)}{\sigma[Z]}$	
Exponential premium p.	$d_i = \int_0^1 \frac{E[X_i \exp(\gamma a Z)]}{E[\exp(\gamma a Z)]} d\gamma$	
Value-at-Risk	$d_i = E[X_i \mid Z = VaR_p(Z)]$	under suitable assumptions on the joint probability distribution of (X_i, Z) (Tasche, 2004).
Expected shortfall	$d_i = E[X_i \mid Z > VaR_p(Z)]$	under suitable assumptions on the joint probability distribution of (X_i, Z) (Tasche, 2004).
Distortion risk m.	$d_i = E[X_i g'(1 - F_Z(Z))]$	under suitable assumptions on cumulative distribution function F_Z and the distortion function g (Tasche, 2004).

3.2 Change of probability measure

Artzner et al. (1999) states that coherent risk measures can be represented by

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}[X]$$

where $\mathcal{P}$ is a set of probability measures. Here, the distribution of random variable X is redistributed with respect to a measure which makes *E[X]* maximum. This can be regarded as worst case scenario. By including comonotonic additivity, one obtains the more specific structure of $\mathcal{P} = \{\mathbb{P}: \mathbb{P}(A) \leq g(\mathbb{P}_0(A)) \text{ for all event sets } A\}$, where $\mathbb{P}_0$ is some given measure, and g is a continuous, increasing, and concave function satisfying $g(0) = 0$ and $g(1) = 1$. Such a function must satisfy $g(p) \geq p$

for $0 \leq p \leq 1$. The set function $g(\mathbb{P}_0(.))$ is called a *distorted probability measure,* and the corresponding risk measures are called *distortion risk measures.* Here, g is a concave distortion function (Wang et al. 1997).[2]

Distortion risk measures are good candidates since they are coherent, comonotonic additive and law invariant. In Table 3, some examples of distortion functions are given. Most of them were introduced in Wang (1996). The name distortion is used since the non-linear function g "distorts" the physical probability measure $\mathbb{P}_0$.

[2] A continuous increasing function $g: [0,1] \mapsto [0,1]$ such that $g(0) = 0$ and $g(1) = 1$ is called distortion function.

Table 3: Examples of distortion functions where $\boldsymbol{p} \in (\mathbf{0}, \mathbf{1})$.[3]

Value-at-Risk	$g_p(x) = 1_{(p,1]}[x]$
Tail Value-at-Risk	$g_p(x) = \min\left\{\frac{x}{p}, 1\right\}$
Proportional hazard transform	$g_p(x) = x^p$
Dual-power transform	$g_p(x) = 1 - (1-x)^{\frac{1}{p}}$
Dennensberg's absolute deviation principle	$g_p(x) = \begin{cases} (1+p)x & \text{for } 0 \le x \le \frac{1}{2} \\ p + (1-p)x & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$
Gini's principle	$g_p(x) = (1+p)x - px^2$
Square-root transform	$g_p(x) = \frac{\sqrt{1-\ln(p)\,x} - 1}{\sqrt{1-\ln(p)} - 1}$
Exponential transform	$g_p(x) = \frac{1 - e^{-\frac{x}{p}}}{1 - e^{-\frac{1}{p}}}$
Logarithmic transform	$g_p(x) = \frac{\ln(1 - \ln p\, x)}{\ln(1 - \ln p)}$

As Tsanakas (2007) mentioned, new risk measures can be generated by re-weighting the probability distribution of the underlying risk

$$\rho(X) = E[X\xi(X)], \qquad (2)$$

where ξ is an increasing function with $E[\xi(X)] = 1$ and hence the expression (2) can be viewed as an expectation under a change of measure.

[3] Source: Darkiewicz et al., 2003

Let a distortion risk measure be given by

$$\rho(X) = \sup_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X]$$
$$\mathcal{P} = \{\mathbb{P} | \mathbb{P}(A) \le g\big(\mathbb{P}_0(A)\big) \text{ for all } A\}.$$

Given a risk Z, there exists $\mathbb{Q} \in \mathcal{P}$ such that $E_{\mathbb{Q}}Z = \rho(Z)$ (the "worst-case measure"). In the case of continuous cumulative density and differentiable distortion, the worst-case measure corresponding to Z is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}_0} = g'\big(1 - F_Z(Z)\big).$$

Here, the random variable $g'\big(1 - F_Z(Z)\big)$ is nonnegative, and satisfies

$$E_{\mathbb{P}_0}\big[g'\big(1 - F_Z(Z)\big)\big] = -\int_{-\infty}^{\infty} g'\big(1 - F_Z(Z)\big) - F_Z{}' dz = -g\big(1 - F(Z)\big)|_{-\infty}^{\infty} = 1$$

Tsanakas and Barnett (2003) showed that the risk measure $\rho(Z)$ corresponding to a reference probability measure $\mathbb{P}_0$ and a distortion function g can be computed as the "Choquet integral"

$$\rho(Z) = \int_{-\infty}^{\infty} \big(g\big(\mathbb{P}_0(Z > t)\big) - 1\big)dt + \int_0^{\infty} g\big(\mathbb{P}_0(Z > t)\big)dt.$$

In case g is differentiable and the cumulative distribution function F_X is continuous, the risk measure may also be computed as

$$\rho(X) = E_{\mathbb{P}_0}[Xg'(1 - F_X(X))] = E_{\mathbb{Q}}[X]. \quad (3)$$

Where $\mathbb{Q}$ is the worst-case measure relating to Z. Now, an explicit formula is found for risk measure with distortion functions, which will be used later on.

3.3 Dynamic Capital Allocation with Using Distortion Risk Measures

As it is discussed in Chapter 3.1, no incentive is produced for any subportfolio to leave the pool. Therefore, Tsanakas (2004) formulised the requirement via the concept of the "fuzzy core" (Aubin, 1981) as:

$$C = \left\{ d \in \mathbb{R}^n \middle| \sum_{j=1}^{n} d_j = \rho(Z) \text{ and } \rho(Z^u) \geq \sum_{j=1}^{n} u_j d_j \, \forall u \in [0,1]^n \right\}$$

where the *fuzzy core*, *C*, will consist of all allocations, d^C, that satisfy (1) and do not allocate more capital to any portfolio than its individual risk assessment, were it not part of the pool. In case of a coherent risk measure, the fuzzy core is convex, compact and non-empty (Aubin, 1981). In addition, if $\rho(Z^w)$ is differentiable at the *n*-vector of ones, $u = \mathbf{1}$, then the fuzzy core consists only of the gradient vector of $\rho(Z^w)$ at $u = \mathbf{1}$ (Aubin, 1981):

$$d_i^C = \left. \frac{\partial \rho(Z^w)}{\partial w_i} \right|_{w_j = 1 \, \forall j}$$

In the case of distortion risk measures, assuming that conditional densities are continuous, then using quantile derivatives (Tasche, 2000b) and using the quantile representation of the Choquet integral (Denneberg, 1994), Tsanakas and Barnett (2003) revealed that $\rho(Z^w)$ is differentiable in u and, by direct calculation, attained the following formula for the unique allocation in the fuzzy core:

$$d_i^C = E[X_i g'(1 - F_Z(Z))] \quad \textit{(4)}$$

3.4 Updating Capital Allocation of Distortion Risk Measures

Distorted probability measures can be interpreted in at least two following ways as an expression of risk aversion (Yaari, 1987) and as an expression of ambiguity (Ellsberg, 1961). These interpretations direct to different "updating rules", i.e. rules for revising risk capital when circumstances change (for instance, part of the business is reinsured; a subsidiary is sold or added, etc.).

The allocation rule for risk capital is based on the collection of probability measures $\mathcal{P}$ described by

$$\mathcal{P} = \{\mathbb{P} \big| \mathbb{P}(A) \leq g\big(\mathbb{P}_0(A)\big) \text{ for all event sets } A\}.$$

Upon the arrival of new information represented by an event set B, the collection $\mathcal{P}$ may be adjusted to

$$\mathcal{P}' = \{\mathbb{P}(\cdot\,|B) | \mathbb{P} \in \mathcal{P}\} \qquad \text{(“Ellsberg”)}$$

or to

$$\mathcal{P}'' = \left\{\mathbb{P} \middle| \mathbb{P}(A) \leq g\big(\mathbb{P}_0(A|B)\big) \text{ for all event sets } A\right\} \qquad \text{(“Yaari”)}$$

In the case of Ellsberg-type conditioning, the rule is $\mathcal{P}' = \{\mathbb{P}(\cdot\,|B)|\mathbb{P} \in \mathcal{P}\}$. The updated risk capital is determined by a Choquet integral in which instead of $g(S_X(x))$ the following term:

$$g\left(S_{X|B}(x); \mathbb{P}_0(B)\right)$$

will be used where $S_X = 1 - F_X$ decumulative distribution function ie,

$$S_{X|B}(x) = \mathbb{P}_0(X > x|B)$$

and the updated distortion function is

$$g_u(s,p) = \frac{g(sp)}{1 + g(sp) - g(1 - p + sp)}, \qquad p = \mathbb{P}_0(B).$$

The updated distortion function has the same characteristics with the original distortion function. It is continuous, nondecreasing, concave, and satisfies $g_u(0,p) = 0$ and $g_u(1,p) = 1$. Additionally, more distortion is observed since $g_u(s;p) \geq g(s)$ for all $p \in [0,1]$ and all $s \in [0,1]$.

If the distortion risk measures are conditioned on an event of probability zero, a limit argument proposed that (Tsanakas, 2004)

$$g_u(s;0) = \lim_{p \downarrow 0} g_u(s;p) \frac{s}{s + (g'(1)/g'(0))(1-s)}. \quad \textit{(5)}$$

$g_u(s;0)$ in equation (5) is a new class of distortion functions determined by $g'(1)/g'(0) \leq 1$. Furthermore, when conditioning a distorted probability on a zero probability event, for any type of differentiable distortion function the updated distortion function will fit in the same class. Also note that, the updated distortion function only depends on the values of the first derivative of the original distortion function at 0 and 1 (Tsanakas, 2004).

3.5 *k*-number Approach

For the cases when the events on which the liability processes are conditioned have zero probability, the updated distortion function (5) can be rewritten as

$$g_u(s;0) = g_k(s;k) = \frac{s}{s + k(1-s)}$$

where $k = g'(1)/g'(0)$. For the cases when $g'(1) \neq 0$ and $g'(0) \neq \infty$, $k \in (0,1)$ must be. This is because $g'(1) \in [0,1]$ and $g'(0) \in [1,\infty]$ as the property of all distortion functions.

In a dynamic capital allocation model, to measure the risk of the portfolio and the allocation of risk capital, updating distortion functions must be used in equation (3) and (4). In case the liability processes are conditioned have zero probability:

$$\rho(X_i) = E\left[X_i g'_u(s;0)\right]$$

$$d_i^C = E[X_i g'_u(z;0)]$$

where $s = S_{X_{t|B}^i}(X_T)$ and $z = S_{Z_{T|B}}(Z_T)$. Then, for these derivatives of updated distortion functions, one can use the derivative of $g_k(s;k)$:

$$g'_k(s;k) = \frac{k}{(s + k(1-s))^2}$$

and hence,

$$\rho(X_i) = E\left[X_i \frac{k}{(s + k(1-s))^2}\right]$$

$$d_i^C = E\left[X_i \frac{k}{\left(z + k(1-z)\right)^2}\right]$$

So, in general, without selecting any specific distortion function, $g'_k(s;k)$ can be used to determine the risk measure $\rho(X_i)$ and the dynamic capital allocation, d_i^C. This representation is simpler in notations. It is no more necessary to deal with distortion functions. All distortion function such that $g(1) \neq 0$ and $g'(0) \neq \infty$, can be mapped into a k-number.

Another observation about $k \in (0,1)$ is that it can be regarded as the risk aversion in sense of diversification benefit. As $k \to 1$

$$E\left[X_i \frac{k}{(s+k(1-s))^2}\right] - E\left[X_i \frac{k}{\left(z+k(1-z)\right)^2}\right] \to 0$$

$$\left(\rho(X_i) - d_i^C\right) \to 0$$

The diversification benefit, which is the difference between measured risk of the liability and the capital allocated to that liability, decreases. This indicates that the company uses a conservative approach towards diversification benefit. A higher *k*-number reflects that the degree of distortion is lower. In the numerical example, this concept would be more clear.

What should be the optimal *k*-number? It depends on the information about liabilities in the portfolio. In case, the dependence structure between the liabilities are unknown or the regulatory authorizes requires more risk capital, then the company should choose a higher *k*-number. For example, it is not always attainable to have correlations of liabilities in a portfolio. Then one might consult for an expert opinion. At that point, more conservative approach towards the diversification benefit would be convenient since correlations are not accurate.

CHAPTER IV

NUMERICAL EXAMPLE

The financial firms such as pension, insurance, etc. collect their liabilities from different business lines into a common pool and then determine their risk capital by using different risk measures such as VaR.

This example is an application of distortion risk measures. Dynamic capital allocation methodology will be applied to the pooled instruments (liabilities) corresponds to correlated Brownian motions with drift, which means a continuous time stochastic process with a trend. As Tsanakas (2004) suggests, by simulating paths of the liability processes, the relationship between liabilities' correlation and capital allocation can be demonstrated. Since Brownian motions' increments are multi-normally distributed, an explicit calculation of the aggregate liability process is possible.

4.1 The Distortion Function

The exponential distortion function that will be used is (see table 3):

$$g(x) = \frac{1 - \exp(-x/p)}{1 - \exp(-1/p)}. \qquad (6)$$

The function has the following derivative:

$$g'(x) = \frac{(1/p)\exp(-x/p)}{1 - \exp(-1/p)}.$$

and since the events on which the liability processes has zero probability, the updated distortion function (5) becomes

$$g_u(s;0) = \frac{s}{s + \exp(-1/p)(1-s)}$$

Figure 1 shows the functions $g(s)$ and $g_u(s;0)$ for $p = 1$.[4]

4.2 Application

For this application by using the distortion function (6), the example of Tsanakas (2004) is reviewed by using different parameters. The pool consists of three liabilities $X_t = [X_t^1\ X_t^2\ X_t^3]'$:

$$X_t = \alpha dt + \beta W_t, \qquad (7)$$

where $W_t = [W_t^1\ W_t^2\ W_t^3]'$ is a three-dimensional Brownian motion such that

[4] Note that for exponential distortion functions $p = 1$ can be chosen.

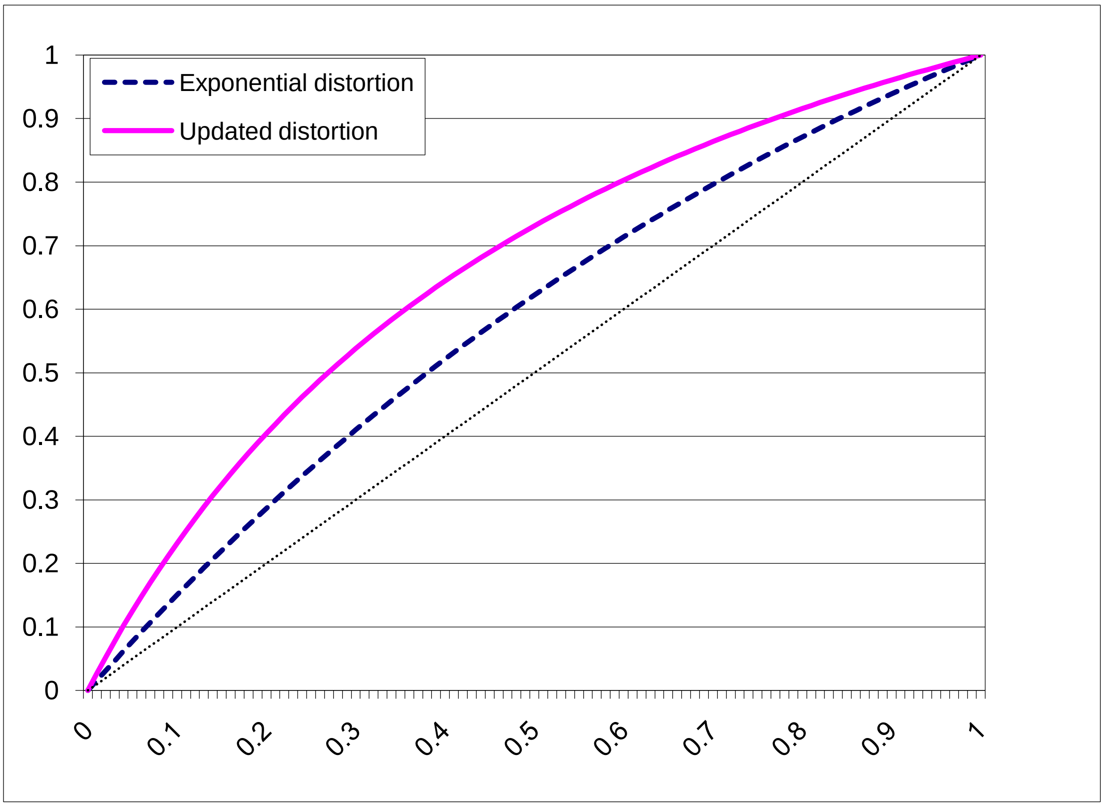

Figure 1: Exponential and updated exponential distortion functions ($\mathbf{p = 1}$).

$$\alpha = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.4 \end{pmatrix}, \quad \beta = \frac{1}{3}\begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Equation (7) means, three liabilities follows a stochastic process where α is the drift term indicating the trend through time and β indicates the volatility effect. In case $\beta = 0$ there exists no randomness.

Hence, each of the individual liability processes X_t^i is a Brownian motion with volatility $\{2/3,\ 2/3,\ \sqrt{3}/3\}$ and drift $\{0.2,\ 0.3,\ 0.4\}$, respectively. The correlations between each individual liability X_t^i and the aggregate liability $Z_t = \sum X_t^i$ at $T$are (See Tsanakas, 2004)

$$r_1 = 0.6786, \qquad r_2 = 0.2811, \qquad r_3 = 0.9459.$$

Applying the equation (3) and (4) by the exponential distortion function with $p = 1$, the risk and the capital allocated to each liability X_t^i can be determined as follows:

$$\rho(X_t^i) = E\left[X_t^i g'_u\left(S_{X_t^i|B_t}(X_T); 0\right)\middle|B_t\right],$$

$$d_t^i = E\left[X_t^i g'_u\left(S_{Z_T|B_t}(Z_T); 0\right)\middle|B_t\right].$$

where B_t is the information known at time t (Tsanakas, 2004).

. The path of each liability X_t^i is simulated with respect to equation (7) with time horizon $T = 5$. The variables $\rho_t(X_t^i)$ and d_t^i are very difficult to calculate directly. So, a simulation approach is used in order to determine the capital allocation and risk measure.[5] In Figure 2 paths of the individual liability processes X_t^1, X_t^2, X_t^3 are illustrated. In Figure 3 the risk measure of the aggregate liability, $\rho_{|B_t}(Z_t)$ and the sum of the risks of the individual liabilities, $\sum \rho_{|B_t}(X_t^i)$ are examined where the benefit of pooling can be observed as the difference between the two lines. Finally, in Figures 4-6, the capital allocated of liability, d_t^i, is compared with the risk of the liability X_t^i, $\rho_{|B_t}(X_t^i)$. The difference between each line indicates the pooling benefit of each liability. The plots also reveal that lower correlation of X_t^i with Z_t derives better benefit from pooling because of diversification effect.

[5] See Appendix for the codes of process.

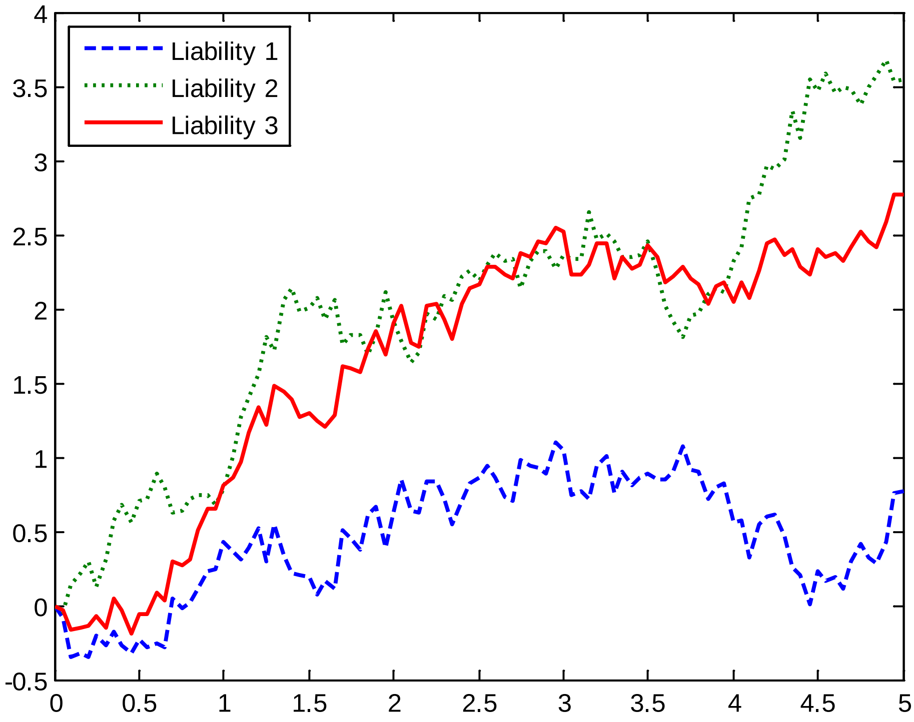

Figure 2: Simulated path of individual liabilities, X_t^1, X_t^2, X_t^3.

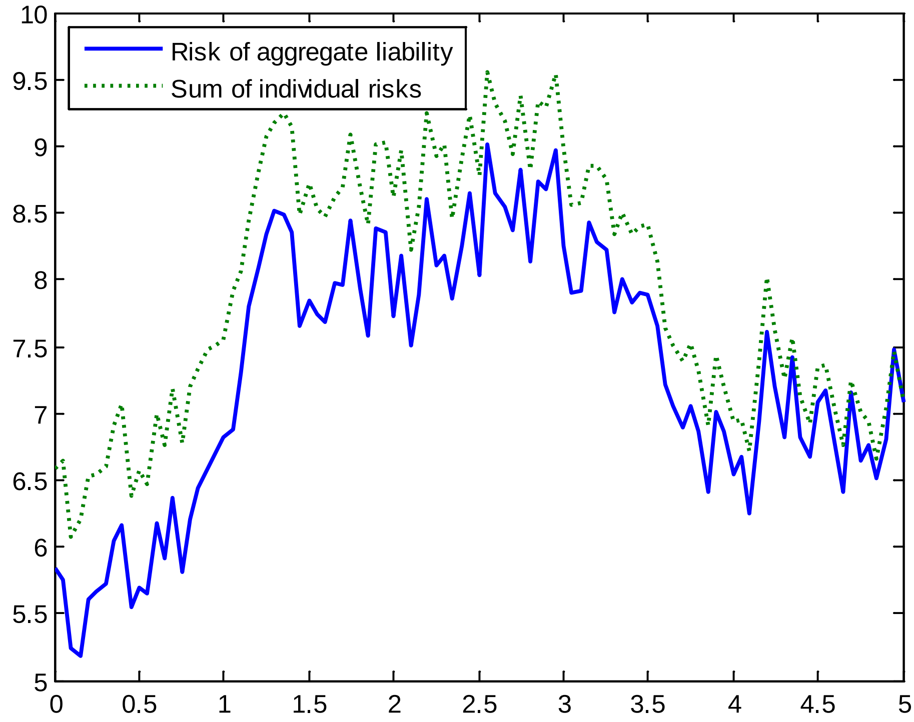

Figure 3: Risk measure of aggregate liability versus sum of risks of individual liabilities.

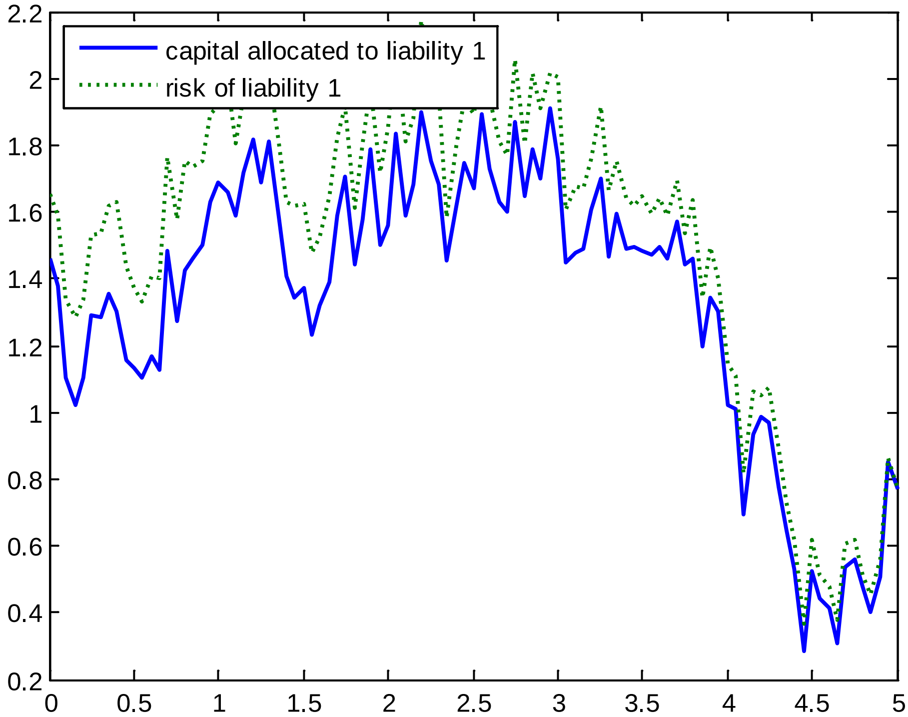

Figure 4: Risk measure, $\boldsymbol{\rho}_{|\boldsymbol{B}_t}(\boldsymbol{X}_t^1)$, and capital, $\boldsymbol{d}_t^1$, allocated to the first liability, $\boldsymbol{X}_t^1$.

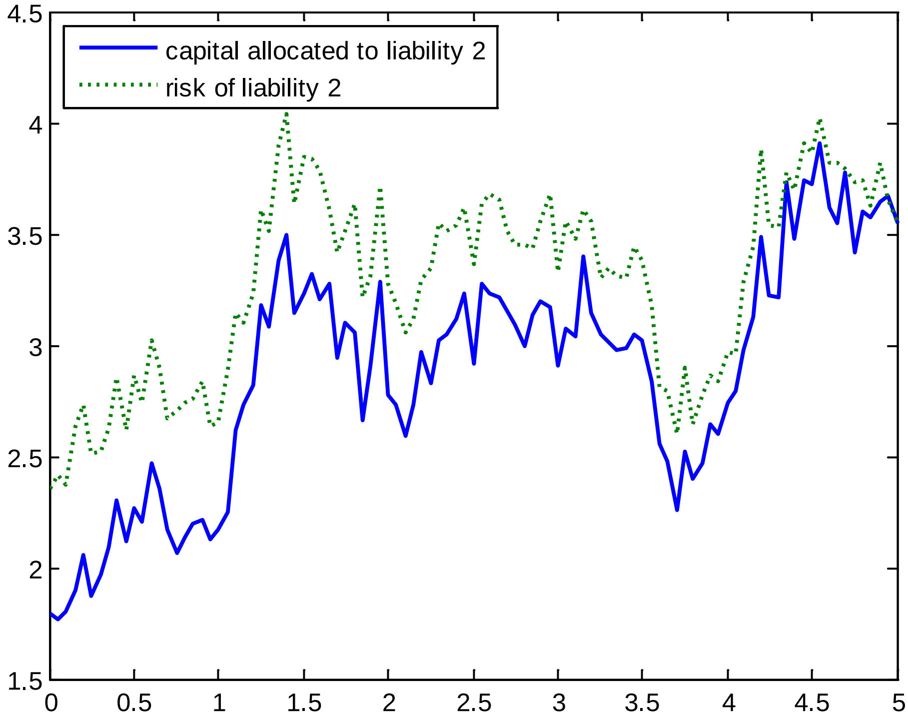

Figure 5: Risk measure, $\boldsymbol{\rho}_{|\boldsymbol{B}_t}(\boldsymbol{X}_t^2)$, and capital, $\boldsymbol{d}_t^2$, allocated to the second liability, $\boldsymbol{X}_t^2$.

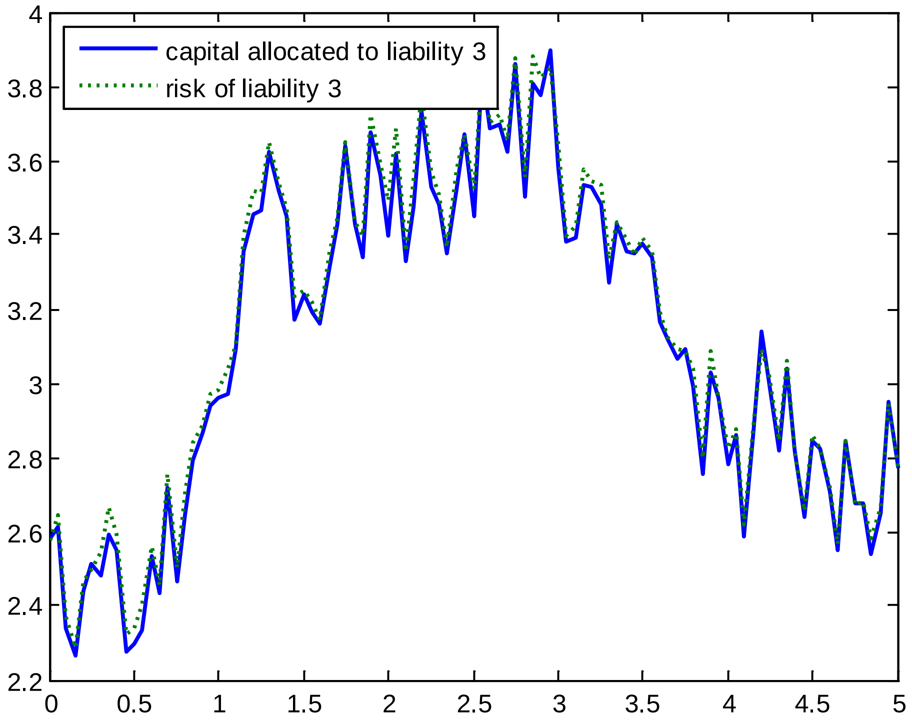

Figure 6: Risk measure, $\boldsymbol{\rho}_{|B_t}(\boldsymbol{X}_t^3)$, and capital, $\boldsymbol{d}_t^3$, allocated to the third liability, $\boldsymbol{X}_t^3$.

For this example, the *k*-number approach could also be used that is $k = 0.3678$ for the exponential distortion function. Then, one gets exactly the same results as above. In Figure 7 and 8, the liability process of X_t^1 is repeated for $k = 0.1$ and $k = 0.9$ respectively. These figures reveal that when k increases the diversification benefit diminishes. Hence, for the cases when correlations among liabilities cannot be determined exactly, for the worst case scenario one can choose a higher *k*-number.

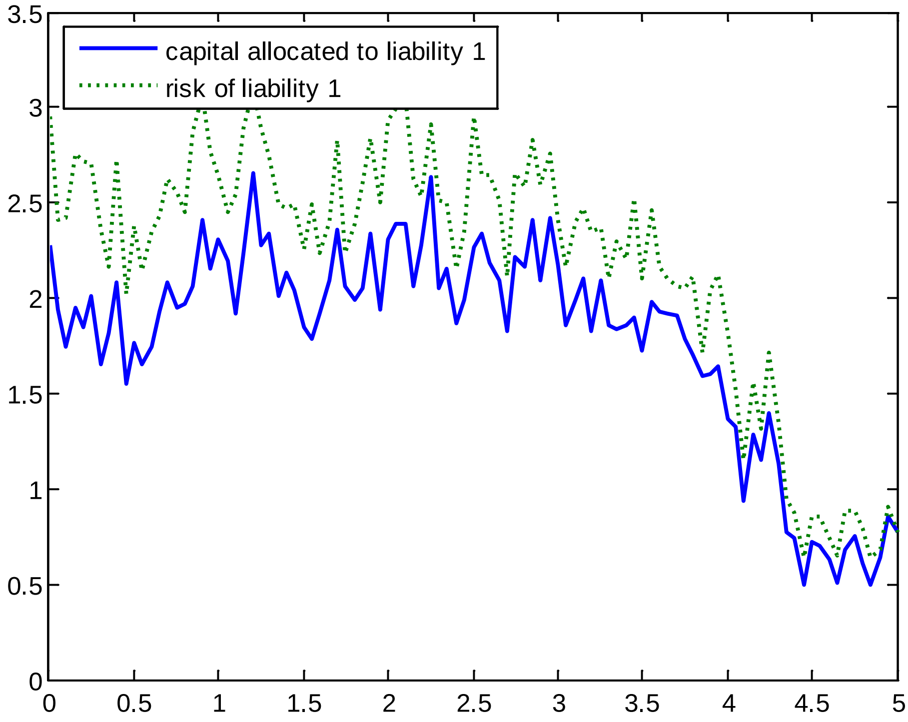

Figure 7: When $\boldsymbol{k = 0.1}$, risk measure and capital allocated to the first liability, $\mathbf{X_t^1}$.

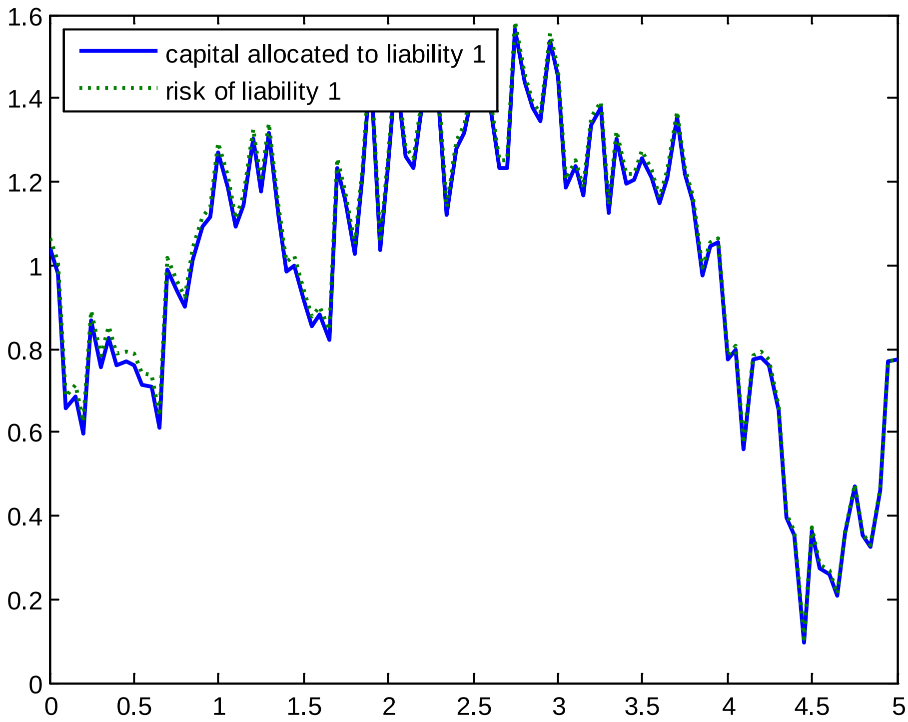

Figure 8: When $\boldsymbol{k = 0.9}$, risk measure and capital allocated to the first liability, $\mathbf{X_t^1}$.

CHAPTER V

CONCLUSION

In this study, the risk capital allocation problem of pooled instruments of risky positions was examined. Main risk measures and allocation methods that are used in the literature are reviewed.

In the previous chapter, an application of distortion measures is studied. It is shown that the allocation amount of a liability strongly depends on its correlation with aggregate liability. While the correlation increases, the benefit from pooling decreases. This means that when adding a risky portfolio into a pooled portfolio, the manager should seek the liabilities with low correlations (with aggregate portfolio) in order to get a diversification benefit.

For the cases when the events on which the liability processes are conditioned have zero probability, the k-number approach could be used without considering any distortion function. The k-number can be seen as risk aversion from diversification benefit. Since in real life, determining correlations among liabilities is a difficult job, one can use a higher k-number for considering worst-case scenario.

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APPENDIX

The Matlab codes of the simulation process used in Chapter IV are provided.

```
%   Dynamic Capital Allocation with Distortion Risk Measures
%   X_t = alpha*dt + beta*W_t

T=5; t(1)=0;
alpha = [0.2;0.3;0.4];          % Drift
beta = (1/3)*[sqrt(2) 0 sqrt(2);-sqrt(2) sqrt(2) 0;1 1 1];  %
Volatility
r = correl(beta)'               % Correlation r_i
X = brownian(100,alpha,beta,T); % Simulated path of individual
liabilities
Z = sum(X,2);                   % Aggregate liability

for i=1:100
    Y(i,:) = allocate(X(i,:),Z(i),alpha,beta,T,t(i),i);
    t(i+1) = i*0.05;
end
Y(101,:)=[X(101,1) X(101,1) X(101,2) X(101,2) X(101,3) X(101,3)
sum(X(101,:))];

X1=[Y(:,1) Y(:,2)];             % The capital allocated to liability
1 and its risk
X2=[Y(:,3) Y(:,4)];
X3=[Y(:,5) Y(:,6)];
% The risk of aggregate liability and total risks of individual
liabilities
X4=[Y(:,7) X1(:,2)+X2(:,2)+X3(:,2)];
```

```
% Correlation r_i between XT and ZT

function r = correl(b)
for i=1:3
    c = varcor(i,b,1,0)
    r(i) = c(1,2)/sqrt(c(1,1)*c(2,2));
end
```

```
% Variance-Covariance matrix

function res = varcor(i,b,T,t)
c(1,1)=sum(b(i,:).^2); c(1,2)=0;
for j=1:3
    for k=1:3
        c(1,2)=c(1,2) + b(i,j)*b(k,j);
    end
end
```

```
c(2,1)=c(1,2);
c(2,2)=sum(b(:,1)).^2+sum(b(:,2)).^2+sum(b(:,3)).^2;
res = (T-t)*c;
```

```
% brownian(N,b,sigma,T) simulates a one-dimensional Brownian motion
on [0,1]
% using normally distributed N steps

function [B s] = brownian(N,a,s,T)

t = (0:1:N)'/N;                          % t is the column vector [0
1/N 2/N ... 1]
W1 = [0; cumsum(randn(N,1))]/sqrt(N);    % Running sum of N(0,1/N)
variables
W2 = [0; cumsum(randn(N,1))]/sqrt(N);
W3 = [0; cumsum(randn(N,1))]/sqrt(N);
W=[W1 W2 W3];
t = t*T;
W = W*sqrt(T);

B = (a*t' + s*W')';                      % The Brownian Motion
s = B(N+1,:);                            % The final value of B.M.
```

```
% Capital Allocation

function s = allocate(X,Z,a,b,T,t,i)

az = sum(a);                             % Drift of Z
c1 = varcor(1,b,T,t);                    % Covariance matrix of
liability 1
c2 = varcor(2,b,T,t);
c3 = varcor(3,b,T,t);
XT = (X'+a*(T-t))';                      % Mean Xi
ZT = Z+az*(T-t);                         % Mean Z

% Now, we simulate 10000 future scenarios for the movement of each
% liabilities and use the final value to calculate the risk and
allocated
% capital for each of them
s1=0;s2=0;s3=0;s4=0;s5=0;s6=0;s7=0;
for q=1:1000
    [B s] = brownian(101-i,a,b,T-(i-1)*0.05);
    Rx = s + X;                          % Simulated XT value
    Rz = sum(s) + Z;                     % Simulated ZT value
    s1 = s1 + Rx(1)*g_u(Rz,ZT,sqrt(c1(2,2)));               % Allocation
of L1
    s2 = s2 + Rx(1)*g_u(Rx(1),XT(1),sqrt(c1(1,1)));         % Risk of L2
    s3 = s3 + Rx(2)*g_u(Rz,ZT,sqrt(c1(2,2)));
    s4 = s4 + Rx(2)*g_u(Rx(2),XT(2),sqrt(c2(1,1)));
    s5 = s5 + Rx(3)*g_u(Rz,ZT,sqrt(c1(2,2)));
    s6 = s6 + Rx(3)*g_u(Rx(3),XT(3),sqrt(c3(1,1)));
    s7 = s7 + Rz*g_u(Rz,ZT,sqrt(c1(2,2)));                  % Risk of
aggregate Liability
end
s = [s1 s2 s3 s4 s5 s6 s7]/1000;            % Expected results of
each X_t
```

```
% The derivative of updated exponential distortion function

function g = g_u(Z,ZT,v)
s = 1- normcdf(Z,ZT,v);         % Decumulative distribution function
of Z
g = (s+exp(-1)*(1-s))^(-1)-s*(1-exp(-1))*((1-exp(-1))*s+exp(-1))^(-2);
```