HUB LOCATION AND HUB NETWORK DESIGN

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by

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June 2009

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ABSTRACT

HUB LOCATION AND HUB NETWORK DESIGN

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The hub location problem deals with finding the location of hub facilities and allocating the demand nodes to these hub facilities so as to effectively route the demand between origin–destination pairs. Hub location problems arise in various application settings in telecommunication and transportation. In the extensive literature on the hub location problem, it has widely been assumed that the subgraph induced by the hub nodes is complete. Throughout this thesis we relax the complete hub network assumption in hub location problems and focus on designing hub networks that are not necessarily complete. We approach to hub location problems from a network design perspective. In addition to the location and allocation decisions, we also study the decision on how the hub network must be designed. We focus on the single allocation version of the problems where each demand center is allocated to a single hub node. We start with introducing the 3-stop hub covering network design problem. In this problem, we aim to design hub networks so that all origin–destination pairs receive service by visiting at most three hubs on a route. Then, we include hub network design decisions in the classical hub location problems introduced in the literature. We introduce the single allocation incomplete *p*-hub median, hub location with fixed costs, hub covering, and *p*-

hub center network design problems to the literature. Lastly, we introduce the multimodal hub location and hub network design problem. We include the possibility of using different hub links, and allow for different transportation modes between hubs, and for different types of service time promises between origin–destination pairs, while designing the hub network in the multimodal problem. In this problem, we jointly consider transportation costs and travel times, which are studied separately in hub location problems presented in the literature. Computational analyses with all of the proposed models are presented on the various instances of the CAB data set and on the Turkish network.

Keywords: Hub location, incomplete hub network design, *p*-hub median, *p*-hub center, hub cover, multimodal hub location.

ÖZET

ANA DAĞITIM ÜSLERİ İÇİN YER SEÇİMİ VE AĞ TASARIMI

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Ana Dağıtım Üssü (ADÜ) yer seçimi problemleri kaynak ve gidilecek yer arasında istenilen servisi sağlamak üzere ADÜ'lerin yerleştirilmesi ve talep noktalarının ADÜ'lere atanması problemlerini içermektedir. ADÜ yer seçimi problemlerinin çok çeşitli uygulamaları mevcuttur. Bu uygulamalar ulaşım ve telekomünikasyon alanlarında yoğunlaşmıştır. ADÜ yer seçimi literatüründeki birçok çalışmada tam serim bir ADÜ ağı varsayılmaktadır. Gerçek hayattaki çok çeşitli uygulamalarda tam serim bir ADÜ ağına gerek duyulmadığı gözlemlenmiştir. Bu çalışmada ADÜ yer seçimi problemlerindeki tam serim ADÜ ağı varsayımı gevşetilmiş ve ADÜ yer seçimi problemlerine ADÜ ağı tasarımı kararları da eklenmiştir. Bu bağlamda ilk olarak üç duraklı ADÜ kaplama problemi üzerinde çalışılmıştır. Bu problemde, kaynak ve gidilecek yer arasındaki servisin belirli bir zaman limiti içerisinde ve en fazla üç ADÜ'ye uğrayarak gerçekleşmesi sağlanmaktadır. Daha sonra, literatürde önerilen temel ADÜ yer seçimi problemlerine ADÜ ağı tasarımı kararları eklenmiştir. Yeni ADÜ yer seçimi ve ağ tasarımı problemleri tanımlanmış ve bu problemlere etkin matematiksel modeller önerilmiştir. Son olarak, çok yollu ADÜ yer seçimi ve ağ tasarımı problemi incelenmiştir. Bu problemde

literatürde ayrı olarak ele alınan maliyet ve servis süreleri birlikte göz önüne alınmış ve daha gerçekçi bir matematiksel model önerilmiştir. Bu model ayrıca, ADÜ'ler arasında farklı taşıma yolları kullanılmasına ve farklı ikililerin farklı servis süreleri içinde servis almasına olanak sağlamaktadır. Önerilen tüm modeller literatürde yaygın olarak kullanılan CAB veri seti ve Türkiye verisi üzerinde denenmiş ve etkili sonuçlar alınmıştır.

Anahtar Kelimeler: ADÜ yer seçimi problemi, ADÜ ağı tasarımı, Modelleme.

Anneme

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I'm dedicating this thesis to my mother Nural Alumur, for all her sacrifice and for raising me to become the person that I am today.

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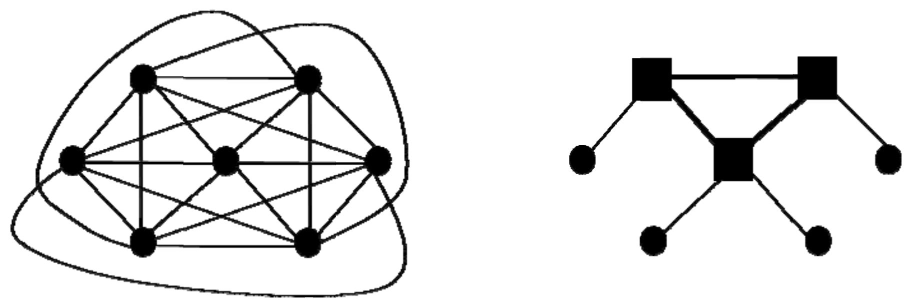
Chapter 1

INTRODUCTION

Hubs are special facilities that serve as switching, transshipment, and sorting points in many-to-many distribution systems. Instead of serving each origin–destination pair with a direct link, hub facilities consolidate and disseminate flow. Establishing hub facilities thus results in a reduction in the number of links in the networks. Flows from the same origin with different destinations are consolidated on their route to the hub and are combined with flows from different origins but same destinations. The consolidation is on the route from the origin to the hub and from the hub to the destination as well as between hubs. This flow consolidation allows the hub facilities to take advantage of economies of scale. Figure 1.1 presents a comparison of a completely interconnected network with a hub network. As it can be observed from this figure, the number of links required to transport flow between demand centers is significantly fewer in hub networks, since the flow is transported via hub facilities.

The hub location problem is concerned with locating hub facilities and allocating non-hub nodes (demand centers) to these located hubs in order to route the flow between origin–destination pairs. The distinguishing features of the hub location problem from the basic facility location problems are thoroughly discussed in O'Kelly (1998).



(a) (b)

Figure 1.1 (a) A completely interconnected network with 7 demand centers, (b) a hub network with 3 hubs and 7 demand centers.

Hub location problems arise in various application settings in telecommunication and transportation. Transportation applications of hub location problems include air passenger travel, air freight travel, express shipments and postal operations. In these applications, demand is usually specified as flows of passengers or goods between origin–destination pairs and these flows are transported with some type of vehicle. Telecommunication applications include computer communication, telephone networks, video teleconferences, and distributed computer processing. In this area, demand is for transmission of information (such as data, voice, video, etc.) which occurs through a variety of media (such as telephone lines, fiber optic cables, etc.). (Campbell et al., 2002).

There are two basic types of hub networks in the literature – single allocation and multiple allocation hub networks. They differ in how non-hub nodes (demand centers) are allocated to hubs. In single allocation hub networks, all the incoming and outgoing traffic of every demand center is routed through a single hub; in multiple allocation hub networks, each demand center can receive and send flow through more than one hub. Since optimal allocations are affected by hub locations and optimal hub locations are affected by allocation decisions, location and allocation problems must be considered together in designing hub networks.

There are three basic assumptions present in standard hub location problems:

1. The hub network is complete with a direct link between every hub pair,

2. there is economies of scale incorporated by a discount factor (usually referred to as α) for using the inter-hub connections, and

3. no direct service (between two non-hub nodes) is allowed; that is, the flow between all origin–destination pairs are to be routed using at least one hub.

Throughout this thesis we relax the first assumption in hub location problems and focus on designing hub networks that are not necessarily complete. We approach hub location problems from a network design perspective. In addition to the location and allocation decisions, we also study the decision on how the hub network must be designed. We only focus on the single allocation version of the problems where each demand center is allocated to a single hub node.

We start with introducing the 3-stop hub covering network design problem. In this problem, motivated by a specific cargo delivery company operating in Turkey, we aim to design hub networks so that all origin–destination pairs receive service by visiting at most three hubs on a route. Then, we include hub network design decisions in the standard hub location problems introduced in the literature. We introduce the single allocation incomplete p-hub median, hub location with fixed costs, hub covering, and p-hub center network design problems to the literature. Lastly, we introduce the multimodal hub location and hub network design problem. We allow for different transportation modes between hubs, and for different types of service time promises between origin–destination pairs, while designing the hub network in the multimodal problem. In this problem, we also consider transportation costs and travel times simultaneously, which are studied separately in hub location problems presented in the literature.

The outline of this thesis is as follows. In the next chapter, we present an overview of the hub location literature. In Chapter 3, we introduce the 3-stop hub covering network design problem. In Chapter 4, we study the incomplete hub network design problems with transportation cost objectives and introduce the incomplete p-hub median and hub location with fixed costs network design problems. Chapter 5 presents the incomplete hub covering and p-hub center network design problems. In Chapter 6, we propose the multimodal hub location and hub network design problem. The thesis concludes with some final remarks and future research directions in Chapter 7.

Chapter 2

THE HUB LOCATION LITERATURE

In this chapter we classify and review the hub location literature. The problem of hub location attracted many researchers over the past two decades. The interest in hub location area is still strong with several papers pending. Recently a special issue of *Computers and Operations Research* is dedicated to new developments on hub location.

In hub location problems, there is a given node set with n nodes and the set of origins, destinations and potential hub locations are identified. The flow between origin–destination pairs, an attribute of interest associated with flows (cost, time, distance, etc.) and the hub-to-hub transportation discount factor α are known. The aim in hub location problems is to find the location of hub nodes and the allocation of demand nodes to these located hub nodes.

Perhaps Goldman (1969) is the first to address the hub location problem. However, interest in hub location began with the pioneering work of O'Kelly (1986a,b, 1987). O'Kelly (1987) presented the first recognized mathematical formulation for a hub location problem by studying airline passenger networks. His formulation is referred to as the *single allocation p-hub median* problem. Given n demand nodes, flow between origin–destination pairs, and the required number of hubs (p), the objective is to minimize the total transportation cost (time, distance, etc.) to serve the given set of flows. He assumed that the hub network is complete with a link between every hub pair; that there is economies of scale incorporated by a discount factor (α) for using the inter-hub connections; and that no direct service (between two non-hub nodes) is allowed.

Let w_{ij} be the flow between nodes i and j, and c_{ij} be the transportation cost of a unit of flow between i and j. Define x_{ik} as 1 if node i is allocated to hub at k, and 0 otherwise; x_{kk} takes on the value 1 if node k is a hub and it is 0 otherwise. The integer programming formulation of the single allocation p-hub median problem given by O'Kelly (1987) is:

$$\text{Minimize} \sum_i \sum_j w_{ij} \left(\sum_k c_{ik} x_{ik} + \sum_m c_{jm} x_{jm} + \alpha \sum_k \sum_m c_{km} x_{ik} x_{jm} \right) \tag{2.1}$$

subject to

$$(n - p + 1) x_{kk} - \sum_i x_{ik} \geq 0 \qquad \forall\, k \tag{2.2}$$

$$\sum_k x_{ik} = 1 \qquad \forall\, i \tag{2.3}$$

$$\sum_k x_{kk} = p \tag{2.4}$$

$$x_{ik} \in \{0,1\} \qquad \forall\, i, k \tag{2.5}$$

The objective function, (2.1), calculates the cost of flow. α in the third term is the economies of scale factor; the cost of flow between the hub facilities must be smaller than the original costs since hub facilities concentrate flow, so $0 \leq \alpha$

< 1. Note that this objective function is quadratic due to the fact that the hub-to-hub discount is a product of the allocation decisions.

Constraint (2.2) ensures that no node is assigned to a location unless a hub is opened at that node. As suggested by Skorin-Kapov et al. (1996) this constraint can be replaced with:

$$x_{ik} \leq x_{kk} \qquad \forall\, i, k \qquad (2.6)$$

Constraints (2.3) and (2.5) guarantee that each node is allocated to exactly one hub, and Constraint (2.4) states that the number of hubs to be located is p.

The quadratic nature of the objective functions in hub location problems distinguishes them from the classical location problems. In standard location problems when the locations of the facilities are determined each demand node receives service from their nearest facility. For hub location problems, when it comes to allocation decisions, the nearest allocation strategy – assigning each demand node to its nearest hub – does not necessarily give optimal solutions. Thus the optimal allocations of demand centers to the located hubs must also be determined.

The earliest reviews on hub location are by O'Kelly and Miller (1994) and Campbell (1994a). O'Kelly and Miller (1994) provided real world examples which violated some of the assumptions of the standard hub location model. They proposed eight classes of hub location problems corresponding to different decisions on allocation, hub interconnection, and non-hub routes; they included references and examples. Campbell (1994a) presented an extensive survey on the hub location problem that included both the transportation and computer-communication oriented models. Klincewicz (1998) offered another extensive review involving facility location, network design, telecommunication, computer systems, and transportation aspects in hub location. O'Kelly (1998) reviewed some distinctive features of hub networks with special attention paid to the contrast between air passenger and air express freight applications. Later, Bryan and O'Kelly (1999) presented an analytical review of the studies on hub networks for passenger airlines and package delivery systems. A comprehensive review of hub location problems is a book chapter by Campbell et al. (2002). More recently, Alumur and Kara (2008a) presented a survey on hub location problems.

O'Kelly (1987) introduced a data set based on the airline passenger interactions between 25 U.S. cities in 1970 evaluated by the Civil Aeronautics Board (CAB). This data set has been used by almost all of the hub location researchers and will be referred to as the CAB data set. (Figure A.1 in Appendix A shows the geographical locations and names of the cities in the CAB data set.) Another commonly used data set is the Australia Post (AP) data set (first used in Ernst and Krishnamoorthy, 1996). AP data set is based on a postal delivery in Sydney, Australia and consists of 200 nodes representing postal districts. The main difference of the AP data set from the CAB data set other than the number of nodes is that the flow matrix of the AP data set is not symmetrical. More recently, a Turkish network data set is introduced (Tan and Kara, 2007, Yaman et al., 2007). (Figure A.2 in Appendix A shows the geographical locations of the 81 demand centers and names of the 16 candidate hub locations on the Turkish network.) The data on the CAB, AP, and Turkish network data sets are all available through the OR library (Beasley, 1990).

The hub location problem is also studied in telecommunication network design (also called backbone/tributary network design). The hub location problem in telecommunication network design differs from the classical hub location literature. The objective in telecommunication network design is to minimize the total costs of building the hub networks rather than the minimization of transportation costs. The reader may refer to Klincewicz (1998) for an extensive review on hub location in network design, telecommunication and computer systems.

Almost all of the hub location models defined in the literature have analogous location versions. Campbell (1994b) defined a *p*-hub median and *p*-hub center on a network analogous to a *p*-median and *p*-center. He introduced four types of hub location problems to the literature: *p*-hub median, hub location with fixed costs, *p*-hub center, and hub covering problems. Our review of the literature follows this classification. The next four sections of this chapter are devoted in turn to the *p*-hub median problem, the hub location problem with fixed costs, the *p*-hub center problem, and hub covering problems. In the last section of this chapter, we present some other hub location studies that do not fit into the previous categories.

2.1 The *p*-hub Median Problem

The objective of the *p*-hub median problem is to minimize the total transportation costs needed to serve the given set of flows, given n demand nodes, flow between origin–destination pairs, and the number of hubs to locate (p). The studies considering the *p*-hub median problem are analyzed here in two different subsections: single allocation and multiple allocation.

2.1.1 Single Allocation

Campbell (1994b) presented the first linear integer programming formulation for the single allocation *p*-hub median problem. If n is the given number of demand nodes, his formulation has $O(n^4)$ variables and constraints. Skorin-

Kapov et al. (1996) stated that the LP relaxation of Campbell's (1994b) formulation resulted in highly fractional solutions. They proposed a new mixed integer formulation with O(n^4) variables and O(n^3) constraints. Skorin-Kapov et al. (1996) also presented the first attempt at optimally solving the single allocation *p*-hub median problem.

O'Kelly et al. (1996) presented a formulation that assumed a symmetric flow data, thus further reducing the size of the problem. An important aspect of O'Kelly et al. (1996) is its discussion of the sensitivity of the solutions to the inter-hub discount factor α. Sohn and Park (1998) formulation presents a further reduction in the number of variables and constraints for the case when the unit flow cost is symmetric and proportional to the distance.

Ernst and Krishnamoorthy (1996) proposed a different linear integer programming formulation which requires fewer variables and constraints in an attempt to solve larger problems. They treated the inter-hub transfers as a multicommodity flow problem where each commodity represents the traffic flow originating from a particular node. The authors observed and modeled how Australia Post uses different discount factors for collection and distribution. Their formulation has O(n^3) variables and O(n^2) constraints. So, they were able to reduce the problem size from the previous formulation by Skorin-Kapov et al. (1996), both in terms of variables and constraints, by a factor of *n*.

Ebery (2001) presented another formulation for the single allocation *p*-hub median problem that requires O(n^2) variables and O(n^2) constraints. This formulation uses fewer variables than all of the other models previously presented in the literature. However, the computational time required to solve this new formulation was greater than that required to solve the formulation in Ernst and Krishnamoorthy (1996).

The p-hub median problem is NP-hard. Moreover, for the single allocation problem, even if the locations of the hubs are fixed, the allocation part of the problem remains NP-hard (Kara, 1999).

Various heuristics are proposed for the single allocation p-hub median problem. Earlier studies include the enumeration based heuristics of O'Kelly (1987), an exchange heuristic based on local improvement by Klincewicz (1991), a tabu search and a GRASP (Greedy Randomized Search Procedure) heuristic by Klincewicz (1992), and a tabu search heuristic by Skorin-Kapov and Skorin-Kapov (1994). O'Kelly et al. (1995) presented a lower bounding technique based on the linearization of the quadratic objective function, where distances are assumed to satisfy the triangle inequality. Campbell (1996) used the idea that the multiple allocation p-hub median solutions provide a lower bound for the optimal solution of the single allocation version to propose two new heuristics for the single allocation problem. Later, Ernst and Krishnamoorthy (1996) developed a simulated annealing heuristic and Pirkul and Schilling (1998) developed an efficient Lagrangean relaxation method, which finds tight upper and lower bounds in a reasonable amount of CPU time. Pirkul and Schilling (1998) were able to obtain the tightest bounds of any heuristic up to that date.

Ernst and Krishnamoorthy (1998b) proposed a branch-and-bound algorithm for the single allocation p-hub median problem in which they solved shortest-path problems to obtain lower bounds. They were able to solve the largest single allocation problems to that date to optimality with their new branch-and-bound algorithm.

As a synthesis of the existing literature, in terms of required number of variables and constraints, Ebery (2001) provided the best mathematical formulation for the single allocation p-hub median problem. However, the best mathematical formulation in terms of empirical computation time requirement is that of Ernst and Krishnamoorthy (1996). The most computationally efficient exact solution procedure is the shortest-path based branch-and-bound algorithm presented in Ernst and Krishnamoorthy (1998b). A very effective heuristic is the Lagrangean relaxation based heuristic presented in Pirkul and Schilling (1998). Finally, among the best metaheuristics are the tabu search heuristic presented in Skorin-Kapov and Skorin-Kapov (1994), and the simulated annealing heuristic presented in Ernst and Krishnamoorthy (1996).

2.1.2 Multiple Allocation

In the multiple allocation problem each demand center can receive and send flow through more than one hub; that is, each demand center can be allocated to more than one hub. In the multiple allocation p-hub median problem, if the hub locations are fixed the allocation decisions are straight forward: each pair of nodes sends flow from their shortest paths via the given hubs. Thus, after the locations of the hubs are determined one may solve the optimal allocation sub-problem by solving an all-pairs shortest path algorithm. However, as already mentioned, for the single allocation version, the allocation problem still remains NP-hard even if the hub locations are fixed (Kara, 1999).

Campbell (1992) was the first to formulate the multiple allocation p-hub median problem as a linear integer program. Skorin-Kapov et al. (1996) presented another formulation resulting in tighter LP relaxations. Ernst and Krishnamoorthy (1998a) proposed a more effective formulation for the multiple allocation *p*-hub median problem based on the idea that they have proposed for the single allocation version in Ernst and Krishnamoorthy (1996). Their new formulation has O(n^3) variables and O(n^2) constraints. Boland et al. (2004) identified some characteristics of the optimal solutions to develop preprocessing techniques and tightening constraints, and applied these to the formulation proposed in Ernst and Krishnamoorthy (1998a).

Ernst and Krishnamoorthy (1998a) and Ernst and Krishnamoorthy (1998b) presented two new branch-and-bound algorithms for the multiple allocation *p*-hub median problem. In Ernst and Krishnamoorthy (1998a), they obtained lower bounds by using LP relaxations, whereas in Ernst and Krishnamoorthy (1998b) they obtained lower bounds by solving shortest path problems rather than LP relaxations. Their second algorithm turned out to be superior in computational analysis.

For the multiple allocation *p*-hub median problem the best formulation in terms of CPU time requirement is the one proposed in Boland et al. (2004) and the best exact solution algorithm is the branch-and-bound method proposed in Ernst and Krishnamoorthy (1998b).

2.2 The Hub Location Problem with Fixed Costs

In *p*-hub median problems, the fixed costs of opening hub facilities are ignored. O'Kelly (1992) introduced the single allocation hub location problem with fixed costs where the number of hubs is a decision variable.

In addition to having single/multiple allocation versions, since the number of hubs is not fixed it is possible to have uncapacitated/capacitated hub location problems with fixed costs. Campbell (1994b) presented the first linear integer programming formulations for single/multiple allocation, uncapacitated/ capacitated hub location problems.

Several studies looked at the uncapacitated single allocation hub location problem. Abdinnour-Helm and Venkataramanan (1998) presented a new quadratic integer formulation based on the idea of multi-commodity flows in networks. Abdinnour-Helm (1998) proposed a heuristic method based on a hybrid of genetic algorithms and tabu search. Labbé and Yaman (2004) derived a family of valid inequalities that generalizes the facet-defining inequalities and that can be separated in polynomial time. Topcuoglu et al. (2005) proposed a genetic algorithm for the uncapacitated single allocation hub location problem. Later, Cunha and Silva (2007) proposed another genetic algorithm combined with a simulated annealing heuristic and Chen (2007) proposed a hybrid heuristic based on simulated annealing and tabu search. Recently, Silva and Cunha (2009) developed a multi start tabu search heuristic and a two-stage tabu search heuristic for the uncapacitated single allocation hub location problem. These heuristics are the best heuristics in terms of solution quality that are proposed for the problem up to this date. Silva and Cunha (2009) were also able to report, for the first time, the optimal solutions of almost all of the benchmark problems given by CAB and AP data sets, by using CPLEX.

For the uncapacitated multiple allocation version, Klincewicz (1996) presented an algorithm based on dual-ascent and dual adjustment techniques within a branch-and-bound scheme. Mayer and Wagner (2002) developed a new branch-and-bound method: the HubLocater. Cánovas et al. (2007) presented a heuristic based on a dual-ascent technique. Through computational analysis using CAB and AP data sets, they solved instances with up to 120 nodes.

These are the best computational results for the uncapacitated multiple allocation hub location problem up to now.

Hamacher et al. (2004) determined the dimension and derived some classes of facets for the polyhedron of the uncapacitated multiple allocation hub location problem. Marín (2005b) presented some facet-defining valid inequalities for the uncapacitated hub location problem with costs satisfying triangle inequality. Marín et al. (2006) presented a new formulation which is a generalization of the earlier formulations and relaxes the assumption of having a cost structure satisfying triangle inequality. By using polyhedral results they were able to tighten and reduce the number of constraints. Their formulation outperformed all of the previous formulations.

Aykin (1994) presented the capacitated version of the hub location problem with fixed costs where hubs have limited capacities. Ernst and Krishnamoorthy (1999) presented two new formulations for the capacitated single allocation hub location problem. Their formulations are modified versions of the previous mixed integer formulations developed for the *p*-hub median problem. They applied the capacity restrictions only to the traffic arriving at the hub directly from non-hub nodes. This capacity definition is usually used in postal service applications in order to represent the sorting capacity of hubs. Ernst and Krishnamoorthy (1999) also proposed two heuristics for the problem. Labbé et al. (2005) studied the capacitated single allocation hub location problem where each hub has a fixed capacity in terms of the traffic that passes through it. They investigated polyhedral properties of this problem and developed a branch-and-cut algorithm.

Costa et al. (2008) suggested a different approach to the capacitated single allocation hub location problem. Instead of using capacity constraints on the amount of flow processed in the hubs the authors introduced a second objective function into their mathematical model, which minimizes the time hubs take to process flows. They considered two different bi-criteria problems. In addition to minimizing total cost in both of the problems, in the first one they minimized the total time of processing the flow (service time) at the hubs and in the second one they minimized the maximum service time on the hubs.

Ebery et al. (2000) considered the multiple allocation version of the capacitated hub location problem and proposed a formulation which is very similar to the one proposed in Ernst and Krishnamoorthy (1998a) for the multiple allocation *p*-hub median problem. Boland et al. (2004) outlined some properties of the optimal solutions for both the uncapacitated and capacitated multiple allocation hub location problems. Marín (2005a) presented a new formulation for the capacitated multiple allocation hub location problem based on the same idea used in Ebery et al. (2000) but exploiting some of the ideas used in Marín et al. (2006) to reduce the size.

Considering that the *p*-hub median models are a special case of the hub location problem with fixed costs, there are more studies on solving the fixed cost problem (both heuristic and exact). For single/multiple allocation and capacitated/uncapacitated versions, different integer programming models, branch-and-bound algorithms, and heuristics have been developed. The *p*-hub median and hub location with fixed costs problems are the most frequently addressed hub location problems in the literature.

2.3 The *p*-hub Center Problem

The *p*-hub center problem is a minimax type problem which is analogous to the *p*-center problem. The aim of the *p*-hub center problem is to locate *p* hubs, and to allocate all non-hub nodes to the located hubs to minimize the maximum cost (time, distance) between origin–destinations pairs.

Campbell (1994b) was first to formulate and discuss the *p*-hub center problem in the hub location literature. Later, Kara and Tansel (2000) provided various linear formulations for the single allocation *p*-hub center problem. They provided three different linearizations of the Campbell (1994b) model together with a new formulation that they proposed. Their new formulation has $O(n^2)$ variables and $O(n^3)$ linear constraints.

Kara and Tansel (2000) also provided a combinatorial formulation of the single allocation *p*-hub center problem and proved that it is *NP*-complete by a reduction from the dominating set problem.

Ernst et al. (2009) developed a new formulation for the single allocation *p*-hub center problem. This formulation has $O(n^2)$ variables and $O(n^2)$ linear constraints. Even though this model has *n* more continuous variables than the model proposed in Kara and Tansel (2000), it has fewer constraints. Computational analysis using CPLEX on the CAB and AP data sets showed that the Ernst et al. (2009) formulation is better in terms of CPU time requirements.

Baumgartner (2003) investigated the polyhedral properties of the single allocation *p*-hub center problem and proposed a branch-and-cut algorithm. Pamuk and Sepil (2001) presented a single-relocation algorithm with tabu-search. Hamacher and Meyer (2006) proposed solving hub covering problems with binary search for the solution of the *p*-hub center problem. Recently, Meyer et al. (2009) proposed a two-phase algorithm for the single allocation *p*-hub center problem. They determined the set of potential optimal hub locations by using a shortest path based branch-and-bound algorithm followed by an allocation phase. They also developed a heuristic based on an ant colony optimization approach to provide good upper bounds for their branch-and-bound algorithm. They were able to solve instances consisting of up to 400 nodes optimally, which are the largest problems solved in the literature to date.

Ernst et al. (2009) also studied the multiple allocation *p*-hub center problem. They proposed a new formulation and proved that the problem is *NP*-hard. For the multiple allocation version they proposed a shortest path based branch-and-bound algorithm which is similar to the algorithm developed for the multiple allocation *p*-hub median problem presented in Ernst and Krishnamoorthy (1998b).

Sim et al. (2009) studied a stochastic version of the *p*-hub center problem where they treated the travel times as random variables. In their model, the probability of providing service within the time limit to be minimized must be higher than a given service level parameter.

Gavriliouk (2009) proposed heuristic procedures for hub location problems based on aggregation techniques. She applied this heuristic for single and multiple allocation *p*-hub center problems.

2.4 Hub Covering Problems

In facility covering problems, demand nodes are considered to be covered if they are within a specified distance of a facility that can serve their demand.

Similarly in hub location, the origin–destination pair (o,d) is covered by hubs k and m if the cost (time, distance) from o to d via k and m does not exceed a specified value. The hub covering problem is to locate hubs and to decide on the allocations to cover all demand such that the cost of opening hub facilities is minimized.

Campbell (1994b) presented the first mixed integer formulation for the hub covering problem. Later, Kara and Tansel (2003) studied the single allocation hub covering problem and proved that it is *NP*-hard. The authors presented and compared three different linearizations of the original quadratic model as well as presenting a new linear model. Wagner (2008) proposed new formulations for both single and multiple allocation hub covering problems. By his proposed preprocessing techniques he rules out some hub assignments and thus the formulations require less number of variables and constraints than that of Kara and Tansel (2003) formulations. He further improved these formulations with a procedure for aggregating some constraints.

Ernst et al. (2005) presented a new formulation for the single allocation hub covering problem. Their new formulation performs better in terms of CPU time requirement than the Kara and Tansel (2003) formulation.

Ernst et al. (2005) also studied the multiple allocation version of the hub covering problem. They proposed two new formulations and an implicit enumerative method for this problem.

Hamacher and Meyer (2006) compared various formulations of the hub covering problem. They analyzed the feasibility polyhedron and identified some facet-defining valid inequalities. They solved the hub covering problem for a given cover radius β and then iteratively reduced β to obtain the optimum solution of the p-hub center problem.

2.5 Other Studies

In classical hub location problems, the hub-to-hub flows are typically discounted by a fixed discount factor α, such that $0 \leq \alpha < 1$. However, the number and location of hubs may be seriously affected by the value chosen for α. Most hub location models have assumed that this inter-hub discount factor is not dependent on the amount of flow using the links. O'Kelly and Bryan (1998) pointed out that "the assumption of flow-independent costs not only miscalculates total network cost but may also erroneously select optimal hub locations and allocations". They proposed a non-linear cost function which allows costs to increase at a decreasing rate as flows increase. There are some other studies proposing different cost functions to apply the economies of scale discount factor more realistically. These studies include Bryan (1998), Horner and O'Kelly (2001), Klincewicz (2002), Kimms (2005), Racunicam and Wynter (2005), and Cunha and Silva (2007).

Considering that the standard hub location models were developed mainly for airline applications, some more cargo-specific models have been developed recently. Kara and Tansel (2001) observed that the time spent at hubs for unloading, loading and sorting operations (transient times) may constitute a significant portion of the total delivery time for cargo delivery systems. They proposed new models, called *the latest arrival hub location problem*, for systems where the transient times are incorporated. Several versions of the latest arrival hub location problem are possible: single or multiple allocation minimax, covering and minisum versions.

The focus in Kara and Tansel (2001) was on the single allocation minimax (center) version. Later, Tan and Kara (2007) studied the latest arrival hub covering problem on an application for the cargo delivery sector in Turkey. Yaman et al. (2007) proposed a latest arrival hub center model which incorporates multiple stopovers and vehicle routes. A paper by Çetiner et al. (2007) studied a combined hubbing and routing problem in postal delivery systems, where they presented a case study using the Turkish postal delivery system data.

Nickel et al. (2001) presented new hub location model applicable to urban public transportation networks. They considered the hub location problem as a network design problem and incurred a fixed cost for locating hub arcs. Podnar et al. (2002) considered a new network design problem where they do not locate hubs but they decide on the links with reduced unit transportation costs. Yoon and Current (2008) studied the multiple allocation incomplete hub network design problem with fixed and variable arc costs. They also considered direct connections between non-hub nodes and incurred variable arc costs associated with demand on the arcs.

Campbell et al. (2005a) introduced a new model called the *hub arc location* model which assumes neither a fully interconnected hub network nor that the flow on every hub-to-hub arc is discounted. Rather than locating hub facilities, their model locates hubs arcs which have reduced unit costs. A companion paper, Campbell et al. (2005b), provided integer programming formulations for four special cases and optimal solution algorithms for these new hub-arc problems. Campbell et al. (2003) implemented the enumeration-based algorithm presented in Campbell et al. (2005b) in a parallel environment in an attempt to optimally solve larger hub arc location problems. Campbell (2009) proposed time definite models for both multiple allocation *p*-hub median and hub arc location problems. He introduced a constraint for the maximum service distance between origin–destination pairs in his models.

Contreras et al. (2009a) introduced the *tree of hubs location problem*. This problem is a variant of the single allocation p-hub median problem in which the network connecting the hub nodes is an undirected tree. They proposed an $O(n^3)$ integer programming formulation for this problem. In a companion paper, Contreras et al. (2009b), the authors introduced an $O(n^4)$ formulation and a Lagrangean relaxation method based on this formulation to obtain tight upper and lower bounds.

Marianov and Serra (2003) modeled a hub network behaving as an M/D/c queuing network. They proposed capacity constraints based on the probability of waiting customers in the system.

Elhedhli and Hu (2005) considered congestion at hubs and proposed a non-linear convex cost function for the objective function of the single allocation p-hub median problem. Via comparison with the non-congestion problem on the CAB data set, the authors stated that the congestion model results in a more balanced distribution of flows through hubs. Similarly, Camargo et al. (2009) explored the congestion effects written as a convex cost function but addressing the multiple allocation hub location problem.

Some studies considered the hub location problem in a competitive environment. These studies include Marianov et al. (1999), Eiselt and Marianov (2009), and Sasaki et al. (2009).

In addition to the hub applications in airline transportation and postal delivery networks, some studies investigated the use of hub networks in marine and railway transportation as well. The difference in railway applications is that the main focus is on routing and scheduling of the trains rather than the location of the hubs. One may refer to Crainic and Laporte (1997) and Cordeau et al. (1998) for reviews related to railway transportation.

Chapter 3

THE 3-STOP HUB COVERING NETWORK DESIGN PROBLEM

In this chapter, we introduce the 3-stop hub covering network design problem. The outline of this chapter is as follows. Section 3.1 provides the motivation and definition of the problem. Section 3.2 presents and explains the proposed mathematical model. In Section 3.3, some linearizations of the model are introduced. The last section compiles the computational analysis on both the CAB data set and the Turkish network, together with some concluding remarks.

3.1 Motivation and Problem Definition

In an attempt to model real life hub location problems encountered in the cargo sector, we aim to provide a tool for designing cost-effective hub networks for cargo companies. In order to observe the real life requirements in this sector, we held many interviews with various cargo companies operating in Turkey. We then found out that many of the hub location problems proposed in the literature lack some real life requirements from this sector. In this section, we present our main observations from the cargo sector and then we will define our problem based on these observations.

CHAPTER 3. THE 3-STOP HUB COVERING NETWORK DESIGN PROBLEM

In cargo applications, the transportation of cargo from origin to destination is handled by operation centers. The journey of a cargo starts from a branch office. A customer either takes his cargo to the branch office of a cargo firm or phones the firm for pick-up. The collected cargo needs to be sorted at operation centers. Thus, branch offices are allocated to operation centers. At the end of each day, a branch office sends its whole cargo to its assigned operation center. At the operation center, the cargo is sorted according to the destination and is loaded into larger and more specialized vehicles based on the destination. When the cargo from every branch office allocated to that operation center is received, the vehicles are sealed and start their routes. These routes are previously determined by the cargo company, so that at the end, each cargo is transported to the operation center of its destination branch office. At the end of the journey, the branch offices pick up their cargo from the operation centers by themselves, and, finally, cargo reaches its destination point.

Because cargo companies use more special, faster, and larger trucks travelling between operation centers, economies of scale are generated by this transportation. This structure used by cargo companies is precisely the same as the hub network structure. Therefore, we identify the branch offices of a cargo company as demand points and the operation centers as hubs. As each branch office is allocated to a single operation center, in most of the cargo firms, we consider a single-allocation structure. In general this hub network structure is similar for most of the cargo companies; however, we note that each cargo company may have its own characteristics or requirements.

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Through interviews with major cargo firms in Turkey, we determined that the cargo firms' main objective is customer satisfaction. Customer satisfaction in this sector is directly related to reliability and guaranteed service time. In practice, the quicker and safer you send the cargo, the more likely the customers are to be satisfied. Most of the national and world wide cargo companies operate on a time basis. They provide different services to customers based on different delivery time guarantees. Thus, in this sector, time is a major concern. In reality, both the establishment of hubs and using hub links incurs some cost, while guaranteeing service time. Because it was observed that service time is the primary objective for cargo companies, in this study, the service time is treated as a hard constraint rather than as an objective.

Truck synchronization is an important concern in designing hub networks for cargo companies. If a cargo truck is to pick the cargo from a hub node on its route, the cargo consolidated at this hub cannot be transported before the truck arrives. Thus, the cargo needs to wait for that truck to arrive. Or conversely, the cargo may not be ready at a hub when the truck arrives. Given that there is an initial service time guarantee, a company needs to consider these waiting times so that the cargo is delivered within the promised service time.

In this study, while building our model, we focused on the needs of a major cargo company operating in Turkey, which we refer to as Company A. The Company does not wish to share its name or the details of its hub network for reasons of confidentiality. Company A is among the largest cargo companies operating in Turkey. The company provides service between every city pair in Turkey. The company administrators believe that building hub facilities increases their service quality. Given that the company uses a high number of hubs, sending separate trucks from a hub to all other hubs is quite costly in terms of investment on the total number of trucks. Thus, they force some trucks to visit intermediate hubs to decrease this total investment cost. Hence, the company currently employs an incomplete hub network structure. Through our interviews with other companies in the region we found that almost all of the cargo firms operating in Turkey employ an incomplete hub network structure. The incomplete hub network design problem is commonly encountered in the cargo sector. Therefore, the basic assumption in the hub location literature of building complete hub networks (Assumption 1 introduced in Chapter 1) is not valid in these applications.

A general concern of the cargo companies is the safety of the cargo. Company A wishes to ensure that the cargo of each customer will arrive at its destination at the guaranteed time without any loss or damage to the cargo. For safety reasons, the cargo trucks travelling between hubs are sealed at the beginning of every route and unsealed at each stop at a hub. While using an incomplete hub network structure, a sealed truck can be unsealed at a hub other than the destination hub of a cargo in that truck. In these in-between stops at hubs a cargo may be mistakenly unloaded resulting in a delay in the service time or may get lost. Even though such instances are rarely met, many precautions are taken by the company to prevent any loss or delay of the cargo. One of their precautions is that they want to minimize the number of intermediate hub stops on any route. In a complete hub network structure, cargo trucks visit at most two hubs on a route. They are sealed in the origin hub and unsealed at the destination hub. In order to reduce the operational costs, Company A uses an incomplete hub network structure. On the other hand, regards to safety, a cargo truck is allowed to make at most 1 additional stop in travelling between two hubs. So, they use a hub network on which each origin–destination pair receives service by visiting at most three hubs on a route.

With these observations, in this study, we propose a new mathematical model. This new model determines the location of hubs, allocates demand centers to these hubs, and designs a hub network by relaxing the assumption of having a fully interconnected hub network. We formulate a single-allocation hub covering model that permits visiting at most three hubs on a route. We have also considered the possible waiting times at the in between hub nodes while modeling the problem. The model minimizes the total costs, including the costs of establishing hubs and hub links, subject to a time limit on the maximum service time.

The proposed model is applicable for all the cargo companies operating on a time basis in addition to the ones operating in Turkey. By the use of our model it may be realized that designing complete hub networks is cost wise inefficient, while there is no contribution to the service time guarantee. On the other hand, using at most a 3-hub stop strategy rather than a complete (2-hub stop) one may decrease the investment on the total number of trucks considerably, while not disregarding safety. Since our model also takes the truck synchronization into account, it is possible to provide the same service, for example, to a network consisting of 4 hubs with 4 trucks in contrast to a complete hub network requiring 12 trucks.

Many additional special cases of building incomplete hub networks are proposed for different applications in the literature. For example, in telecommunication literature designing different hub network (usually referred as backbone network) topologies such as star, ring, tree, and path are considered. The reader may refer to Klincewicz (1998) for such applications. For some applications, it is desirable to use paths with few numbers of edges in telecommunication. Dahl (1999) and Dahl and Johannessen (2004) pointed out the need for using few edges in paths in order to avoid unacceptable delay and to increase reliability. The constraint on the number of edges to be visited in between any origin–destination pair is referred to as hop-constraints. Dahl (1999) studied the *k*-hop constrained problem and the related polyhedra. Dahl and Johannessen (2004), on the other hand, studied the 2-hop constraint problem on a given network and proved its NP-hardness. They provided a path based formulation of the problem and studied its polyhedral. The 2-hop constraint idea is very similar to our 3-hub stop idea. The former restricts the number of edges to be visited to two while we restrict the number of hub nodes to be visited to three, equivalently number of hub edges to be two.

In this study, some computational analysis on the Turkish network is provided. The model was also tested on the CAB data set, which is a benchmark data set used for hub location problems. It was shown through application of the well-known CAB data set that, in some cases, there is no need for a complete hub network, even for the tightest values of service time requirements.

3.2 Mathematical Model

Our problem is to find the location of hub nodes, to allocate the demand nodes to the located hub nodes, and to determine which links are to be established between hub nodes in order to provide service within a given time bound and allowing for at most three hub stops on any route. Let N be the set of demand nodes, and let $H \subseteq N$ be the candidate set of hub nodes.

The parameters of our mathematical model are as follows.

FH_j = fixed cost of opening a hub at node $j \in H$
FL_{ij} = fixed cost of opening a hub link between hubs $i \in H$ and $j \in H$
t_{ij} = travel time between nodes $i \in N$ and $j \in N$
β = maximum service time requirement

The decision variables of the mathematical model, in addition to the x variables defined in Chapter 2 in the formulation given by O'Kelly (1987), are:

r_j = ready time of cargo at hub $j \in H$
z_{ij} = 1 if a hub link is established between hubs $i \in H$ and $j \in H$; 0 otherwise
y_{ikj} = 1 if hub $k \in H$ is used when travelling from hub $i \in H$ to hub $j \in H$; 0 otherwise

The decision variables of the model are schematically shown in Figure 3.1.

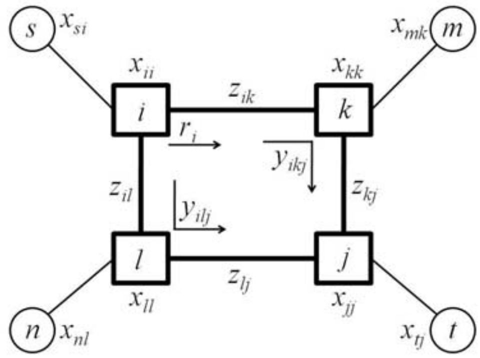


Figure 3.1 Decision variables of the mathematical model.

An integer programming formulation of the problem (*3-stop-0)* defined above is as follows:

$$\text{Minimize } \sum_{j\in H} FH_j x_{jj} + \sum_{j\in H}\sum_{i\in H} FL_{ij} z_{ij} \tag{3.1}$$

subject to

$$\sum_{j\in H} x_{ij} = 1 \qquad \forall\, i \in N \tag{2.3}$$

$$x_{ij} \le x_{jj} \qquad \forall\, i \in N, j \in H \tag{2.6}$$

$$2z_{ij} \le x_{ii} + x_{jj} \qquad \forall\, i,j \in H: i \ne j \tag{3.2}$$

$$\left(1 - z_{ij}\right) - \sum_{l\in H: l\ne i} x_{il} - \sum_{l\in H: l\ne j} x_{jl} \le \sum_{k\in H: k\ne i, k\ne j} y_{ikj} \qquad \forall\, i,j \in H: i \ne j \tag{3.3}$$

$$2y_{ikj} \le z_{ik} + z_{kj} \qquad \forall\, i,j,k \in H: i \ne j, i \ne k, j \ne k \tag{3.4}$$

$$\sum_{k \in H} y_{ikj} \leq (1 - z_{ij}) \qquad \forall\, i, j \in H: i \neq j \qquad (3.5)$$

$$z_{ij} = z_{ji} \qquad \forall\, i, j \in H, \qquad (3.6)$$

$$r_j \geq t_{ij} x_{ij} \qquad \forall\, i \in N, j \in H \qquad (3.7)$$

$$(r_j + r_i + \alpha t_{ij}) z_{ij} \leq \beta \qquad \forall\, i, j \in H, \qquad (3.8)$$

$$(r_j + \mathrm{Max}\{r_k, r_i + \alpha t_{ik}\} + \alpha t_{kj}) y_{ikj} \leq \beta \qquad \forall\, i, j, k \in H \qquad (3.9)$$

$$x_{ij} \in \{0,1\} \qquad \forall\, i \in N, j \in H \qquad (2.5)$$

$$z_{ij} \in \{0,1\} \qquad \forall\, i, j \in H: i \neq j \qquad (3.10)$$

$$y_{ikj} \in \{0,1\} \qquad \forall\, i, j, k \in H \qquad (3.11)$$

The objective function (3.1) minimizes the total cost of establishing the hub network. The total cost term includes the fixed cost of locating hubs and establishing hub links.

Constraint (3.2) links *x* variables to *z* variables and ensures that a hub link can only be opened between two established hubs. We force the model via Constraint (3.3) so that if a direct link does not exist between two hub nodes, these two hubs must be reachable via stopping at a hub node in between. Thus, every two demand centers can receive service via at most three hubs on a route. Note that for given *i* and *j*, the summations $\sum_{l \in H: l \neq i} x_{il}$ and $\sum_{l \in H: l \neq j} x_{jl}$ on the left hand side of the Constraint (3.3) both take on the value 0 if *i* and *j* are both hub nodes; i.e., if $x_{ii} = 1$ and $x_{jj} = 1$. Thus, the left-hand side of the Constraint (3.3) takes on the value 1, if a direct hub link is not established between two established hubs and forces the *y* variable to take on the value 1 for some hub *k*. Constraints (3.4) and (3.5) are logical constraints linking *y* and *z* variables. The in-between hub can only be used if a direct hub link exists from both of the hubs (Constraint (3.4)). We do not need to use an in-between hub, if a direct hub link connection between two hubs exists, and exactly one hub must be used in travel between two hubs (Constraint (3.5)).

The case for an in-between hub is illustrated in Figure 3.1. Because $z_{ij} = 0$ for $x_{ii} = 1$ and $x_{jj} = 1$, the left hand side of Constraint (3.3) takes on the value 1 forcing the model to use another hub in between hubs i and j. By Constraints (3.4) and (3.5) y_{ikj} must be equal to 1 for some k such that $x_{kk} = 1$, $z_{ik} = 1$, and $z_{kj} = 1$. Then in the figure either y_{ikj} or y_{ilj} must be equal to one. Note that by Constraint (3.5) only one of y_{ikj} or y_{ilj} can take on the value 1.

We establish an undirected hub network so that if a hub link is opened in one direction it should also be opened in the other direction (Constraint (3.6)). Constraint (3.7) ensures that the ready time of the cargo at a hub is greater than the time needed to travel from all the demand points allocated to that hub. Remember that in cargo applications, a hub waits for all the cargo coming from demand centers that is allocated to that hub before sending the cargo to another hub or demand center. The left hand side of Constraint (3.8) calculates the maximum travel time between demand centers allocated to two different hubs, when a direct hub link is established between these two hubs whereas, the maximum travel time between demand centers allocated to two different hubs when a direct hub link is not established in between is calculated in Constraint (3.9). Recall that if there is not a direct hub link between two hubs, there is a known hub to be visited in between, which is obtained by y variables. Note that the ready time of the cargo at the in-between hub may be greater than the time required to travel from the origin to the in-between hub. Thus, we need the maximum operator on the left hand side of Constraint (3.9). Constraints (2.5), (3.10) and (3.11) are the constraints that define binary variables.

This mathematical model is a nonlinear binary programming model due to Constraints (3.8) and (3.9). If we let $|N| = n$ and $|H| = h$ the model has $(h^3 + h^2 + nh)$ binary variables and $(2h^3 + 5h^2 + 2nh + n)$ constraints.

3.3 Linearizations

We propose Constraint (3.8a) below for the linearization of Constraint (3.8).

$$r_j + r_i + \alpha t_{ij} z_{ij} \le \beta \qquad \forall\, i \in H, j \in H \qquad (3.8a)$$

Let us refer to the new formulation by replacing Constraint (3.8) with (3.8a) as (*3-stop-1)*.

Theorem 3.1 *Any feasible solution to (3-stop-0) is a feasible solution to (3-stop-1) and vice versa.*

Proof Let $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ be a feasible solution to (*3-stop-0*). Let us show that $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ is also feasible to (*3-stop-1*). As all constraints other than Constraint (3.8) are common to both, it suffices to show that $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ is feasible to (3.8a). Consider the equation (3.8a) associated with nodes i and j. There are two cases depending on the value of z_{ij}.

- *Case 1*: $z_{ij} = 1$. Then, Constraints (3.8) and (3.8a) yield the same left hand side.
- *Case 2*: $z_{ij} = 0$. The left hand side of the Constraint (3.8) yields 0; however, the left hand side of the Constraint (3.8a) yields $r_j + r_i$. It suffices to show that $r_j + r_i$ is less than or equal to β. Note that when $i = j$,

Constraint (3.8a) yields $2r_i \leq \beta$ because $t_{ii} = 0$ and $z_{ii} = 0$. Thus, we have both $r_j \leq \frac{\beta}{2}$ and $r_i \leq \frac{\beta}{2}$. By summing these two, we obtain $r_j + r_i \leq \beta$. Thus, Constraint (3.8a) is satisfied.

To prove the converse, observe that the left hand side of (3.8) is always less than or equal to the left hand side of (3.8a); that is,

$$(r_j + r_i + \alpha t_{ij})z_{ij} \leq r_j + r_i + \alpha t_{ij} z_{ij} \qquad \forall\, i \in H, j \in H.$$

Therefore, any feasible solution to (*3-stop-1*) is also feasible to (*3-stop-0*). □

For the linearization of Constraint (3.9), we provide two sets of constraints below:

$$r_j + r_k + \alpha t_{kj} y_{ikj} \leq \beta \qquad \forall\, i \in H, j \in H, k \in H \qquad (3.9a)$$

$$r_j + r_i + \alpha(t_{ik} + t_{kj}) y_{ikj} \leq \beta \qquad \forall\, i \in H, j \in H, k \in H \qquad (3.9b)$$

Let us refer to the new formulation by replacing Constraint (3.9) with (3.9a) and (3.9b) in (*3-stop-1*) as (*3-stop-2*).

Theorem 3.2 *Any feasible solution to (3-stop-1) is a feasible solution to (3-stop-2) and vice versa.*

Proof Let $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ be a feasible solution to (*3-stop-1*). Let us show that $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ is also feasible to (*3-stop-2*). Because all constraints other than Constraint (3.9) are common to both models, it suffices to show that $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ is feasible to (3.9a) and (3.9b). Consider the equation (3.9a) and (3.9b) associated with nodes *i, j,* and *k*. There are three cases, depending on the values of r_i, r_k, and y_{ikj}.

- *Case 1*: $y_{ikj} = 1$
  - *Case 1a*: $r_k \geq r_i + \alpha t_{ik}$. Then, Constraints (3.9) and (3.9a) yield the same left hand side. However, the left hand side of Constraint (3.9b) yields $r_j + r_i + \alpha(t_{ik} + t_{kj}) \leq \beta$. But as $r_j + r_i + \alpha t_{ik} + \alpha t_{kj} \leq r_j + r_k + \alpha t_{kj} \leq \beta$, Constraint (3.9b) is also satisfied.
  - *Case 1b*: $r_k < r_i + \alpha t_{ik}$. Then, Constraints (3.9) and (3.9b) yield the same left hand side. The left hand side of Constraint (3.9a) yields $r_j + r_k + \alpha t_{kj} \leq \beta$. But as, $r_j + r_k + \alpha t_{kj} < r_j + r_i + \alpha t_{ik} + \alpha t_{kj} \leq \beta$, the Constraint (3.9a) is also satisfied.
- *Case 2*: $y_{ikj} = 0$. The left hand side of the Constraint (3.9) yields 0; however, the left hand side of the Constraint (3.9a) yields $r_j + r_k$, and the left hand side of the Constraint (3.9b) yields $r_j + r_i$. It suffices to show that both $r_j + r_k$ and $r_j + r_i$ are less than or equal to β. From Constraint (3.8a) and the argument in the proof of Theorem 3.1, we know that $r_j \leq \frac{\beta}{2}$, $r_k \leq \frac{\beta}{2}$, and $r_i \leq \frac{\beta}{2}$. By summing these constraints, we obtain $r_j + r_k \leq \beta$ and $r_j + r_i \leq \beta$. Thus, both Constraints (3.9a) and (3.9b) are satisfied.

Thus, we conclude that $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ is also feasible to (*3-stop-2*).

To prove the converse, observe that the left hand side of (3.9) is either equal to the left hand side of (3.9a) or (3.9b) or less than both of them. So any feasible solution to (*3-stop-2*) is also feasible to (*3-stop-1*). □

Now, let us state the linearized mathematical model (*3-stop*):

Minimize (3.1)

subject to

(2.3), (2.6), (3.2)–(3.7), (3.8a), (3.9a), (3.9b), (2.5), (3.10), (3.11).

Corollary 1 *Any feasible solution to (3-stop-0) is a feasible solution to (3-stop) and vice versa.*

Corollary 2 *An optimum solution to (3-stop) is also an optimum solution to (3-stop-0) and vice versa.*

(*3-stop*) is a strong linearization of (*3-stop-0*) in three ways: (1) it uses precisely the same set of variables as in (*3-stop-0*), that is, there is no change in the dimension of the space; (2) the feasible sets are exactly the same; and (3) the optimal sets are the same.

For our applications we added Constraint (3.12):

$$y_{ikj} \leq (1 - z_{ij}) \qquad \forall\, i \in H, j \in H, k \in H \qquad (3.12)$$

to the model (*3-stop*) in order to have tighter LP relaxations.

3.4 Computational Results

The model is first applied on the Turkish network. On this network, 81 cities are considered as demand centers. We took 16 candidate sites for hub locations among these demand centers: the most populated and industrialized cities in Turkey suitable for hub location (Yaman et al., 2007). Figure A.2 in Appendix A shows the geographical locations of the demand centers and candidate hub locations on a map of Turkey and presents the names of the candidate hub locations.

Our problem parameters for this Turkish network are summarized in Table 3.1. The travel times (t_{ij}) between all nodes on the network can be obtained from Beasley (1990). The fixed costs for locating hub facilities (FH_j) are taken from a previous study by Tan and Kara (2007). Various factors, such as the industrialization level, the in and out cargo intensity, land price, and the highway intensity of different cities have been considered in determining these fixed costs.

In addition to the fixed cost of opening hubs, the total cost term in the objective function includes the costs for establishing hub links (FL_{ij}). In order to propose a general model, we allowed for the costs of establishing hub links to differentiate each link in the model. However, through our interviews with cargo firms we observed that the costs for using inter-hub links are actually fixed, are the same for all links, and are not proportional to distances, i.e., that $FL_{ij} = FL$ for all $i \in H$ and $j \in H$. Thus, we take the link costs to be fixed in our computations. In order to observe the changes on the hub network with respect to these cost values, we took two different fixed cost values for hub links: low link cost and high link cost. The low link cost value is a fixed value that is taken as relatively lower than the average fixed hub costs, and the high link cost value is taken as approximately the average of fixed hub cost values.

Through our interviews with cargo firms, the hub-to-hub transportation time discount factor (α) was found to be 0.9 on ground transportation in Turkey. Thus, we took α to be 0.9 in all of our computations.

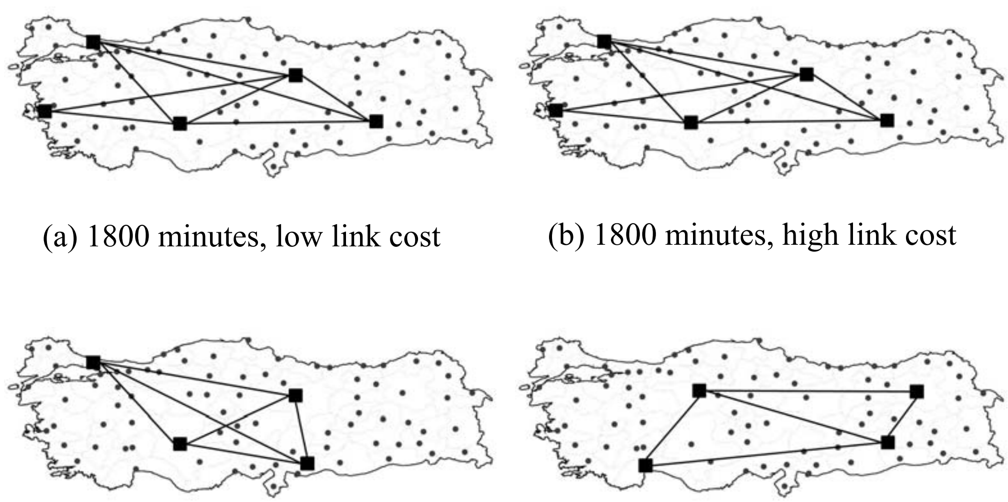
In this Turkish network, with a 0.9 discount factor, the tightest possible service time value between two demand centers is about 30 hours, i.e., 1800 minutes. We varied the service time values (β) between 30 and 33 hours (1800 to 1980 minutes) with ten-minute time intervals. In order to comment on the computational times more realistically, we divided the β values into four intervals. The first interval (Interval-1) starts from the β value of 1800 minutes, which results in opening five hubs on the Turkish network. All of the tested β values in Interval-1 lead to opening five hubs. Interval-2 starts from the first β value leading to opening four hubs, which is 1830 minutes. Thus, with ten-minute time intervals, 1830 minutes is the tightest possible β value for opening four hubs on this network. Similarly, the β values in Interval-3 and Interval-4 lead to opening three and two hubs respectively. The summary of the β values and intervals are listed in Table 3.1.

Table 3.1 Parameters for the Turkish network.

Parameter		Value
	$\|N\|$	81
	$\|H\|$	16
	t_{ij}, FH_j	From Tan and Kara (2007)
	α	0.9
	FL	Low and high
	Interval-1: 5 hubs	1800–1820
β	Interval-2: 4 hubs	1830–1850
(min)	Interval-3: 3 hubs	1860–1920
	Interval-4: 2 hubs	1930–1980

We took our runs on CPLEX 8.1, on an AMD Opteron 252, 2.6 GHz server with 2GB RAM. All the runs are solved to optimality.

Figures 3.2a–d schematically show some of our computational results. The location of the hubs and established hub links are shown in these figures. In order to avoid complications, we did not show the allocation of demand nodes in these figures. From Figures 3.2a and b, with the service time bound of 1800 minutes, observe that even though the link costs are different, there is no change in either the location of the hubs or the established hub links. However, note that the resulting hub network in both of the solutions is incomplete. In Figure 3.2c, we obtained a complete hub network with 1850 minutes of service time bound and with low link costs. On the other hand, when we increased the link costs (Figure 3.2d) the hubs are opened in different locations in order to reduce the total link cost, and we obtained an incomplete hub network. In general we discovered that, except in a few instances, our solutions on the Turkish network were insensitive to the link costs.



(a) 1800 minutes, low link cost (b) 1800 minutes, high link cost

(c) 1850 minutes, low link cost (d) 1850 minutes, high link cost

Figure 3.2 Computational results on the Turkish network.

Table 3.2 shows our computational times obtained by using CPLEX 8.1. This table lists the minimum, maximum, and average CPU times obtained, corresponding to four different β intervals. In Interval-1, we have an instance that lasted more than 7.5 hours. This is the highest CPU time that we observed on this network, and it corresponds to the instance with the tightest β, shown in Figure 3.2b. However, even for Interval-1 with tight β values we have results in an average of about 2.5 hours. This value decreases to approximately 45 seconds with loose β values in Interval-4.

Table 3.2 CPU times on the Turkish network.

β	Min	Max	Average
Interval-1	43.13 min	7.79 hr	2.47 hr
Interval-2	19.38 min	1.22 hr	43.46 min
Interval-3	1.34 min	6.23 min	2.96 min
Interval-4	7.62 sec	1.27 min	45.47 sec

In order to discuss and compare our results with the hub location literature, we also tested our model on the well-known CAB data set introduced in O'Kelly (1987). Figure A.1 in Appendix A shows the names and geographical locations of these 25 cities on the CAB data set.

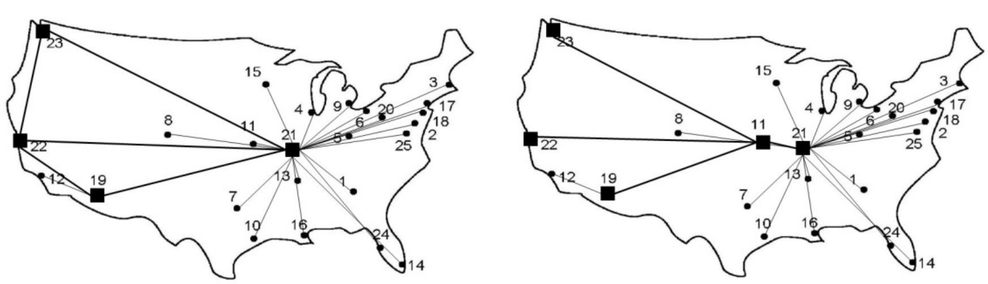
The parameters taken for the instances on the CAB data set are listed in Table 3.3. There are 25 nodes in the CAB network, and we took all nodes to be the candidate hub locations. Because there are no real travel-time values presented in the literature on the CAB data set, as customarily done in the literature, we took travel times equal to travel distances. Again, no real data on fixed hub costs is reported in the literature, so we took the fixed hub cost value to be 100 for all locations (O'Kelly, 1992). To observe any changes on the hub network we tested two different link cost values. We took link costs to be 1 and 100, where 1 corresponds to low link costs and 100 to high.

Similar to all of the applications on the CAB data set in the literature, we varied the α values. We took α to be 0.2, 0.4, 0.6, and 0.8. We varied β according to the optimum p-hub center solutions found in Kara and Tansel (2001), corresponding to locating four, three, and two hubs for each α value. We again took our runs on CPLEX 8.1 on the same server.

Table 3.3 Parameters for the CAB network.

Parameter	Value
$\lvert N \rvert$	25
$\lvert H \rvert$	25
t_{ij}	distance_{ij}
FH	100
FL	1, 100
α	0.2, 0.4, 0.6, 0.8
β	Tightest possible distance for each α value corresponding to locating 4, 3, and 2 hubs

Figure 3.3 presents two results from the CAB data set. In both of these results, the α value is taken to be 0.8, and the β value is the tightest possible service distance on this network with four hubs, corresponding to this α value, which is 2457. Figures 3.3a and b show the corresponding results with low and high link costs, respectively. When we increased the fixed link costs, the model locates one more hub, and the link number is reduced by one. Note that in both of these figures the resulting hub network is incomplete, even though the service distance requirement is at its minimum possible value. The results from the CAB data set also prove that the complete hub network assumption presented in most of the hub location models is not necessary in application. In all of the CAB data set instances that corresponded to opening four hubs, all solutions resulted in incomplete hub networks.



(a) $\alpha = 0.8$, $\beta = 2457$, low link cost (b) $\alpha = 0.8$, $\beta = 2457$, high link cost

Figure 3.3 Computational results on the CAB data set.

We listed our computational times on the CAB data set in Table 3.4. For each α value (0.2, 0.4, 0.6, and 0.8) we took three different β values and two different link costs (low and high).

Table 3.4 The CPU times for the CAB data set.

α	β	Low link cost	High link cost
0.2	**1617**	8.81 min	3.05 hr
	1913	2.34 min	5.06 min
	2137	1.10 min	26.01 sec
0.4	**1881**	10.42 min	43.11 min
	2099	5.96 min	14.45 min
	2401	0.69 min	5.23 min
0.6	**2184**	13.84 min	1.35 hr
	2336	27.09 min	57.64 min
	2557	10.15 min	12.73 min
0.8	**2457**	1.57 hr	10.79 hr
	2552	59.25 min	1.90 hr
	2713	3.05 min	3.56 min

In the CAB data set, the problems with low link costs tend to be solvable quicker than the corresponding high link-cost instances. Also, the solution times tend to increase as α increases. The worst case performances of our model on this data set is obtained when the value of α is 0.8. The highest CPU time that we observed on this data set is below 11 hours (for the instance shown in Figure 3.3b), and the lowest is about 26 seconds.

With both the Turkish network and the CAB data set, we obtained optimal solutions with our proposed model in reasonable CPU times. Since we tested the tightest possible β values in both of the data sets, we presume that these are among the hardest instances on these data sets. When we compare the average CPU time requirements, the CAB data set instances turned out to be a little bit harder than the Turkish network instances. This is due to the increase in the number of candidate hub locations; the CAB data set contains 25 candidate hub locations, whereas the Turkish network contains 16. On the other hand, we observed that the increase in the number of demand centers did not lead to a significant increase in the CPU time requirements compared to the increase in the CPU time with the increase in the number of candidate hub locations. In both of the data sets, except in a few instances, the solutions turned out to be insensitive to the link costs. However, again in both of the data sets, the low link cost instances required less CPU time than the corresponding high link cost instances.

Even though we have tested the tightest possible β values in both of the data sets, the model resulted in building incomplete hub networks in most of the instances. This shows that the service that is provided with a complete hub network can also be provided with an efficiently designed incomplete one with less investment cost requirements.

A generalization of the 3-stop hub covering network design problem introduced in this chapter is the incomplete hub covering network design problem with no restriction on the hub network other than connectivity. We study this problem in Chapter 5 of this thesis. In the next chapter, we introduce incomplete hub network design models with the minimization of total transportation cost objective.

Chapter 4

MINIMIZATION OF TOTAL TRANSPORTATION COSTS IN DESIGNING INCOMPLETE HUB NETWORKS

This chapter studies the single allocation incomplete hub network design problems with the minimization of transportation cost objective. These are namely the single allocation incomplete *p*-hub median and hub location with fixed costs network design problems. After presenting the motivation in Section 4.1, we propose the integer programming formulations for these problems in Section 4.2 and Section 4.3 of this chapter, respectively. Section 4.4 compiles the computational analysis and Section 4.5 presents some concluding remarks.

4.1 Motivation

In hub location studies, it is typically assumed that the hub network is complete with the presence of a direct hub link between every hub pair. This assumption eliminates hub network design decisions and simplifies the routing of flow in hub location problems. However, in reality, many less-than-truckload and telecommunication networks do not operate on a complete hub network structure. As it is also stated in Chapter 3 of this thesis, we have observed that many cargo companies employ incomplete hub network structures. Since, in general, establishment of complete hub networks unnecessarily increase the total investment costs in designing hub networks.

As a result of the general assumption presented in hub location problems, the cost of flow transported between hubs is discounted by using economies of scale discount factor (α). Thus in hub location, it is desirable to have larger flow on hub links in order to account for the economies of scale discount factor more realistically. We observed that, with complete hub networks the amount of flow transported on some hub links may not be high enough to justify the discount from economies of scale. Campbell (2005b) also presented such an argument. It is observed that, when compared with complete hub networks, the incomplete hub networks allow the solutions to adapt to use hub links with larger flows.

Almost all of the hub location research focused on building complete hub networks. There are only a few studies focusing on the design of incomplete hub networks and relaxing the complete hub network assumption in hub location problems. O'Kelly and Miller (1994) mentioned the possibility of using different hub network design protocols, one being the incomplete hub network design. Nickel et al. (2001) proposed a model for the hub location problem arising in urban public transportation networks. The authors minimized the total transportation costs plus the fixed costs of locating hubs and building hub links. They considered the multiple allocation structure in which a non-hub node can be allocated to more than one hub. They proposed a mathematical formulation with $O(n^4)$ variables and constraints and were able to solve and illustrate their results only on a 10-node problem.

Campbell et al. (2005a, 2005b) proposed hub arc location problems to the literature. Such problems locate hub arcs with reduced unit costs, rather than locating hub facilities. A fixed number of hub arcs is located while minimizing the total transportation costs. The resulting hub arc network in these problems does not need to be connected. Campbell et al. (2005b) presented integer programming formulations for four special cases of the general multiple allocation hub arc location model and presented an enumeration-based algorithm. Campbell et al. (2003) presented a parallel implementation of this algorithm in an attempt to solve larger hub arc location problems.

Yoon and Current (2008) studied the multiple allocation incomplete hub network design problem with fixed and variable arc costs. Their proposed model minimized the total transportation costs and the fixed costs of locating hubs and hub links. They also considered direct connections between the non-hub nodes and incurred variable arc costs associated with demand on the arcs. They developed a dual-based heuristic algorithm for the solution of the problem.

In this chapter, we focus on designing hub networks that are not necessarily complete for hub location problems with minimization of total transportation cost objectives. Unlike the 3-stop hub covering model introduced in Chapter 3, we do not impose any structure on the hub networks, such as a hub network with at most there hub stops, other than connectivity. We focus on the single allocation version of the problems where each demand node is allocated exactly to a single hub. We propose efficient mathematical formulations with $O(n^3)$ variables and constraints for the single allocation incomplete p-hub median and hub location with fixed costs network design problems. The aim in both of the models is to find the location of hub nodes, the allocation of non-hub nodes to these hub nodes, and which hub links to establish between the hub nodes. Unlike the hub covering problem introduced in Chapter 3, both of the models introduced in this chapter do not consider travel times. The transportation cost model is tested on the various instances of the CAB data set and on the Turkish network using the optimization software CPLEX 11.2.

All of the studies in the literature modeling incomplete hub networks with transportation cost objectives (Nickel et al. (2001), Campbell et al. (2005a, 2005b), and Yoon and Current (2008)) focused on modeling only the multiple allocation variant of the problem which is computationally easier to solve than its single allocation counterpart. For the multiple allocation problem, if the locations of the hubs and hub links are fixed, the allocation decision is straightforward: each pair of demand nodes send flow from their shortest paths via the given hub network. However, for the single allocation version the allocation sub-problem still remains NP-hard even if the locations of the hubs and hub links are fixed. On the other hand, the papers by Nickel et al. (2001) and Yoon and Current (2008), both of which studied the incomplete hub network design problem with the minimization of total transportation cost objective on a connected hub network, proposed integer programming formulations with $O(n^4)$ variables and constraints. Thus in this study, in addition to modeling the single allocation version of the problems, we have reduced the model size from the previous formulations, which are only proposed for the multiple allocation variant, by a factor of n (the order of the number of nodes).

In the second and third sections of this chapter, we present our mathematical formulations for the single allocation incomplete p-hub median network design problem, and the single allocation incomplete hub location with fixed costs network design problem, respectively.

4.2 The Incomplete *p*-hub Median Network Design Problem

The single allocation incomplete p-hub median network design problem is to locate p hubs, to allocate each non-hub node to a single hub, and to determine which q hub links to establish between hubs such that the total transportation cost is minimized.

Our aim is to decide on the location of hub nodes, the allocation of non-hub nodes to these hub nodes, and which hub links to establish between the hub nodes, while minimizing total transportation costs. We aim to design our network so that any node can send flow to any other node in the network. In order to ensure this, the hub network to be established must be connected.

It may be straightforward to model the problem at hand using multi-commodity flow balance constraints with $O(n^4)$ variables. For example, we could use f_{ij}^{kl} variables to represent the flow on hub arc (i, j) originating from node k destined to node l, similar to the multiple allocation formulations proposed by Nickel et al. (2001) and Yoon and Current (2008). In this study, we developed a formulation with $O(n^3)$ variables.

Since all the flow originating from a non-hub node must visit the single hub that the non-hub node is allocated to, the calculation of the total flow from a non-hub node to its hub node is trivial. Thus, for each non-hub node, this value is exactly the total amount of flow originating from the non-hub node. Similarly, the total flow from a hub node to a non-hub node is exactly the total amount of flow destined to this non-hub node. What is challenging is the calculation of the total flow between hub nodes over an incomplete hub network with few variables. In the following we introduce one such model.

In order to present the mathematical formulation for the single allocation incomplete p-hub median network design problem, in addition to the previously defined parameters let q be the number of hub links to be established. Also, let $O_i = \sum_j w_{ij}$ be the total amount of flow originating from node i and $D_j = \sum_i w_{ij}$ be the total amount of flow destined to node j.

In addition to the x_{ij} variables, defined in Chapter 2, and z_{ij} variables, defined in Chapter 3, we define:

f_{ij}^k = total amount of flow originating from node $k \in N$ to be routed on hub link $\{i, j\}$ in the direction from $i \in H$ to $j \in H$.

The single allocation incomplete p-hub median network design problem is modeled as:

$$\text{Minimize} \sum_{i\in N}\sum_{k\in H} c_{ik}O_i x_{ik} + \sum_{i\in H}\sum_{j\in H}\sum_{k\in N} \alpha c_{ij} f_{ij}^k + \sum_{i\in N}\sum_{k\in H} c_{ki}D_i x_{ik} \tag{4.1}$$

subject to

$$\sum_{j\in H} x_{ij} = 1 \qquad \forall\, i \in N \tag{2.3}$$

$$\sum_{j\in H} x_{jj} = p \tag{2.4}$$

$$x_{ij} \le x_{jj} \qquad \forall\, i \in N, j \in H \tag{2.6}$$

$$z_{ij} \le x_{ii} \qquad \forall\, i, j \in H: i < j \tag{4.2}$$

$$z_{ij} \le x_{jj} \qquad \forall\, i, j \in H: i < j \tag{4.3}$$

$$\sum_{i\in H}\sum_{j\in H: j>i} z_{ij} = q \tag{4.4}$$

$$\sum_{j \in H: j \neq i} f_{ji}^k + O_k x_{ki} = \sum_{j \in H: j \neq i} f_{ij}^k + \sum_{l \in N} w_{kl} x_{li} \qquad \forall\, i \in H, k \in N \tag{4.5}$$

$$f_{ij}^k + f_{ji}^k \leq O_k z_{ij} \qquad \forall\, i, j \in H: i < j, k \in N \tag{4.6}$$

$$f_{ij}^k \geq 0 \qquad \forall\, i, j \in H: i \neq j, k \in N \tag{4.7}$$

$$x_{ij} \in \{0,1\} \qquad \forall\, i \in N, j \in H \tag{2.5}$$

$$z_{ij} \in \{0,1\} \qquad \forall\, i, j \in H: i < j \tag{3.10}$$

The objective function (4.1) minimizes the total transportation costs. The first term in the objective function calculates the total cost of transportation from non-hub nodes to hub nodes, the second term calculates the total discounted cost of transportation in the hub network, and the third term calculates the total cost of transportation from hub nodes to non-hub nodes.

Constraints (4.2) and (4.3) guarantee that a hub link can only be established if both of the end nodes of that link are hub nodes. Note that since we are designing an undirected hub network we defined z_{ij} variables only for $i < j$. Constraint (4.4) exactly q hub links are to be established.

We calculate the amount of flow to be routed on hub link $\{i, j\}$ in either orientation from any node k in the network via Constraint (4.5). Note that this is the divergence equation introduced by Ernst and Krishnamoorthy (1996) for the single allocation p-hub median problem with complete hub networks. By Constraint (4.5), for each node k and hub node i, the total flow originating from the node k entering into the hub node i must be equal to the outgoing flow. The first terms on the left and right hand sides of Constraint (4.5) calculate the flow within the hub network (f variables), whereas the second terms correspond to the flows via the allocations. By Constraint (4.6), we restrict the f variables to be positive only on the established hub links.

The rest of the constraints of the model represent non-negativity and binary requirements.

In the worst case $h = n$ and the model has $(\frac{3}{2} n^2)$ binary variables and (n^3) real variables. The number of constraints of our model is $(n^3 + 3\ n^2 + n + 2)$. Hence, in total we have $O(n^3)$ variables and constraints in the single allocation incomplete p-hub median network design problem.

In the next section, we proceed with defining the incomplete hub location with fixed costs network design problem and present an integer programming formulation.

4.3 The Incomplete Hub Location with Fixed Costs Network Design Problem

In the hub location literature, the problem in which the number of hubs to be established is taken as a decision variable in the p-hub median formulation is commonly referred to as the hub location problem with fixed costs.

More formally, the single allocation incomplete hub location with fixed costs network design problem is to locate hubs, to allocate each non-hub node to a single hub, and to determine which hub links to establish between hubs such that the total transportation costs plus the fixed costs of building the hub network is minimized.

The integer programming formulation for the single allocation incomplete hub location with fixed costs network design problem is very similar to the corresponding p-hub median version. The only differences are that the objective function includes both the fixed hub and hub link establishment costs and that the number of hubs and hub links to be located are now determined by the model.

With the previously defined decision variables and constraints, the single allocation incomplete hub location with fixed costs network design problem is modeled as:

$$\begin{aligned}\text{Minimize} \sum_{k\in H} FH_k x_{kk} + \sum_{i\in H}\sum_{j\in H: j>i} FL_{ij} z_{ij} + \sum_{i\in N}\sum_{k\in H} c_{ik} O_i x_{ik} \\ + \sum_{i\in H}\sum_{j\in H}\sum_{k\in N} \alpha c_{ij} f_{ij}^k + \sum_{i\in N}\sum_{k\in H} c_{ki} D_i x_{ik}\end{aligned} \tag{4.8}$$

subject to

(2.3), (2.6), (4.2), (4.3), (4.5)–(4.7), (2.5), (3.10).

In addition to the total transportation costs, the objective function (4.8) also includes the total cost of building hubs and the fixed costs of building the hub links. The rest of the constraints of the model comply with the incomplete p-hub median formulation, except the number of hubs and hub links to be located are now determined by the model.

Though not considered in this thesis, one should note that both the incomplete p-hub median and the hub location with fixed costs network design models are readily extendible to take care of the capacity restrictions by replacing O_k, the total amount of flow originating from node *k*, with the capacity of link $\{i, j\}$ in Constraint (4.6). More formally, the capacity constraints can be written as:

$$f_{ij}^k + f_{ji}^k \leq Capacity_{ij} z_{ij} \quad \forall\, i, j \in H: i < j, k \in N \qquad (4.6^*)$$

In the following section we present some computational analysis with the minimization of total transportation cost objective.

4.4 Computational Analysis

We tested the performance of our models on the CAB and Turkish network data sets, previously introduced in Chapter 3 of this thesis.

For both of these data sets, no real data on the fixed cost of establishing hub links has been introduced in the literature. In order to observe and compare the results for different fixed cost values, we tested three different ways for establishing hub link costs. For the first one we took the fixed costs for all hub links to be the same. In the second cost pattern, the fixed costs are directly proportional to the length of the hub link, and finally in the third one the structure from Calik et al. (2009) where the fixed costs are directly proportional to the length of the hub link and inversely proportional to flow between the nodes are used. The formula for calculating the fixed costs for hub links presented in Calik et al. (2009) is as follows:

$$FL_{ij} = \frac{\frac{Distance_{ij}}{w_{ij}}}{\max\limits_{i,j} \frac{Distance_{ij}}{w_{ij}}} \times 100$$

where $Distance_{ij}$ is the distance between nodes i and j, and w_{ij} is the amount of flow between nodes i and j.

During our computational analysis, we utilized the optimization software CPLEX version 11.2. We took our runs on a server with a 2.66 GHz Intel Xeon processor and 8GB of RAM.

We first tested our incomplete hub location with fixed costs network design formulation on the CAB data set with the mentioned three possible fixed cost patterns. However, since flow costs are high when compared to fixed hub link costs (under all scenarios) on the CAB data set, the model resulted in establishing complete hub networks in all of the test instances. Since fixing the number of hubs and hub links to be located converts the fixed cost problem to the incomplete p-hub median problem, we do not present any computational results for the fixed cost model.

For the CAB data set with p ranging from 2 to 5, we tested differing q values for our incomplete p-hub median network design formulation. As customarily done in the literature, we took α value to be 0.2, 0.4, 0.6, and 0.8.

We report our results on the CAB data set for the incomplete p-hub median problem in Table 4.1. For each instance, Table 4.1 reports the required CPU time in seconds, the locations of the hub nodes, and presents the percentage of increase in transportation costs with respect to establishing a complete hub network. We report these percentages, since in incomplete hub networks the flow between hubs is sometimes not transported directly, the total transportation cost may increase. In order to calculate this percentage of increase, we subtracted the total transportation cost of the complete hub network from the observed transportation cost, divided this value to the transportation cost of the complete hub network and multiplied by 100.

On the average the model is solved within 3 minutes of CPU time. The minimum CPU time requirement was about 1 second, whereas the maximum was just below 41 minutes. Observe from Table 4.1 that the instances with greater values of α turned out to be harder. Also, the instances in which we forced the hub networks to be sparse turned out to be harder than the instances with almost complete hub networks. The hardest of these 44 instances was when α=0.8, p=5, and q=6 which required about 40.7 minutes of CPU time.

Looking at the locations of the hub nodes in Table 4.1, we observe that Los Angeles (12) is always selected as a hub node and at the instances where we located 3 or more hub nodes, Chicago (4) is always selected. While locating 4 or more hubs either New York (17) or Philadelphia (18) is always present in the hub set. This is mainly due to the generation of high flow from these cities. When we rank the flow generated from the nodes in the CAB data set New York (17) has the largest flow, and Chicago (4) and Los Angeles (12) are second and third, respectively.

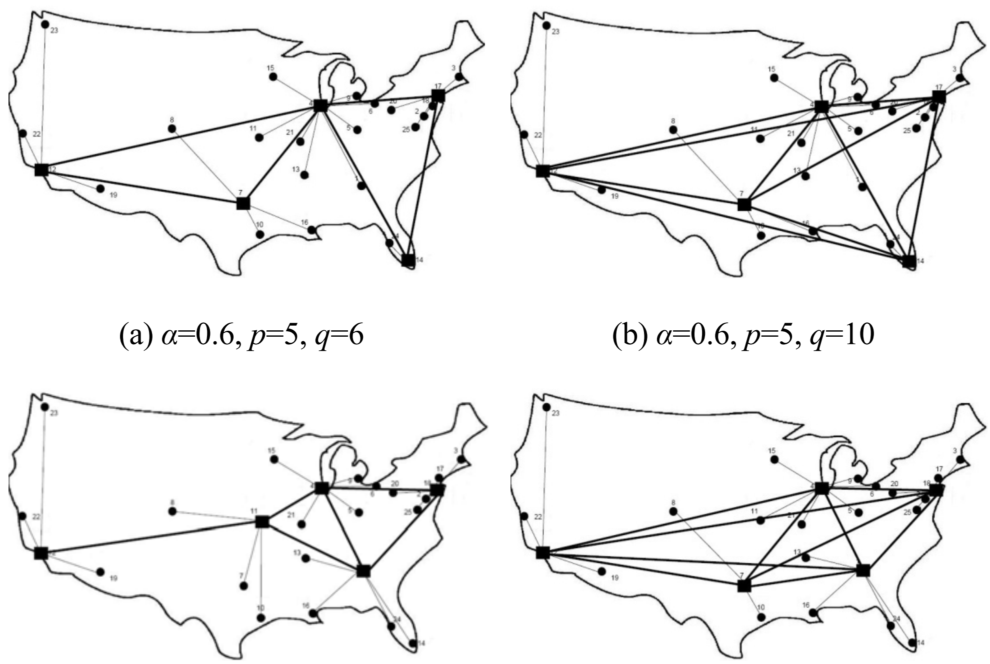
In general, the hub locations were insensitive to the number of hub links to be established. Except at two instances, the hub locations were exactly the same as the complete hub network solutions. At these two instances (α=0.6, p = 3, q=2 and α=0.8, p=5, q=6) only one hub node differs (Philadelphia (18) and Kansas City (11), respectively) from the complete hub network solution.

Table 4.1 The results on the CAB data set with the incomplete *p*-hub median problem.

α	*p*	*q*	CPU time (sec)	Hub locations	% increase in transportation costs
0.2	**2**	**1**	0.66	12,20	0
0.2	**3**	**2**	8.92	4,12,17	0.020
0.2	**3**	**3**	3.99	4,12,17	0
0.2	**4**	**4**	7.40	4,12,17,24	0.507
0.2	**4**	**5**	2.65	4,12,17,24	0.022
0.2	**4**	**6**	2.28	4,12,17,24	0.000
0.2	**5**	**6**	7.03	4,7,12,14,17	0.867
0.2	**5**	**7**	1.84	4,7,12,14,17	0.327
0.2	**5**	**8**	1.78	4,7,12,14,17	0.031
0.2	**5**	**9**	1.91	4,7,12,14,17	0.0044
0.2	**5**	**10**	1.34	4,7,12,14,17	0
Average			3.62		0.162
0.4	**2**	**1**	4.35	12,20	0
0.4	**3**	**2**	32.54	4,12,18	0.082
0.4	**3**	**3**	10.50	4,12,18	0
0.4	**4**	**4**	43.98	1,4,12,17	0.866
0.4	**4**	**5**	19.23	1,4,12,17	0.036
0.4	**4**	**6**	12.99	1,4,12,17	0
0.4	**5**	**6**	40.06	4,7,12,14,17	1.209
0.4	**5**	**7**	18.51	4,7,12,14,17	0.4491
0.4	**5**	**8**	8.25	4,7,12,14,17	0.047
0.4	**5**	**9**	6.39	4,7,12,14,17	0.007
0.4	**5**	**10**	6.01	4,7,12,14,17	0
Average			18.44		0.245
0.6	**2**	**1**	6.21	12,20	0
0.6	**3**	**2**	87.43	4,12,18	0.177
0.6	**3**	**3**	20.43	2,4,12	0
0.6	**4**	**4**	137.18	1,4,12,17	1.090
0.6	**4**	**5**	61.25	1,4,12,17	0.045
0.6	**4**	**6**	27.74	1,4,12,17	0
0.6	**5**	**6**	666.43	4,7,12,14,17	1.466
0.6	**5**	**7**	233.15	4,7,12,14,17	0.544
0.6	**5**	**8**	65.37	4,7,12,14,17	0.057
0.6	**5**	**9**	74.87	4,7,12,14,17	0.008
0.6	**5**	**10**	42.61	4,7,12,14,17	0
Average			129.33		0.308
0.8	**2**	**1**	18.32	12,20	0
0.8	**3**	**2**	185.66	2,4,12	0.269
0.8	**3**	**3**	85.34	2,4,12	0
0.8	**4**	**4**	970.26	1,4,12,18	1.2865
0.8	**4**	**5**	432.11	1,4,12,18	0.124
0.8	**4**	**6**	122.06	1,4,12,18	0
0.8	**5**	**6**	2441.97	1,4,11,12,18	1.907
0.8	**5**	**7**	574.03	1,4,7,12,18	0.430
0.8	**5**	**8**	482.37	1,4,7,12,18	0.165
0.8	**5**	**9**	394.34	1,4,7,12,18	0.034
0.8	**5**	**10**	269.66	1,4,7,12,18	0
Average			543.28		0.383

With lower values of α, hub nodes are located near the peripheries of the region. Since in the CAB data set $c_{ij} = Distance_{ij}$, by locating hub nodes near the peripheries the higher cost values get higher advantage from the economies of scale. When α increases, hub locations are concentrated in the center of the region. The percentage of increase in transportation costs is reported as 0 for the instances with complete hub networks. As expected, this percentage increases as the hub network is forced to be sparser. The highest increase we obtained at the CAB instances in Table 4.1 was 1.907%. We also observed from Table 4.1 that the percentages of increase in the transportation costs are lower when α value is lower. When we took the average of percent increases for different values of α, we obtain 0.162% for α=0.2 and 0.245%, 0.308%, and 0.383% for α equal to 0.4, 0.6, and 0.8, respectively.

Figure 4.1 illustrates a sample of solutions on the CAB data set. In order to analyze the flow behavior of the designed network links (both hub links and spoke links), we explored the flow data with α=0.6 and p=5 corresponding to instances a and b of Figure 4.1. This is one of the common instances where hub locations and allocation decisions coincide for both sparse (q=6) and complete (q=10) hub network structures. In both of the resulting designs there are 20 spoke links. The complete design contains 10 hub links, whereas the sparse design has only six hub links. For both of the designs, we ranked the resulting flow on the links of the network in a descending order. Under both sparse and complete designs, the largest flow in the network is between hubs Chicago (4) and New York (17). For both of the networks, the spoke links Boston (3)–New York and Washington (25)–New York are in top 5. This is in sync with our expectations to have high flows on some spoke links under single allocation hub network structure. It is interesting to note that in the complete design not all of the hub links carry large flows. In particular, only 4 out of 10 hub links are among the first 10 highest flow carrying links. On the other hand, two of the remaining hub links are the two lowest flow carrying links. In contrast, in the sparse design 5 out of 6 hub links appear in the top 10 list. The remaining hub link Dallas (7)–Los Angeles (12) has the 20^{th} largest flow.



(a) α=0.6, p=5, q=6 (b) α=0.6, p=5, q=10

(c) α=0.8, p=5, q=6 (d) α=0.8, p=5, q=10

Figure 4.1 CAB data set results with the transportation cost objective.

The flow behavior of the above example is typical of many that we encountered during our experimentation. In the absence of a direct hub link between some hubs, the flow between these hubs is to be routed on more than one hub link. Thus, it is expected to have more flow on hub links in incomplete hub networks compared to complete ones. In hub location, it is desirable to have larger flow between hubs in order to account for the economies of scale discount factor more realistically. When compared with complete hub networks, the incomplete hub networks allow the solutions to adapt to use hub links with larger flows. However, in the incomplete hub networks since the flow between hubs is sometimes not transported directly, the total transportation cost may increase.

In order to observe the increase in transportation costs with respect to the number of established hub links, we decided to draw a trade-off curve. For the curve we analyzed the instance with α= 0.8 and p=5, which is the hardest of our instances and the instance with the greatest percentage increase. For this, we calculated the percent increase with all possible number of established hub links, starting from a tree-hub network to a complete one. Figure 4.2 depicts the resulting trade-off curve.

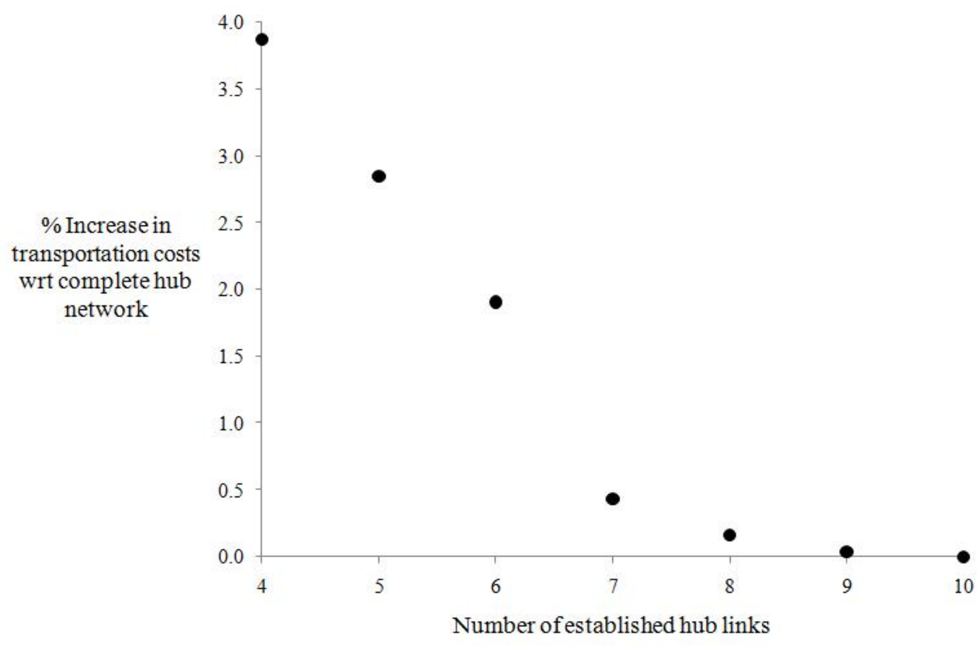


Figure 4.2 Trade-off curve with α=0.8 and p=5.

In Figure 4.2, when we forced the model to establish a tree-hub network with four hub links the percent increase in transportation costs was about 3.87%. This value was about 0.03% when we reduced one hub link from the complete hub network (q=9). Observe that, there is a steep increase in the curve below q=7.

By analyzing the curve, the decision maker can observe the trade-off between establishing an incomplete hub network versus the increase in transportation costs. The main drawback of building incomplete hub networks is the increase in the total transportation costs. However, we observed that the increase in the total transportation costs with respect to building complete hub networks is not very significant. If the decision maker considers the fixed costs of building hub links, for example for assigning new aircrafts between two hub nodes, this increase in transportation costs can be tolerable.

In a similar fashion, we tested the incomplete p-hub median network design model with the Turkish network. We again analyzed the increase in transportation costs with respect to complete hub networks. The results are provided in Table 4.2.

Our findings on the CAB data set placed larger values of α to be most challenging thus on the Turkish network we tested the two largest possible values for α. On the Turkish network, for α=0.6, and 0.8, and for each p=4, 6, 8, and 10, we tested three different q values, corresponding to sparse, medium, and complete hub networks. We report the CPU time requirements given by CPLEX in the fourth column of Table 4.2. Considering the fact that this is a decision problem of strategic nature the CPU times are reasonable.

Table 4.2 Incomplete *p*-hub median results on the Turkish network.

α	p	q	CPU time (sec)	% increase in transportation costs
0.6	**4**	**3**	3555.36	2.046
0.6	**4**	**4**	729.32	0.024
0.6	**4**	**6**	377.40	0
0.6	**6**	**8**	86400 (0.33%)	0.743
0.6	**6**	**10**	541.37	0.037
0.6	**6**	**15**	302.62	0
0.6	**8**	**15**	8120.42	0.289
0.6	**8**	**20**	2139.91	0.027
0.6	**8**	**28**	564.68	0
0.6	**10**	**20**	86400 (0.39%)	0.773
0.6	**10**	**30**	4038.39	0.058
0.6	**10**	**45**	503.44	0
0.8	**4**	**3**	6864.23	2.813
0.8	**4**	**4**	599.14	0.038
0.8	**4**	**6**	495.05	0
0.8	**6**	**8**	86400 (0.44%)	0.797
0.8	**6**	**10**	1685.50	0.033
0.8	**6**	**15**	1137.39	0
0.8	**8**	**15**	15541.44	0.309
0.8	**8**	**20**	1611.45	0.027
0.8	**8**	**28**	697.62	0
0.8	**10**	**20**	86400 (0.79%)	0.746
0.8	**10**	**30**	62260.15	0.060
0.8	**10**	**45**	4544.15	0
Average			5815.45	0.367

Observe from Table 4.2 that some instances corresponding to sparse hub networks could not be solved to optimality within 24 hours (86,400 seconds). At these instances, we present the gap reported by CPLEX in parenthesis and calculated the percent increase in transportation costs by using the best integer solution found by CPLEX at the end of 24 hours. Thus, the percentages reported for the instances that took longer than 24 hours are pessimistic.

Excluding the two instances with p=4 and q=3, the percent increases in transportation costs with respect to complete hub networks, reported in the last column of Table 4.2, are all below 1%. The average of the percent increases of all the instances is 0.367%. This again shows that when building hub networks, the difference in transportation costs with respect to complete hub networks can be tolerable when the fixed costs of establishing hub links are considered.

4.5 Conclusions

In this chapter, we studied the single allocation incomplete *p*-hub median and hub location with fixed costs network design problems. The problems were motivated from real-life observations of many hub networks. We presented novel O(n^3) mathematical formulations for both of the problems. The models are readily extendible to handle capacity restrictions. Computational analysis demonstrated that the proposed integer programming formulations are very efficient. We were able to solve all CAB instances within 45 minutes of CPU time. In order to observe the performance of the model on larger networks, we presented some computational results on the Turkish network. These models are among the first single allocation incomplete hub network design models with minimization of total transportation cost objective.

We were able to show that the increase in transportation costs with respect to building complete hub networks can be negligible when the fixed costs of building hub links are considered. We were also able to show that the incomplete hub network solutions allow using hub links with larger flows accounting for the economies of scale discount factor more realistically.

In the next chapter, we study the single allocation incomplete hub network design problems with center and cover type objectives. Explicitly, we introduce the incomplete hub covering and *p*-hub center network design problems.

Chapter 5

INCOMPLETE HUB COVERING AND *P*-HUB CENTER NETWORK DESIGN PROBLEMS

This chapter focuses on covering and center type incomplete hub network design problems. Unlike the problems with the minimization of total transportation cost objectives introduced in Chapter 4, center and covering type incomplete hub network design models do not consider total transportation costs.

In Section 5.1, the single allocation incomplete hub covering network design problem is introduced. This section presents the mathematical formulation, some valid inequalities for the proposed model, and the computational analysis. Section 5.2 studies the single allocation incomplete p-hub center network design problem and presents a mathematical model together with the computational analysis.

5.1 The Incomplete Hub Covering Network Design Problem

The single allocation incomplete hub covering network design problem is to find the location of hub nodes, the allocation of non-hub nodes to these hub nodes, and which hub links to establish between the hub nodes, while providing service between every origin–destination pair within the given time bound and while minimizing the fixed costs of building the hub network.

The incomplete hub covering network design problem is important for the systems where time is a major concern. As Campbell (1994) pointed out, hub covering and p-hub center type problems are important for hub systems involving perishable or time sensitive items. A major application area for hub covering problems is the cargo sector, where there is a promised service time between origin–destination pairs. In the hub covering and p-hub center type problems, the economies of scale discount factor α is applied to time instead of cost. The time discount factor α is again a number between 0 and 1, however it is most likely to be higher than the cost discount factor and it is expected to be a number closer to 1. Travel time is discounted when the flow is consolidated to be transported with bigger and faster vehicles. For example, for a cargo application in Turkey, Tan and Kara (2007) found out that there is a time discount factor of 0.9 when using inter-hub connections.

Our incomplete hub covering network design problem is a generalization of the 3-stop hub covering problem proposed in Chapter 3. In the incomplete hub covering network design problem, we no longer put a restriction on the number of hub stops on the route between origin–destination pairs and we do not impose any structure on the hub network other than connectivity. Unlike the 3-stop problem, we do not model the synchronization of trucks.

To the best of our knowledge, there is a single paper studying the incomplete hub covering network design problem in the literature. Recently, Calik et al. (2009) provided an $O(n^4)$ mathematical formulation and a tabu-search based heuristic for the problem. The authors were able to solve their model optimally using CPLEX on the CAB data set with up to 20 nodes. In some instances with 20 nodes, CPLEX was not able to find even a feasible solution. They applied the heuristic algorithm to the Turkish network.

In this thesis, we formulate the single allocation incomplete hub covering network design problem with O(n^3) binary variables and constraints. Section 5.1.1 presents our mathematical formulation. To increase the exact solution potential we proposed and compared some valid inequalities for this model in Section 5.1.2. Computational analysis presented in Section 5.1.3 show that the model is very efficiently solved in seconds using CPLEX on both the CAB and the Turkish network data sets.

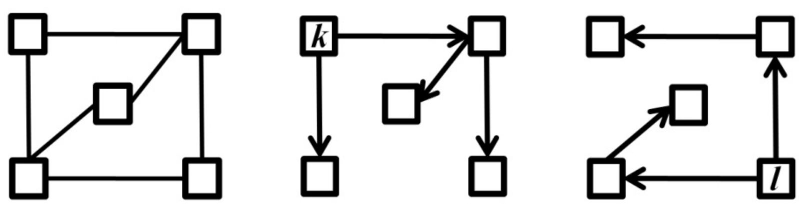
5.1.1 Mathematical Formulation for the Incomplete Hub Covering Network Design Problem

For the modeling of the incomplete hub covering network design problem, we no longer require any parameters and variables associated with flow, as previously presented for the incomplete *p*-hub median and fixed cost models in Chapter 4, since we do not consider the transportation costs. However, we need to know which hub links are used on the path from any origin to destination to calculate the travel time. Similar to the formulation used in Calik et al. (2009), it is possible to model the problem with O(n^4) variables. For example, we could use y_{ij}^{kl} variables and set them to one if a hub link $\{i,j\}$ is used on the path from node k to node l. In this thesis, we developed a novel formulation with O(n^3) binary variables.

Since every pair of demand nodes must receive service, the hub network to be established must be connected. The idea behind our model is that, for each established hub, we would like to find a spanning tree rooted at this hub that visits every other hub in the hub network using only the established hub links.

Employing these spanning trees for each hub node ensures the pair-wise connectivity of the underlying hub network. By the use of the spanning trees there is a unique path from the root of the spanning tree to every other hub node within the hub network. We calculate the travel time between all pairs of hubs by using the spanning trees. We then ensure that the travel time between all origin–destination pairs are within the given time bound.

Figure 5.1 demonstrates the spanning tree idea. Figure 5.1a represents a potential hub network solution of the incomplete hub covering network design problem. Figures 5.1a and b show the possible spanning trees formed for hubs k and l. The directed tree rooted at hub k in Figure 5.1b illustrates the unique paths from hub k to all the other hubs in the hub network. Similarly, Figure 5.1b demonstrates paths from hub l to all the other hubs. Note that the spanning trees associated with each hub node can only use the established hub links. We calculate the travel time between all pairs of hubs by using these unique paths formed by the spanning trees.



(a) (b) (c)

Figure 5.1 Spanning tree idea.

For the mathematical model, we again need a given node set N consisting of n demand nodes and a potential hub set H such that $H \subseteq N$ with h nodes. We assume throughout our model that the triangle inequality is satisfied. The new decision variables required for the mathematical model are:

y_{ijk} = 1 if the spanning tree rooted at hub $k \in H$ uses the hub link $\{i,j\}$ from hub $i \in H$ to hub $j \in H$; otherwise.

r_j = radius of hub $j \in H$, i.e., the maximum travel time between hub j and the nodes that are allocated to hub j.

d_{ij} = travel time from hub $i \in H$ to hub $j \in H$ in the hub network (not discounted).

An integer programming formulation of the single allocation incomplete hub covering network design problem defined above is as follows:

$$\text{Minimize} \sum_{k \in H} FH_k x_{kk} + \sum_{i \in H} \sum_{j \in H: j > i} FL_{ij} z_{ij} \tag{5.1}$$

subject to

$$\sum_{j \in H} x_{ij} = 1 \qquad \forall\, i \in N \tag{2.3}$$

$$x_{ij} \leq x_{jj} \qquad \forall\, i \in N, j \in H \tag{2.6}$$

$$z_{ij} \leq x_{ii} \qquad \forall\, i, j \in H: i < j \tag{4.2}$$

$$z_{ij} \leq x_{jj} \qquad \forall\, i, j \in H: i < j \tag{4.3}$$

$$\sum_{i \in H: i \neq j} y_{ijk} \geq x_{kk} + x_{jj} - 1 \qquad \forall\, j, k \in H: j \neq k \tag{5.2}$$

$$\sum_{i \in H: i \neq j} y_{ijk} \leq x_{kk} \qquad \forall\, j, k \in H: j \neq k \tag{5.3}$$

$$y_{ijk} + y_{jik} \leq z_{ij} \qquad \forall\, i, j, k \in H: i < j \tag{5.4}$$

$$d_{kj} \geq (d_{ki} + t_{ij}) y_{ijk} \qquad \forall\, i, j, k \in H: i \neq j, j \neq k \tag{5.5}$$

$$d_{ij} = d_{ji} \qquad \forall\, i, j \in H: i \neq j \tag{5.6}$$

$$d_{kk} = 0 \qquad \forall\, k \in H \tag{5.7}$$

$$r_j \geq t_{ij} x_{ij} \qquad \forall\, i \in N, j \in H \qquad (3.7)$$

$$r_k + \alpha d_{kj} + r_j \leq \beta \qquad \forall\, j, k \in H \qquad (5.8)$$

$$x_{ij} \in \{0,1\} \qquad \forall\, i \in N, j \in H \qquad (2.5)$$

$$z_{ij} \in \{0,1\} \qquad \forall\, i, j \in H: i < j \qquad (3.10)$$

$$y_{ijk} \in \{0,1\} \qquad \forall\, i, j, k \in H: i \neq j, j \neq k \qquad (5.9)$$

$$d_{ij} \geq 0 \qquad \forall\, i, j \in H \qquad (5.10)$$

The first term in the objective function (5.1) calculates the total cost of establishing hub links and the second term, the total cost of establishing hubs. In the objective function, we minimize the overall cost of establishing an incomplete hub network.

For the case when there will be more than one hub established in the network, Constraint (5.2) ensures that the degree for each hub node is at least one, so that every hub node is an end node for at least one hub link. Through this constraint, the model guarantees that the tree rooted at hub k will have an entering arc into every other hub j. Constraint (5.3) assures that each spanning tree rooted at hub k can have at most one entering arc into another hub node j and forces the spanning tree arcs associated with a non-hub node to take zero values. Constraint (5.4) causes the spanning tree arcs to be hub arcs.

Constraint (5.5) calculates the time needed to travel from one hub node to another using the established spanning tree arcs in the hub network. Note that the spanning tree formed for each hub node in the network does not need to be the minimum spanning tree; thus, the time calculated in Constraint (5.5) does not need to be the minimum traveling time. We linearized Constraint (5.5) with a BigM type linearization as follows:

$$d_{kj} \geq d_{ki} + t_{ij} y_{ijk} - \frac{\beta}{\alpha}\left(1 - y_{ijk}\right) \qquad \forall\, i, j, k \in H: i \neq j, j \neq k \qquad (5.5^*)$$

For given i, j and k, when y_{ijk}=1 both of the Constraints (5.5) and (5.5*) yield the same right-hand side. Note that the discounted maximum travel time between two hubs in the network cannot be greater than β. Thus, the travel time between two hub nodes in the network can be at most β/α. So when y_{ijk}=0, the right-hand side of Constraint (5.5) yields 0, whereas the right-hand side of Constraint (5.5*) yields a number less than or equal to zero. Because $d_{ij} \geq 0$ by Constraint (5.10), we conclude that Constraint (5.5*) correctly linearizes Constraint (5.5).

Constraints (5.5) or (5.5*) also act as subtour breaking constraints for the possible values that y variables can take. Note that the previous constraints only guarantee that each hub node has an outgoing arc as well as an incoming arc to every other hub node. We would not necessarily end up with a rooted spanning tree, were it not for these constraints.

We assume symmetric time data; thus, d variables will also be symmetric (Constraint (5.6)), and the distance from a node to itself will be zero (Constraint (5.7)).

The maximum travel time needed from a demand node to its assigned hub, the radius of a hub, is calculated in Constraint (3.7) for each hub. Constraint (5.8) ensures that the travel time between any two nodes in the network is less than the given time limit. Because d variables calculate the travel time between hubs, we need to discount this travel time by the economies of scale time discount factor α.

Our linear integer programming formulation of the single allocation incomplete hub covering network design problem consists of the objective function (5.1) and the constraints (2.3), (2.5), (2.6), (3.7), (3.10), (4.2), (4.3), (5.2)–(5.4), (5.5*), (5.6)–(5.10). If $h = n$, then the model has $(n^3 + \frac{3}{2}n^2)$ binary variables and $(n^2 + n)$ real variables. The number of constraints of our model is $(2n^3 + 7n^2 + 2n)$. Hence, in total we have O(n^3) variables and constraints.

5.1.2 Incorporating Valid Inequalities

In this section, we present some inequalities that are valid for our incomplete hub covering network design model. The aim in providing these inequalities is to reduce the time needed to solve our model to optimality.

The first valid inequality that we introduce is Constraint (*A*):

$$d_{ij} \geq t_{ij} \qquad \forall\, i, j \in H: i \neq j \qquad (A)$$

Constraint (*A*) states that the *d* variable associated with two candidate hub nodes *i* and *j* in the hub network is at least the direct travel time between these nodes.

For the second valid inequality, we need to define a second travel time parameter $\bar{t}_{ij}$ calculated for all pairs of potential hub nodes as $\bar{t}_{ij} = \min_{k \in H: k \neq i,j}(t_{ik} + t_{kj})\ \forall\, i, j \in H: i \neq j$. The second travel time parameter between nodes *i* and *j* is the minimum travel time from node *i* to node *j* using exactly one node *k* in between. With the definition of the second travel time parameter, we present our second valid inequality:

$$d_{ij} \geq \bar{t}_{ij}\left(1 - z_{ij}\right) \qquad \forall\, i, j \in H: i < j \qquad (B)$$

By Constraint (*B*), if there is not a direct hub link established between two candidate hub nodes i and j, then the d variable associated with them in the hub network is at least the second travel time between these nodes.

Our third valid inequality is based on the z variables:

$$\sum_{i \in H} \sum_{j \in H: j > i} z_{ij} \geq \sum_{k \in H} x_{kk} - 1 \qquad (C)$$

Note that we are building a connected hub network in this problem, so that every node in the network can send flow to any other node. The minimum number of edges in a connected network is (number of nodes in the network – 1). By using this fact, Constraint (*C*) states that the number of hub links to be established is at least the total number of hub nodes to be established, minus one.

Our fourth valid inequality links d and z variables:

$$d_{ij} \leq t_{ij} z_{ij} + \frac{\beta}{\alpha}(1 - z_{ij}) \qquad \forall\, i, j \in H: i < j \qquad (D)$$

By Constraint (*D*) we ensure that, for a given node pair i and j, if there is a direct hub link established between these nodes, the d variable associated with them is at most the travel time between them. Otherwise, because the discounted maximum travel time between any two hubs in the network cannot be greater than β, we do not put any limit on the d variable.

We tested the performance of these valid inequalities (*A*)–(*D*) both individually and collectively. For this, we used both the CAB and the Turkish network data sets.

During our computational analysis with the valid inequalities, we utilized the optimization software CPLEX version 10.1. We took our runs on a personal computer with a 2.00 GHz Intel Core 2 Duo processor and 3GB of RAM.

We took the α value to be 0.8 for the CAB data set instances and calculated the minimum possible β value with this α value on this network as 2180. In order to observe the performance of our valid inequalities on larger networks we generated two networks from the Turkish network by taking both 50 and 75 randomly chosen candidate hub locations from 81 demand centers on this network. We calculated the minimum possible β values with α= 0.9 on both of the networks with 50 and 75 candidate hub nodes as 1697.4. To observe the performance of all the valid inequalities with both tight and loose β values we tested three different β values, one being the tightest, for all of the three networks. The test bed for the valid inequalities is shown in Table 5.1.

Table 5.1 Test bed for valid inequalities.

Instance	Data set	|N|	|H|	α	β
1	CAB	25	25	0.8	2180
2	CAB	25	25	0.8	2490
3	CAB	25	25	0.8	2800
4	Turkish network	81	50	0.9	1697.4
5	Turkish network	81	50	0.9	1798.7
6	Turkish network	81	50	0.9	1900
7	Turkish network	81	75	0.9	1697.4
8	Turkish network	81	75	0.9	1848.7
9	Turkish network	81	75	0.9	2000

While testing the valid inequalities, we put a time limit of 1 hour on CPLEX. The solution times obtained for the instances shown in Table 5.1 are listed in Table 5.2.

Table 5.2 Solution times (in seconds) with valid inequalities.

	Instance								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
None	3600	3600	44.765	3600	3600	3113.010	Mem.	Mem.	Mem.
A	3.062	6.894	2.796	390.934	**126.689**	44.015	828.673	1417.761	225.347
B	3600	2202.179	9.290	3600	3600	926.988	Mem.	Mem.	Mem.
C	3600	3600	69.874	3600	3600	3600	Mem.	Mem.	Mem.
D	3600	3600	25.919	3600	3600	2082.668	Mem.	Mem.	Mem.
AB	3.095	6.462	2.978	165.733	290.069	107.217	925.867	2523.132	356.852
AC	3.072	10.781	**2.530**	109.917	275.801	**40.914**	978.899	3600	234.935
AD	3.285	7.327	3.102	184.993	699.475	43.435	561.294	**1376.962**	**168.460**
BC	3600	1455.526	6.587	3600	3600	1477.142	Mem.	Mem.	Mem.
BD	3600	2623.513	7.828	3600	3600	1793.194	Mem.	Mem.	Mem.
CD	3600	3600	71.713	3600	3600	3015.396	Mem.	Mem.	Mem.
ABC	**2.954**	8.460	2.904	189.627	260.004	111.057	1191.540	3600	180.840
ABD	3.498	7.668	3.292	382.307	232.656	106.304	3600	3357.993	446.905
ACD	3.034	**4.847**	2.734	**87.299**	443.747	42.706	**538.013**	3600	235.238
BCD	3600	1190.758	4.449	3600	3600	729.716	Mem.	Mem.	Mem.
ABCD	3.355	7.672	2.894	178.463	316.412	108.385	1304.489	3600	551.354

CPLEX reported an "Out of memory" error at the runs marked with "Mem." in Table 5.2 on our computer. On the other hand, some runs, indicated by 3600 seconds in Table 5.2, could not be solved to optimality in 1 hour. At some of these instances CPLEX could not even find an initial feasible solution. In Table 5.2, we reported the best CPU time obtained for a given instance in bold. Note from Table 5.2 that none of the instances including valid inequality (*A*) suffered from the memory problem and, except 5 runs from instances 7 and 8 with 75 candidate hub nodes, all of the runs including valid inequality (*A*) were completed in optimality within 1 hour. Utilizing inequality (*A*) alone, all instances were solved to optimality within 24 minutes. This shows us that the valid inequality (*A*) is very effective. Even though using valid inequality (*A*) alone did not always result in the quickest solution times (for example, at instances 4, 7, and 9), the average CPU time is the lowest. On the other hand, in the average CPU times, valid inequalities (*A*) and (*D*) together has the second smallest CPU time requirement. The difference between (*A*) and (*AD*) is in fact negligible. However, note that neither valid inequality (*D*) alone nor the addition of valid inequality (*D*) to other inequalities is as effective as the valid inequality (*A*).

With valid inequalities, we also compared the linear programming (LP) relaxations of the model and the LP relaxations at the root node reported by CPLEX. It turns out that none of the valid inequalities resulted in different LP relaxation values. However, the use of valid inequality (*C*) slightly increases (less than 8.5%) the LP relaxation at the root node reported by CPLEX.

In light of our observations, we decided to use our model together with valid inequality (*A*) alone for our computational analysis with the incomplete hub covering model. The reader should note that different data sets may also make the other valid inequalities effective.

5.1.3 Computational Analysis

We tested the performance of our incomplete hub covering network design formulation on the CAB and the Turkish network data sets. No time data has been provided for the CAB data set, thus, similar to other hub covering studies in the literature, we took $t_{ij} = Distance_{ij}$ for this data set. For the fixed costs of opening hubs we took FH_i=100 for all nodes (O'Kelly, 1992). We varied α from 0.2 to 1.

On the CAB data set, we tested three different fixed cost values for opening hub links. In the first one we let FL_{ij}=10 for all i, j. For the second one we took

FL_{ij}= $Distance_{ij}$ and for the last one we used the values from Calik et al. (2009) in which they calculated the fixed costs for hub links as follows:

$$FL_{ij} = \frac{\frac{Distance_{ij}}{Flow_{ij}}}{\max_{i,j} \frac{Distance_{ij}}{Flow_{ij}}} \times 100$$

where $Distance_{ij}$ is the distance between nodes i and j, and $Flow_{ij}$ is the amount of flow between nodes i and j.

We tested our incomplete hub covering network design formulation with valid inequality (*A*) using CPLEX 11.2 on a server with a 2.66 GHz Intel Xeon processor and 8GB of RAM.

In order to obtain the tightest possible β values on the CAB data set, we first solved p-hub center problems with complete hub networks (Ernst et al., 2009) with 2, 3, 4, and 5 hubs and with different possible α values. Then we tested our model with three different fixed link cost values on these instances. The test parameters (shown in bold) and the corresponding results for different fixed link cost values are provided in Tables 5.3.

When we look at the CPU times obtained in Table 5.3, we observed that CPU times are in fact dependent on the test data. The average CPU times that we obtained on the CAB instances in Table 5.3 with three different fixed link cost values are 2.7 seconds, 24.2 minutes, and 3.0 seconds respectively. In general all of the instances are solved in reasonable CPU times. Because we used the tightest β values in all of these instances, it is reasonable to presume that these are among the hardest instances on this data set.

Table 5.3 Incomplete hub covering results on the CAB data set.

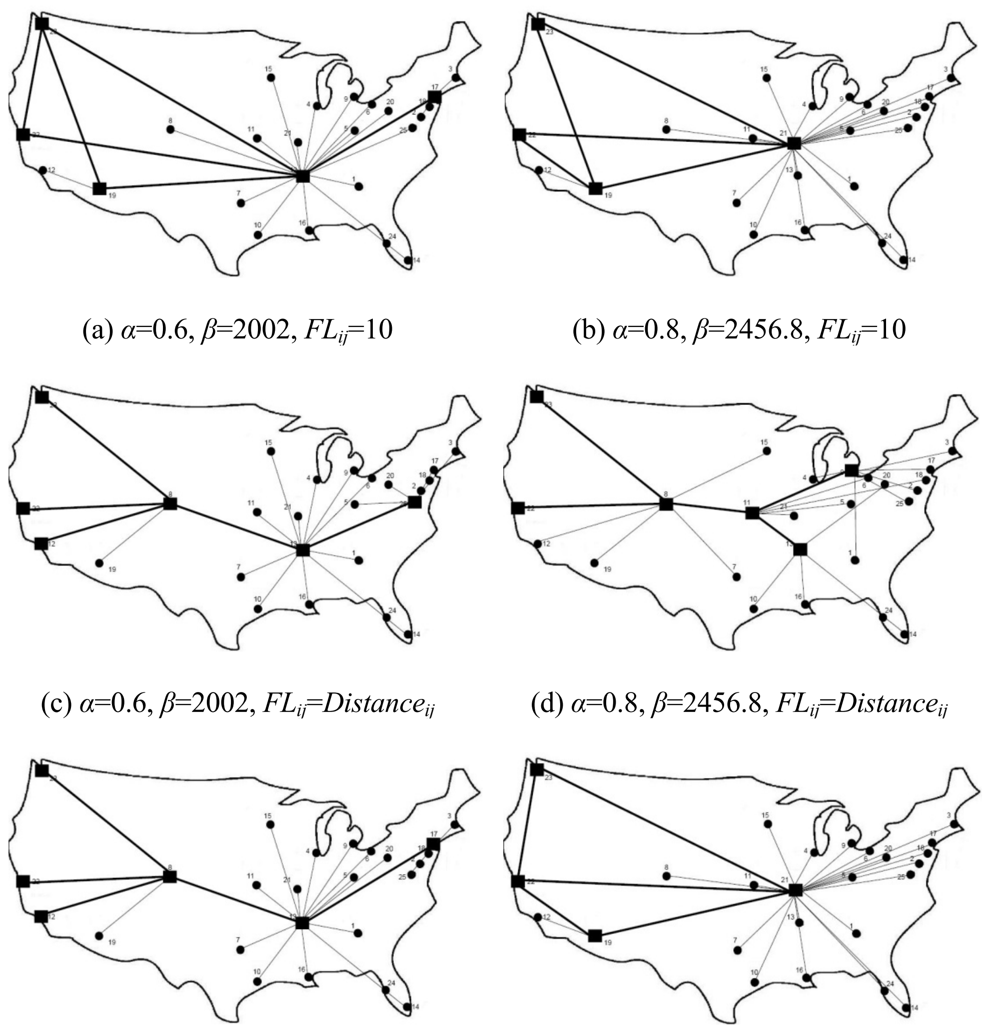
		FL_{ij}=10			FL_{ij}=$Distance_{ij}$			FL_{ij} from Calik et al.		
α	β	CPU time (sec)	Hub Locations	Hub links	CPU time (sec)	Hub Locations	Hub links	CPU time (sec)	Hub Locations	Hub links
0.2	**2136**	0.86	21,22	1	6.93	5,8,13	2	1.42	21,22	1
0.2	**1912.8**	2.69	3,13,22	2	171.41	5,13,22	2	2.4	13,17,22	2
0.2	**1616.2**	4.86	9,16,19,23	4	1721.51	1,5,8,22,23	4	1.86	9,16,19,23	4
0.2	**1346**	3.42	2,11,12,23,24	4	3609.98	1,11,13,19,20,22,23	6	6.53	11,12,17,23,24	4
0.4	**2400.4**	4.64	5,8	1	52.46	8,21	1	3.51	8,21	1
0.4	**2098.2**	2.66	1,2,8	2	282.16	1,8,18,20	3	2.17	1,8,25	2
0.4	**1880.4**	2.17	3,12,13,23	3	2365.12	1,7,22,23,25	4	3.24	12,13,17,23	3
0.4	**1596.8**	3.75	11,12,14,18,23	4	14149.08	11,12,13,17,21,23,24	6	6.48	11,12,14,18,23	4
0.6	**2556.6**	1.33	8,21	1	1.79	8,21	1	1.37	8,21	1
0.6	**2335.2**	4.26	8,16,25	3	278.69	8,20,21,24	3	4.32	8,16,25	3
0.6	**2183.2**	5.57	19,21,22,23	3	380.36	3,6,8,11,24	4	2.62	19,21,22,23	3
0.6	**2002**	4.47	13,17,19,22,23	6	3304.79	8,12,13,22,23,25	5	12.54	8,12,13,17,22,23	5
0.8	**2712.8**	0.83	8,21	1	3.00	8,21	1	0.93	8,21	1
0.8	**2551.6**	2.48	6,8,16	3	186.50	8,11,22,23	3	2.82	6,8,16	3
0.8	**2456.8**	4.38	19,21,22,23	5	474.77	8,9,11,13,22,23	5	2.35	19,21,22,23	5
0.8	**2370.6**	3.18	6,8,16,22,23	5	1709.71	8,17,21,22,23,24	5	2.19	6,8,16,22,23	5
1	**2826**	0.29	8,11	1	1.10	8,11	1	0.63	8,11	1
1	**2762**	0.61	8,11,23	3	13.43	8,11,23	3	0.55	8,11,23	3
1	**2726**	0.62	4,8,23,24	6	162.17	4,8,13,23,24	7	0.83	4,8,23,24	6
1	**2725**	1.63	7,8,9,14,23	7	176.32	4,8,13,14,23	7	1.07	4,7,8,14,23	8
Average		2.74			1452.56			2.99		

Among the three fixed cost values, the instances with FL_{ij}=$Distance_{ij}$ turned out to be the hardest. In general, there is not a significant CPU time difference between the instances with the first and third fixed link cost values. The reason why the FL_{ij}=$Distance_{ij}$ instances lasted longer is because FL_{ij} values are higher when compared to fixed hub costs.

Table 5.3 also provides the location of the hub nodes and the number of hub links in the established hub networks. Except for a few instances, the results with the second fixed link cost structure turned out to be very different than the other two sets of results. In these instances with FL_{ij}=$Distance_{ij}$, the model resulted in establishing more hubs to reduce the total fixed costs of establishing hub links. For example, at the first instance in Table 5.3 when α is

0.2 and β value is 2136, both the first and the third fixed link cost values resulted in opening two hubs at nodes St. Louis (21) and San Francisco (22) with one hub link {21,22}, whereas the second one resulted in opening three hubs at nodes Cincinnati (5), Denver (8), and Memphis (13) with two hub links {5,13} and {8,13}. This is because in the CAB data set ($Distance_{\{21,22\}}$=1736) > ($Distance_{\{5,13\}}$ = 402) + ($Distance_{\{8,13\}}$=880), and thus the difference $FL_{\{21,22\}} - (FL_{\{5,13\}} + FL_{\{8,13\}}) > FH$=100. Hence the objective function value of the model with the second fixed hub link cost value is higher with opening two hubs St. Louis (21) and San Francisco (22), compared to opening hubs Cincinnati (5), Denver (8), and Memphis (13).

The number of established hub links reported in Table 5.3 indicates how sparse the hub network is. Note that 14 of the 60 instances listed in Table 5.3 correspond to locating two hubs, where an incomplete hub network solution is not possible. Even though we used the tightest possible β values, we obtained incomplete hub networks in 37 of the remaining 46 instances. Especially for the instances with FL_{ij}=$Distance_{ij}$ all of the solutions with three or more hubs, with one exception when α is 1 and β value is 2762, resulted in incomplete hub networks. For the other two fixed cost structures, the respective numbers of established hub links are usually the same though sometimes with the establishment of different hub nodes. In all of the instances, where the model resulted in opening four or more hubs, all of the hub networks were incomplete with one exception when α=1 and β=2726. Figure 5.1 depicts incomplete hub network solutions of some instances from Table 5.3 on the CAB data set.



(a) α=0.6, β=2002, FL_{ij}=10 (b) α=0.8, β=2456.8, FL_{ij}=10

(c) α=0.6, β=2002, FL_{ij}=$Distance_{ij}$ (d) α=0.8, β=2456.8, FL_{ij}=$Distance_{ij}$

(e) α=0.6, β=2002, FL_{ij} from Calik et al. (f) α=0.8, β=2456.8, FL_{ij} from Calik et al.

Figure 5.2 Incomplete hub covering results with the CAB data set.

Because we are using the tightest possible time bounds on this network, in order to discount the travel time, the nodes that are located near the periphery are often selected as hub nodes (such as San Francisco (22) and Seattle (23)) especially when α increases. In almost all of the solutions presented in Figure

5.1, at least one central hub node is established, and most of the non-hub nodes are allocated to this central hub node.

In order to observe the performance of our model on larger networks, we tested it on the Turkish network. For the Turkish network data set, we took the fixed cost values for opening hubs from Tan and Kara (2007), for opening hub links from Beasley (1990) and the α value to be 0.9. On this network with 16 candidate hub locations, the tightest possible β value corresponding to α=0.9 is 1783.1 minutes. In addition to the tightest possible β, we tested β values between 1800 and 2100 minutes at 30-minute intervals. The summary of our results on the Turkish network is provided in Table 5.4.

Table 5.4 Incomplete hub covering results on the Turkish network.

α	β	CPU time (sec)	Hub Locations	Number of hub links
0.9	**1783.1**	0.31	21,34,35,42,58	8
	1800	0.43	21,34,35,42,58	8
	1830	0.53	21,34,42,58	5
	1860	0.41	6,21,25	3
	1890	0.48	6,21,61	3
	1920	0.46	6,7,58	3
	1950	0.39	6,58	1
	1980	0.46	6,58	1
	2100	0.15	55	0
Average		0.40		

All instances with the Turkish network in Table 5.4 are solved within 1 second due to having 16 potential hub locations. All of the solutions with four or more hubs resulted in incomplete hub networks. We illustrated two of these results in Figure 5.2.

(a) β=1800 (b) β=1830

Figure 5.3 Incomplete hub covering results with the Turkish network.

Similar to the observations from the CAB data set, we conclude from the Turkish network solutions that building complete hub networks is neither necessary nor cost efficient when building high number of hubs (for example more than three hubs) for providing service within a given time bound.

Lastly, in order to challenge the solution potential of our model with CPLEX, we fixed β and increased the number of candidate hub locations on the Turkish network. We report our results in Table 5.5. As expected, the CPU time requirement grows rapidly with the increase on the candidate number of hub nodes. With our proposed model, we were able to solve even the largest instance with 81 candidate hub locations within 2.53 minutes. These are the largest instances of incomplete hub network solutions solved to optimality in the literature up to now.

From our computational analysis on both the CAB data set with 25 nodes and the Turkish network with 81 nodes, we obtained optimal solutions of our model with CPLEX in few minutes, proving the effectiveness of our proposed mathematical model.

Table 5.5 Performance of the hub covering model with CPLEX on large networks.

|N|	α	β	|H|	CPU time (sec)
81	**0.9**	**1800**	16	0.43
			25	1.86
			35	4.17
			45	12.27
			55	24.63
			65	83.90
			75	108.47
			81	151.41

In the next section, we study the single allocation incomplete p-hub center network design problem. We propose a mathematical formulation for the problem and present some computational analysis.

5.2 The Incomplete p-hub Center Network Design Problem

The p-hub center problem locates p hubs, such that the maximum travel time between any origin–destination pair is minimized. On the other hand, we define the incomplete p-hub center network design problem as one that additionally determines which q hub links to establish in the p-hub center problem.

More formally, we define the single allocation incomplete p-hub center network design problem as locating p hubs, allocating each non-hub node to a single hub, and determining which q hub links to establish between hubs such that the maximum travel time between any origin–destination pair is minimized.

In Section 5.2.1 we propose a mathematical formulation for the single allocation incomplete p-hub center network design problem. Section 5.2.2

compiles the computational analysis with the incomplete p-hub center formulation.

5.2.1 Mathematical Formulation

Similar to the hub covering version, it is possible to model the single allocation incomplete p-hub center network design problem with $O(n^3)$ decision variables and constraints. The number of hubs, p, and hub links, q, to be located are now given, and the service time parameter introduced in the hub covering formulation is to be treated as a decision variable. The new decision variable, also named as β, is defined as the maximum travel time between any origin–destination pair.

With the previously defined decision variables and parameters, the problem can be modeled as:

$$\text{Minimize } \beta \tag{5.11}$$

subject to

(2.3)–(2.6), (3.7), (3.10), (4.2)–(4.4), (5.2)–(5.10),

$$\beta \geq 0 \tag{5.12}$$

For the linearization of Constraint (5.5), because β is now a decision variable, we suggest using BigM instead of using $\frac{\beta}{\alpha}$ in Constraint (5.5*).

Note that all valid inequalities introduced for the incomplete hub covering network design problem are also valid for the p-hub center version. However, for the valid inequality (D), we again suggest using BigM instead of using $\frac{\beta}{\alpha}$.

5.2.2 Computational Analysis

We again performed computational analysis with our single allocation incomplete p-hub center network design formulation with CPLEX 11.2 on the same server using the CAB data set. We included valid inequality (A) to our formulation and varied α, p, and q. We compile our results in Table 5.6.

We report the optimum β value, the CPU time requirement, and the hub locations in Table 5.6. Note that for a given α and p, even though we increase the number of hub links to be established, after a while the optimum β value stays constant. These β values are the tightest possible β values corresponding to a given α and p. Also note that these are the β values that we used in our hub covering instances. We increased q in the test instances until we reach the tightest β value, i.e. until the optimum β value stays constant. Observe from Table 5.6 that, excluding the 5 instances locating two hub nodes, the model did not need to establish complete hub networks in 11 out of 15 instances (with five different α and three different p values) in order to obtain the tightest possible β values.

Table 5.6 Incomplete p-hub center results on the CAB data set.

α	p	q	Optimum β value	CPU time (sec)	Hub locations
0.2	**2**	**1**	2136.0	2.25	21,22
0.2	**3**	**2**	1912.8	3.86	13,18,22
0.2	**3**	**3**	1912.8	4.45	5,13,22
0.2	**4**	**3**	1648.4	13.94	11,20,22,24
0.2	**4**	**4**	1616.2	9.09	9,16,19,23
0.2	**4**	**5**	1616.2	8.82	9,16,19,23
0.2	**5**	**4**	1346.0	77.74	11,12,23,24,25
0.2	**5**	**5**	1346.0	97.17	11,12,23,24,25
0.4	**2**	**1**	2400.4	2.46	8,21
0.4	**3**	**2**	2098.2	9.44	1,8,20
0.4	**3**	**3**	2098.2	5.53	1,8,20
0.4	**4**	**3**	1880.4	97.85	2,12,13,23
0.4	**4**	**4**	1880.4	13.75	12,13,20,23
0.4	**5**	**4**	1596.8	126.01	11,12,18,23,24
0.4	**5**	**5**	1596.8	133.87	11,12,14,18,23
0.6	**2**	**1**	2556.6	2.69	8,21
0.6	**3**	**2**	2374.6	7.72	12,21,23
0.6	**3**	**3**	2335.2	5.57	8,16,20
0.6	**4**	**3**	2183.2	91.27	19,21,22,23
0.6	**4**	**4**	2183.2	24.15	19,21,22,23
0.6	**5**	**4**	2072.8	234.95	11,12,18,23,24
0.6	**5**	**5**	2032.0	138.95	13,18,19,22,23
0.6	**5**	**6**	2002.0	111.83	13,19,22,23,25
0.6	**5**	**7**	2002.0	93.16	13,19,22,23,25
0.8	**2**	**1**	2712.8	2.26	8,21
0.8	**3**	**2**	2648.8	217.82	8,21,23
0.8	**3**	**3**	2551.6	4.15	6,8,16
0.8	**4**	**3**	2508.0	68.24	11,19,22,23
0.8	**4**	**4**	2487.6	27.56	6,8,16,23
0.8	**4**	**5**	2456.8	26.60	19,21,22,23
0.8	**4**	**6**	2456.8	17.01	19,21,22,23
0.8	**5**	**4**	2456.8	319.30	11,19,21,22,23
0.8	**5**	**5**	2370.6	137.04	6,8,16,22,23
0.8	**5**	**6**	2370.6	105.10	6,8,16,22,23
1	**2**	**1**	2826.0	1.64	8,11
1	**3**	**2**	2826.0	7.72	8,11,24
1	**3**	**3**	2762.0	2.72	8,11,23
1	**4**	**3**	2826.0	335.51	8,11,18,25
1	**4**	**4**	2762.0	67.16	8,11,21,23
1	**4**	**5**	2739.0	11.56	4,8,23,24
1	**4**	**6**	2726.0	6.63	4,8,23,24
1	**5**	**4**	2826.0	9844.53	6,8,11,13,23
1	**5**	**5**	2762.0	1241.64	3,4,8,11,23
1	**5**	**6**	2739.0	1614.08	4,8,23,24,25
1	**5**	**7**	2725.0	103.75	4,8,14,23,24
1	**5**	**8**	2725.0	19.90	8,9,14,21,23
Average				336.92	

When we compare Tables 5.3 and 5.6 we observe similarities. For example, in Table 6, α=0.8, p=4 and q=5 yielded opening hub nodes Phoenix (19), St. Louis (21), San Francisco (22), and Seattle (23) with an optimum β value of 2456.8, and in Table 5.3 same hubs are established with α=0.8 and β=2456.8 with both the first and third fixed link cost patterns. In almost all of the instances that required complete hub networks in the hub covering solutions, the tightest β values are again obtained on complete hub networks in the p-hub center solutions. On the other hand, there are some p-hub center solutions in which we obtained the tightest β values, i.e. the values that we used in the hub covering version, that resulted in establishing different hub nodes. For example, at the instance with α=0.6, p=5 and q=6 the p-hub center model established hubs at nodes Memphis (13), Phoenix (19), San Francisco (22), Seattle (23) and Washington (25); however, we did not obtain the same solution with the hub covering formulation when α=0.6 and β=2002. This provides an example of having multiple optimal solutions with our incomplete p-hub center network design problem. In sync with the complete or location analogous versions we observed multiple optimal solutions with this model, but only reported the first optimal solution found by CPLEX in our computational analysis.

When we look at the CPU times on the CAB data set reported by CPLEX, the hardest instance in Table 5.6 lasted about 2.7 hours, and the average CPU time was approximately 5.5 minutes. The instances in which we forced the model to obtain sparse hub networks turned out to be harder.

We also tested the incomplete p-hub center network design problem on the Turkish network. We provide the results in Table 5.7. The tightest possible β value in the Turkish network with 16 candidate hub locations and α=0.9 is 1783.1; i.e., when we establish a complete hub network with 16 hubs. This β

value can be obtained with locating five hubs and establishing seven hub links or six hubs and eight hub links. Observe from Table 5.7 that, in all of the instances with locating four or more hubs we obtained the tightest possible β values on incomplete hub networks.

Table 5.7 Incomplete *p*-hub center results on the Turkish network.

α	p	q	Optimum β value	CPU time (sec)	Hub locations
0.9	**2**	**1**	1923.7	2.96	6,58
0.9	**3**	**2**	1878.8	20.11	6,21,58
0.9	**3**	**3**	1856.3	12.91	6,21,25
0.9	**4**	**3**	1878.8	103.93	6,16,21,58
0.9	**4**	**4**	1856.3	33.00	6,21,45,58
0.9	**4**	**5**	1821.5	17.95	21,34,42,58
0.9	**4**	**6**	1821.5	21.96	16,21,42,58
0.9	**5**	**4**	1878.8	1120.34	6,21,25,33,58
0.9	**5**	**5**	1849.8	240.05	6,21,42,45,58
0.9	**5**	**6**	1805.6	169.07	21,34,42,45,58
0.9	**5**	**7**	1783.1	32.35	21,25,34,42,58
0.9	**5**	**8**	1783.1	28.22	21,25,34,42,58
0.9	**6**	**5**	1878.8	5767.56	6,21,25,58,61,63
0.9	**6**	**6**	1849.6	1488.71	6,21,34,35,42,58
0.9	**6**	**7**	1805.6	304.90	7,21,25,34,42,58
0.9	**6**	**8**	1783.1	123.29	21,25,34,42,55,58
0.9	**6**	**9**	1783.1	131.48	6,21,25,34,42,55
Average				565.81	

The CPU time requirement with the Turkish network was higher than the CAB instances. Nonetheless, the running times reported by CPLEX are still reasonable. On the average the model is solved within 10 minutes, where the hardest instance lasted about 1.6 hours.

5.3 Conclusions

In this chapter we studied incomplete hub network design problems with objectives related to time. We introduced the single allocation incomplete hub covering and *p*-hub center network design problems. We proposed novel O(n^3) integer programming formulations for both of the problems. To increase the exact solution potential, we proposed and compared four different valid inequalities for the hub covering model. Computational analyses with these formulations on the CAB and the Turkish network data sets are also presented. All test instances were very efficiently solved with our models.

We were able to show that, in some instances, the service that is provided with a complete hub network can also be provided with an efficiently designed incomplete hub network. Thus, the contribution to the quality of service with complete hub network designs may not be high enough to justify their high costs.

In the next chapter we study the multimodal hub location and hub network design problem. We consider both the transportation costs and the travel times between origin–destination pairs within this model. This model also allows using different types of hub links between hub nodes and different service time requirements between origin–destination pairs.

Chapter 6

MULTIMODAL HUB LOCATION AND HUB NETWORK DESIGN PROBLEM

In this chapter, we propose the multimodal hub location and hub network design problem. Section 6.1 presents the motivation and the problem definition. In Section 6.2, a mathematical formulation for the problem is introduced. Section 6.3 proposes some methods for enhancing the model and Section 6.4 compiles the computational analysis. Lastly, Section 6.5 presents some concluding remarks.

6.1 Motivation and Problem Definition

In earlier chapters of this thesis, the motivation for designing incomplete hub networks is presented. As it is stated before, in reality, many less-than truckload and telecommunication networks operate on an incomplete hub network structure. The computational results in Chapter 4 showed that in the models with the minimization of total transportation cost objective, the increase in transportation costs can be considered negligible when compared with the fixed costs of establishing hub links. Also, building complete hub networks may result in some hub links carrying low amounts of flow yet employing economies of scale discount factor. In the models considering travel time, Chapters 3 and 5 showed that, in most of the instances, there is no need to establish a complete hub network to provide service within a given service time bound.

In hub location literature, in the models that consider transportation costs (*p*-hub median and fixed cost models) service times between origin–destination pairs are neglected. On the other hand, in the models that consider travel times (*p*-hub center and hub covering models) transportation costs are ignored. In reality, for many less-than truckload networks, the transportation costs and service times are both very significant. For example for cargo delivery sector, there is usually a given service time promise for origin–destination pairs. On the other hand, the cargo firms would like to minimize both the fixed costs of establishing the hub network and the total costs of transportation for providing service within the promised service times.

The hub location models proposed in the literature do not consider transportation cost and travel time simultaneously. A recent study by Campbell (2009) also addresses this deficiency. In this paper, the author proposed time definite models for multiple allocation *p*-hub median and hub arc location problems. For both of the problems, a constraint is introduced on the maximum travel distance for each origin–destination pair. To the best of our knowledge, Campbell (2009) is the only study minimizing transportation costs subject to a constraint on the service level in hub location problems.

Most of the less-than truckload firms offer different delivery schedules for their customers. For example, cargo delivery firms offer different types of service time promises, such as overnight delivery, second day delivery etc., for different origin–destination pairs. Even the studies focusing on cargo delivery applications do not consider the possibility of having different service time promises in hub location problems.

Another important aspect in designing hub networks is the choice for mode of transportation. It is assumed that there is only one hub type and one type of transportation mode in hub location models presented in the literature. However, there may be a choice between air, ground and water transportation systems. For example, it is observed that various cargo firms operating in Turkey employ two different transportation modes, namely, air and ground transportation.

In this chapter, we aim to introduce a model that addresses all of the above stated observations from real-life hub networks. The model decides on the location of different types of hubs, the allocation of the non-hub nodes to the located hubs, and which hub links to establish between hubs with which type of transportation mode. There are given service time parameters which may differ for each origin–destination pair. The objective function of the model minimizes the fixed costs of establishing the hub network and the total costs of transportation. We name the problem with these specifications as the *multimodal hub location and hub network design problem.*

We present a mathematical formulation of this problem with two types of hubs and transportation modes and two different service time parameters in the next section of this chapter.

6.2 Mathematical Model

The aim of the model is to decide on the location of hub nodes, the allocation of non-hub nodes to these hub nodes, and which types of hub links to establish between the hub nodes. We aim to design our network so that each origin–destination pair receives service within its given service time bound. The objective of the model is to minimize the total costs which include the fixed costs of building the hub network and the total costs of transportation.

For the mathematical model, we need a given node set N consisting of n demand nodes and a potential hub set H such that $H \subseteq N$ with h nodes. We assume that there are two different types of transportation modes: air (denoted by a) and ground (denoted by g). There are two types of hubs and hub links that can be established based on these two transportation modes. It is assumed that, at most one type of hub link can be established between two hub nodes. The unit transportation costs and travel times are dependent on the mode of transportation. The economies of scale cost and time discount factors are also dependent on the choice of mode of transportation. It is assumed that, different transportation modes are only allowed to be used within the hub network; that is, via travelling between the hub nodes. For the allocation decisions, only ground transportation is employed. We assume throughout our model that the triangle inequality is satisfied for the parameters related to unit transportation cost and travel time.

There are two different types of service time promises: tight (β_1) and loose (β_2), such that $\beta_1 \leq \beta_2$. A given set of origin–destination pairs, say S_{β_1}, must receive service within the tight service time bound and the rest of the origin-destination pairs, say S_{β_2}, must receive service within the loose service time bound. It is assumed that the service levels are symmetric; that is, if a given (o, d) $\in S_{\beta_1}$, then (d, o) $\in S_{\beta_1}$ Even though it is assumed that there are two modes of transportation and two service time parameters, the model is readily extendible to include more.

The additional parameters required for the mathematical model are listed below:

$c_{ij}^{a}(c_{ij}^{g})$ = transportation cost of a unit of flow between nodes $i \in N$ and $j \in N$ using air (ground) transportation mode.

$FH_j^a(FH_j^g)$ = fixed cost of opening an air (ground) hub at node $j \in H$.

$FL_{ij}^a(FL_{ij}^g)$ = fixed cost of opening an air (ground) hub link between hubs $i \in H$ and $j \in H$.

$t_{ij}^a(t_{ij}^g)$ = travel time between nodes $i \in N$ and $j \in N$ using air (ground) transportation mode.

β_1 = tight service time bound.

β_2= loose service time bound.

S_{β_1}= set of origin–destination pairs requiring service within the tight service time bound.

S_{β_2}= set of origin–destination pairs requiring service within the loose service time bound.

$\alpha_c^a(\alpha_c^g)$= hub-to-hub transportation cost discount factor using air (ground) transportation.

$\alpha_t^a(\alpha_t^g)$= hub-to-hub transportation time discount factor using air (ground) transportation.

q = number of air hub links to be established.

Also let $M = \beta_2 - \beta_1 + \max_{i \in N, j \in H}\left(t_{ij}^g\right)$.

Through our observations from the cargo applications, we remark that using air transportation is more expensive but faster compared with ground transportation. Thus, we expect the cost parameters associated with air transportation to be higher than the corresponding cost parameters of using ground transportation and the travel times using air transportation to be lower than the travel times using ground transportation.

It is possible to model the multimodal hub location and hub network design problem with O(n^3) variables and constraints. For this, we employ some ideas from the incomplete hub network design models previously introduced in this thesis. Similar to the incomplete hub covering model introduced in Chapter 5, for each established hub, we find a spanning tree rooted at this hub that visits every other hub in the hub network using only the established hub links. The connectivity of the hub network is assured by employment of such spanning trees. We then calculate the travel time and the transportation costs between all pairs of hubs using these spanning trees. Since, there are two types of service time parameters we introduce two radii for each hub node to ensure that all origin–destination pairs receive service within their service time bounds.

The new decision variables of the mathematical model are:

$H_k^a (H_k^g)$= 1 if an air (ground) hub is established at node $k \in H$; 0 otherwise.

$Z_{ij}^a (Z_{ij}^g)$ = 1 if an air (ground) hub link is established between hubs $i \in H$ and $j \in H$; 0 otherwise.

T_{ij}= discounted travel time from hub $i \in H$ to hub $j \in H$ in the hub network.

C_{ij}= discounted unit transportation cost from node $i \in N$ to node $j \in N$ on the designed hub network.

R_j^1 = radius of hub $j \in H$ for tight service time bound.

R_j^2 = radius of hub $j \in H$ for loose service time bound.

The multimodal hub location and hub network design problem is modeled as:

$$\text{Minimize} \sum_{k \in H} FH_k^a H_k^a + \sum_{k \in H} FH_k^g H_k^g + \sum_{i \in H} \sum_{j \in H: j > i} FL_{ij}^a Z_{ij}^a + \sum_{i \in H} \sum_{j \in H: j > i} FL_{ij}^g Z_{ij}^g + \sum_i \sum_j w_{ij} C_{ij} \tag{6.1}$$

subject to

$$\sum_{j \in H} X_{ij} = 1 \qquad \forall\, i \in N \tag{2.3}$$

$$\sum_{j \in H} X_{jj} = p \tag{2.4}$$

$$X_{ij} \le X_{jj} \qquad \forall\, i \in N, j \in H \tag{2.6}$$

$$X_{jj} = H_j^a + H_j^g \qquad \forall\, j \in H \tag{6.2}$$

$$\sum_{i \in H} \sum_{j \in H: j > i} Z_{ij}^a = q \tag{6.3}$$

$$Z_{ij}^a \le H_i^a \qquad \forall\, i, j \in H: i < j \tag{6.4}$$

$$Z_{ij}^a \le H_j^a \qquad \forall\, i, j \in H: i < j \tag{6.5}$$

$$Z_{ij}^g \le H_i^a + H_i^g \qquad \forall\, i, j \in H: i < j \tag{6.6}$$

$$Z_{ij}^g \le H_j^a + H_j^g \qquad \forall\, i, j \in H: i < j \tag{6.7}$$

$$Z_{ij}^{a} + Z_{ij}^{g} \leq 1 \qquad \forall\, i,j \in H: i < j \qquad (6.8)$$

$$\sum_{i \in H: i \neq j} Y_{ijk} \geq X_{kk} + X_{jj} - 1 \qquad \forall\, j,k \in H: j \neq k \qquad (5.2)$$

$$\sum_{i \in H: i \neq j} Y_{ijk} \leq X_{kk} \qquad \forall\, j,k \in H: j \neq k \qquad (5.3)$$

$$Y_{ijk} + Y_{jik} \leq Z_{ij}^{a} + Z_{ij}^{g} \qquad \forall\, i,j,k \in H: i < j \qquad (6.9)$$

$$T_{kj} \geq \Big(T_{ki} + \alpha_t^a t_{ij}^a \big(Z_{ij}^a + Z_{ji}^a\big) + \alpha_t^g t_{ij}^g \big(Z_{ij}^g + Z_{ji}^g\big)\Big) Y_{ijk} \qquad \forall\, i,j,k \in H: i \neq j, j \neq k \qquad (6.10)$$

$$C_{kj} \geq \Big(C_{ki} + \alpha_c^a c_{ij}^a \big(Z_{ij}^a + Z_{ji}^a\big) + \alpha_c^g c_{ij}^g \big(Z_{ij}^g + Z_{ji}^g\big)\Big) Y_{ijk} \qquad \forall\, i,j,k \in H: i \neq j, j \neq k \qquad (6.11)$$

$$R_j^1 \geq t_{ij}^g X_{ij} - M(1 - X_{ij}) \qquad \forall\, i \in N, j \in H: \exists\, k: (i,k) \in S_{\beta_1} \qquad (6.12)$$

$$R_j^1 + T_{jk} + R_k^1 \leq \beta_1 \qquad \forall\, j,k \in H \qquad (6.13)$$

$$R_j^2 \geq t_{ij}^g X_{ij} \qquad \forall\, i \in N, j \in H \qquad (6.14)$$

$$R_j^2 + T_{jk} + R_k^2 \leq \beta_2 \qquad \forall\, j,k \in H \qquad (6.15)$$

$$C_{ij} \geq \big(C_{ik} + c_{kj}^g\big) X_{jk} \qquad \forall\, i,j \in N: i \neq j, k \in H \qquad (6.16)$$

$$C_{ij} = C_{ji} \qquad \forall\, i,j \in N \qquad (6.17)$$

$$C_{ij} \geq 0 \qquad \forall\, i,j \in N \qquad (6.18)$$

$$T_{ij} \geq 0 \qquad \forall\, i,j \in H \qquad (6.19)$$

$$X_{ij} \in \{0,1\} \qquad \forall\, i \in N, j \in H \qquad (2.5)$$

$$H_j^a, H_j^g \in \{0,1\} \qquad \forall\, j \in H \qquad (6.20)$$

$$Z_{ij}^a, Z_{ij}^g \in \{0,1\} \qquad \forall\, i,j \in H: i < j \qquad (6.21)$$

$$Y_{ijk} \in \{0,1\} \qquad \forall\, i,j,k \in H: i \neq j, j \neq k \qquad (5.9)$$

In the objective function (6.1), the first term calculates the fixed costs of establishing air and ground hubs; the second term calculates the fixed costs of establishing hub links with air and ground transportation modes. The last term in the objective function calculates the total cost of transportation. While calculating the transportation costs, the amount of flow to be routed between all pairs of nodes are multiplied with C_{ij} variables. The values of the C_{ij} variables are calculated within the model using the hub network to be designed.

Since there are two types of hub nodes to be located, Constraint (6.2) assures that if a hub is opened at a node then either a ground or an air hub must be established at that node.

By Constraint (2.4), the model establishes p hub nodes. Via Constraint (6.3), we restrict the number of air hub links to q. These two constraints can be removed since there are fixed cost terms associated with them in the objective function. However, we observed that cargo firms usually have a limited number of airplanes to employ. In order to differentiate between the hub links that may employ airplanes, the fixed cost term is added to the objective function. These constraints also allow us to parametrically analyze the results of the model.

We assumed a symmetric time data thus defined Z_{ij}^{a} and Z_{ij}^{g}variables with $i < j$ convention. Constraints (6.4) and (6.5) ensure that an air hub link can only be established if both of the end nodes of that link are air hubs. On the other hand, ground hub links can be established between both air and ground hub nodes via Constraints (6.6) and (6.7). This distinction is because air transportation can only be utilized when there are airports at hubs, whereas ground transportation is available between every hub. By Constraint (6.8), only one type of hub link can be established between two hub nodes.

Constraints (5.2) and (5.3) establish the spanning trees. Constraint (6.9) guarantees that the spanning trees can only use the established hub links.

Constraint (6.10) calculates the discounted travel time between every hub node in the hub network using the spanning trees. Similarly Constraint (6.11) calculates the discounted unit transportation cost between every hub node in the hub network using the spanning trees.

We defined two radii for each hub node to be established. The first radius of a hub node calculates the travel time from all non-hub nodes that are allocated to this hub node and which are an origin requiring service within β_1 to any destination node. Constraint (6.12) calculates the first radius of a hub. It suffices to calculate the first radius of a hub node using only the origins of the set S_{β_1}, since it is assumed that service levels are symmetric for origin–destination pairs. If there is no node allocated to a hub node requiring service within β_1, then the tight radius associated with that hub node is bounded below by a negative number by the definition of *M*. Constraint (6.13) ensures that the travel time between origin–destination pairs requiring service within β_1 are satisfied.

For the second radius, since all origin–destination pairs must either receive service within β_1 or β_2, and $\beta_1 \leq \beta_2$, the second radius is calculated in Constraint (6.14) as the maximum travel time from a non-hub node to its allocated hub. We ensure by Constraint (6.15) that travel time between any two nodes in the network is less than β_2.

Note that by the employment of the radii variables, R_j^1 and R_j^2, it is sufficient to calculate the travel time only between the hub nodes.

The discounted unit transportation costs are calculated within all pairs of hub nodes in Constraint (6.11). Constraints (6.16) and (6.17) calculate the unit transportation costs between all pairs of non-hub nodes in the network. We assumed that the flow costs are symmetric. However, replacement of Constraint (6.17) with $C_{ij} \geq \left(c_{ik}^{g} + C_{kj}\right)X_{ik}$ handles the non-symmetric case.

The rest of the constraints of the model, are the non-negativity constraints and the constraints defining binary variables.

This is a non-linear programming model due to constraints (6.10), (6.11), and (6.16). In the next section we present some enhancements for this mathematical model. We provide some linearizations, valid inequalities, and upper and lower bounding techniques.

6.3 Enhancing the Model

In this section, we aim to enhance the mathematical model presented for the multimodal hub location and hub network design problem introduced in the previous section. In Section 6.3.1 we provide linearizations for the non-linear constraints, and in Section 6.3.2 we propose some effective valid inequalities. We also present methods to obtain a lower and an upper bound on the objective function value of the model in Sections 6.3.3 and 6.3.4, respectively.

6.3.1 Linearizations

This section presents linearizations for all of the non-linear constraints, (6.10), (6.11), and (6.16), introduced in the mathematical model.

First, we propose Constraint (6.10*) below for the linearization of Constraint (6.10).

$$T_{kj} \geq T_{ki} + \alpha_t^a t_{ij}^a\left(Z_{ij}^a + Z_{ji}^a\right) + \alpha_t^g t_{ij}^g\left(Z_{ij}^g + Z_{ji}^g\right) - M_1\left(1 - Y_{ijk}\right) \quad \forall\, i,j,k \in H: i \neq j, j \neq k \qquad (6.10^*)$$

where $M_1 = \beta_2 + \max_{i,j\in H}\left(\alpha_t^a t_{ij}^a, \alpha_t^g t_{ij}^g\right)$.

Proposition 6.1 *Constraint* (6.10*) *correctly linearizes Constraint* (6.10).

Proof Consider the Constraints (6.10) and (6.10*) associated with nodes $i,j,k \in H: i \neq j, j \neq k$. There are two cases depending on the value of Y_{ijk}.

- *Case 1*: $Y_{ijk} = 1$. Then Constraints (6.10) and (6.10*) yield the same right- hand side.
- *Case 2*: $Y_{ijk} = 0$. The right-hand side of the Constraint (6.10) yields 0; however, the right-hand side of the Constraint (6.10*) yields: $T_{ki} + \alpha_t^a t_{ij}^a\left(Z_{ij}^a + Z_{ji}^a\right) + \alpha_t^g t_{ij}^g\left(Z_{ij}^g + Z_{ji}^g\right) - \beta_2 - \max_{i,j\in H}\left(\alpha_t^a t_{ij}^a, \alpha_t^g t_{ij}^g\right)$. Since $T_{kj} \geq 0$ by Constraint (6.19) it suffices to show that $T_{ki} + \alpha_t^a t_{ij}^a\left(Z_{ij}^a + Z_{ji}^a\right) + \alpha_t^g t_{ij}^g\left(Z_{ij}^g + Z_{ji}^g\right) \leq \beta_2 + \max_{i,j\in H}\left(\alpha_t^a t_{ij}^a, \alpha_t^g t_{ij}^g\right)$. By Constraint (6.15), $T_{ki} \leq \beta_2$. By Constraint (6.8), either $\left(Z_{ij}^a + Z_{ji}^a\right)$ or $\left(Z_{ij}^g + Z_{ji}^g\right)$ can take the value one. Thus, $\alpha_t^a t_{ij}^a\left(Z_{ij}^a + Z_{ji}^a\right) + \alpha_t^g t_{ij}^g\left(Z_{ij}^g + Z_{ji}^g\right) \leq \max_{i,j\in H}\left(\alpha_t^a t_{ij}^a, \alpha_t^g t_{ij}^g\right)$. So we conclude that the

right-hand side of the Constraint (6.10*) yields a number less than or equal to zero.

Hence, Constraint (6.10*) correctly linearizes Constraint (6.10). □

For the linearizations of constraints related to cost, let E be the set of hub links that can be established. More formally,

$$E = \{e = \{i,j\}: i,j \in H: i < j\} = \left\{e_1, e_2, \dots, e_{\frac{h(h-1)}{2}}\right\}.$$

Also let,

$$\gamma_e = \max\left\{\alpha_c^a c_{ij}^a, \alpha_c^g c_{ij}^g\right\} \text{ for all } e = \{i,j\} \in E.$$

Assume without loss of generality that γ_e values are sorted as

$$\gamma_{e_1} \geq \gamma_{e_2} \geq \cdots \geq \gamma_{e_{\frac{h(h-1)}{2}}}.$$

We propose Constraint (6.11*) below for the linearization of Constraint (6.11).

$$C_{kj} \geq C_{ki} + \alpha_c^a c_{ij}^a\left(Z_{ij}^a + Z_{ji}^a\right) + \alpha_c^g c_{ij}^g\left(Z_{ij}^g + Z_{ji}^g\right) - M_2\left(1 - Y_{ijk}\right) \quad \forall\, i,j,k \in H: i \neq j, j \neq k \qquad (6.11^*)$$

where $M_2 = 2\gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$.

Proposition 6.2 *Constraint* (6.11*) *correctly linearizes Constraint* (6.11).

Proof Consider the Constraints (6.11) and (6.11*) associated with nodes $i,j,k \in H: i \neq j, j \neq k$. There are two cases depending on the value of Y_{ijk} .

- *Case 1*: $Y_{ijk} = 1$. Then, Constraints (6.11) and (6.11*) yield the same right- hand side.
- *Case 2*: $Y_{ijk} = 0$. The right-hand side of Constraint (6.11) yields 0; however, the right-hand side of the Constraint (6.11*) yields: $C_{ki} + \alpha_c^a c_{ij}^a(Z_{ij}^a + Z_{ji}^a) + \alpha_c^g c_{ij}^g(Z_{ij}^g + Z_{ji}^g) - M_2$. Since $C_{kj} \geq 0$ by Constraint (6.18), it suffices to show that $C_{ki} + \alpha_c^a c_{ij}^a(Z_{ij}^a + Z_{ji}^a) + \alpha_c^g c_{ij}^g(Z_{ij}^g + Z_{ji}^g) \leq 2\gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$. By Constraint (6.8), either $(Z_{ij}^a + Z_{ji}^a)$ or $(Z_{ij}^g + Z_{ji}^g)$ can take the value one thus, $\alpha_c^a c_{ij}^a(Z_{ij}^a + Z_{ji}^a) + \alpha_c^g c_{ij}^g(Z_{ij}^g + Z_{ji}^g) \leq \max_{\{i,j\}\in E}(\alpha_c^a c_{ij}^a, \alpha_c^g c_{ij}^g) = \gamma_{e_1}$. Since the hub network to be designed consists of *p* hub nodes, at most *p*–1 hub links can be traversed when travelling from hub k to hub *i*. The maximum discounted total unit transportation cost of traversing *p*–1 hub links is $\gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$. Thus, $C_{ki} \leq \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$. Since $\alpha_c^a c_{ij}^a(Z_{ij}^a + Z_{ji}^a) + \alpha_c^g c_{ij}^g(Z_{ij}^g + Z_{ji}^g) \leq \gamma_{e_1}$ and $C_{ki} \leq \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$, by summing these two we obtain $C_{ki} + \alpha_c^a c_{ij}^a(Z_{ij}^a + Z_{ji}^a) + \alpha_c^g c_{ij}^g(Z_{ij}^g + Z_{ji}^g) \leq M_2$. So we conclude that the right-hand side of the Constraint (6.11*) yields a number less than or equal to zero.

Hence, Constraint (6.11*) correctly linearizes Constraint (6.11). □

We propose Constraint (6.16*) below for the linearization of Constraint (6.16).

$$C_{ij} \geq C_{ik} + c_{kj}^g X_{jk} - M_3(1 - X_{jk}) \qquad \forall\, i,j \in N: i \neq j, k \in H \tag{6.16*}$$

where $M_3 = \max_{i\in N, j\in H}(c_{ij}^g) + \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$.

Proposition 6.3 *Constraint* (6.16*) *correctly linearizes Constraint* (6.16).

Proof Consider the Constraints (6.16) and (6.16*) associated with nodes $i, j \in N: i \neq j, k \in H$. There are two cases depending on the value of X_{jk}.

- *Case 1*: $X_{jk} = 1$. Then, Constraints (6.16) and (6.16*) yield the same right- hand side.
- *Case 2*: $X_{jk} = 0$. The right-hand side of the Constraint (6.16) yields 0; however, the right-hand side of the Constraint (6.16*) yields: $C_{ik} - M_3$. Since $C_{ij} \geq 0$ by Constraint (6.18) it suffices to show that $C_{ij} \leq \max_{i \in N, j \in H}(c_{ij}^g) + \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$ From the argument in the proof of the Proposition 6.2 we know that, if $k, l \in H$ then $C_{lk} \leq \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$. Since $i \in N$, we add $\max_{i \in N, j \in H}(c_{ij}^g)$ to the summation, which is the maximum unit transportation cost from a non-hub node to a hub node. Then, we have $C_{ij} \leq \max_{i \in N, j \in H}(c_{ij}^g) + \gamma_{e_1} + \gamma_{e_2} + \cdots + \gamma_{e_{(p-1)}}$, so that the right-hand side of the Constraint (6.16*) yields a number less than or equal to zero.

We conclude that Constraint (6.16*) correctly linearizes Constraint (6.16). □

A linear integer programming formulation of the multimodal hub location and hub network design problem consists of the objective function (6.1) and constraints (2.3)–(2.6), (5.2), (5.3), (5.9), (6.2)–(6.9), (6.10*), (6.11*), (6.12)–(6.15), (6.16*), (6.17)–(6.21). In the worst case $h = n$ and the model has (n^3 + $2n^2$ + $2n$) binary variables and ($2n^2$+ $2n$) real variables. The number of constraints of our model is ($4n^3 + 10n^2 + n^2/2 + 2n + 2$). Hence, in total we have O(n^3) variables and constraints in the linear integer programming formulation of the multimodal hub location and hub network design problem.

6.3.2 Valid Inequalities

In this section we derive several families of valid inequalities for our mathematical model. The methodology borrows ideas from Yaman et al. (2009).

Let F be the feasible set for the multimodal hub location and hub network design problem. That is, F is the set of solutions satisfying the constraints (2.3)–(2.6), (5.2), (5.3), (5.9), (6.2)–(6.9), (6.10*), (6.11*), (6.12)–(6.15), (6.16*), (6.17)–(6.21).

As already stated before, it is assumed that triangle inequality is satisfied for both the unit transportation costs and the travel times for the mathematical model. For the sets of valid inequalities to be introduced, we additionally assume that $\alpha_c^g c_{ij}^g \leq \alpha_c^a c_{ij}^a$ and $\alpha_t^a t_{ij}^a \leq \alpha_t^g t_{ij}^g$ for all $i, j \in N$. That is, the unit discounted transportation cost using air transportation is assumed to be higher than the unit discounted transportation cost of using ground transportation between any two nodes, and the discounted travel time using air transportation is assumed to be lower than the discounted travel time using ground transportation between any two nodes.

Recall from Section 6.2 that we expect the unit transportation costs associated with ground transportation to be lower than the corresponding unit transportation costs of using air transportation ($c_{ij}^g \leq c_{ij}^a$ for all $i, j \in N$) and the travel times using air transportation to be lower than the travel times using ground transportation ($t_{ij}^a \leq t_{ij}^g$ for all $i, j \in N$). We observed through our observations from the cargo applications that the unit transportation costs by using air transportation is order of magnitudes higher than that of using ground transportation. Similarly, the transportation times by using air transportation is order of magnitudes lower than that of using ground transportation. Thus, the assumption of having $\alpha_c^g c_{ij}^g \leq \alpha_c^a c_{ij}^a$ and $\alpha_t^a t_{ij}^a \leq \alpha_t^a t_{ij}^g$ for all $i, j \in N$ is reasonable.

We first define some new parameters required for the definition of our valid inequalities. For $i \in N$ and $j \in N \backslash \{i\}$, let

$$\tau_{ij}^1 = \min_{k \in H \backslash \{j\}} \left(\alpha_c^g c_{ik}^g + c_{kj}^g \right)$$

$$\tau_{ij}^2 = \min_{k \in H \backslash \{i\}} \left(c_{ik}^g + \alpha_c^g c_{kj}^g \right)$$

$$\tau_{ij}^3 = \min_{k \in H \backslash \{i,j\}} \min_{l \in H \backslash \{i,j\}} \left(c_{ik}^g + \alpha_c^g c_{kl}^g + c_{lj}^g \right).$$

Since triangle inequality holds for unit transportation costs and $\alpha_c^g c_{ij}^g \leq \alpha_c^a c_{ij}^a$ for all $i, j \in N$ by assumption, if node *i* is a hub and node *j* is not a hub, then the minimum unit transportation cost from node *i* to node *j* is τ_{ij}^1. (Note that, since node *k* may be equal to node *i* in the minimum operator, the case when node *j* is allocated to hub *i* is also covered.) Conversely, if node *j* is a hub and node *i* is not a hub, then τ_{ij}^2 is a lower bound on the unit transportation cost from node *i* to node *j*. Finally, τ_{ij}^3 provides a lower bound on the unit transportation cost from node *i* to node *j* if neither of them are hub nodes.

Proposition 6.4 *For* $i \in H$ *and* $j \in H \backslash \{i\}$, *inequalities*

$$\begin{aligned} C_{ij} \geq \alpha_c^g c_{ij}^g &+ \left(\tau_{ij}^1 - \alpha_c^g c_{ij}^g \right) \left(1 - X_{jj} \right) \\ &+ \min\left\{ \tau_{ij}^3 - \tau_{ij}^1, \tau_{ij}^2 - \alpha_c^g c_{ij}^g \right\} \left(1 - X_{ii} \right) \end{aligned} \tag{A.1}$$

and

$$C_{ij} \geq \tau_{ij}^3 + \left(\tau_{ij}^2 - \tau_{ij}^3\right)X_{jj} + \min\left\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\right\}X_{ii} \tag{A.2}$$

are valid for F.

For $i \in H$ *and* $j \in N\backslash H$, *the inequality*

$$C_{ij} \geq \tau_{ij}^1 + \left(\tau_{ij}^3 - \tau_{ij}^1\right)(1 - X_{ii}) \tag{A.3}$$

is valid for F.

For $i \in N\backslash H$ *and* $j \in H$, *the inequality*

$$C_{ij} \geq \tau_{ij}^2 + \left(\tau_{ij}^3 - \tau_{ij}^2\right)\left(1 - X_{jj}\right) \tag{A.4}$$

is valid for F.

For $i \in N\backslash H$ *and* $j \in N\backslash H$, *the inequality*

$$C_{ij} \geq \tau_{ij}^3 \tag{A.5}$$

is valid for F.

Proof First we prove the validity of inequality (A.1). Consider the inequality (A.1) associated with nodes $i, j \in H: i \neq j$. There are four cases depending on the values of X_{ii} and X_{jj}.

- *Case 1*: $X_{ii} = 1$ and $X_{jj} = 1$. Then, both of the nodes *i* and *j* are hubs and inequality (A.1) reduces to $C_{ij} \geq \alpha_c^g c_{ij}^g$. Since triangle inequality holds and $\alpha_c^g c_{ij}^g \leq \alpha_c^a c_{ij}^a$ for all $i, j \in N$ by assumption, the unit transportation cost from hub *i* to hub *j* is at least $\alpha_c^g c_{ij}^g$, and (A.1) is valid.

- *Case 2*: $X_{ii} = 1$ and $X_{jj} = 0$. If node i is a hub and node j is not a hub, then the inequality (A.1) simplifies to $C_{ij} \geq \tau_{ij}^1$. It is then valid by the definition of τ_{ij}^1.

- *Case 3*: $X_{ii} = 0$ and $X_{jj} = 1$. If node j is a hub and node i is not a hub, then by the definition of τ_{ij}^2 we know that $C_{ij} \geq \tau_{ij}^2$. If min $\{\tau_{ij}^3 - \tau_{ij}^1, \tau_{ij}^2 - \alpha_c^g c_{ij}^g\} = \tau_{ij}^2 - \alpha_c^g c_{ij}^g$, then inequality (A.1) simplifies to $C_{ij} \geq \tau_{ij}^2$ and it is satisfied. If min $\{\tau_{ij}^3 - \tau_{ij}^1, \tau_{ij}^2 - \alpha_c^g c_{ij}^g\} = \tau_{ij}^3 - \tau_{ij}^1$, then $\tau_{ij}^2 - \alpha_c^g c_{ij}^g + \alpha_c^g c_{ij}^g \geq \tau_{ij}^3 - \tau_{ij}^1 + \alpha_c^g c_{ij}^g$ and $C_{ij} \geq \tau_{ij}^2 \geq \alpha_c^g c_{ij}^g + \tau_{ij}^3 - \tau_{ij}^1$ and inequality (A.1) is valid.

- *Case 4*: $X_{ii} = 0$ and $X_{jj} = 0$. If neither node i nor node j is a hub; , then by the definition of τ_{ij}^3, $C_{ij} \geq \tau_{ij}^3$. If min $\{\tau_{ij}^3 - \tau_{ij}^1, \tau_{ij}^2 - \alpha_c^g c_{ij}^g\} = \tau_{ij}^3 - \tau_{ij}^1$, then inequality (A.1) simplifies to $C_{ij} \geq \tau_{ij}^3$ and it is satisfied. If min $\{\tau_{ij}^3 - \tau_{ij}^1, \tau_{ij}^2 - \alpha_c^g c_{ij}^g\} = \tau_{ij}^2 - \alpha_c^g c_{ij}^g$, then inequality (A.1) reduces to $C_{ij} \geq \tau_{ij}^1 + \tau_{ij}^2 - \alpha_c^g c_{ij}^g$. Since, $\tau_{ij}^3 - \tau_{ij}^1 \geq \tau_{ij}^2 - \alpha_c^g c_{ij}^g$, and $\tau_{ij}^3 \geq \tau_{ij}^1 + \tau_{ij}^2 - \alpha_c^g c_{ij}^g$, then $C_{ij} \geq \tau_{ij}^3 \geq \tau_{ij}^1 + \tau_{ij}^2 - \alpha_c^g c_{ij}^g$ and inequality (A.1) is valid.

Since inequality (A.1) is satisfied in all of the four possible cases, we conclude that inequality (A.1) is valid.

We now prove the validity of inequality (A.2). Similarly, there are four cases depending on the values of X_{ii} and X_{jj}.

- *Case 1*: $X_{ii} = 0$ and $X_{jj} = 0$. If neither of the nodes i and j are hub, the inequality (A.2) reduces to $C_{ij} \geq \tau_{ij}^3$ and it is satisfied by the definition of τ_{ij}^3.

- *Case 2*: $X_{ii} = 0$ and $X_{jj} = 1$. If node i is not a hub and node j is a hub, then the inequality simplifies to $C_{ij} \geq \tau_{ij}^2$ and it is again valid by definition.

- *Case 3*: $X_{ii} = 1$ and $X_{jj} = 0$. If node i is a hub and node j is not a hub, then we know that $C_{ij} \geq \tau_{ij}^1$ by definition. If min $\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\} = \tau_{ij}^1 - \tau_{ij}^3$, the inequality (A.2) simplifies to $C_{ij} \geq \tau_{ij}^1$ and it is valid. If min $\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\} = \alpha_c^g c_{ij}^g - \tau_{ij}^2$, then $\tau_{ij}^1 - \tau_{ij}^3 \geq \alpha_c^g c_{ij}^g - \tau_{ij}^2$ and $C_{ij} \geq \tau_{ij}^1 \geq \tau_{ij}^3 + \alpha_c^g c_{ij}^g - \tau_{ij}^2$. Hence, inequality (A.2) is valid.

- *Case 4*: $X_{ii} = 1$ and $X_{jj} = 1$. Then, inequality (A.2) reduces to $C_{ij} \geq \tau_{ij}^2 +$ min $\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\}$. If both of the nodes i and j are hubs, we know that $C_{ij} \geq \alpha_c^g c_{ij}^g$ by assumption. If min $\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\} = \alpha_c^g c_{ij}^g - \tau_{ij}^2$, then inequality reduces to $C_{ij} \geq \alpha_c^g c_{ij}^g$ and it is satisfied. If min $\{\tau_{ij}^1 - \tau_{ij}^3, \alpha_c^g c_{ij}^g - \tau_{ij}^2\} = \tau_{ij}^1 - \tau_{ij}^3$, then $\alpha_c^g c_{ij}^g - \tau_{ij}^2 \geq \tau_{ij}^1 - \tau_{ij}^3$. So, $C_{ij} \geq \alpha_c^g c_{ij}^g \geq \tau_{ij}^2 + \tau_{ij}^1 - \tau_{ij}^3$ and the inequality is satisfied.

Inequality (A.2) is satisfied in all of the four possible cases. So, we conclude that inequality (A.2) is valid.

If node i is a hub, inequality (A.3) simplifies to $C_{ij} \geq \tau_{ij}^1$ and it is valid by definition since $j \in N\backslash H$. If i is not a hub it reduces to $C_{ij} \geq \tau_{ij}^3$ and it is valid again by definition. Similarly, inequality (A.4) for $i \in N\backslash H$, reduces to

$C_{ij} \geq \tau_{ij}^2$ when j is a hub, and to $C_{ij} \geq \tau_{ij}^3$ when j is not a hub and valid for both of the cases. For inequality (A.5), since $i, j \in N \backslash H$, $C_{ij} \geq \tau_{ij}^3$ by definition.

Hence, inequalities (A.1)–(A.5) are valid.□

The inequalities (A.1)–(A.5) are derived based on the information that a node becomes a hub or not to obtain lower bounds on the unit transportation costs. In the next set of valid inequalities, we again obtain lower bounds for the unit transportation costs but this time use the information that if a node is not a hub then it must be allocated to a hub node.

Proposition 6.5 *For* $i \in H$ *and* $j \in N \backslash \{i\}$, *the inequality*

$$C_{ij} \geq \sum_{h \in H \backslash \{i\}} \left(c_{ih}^g + \min_{k \in H \backslash \{i\}} \left(\alpha_c^g c_{hk}^g + c_{kj}^g \right) \right) X_{ih} + \min_{k \in H} \left(\alpha_c^g c_{ik}^g + c_{kj}^g \right) X_{ii} \tag{B.1}$$

is valid for F.

For $i \in N \backslash H$ *and* $j \in N \backslash \{i\}$, *the inequality*

$$C_{ij} \geq \sum_{h \in H} \left(c_{ih}^g + \min_{k \in H} \left(\alpha_c^g c_{hk}^g + c_{kj}^g \right) \right) X_{ih} \tag{B.2}$$

is valid for F.

For $i \in N$ *and* $j \in H \backslash \{i\}$, *the inequality*

$$C_{ij} \geq \sum_{h \in H \setminus \{j\}} \left(\min_{k \in H \setminus \{j\}} \left(c_{ik}^{g} + \alpha_c^{g} c_{kh}^{g} \right) + c_{hj}^{g} \right) X_{jh} + \min_{k \in H} \left(c_{ik}^{g} + \alpha_c^{g} c_{kj}^{g} \right) X_{jj} \tag{B.3}$$

is valid for F.

For $i \in N$ *and* $j \in N \setminus (H \cup \{i\})$, *the inequality*

$$C_{ij} \geq \sum_{h \in H} \left(\min_{k \in H} \left(c_{ik}^{g} + \alpha_c^{g} c_{kh}^{g} \right) + c_{hj}^{g} \right) X_{jh} \tag{B.4}$$

is valid for F.

Proof Let $i \in H$ and $j \in N \setminus \{i\}$. If $X_{ii} = 1$, then $\sum_{h \in H \setminus \{i\}} X_{ih} = 0$ and inequality (B.1) reduces to $C_{ij} \geq \min_{k \in H} \left(\alpha_c^{g} c_{ik}^{g} + c_{kj}^{g} \right)$. Independent of whether node *j* is a hub or not, since node *k* may be equal to node *j*, $\min_{k \in H} \left(\alpha_c^{g} c_{ik}^{g} + c_{kj}^{g} \right)$ provides a lower bound on the unit transportation cost from node *i* to node *j*, by triangle inequality and by the assumption that $\alpha_c^{g} c_{ij}^{g} \leq \alpha_c^{a} c_{ij}^{a}$ for all $i, j \in N$. So, inequality (B.1) is satisfied. If $X_{ii} = 0$, then node *i* must be allocated to a hub, say *h*, such that $X_{ih} = 1$ (node *h* is allowed to be the same node as *j*), then the unit transportation cost from hub *h* to node *j* is at least $\min_{k \in H \setminus \{i\}} \left(\alpha_c^{g} c_{hk}^{g} + c_{kj}^{g} \right)$. So, $C_{ij} \geq c_{ih}^{g} + \min_{k \in H \setminus \{i\}} \left(\alpha_c^{g} c_{hk}^{g} + c_{kj}^{g} \right)$. Hence, inequality (B.1) is valid.

For (B.2), $i \in N \setminus H$ and $j \in N \setminus \{i\}$, and thus $X_{ih} = 1$ for some hub *h*. Then from the above argument of the proof of inequality (B.1) the unit transportation cost from hub *h* to node *j* is at least $\min_{k \in H \setminus \{i\}} \left(\alpha_c^{g} c_{hk}^{g} + c_{kj}^{g} \right)$. So, $C_{ij} \geq \sum_{h \in H} \left(c_{ih}^{g} + \min_{k \in H} \left(\alpha_c^{g} c_{hk}^{g} + c_{kj}^{g} \right) \right) X_{ih}$ and (B.2) is valid.

For $i \in N$ and $j \in H\backslash\{i\}$, if $X_{jj} = 1$, then $\sum_{h \in H\backslash\{j\}} X_{jh} = 0$ and it takes at least $\min_{k \in H}\left(c_{ik}^g + \alpha_c^g c_{kj}^g\right)$ units of transportation cost to travel from node *i* to node *j*. Hence, $C_{ij} \geq \min_{k \in H}\left(c_{ik}^g + \alpha_c^g c_{kj}^g\right)$ and inequality (B.3) is satisfied. If $X_{jj} = 0$, then it is allocated to a hub, say h, and the unit transportation cost from node i to hub h is at least $\min_{k \in H\backslash\{j\}}\left(c_{ik}^g + \alpha_c^g c_{kh}^g\right)$. Hence, $C_{ij} \geq \min_{k \in H\backslash\{j\}}\left(c_{ik}^g + \alpha_c^g c_{kh}^g\right) + c_{hj}^g$ and inequality (B.3) is valid. Similarly for (B.4), since $X_{jj} = 0$, $C_{ij} \geq \sum_{h \in H}\left(\min_{k \in H}\left(c_{ik}^g + \alpha_c^g c_{kh}^g\right) + c_{hj}^g\right) X_{jh}$ from the same argument. Hence, (B.4) is also valid.

So, inequalities (B.1)–(B.4) are valid.□

Previous sets of valid inequalities, (A) and (B), are both derived to obtain lower bounds for the unit transportation costs between nodes. In the sequel, we introduce the valid inequalities related to travel time. For $i \in N$ and $j \in N\backslash\{i\}$, let

$$\mu_{ij}^1 = \min_{k \in H\backslash\{j\}}\left(\alpha_t^a t_{ik}^a + t_{kj}^g\right)$$

$$\mu_{ij}^2 = \min_{k \in H\backslash\{i\}}\left(t_{ik}^g + \alpha_t^a t_{kj}^a\right)$$

$$\mu_{ij}^3 = \min_{k \in H\backslash\{i,j\}} \min_{l \in H\backslash\{i,j\}}\left(t_{ik}^g + \alpha_t^a t_{kl}^a + t_{lj}^g\right).$$

Since triangle inequality holds for travel times and $\alpha_t^a t_{ij}^a \leq \alpha_t^g t_{ij}^g$ for all $i, j \in H$ by assumption, if node i is a hub and node *j* is not a hub, then the minimum travel time from node i to node *j* is μ_{ij}^1. Conversely, if node *j* is a hub and node *i* is not a hub, then μ_{ij}^2 is a lower bound on the travel time from node i to node j. Finally, μ_{ij}^3 provides a lower bound on the travel time from node i to node j if none of these nodes is a hub.

Proposition 6.6 *For* $i \in H$ *and* $j \in H\backslash\{i\}$, *inequalities*

$$T_{ij} \geq \alpha_t^a t_{ij}^a + \left(\mu_{ij}^1 - \alpha_t^a t_{ij}^a\right)\left(1 - X_{jj}\right) + \min\left\{\mu_{ij}^3 - \mu_{ij}^1, \mu_{ij}^2 - \alpha_t^a t_{ij}^a\right\}\left(1 - X_{ii}\right) \tag{C.1}$$

$$T_{ij} \geq \mu_{ij}^3 + \left(\mu_{ij}^2 - \mu_{ij}^3\right)X_{jj} + \min\left\{\mu_{ij}^1 - \mu_{ij}^3, \alpha_t^a t_{ij}^a - \mu_{ij}^2\right\}X_{ii} \tag{C.2}$$

$$T_{ij} \geq \sum_{h \in H\backslash\{i\}} \left(t_{ih}^g + \min_{k \in H\backslash\{i\}}\left(\alpha_t^a t_{hk}^a + t_{kj}^g\right)\right)X_{ih} + \min_{k \in H}\left(\alpha_t^a t_{ik}^a + t_{kj}^g\right)X_{ii} \tag{C.3}$$

$$T_{ij} \geq \sum_{h \in H\backslash\{j\}} \left(\min_{k \in H\backslash\{j\}}\left(t_{ik}^g + \alpha_t^a t_{kh}^a\right) + t_{hj}^g\right)X_{jh} + \min_{k \in H}\left(t_{ik}^g + \alpha_t^a t_{kj}^a\right)X_{jj} \tag{C.4}$$

are valid for F.

Proof The inequalities (C.1) and (C.2) are very similar in structure to the inequalities (A.1) and (A.2). The only difference is that the unit transportation cost values are appropriately replaced with travel times. Thus, the proofs of the validity of the inequalities (C.1) and (C.2) follow from the proofs of the valid inequalities (A.1) and (A.2) from Proposition 6.4. Similarly, the inequalities (C.3) and (C.4) are similar in structure with the inequalities (B.1) and (B.3), and the validity of these inequalities follow from the proofs of the valid inequalities (B.1) and (B.3) from Proposition 6.5. □

We tested the performance of these three sets of valid inequalities, (A), (B), and (C), both individually and collectively. The results are provided in Section 6.4 of this chapter. Now, we proceed with providing a lower bounding technique for the mathematical model.

6.3.3 Lower Bound

In this section, we propose a lower bound for the objective function of our multimodal hub location and hub network design problem. The aim in providing the lower bound is to increase the exact solution potential of our model.

Note that for a given number of air hub links to be established, q, there is a corresponding minimum number of air hubs that must be established. Let p^a denote the minimum number of air hubs to be established corresponding to a given q value. Then,

$$p^a = \min\left\{k \in \mathrm{Z}^+ : \frac{k(k-1)}{2} \geq q\right\},$$

where Z^+ defines the set of non-negative integers.

For example, if q = 1, then $p^a = 2$ meaning that at least two air hubs must be established to establish an air hub link in between, if q = 2 then $p^a = 3$, and so on.

For the lower bound, as stated in Section 6.2., we expect the fixed costs of establishing air hubs to be higher than the fixed costs of establishing ground hubs for each candidate hub node, that is, $FH_k^a \geq FH_k^g$ for each $k \in H$. Also, the fixed costs of establishing air hub links to be higher than the fixed costs of establishing ground hub links, i.e., $FL_{ij}^a \geq FL_{ij}^g$ for each $i, j \in H: i < j$. Let, the candidate set of hub nodes be $H = \{h_1, h_2, \ldots, h_h\}$. Recall from section 6.3.1 that E is the set of hub links that can be established, i.e., $E = \{e = \{i, j\}: i, j \in H: i < j\} = \left\{e_1, e_2, \ldots, e_{\frac{h(h-1)}{2}}\right\}$.

Without loss of generality, assume that the fixed costs of opening hubs and hub links are sorted as follows:

$$FH_{h_{i_1}}^a \leq FH_{h_{i_2}}^a \leq \cdots \leq FH_{h_{i_h}}^a,$$

$$FH_{h_{j_1}}^g \leq FH_{h_{j_2}}^g \leq \cdots \leq FH_{h_{j_h}}^g,$$

$$FL_{e_{k_1}}^a \leq FL_{e_{k_2}}^a \leq \cdots \leq FL_{e_{k_{\frac{h(h-1)}{2}}}}^a,$$

$$FL_{e_{l_1}}^g \leq FL_{e_{l_2}}^g \leq \cdots \leq FL_{e_{l_{\frac{h(h-1)}{2}}}}^g.$$

Let,

$$LB = \sum_{m=1}^{p^a} FH_{h_{i_m}}^a + \sum_{m=1}^{p-p^a} FH_{h_{j_m}}^g + \sum_{m=1}^{q} FL_{e_{k_m}}^a + \sum_{m=1}^{(p-1)-q} FL_{e_{l_m}}^g + \sum_{i \in N} \sum_{j \in N} w_{ij} \min\{\alpha_c^a c_{ij}^a, \alpha_c^g c_{ij}^g\}$$

Theorem 6.1: *Let z be the objective function value of any feasible solution in set* F*, then LB* $\leq$ *z.*

Proof Let $\bar{z}$ be the objective function value of a feasible solution in the set F If we let $(\bar{H}, \bar{Z}, \bar{C})$ represent the vector of the values of the variables *H*, *Z*, and *C* in the feasible solution associated with $\bar{z}$, then

$$\bar{z} = \sum_{k \in H} FH_k^a \bar{H}_k^a + \sum_{k \in H} FH_k^g \bar{H}_k^g + \sum_{i \in H} \sum_{j \in H: j > i} FL_{ij}^a \bar{Z}_{ij}^a + \sum_{i \in H} \sum_{j \in H: j > i} FL_{ij}^g \bar{Z}_{ij}^g + \sum_{i \in N} \sum_{j \in N} w_{ij} \bar{C}_{ij}$$

Since $\bar{H}$ is feasible, by Constraints (2.3) and (6.2) we know that $\sum_{k \in H} (\bar{H}_k^a + \bar{H}_k^g) = p$. By definition p^a is the minimum number of air hubs to be established for a given q value, thus, $\sum_{m=1}^{p^a} FH_{h_{i_m}}^a$ calculates the minimum cost of establishing p^a air hubs. As we assumed that $FH_k^a \geq FH_k^g$ for all $k \in H$, $\sum_{m=1}^{p-p^a} FH_{h_{j_m}}^g$ calculates the minimum cost of establishing remaining $(p - p^a)$ hubs. Then clearly,

$$\sum_{m=1}^{p^a} FH_{h_{i_m}}^a + \sum_{m=1}^{p-p^a} FH_{h_{j_m}}^g \leq \sum_{k \in H} FH_k^a \bar{H}_k^a + \sum_{k \in H} FH_k^g \bar{H}_k^g.$$

Similarly, $\sum_{m=1}^{q} FL_{e_{k_m}}^a$ represents the minimum cost of building q air hub links. The minimum number of links in a connected network with *n* nodes is $(n-1)$, which is possible when the network is a tree. Similarly, on a connected hub network with *p* hub nodes, $(p-1)$ is the lower bound on the number of hub links to be established. Remember that the spanning tree variables ensure the connectivity of the hub network in our formulation.

Thus, we know that any feasible hub network solution is connected. By the constraints of the model, q air hub links are to be established, thus the minimum number of ground hub links to be established is $\max\{0, (p-1)-q\}$. The minimum cost of establishing $(p-1)-q$ ground hub links is given by $\sum_{m=1}^{(p-1)-q} FL_{e_{l_m}}^{g}$. Thus,

$$\sum_{m=1}^{q} FL_{e_{k_m}}^{a} + \sum_{m=1}^{(p-1)-q} FL_{e_{l_m}}^{g} \leq \sum_{i\in H}\sum_{j\in H:j>i} FL_{ij}^{a}\bar{Z}_{ij}^{a} + \sum_{i\in H}\sum_{j\in H:j>i} FL_{ij}^{g}\bar{Z}_{ij}^{g}$$

Remember that *C* variables are defined as the unit discounted transportation cost between all nodes on the designed hub network. Since, we assumed that triangle inequality is satisfied it is clear that $\bar{C}_{ij} \geq \min\ \{\alpha_c^a c_{ij}^a, \alpha_c^g c_{ij}^g\}$ for all $i,j \in N$. Thus, $\sum_{i\in N}\sum_{j\in N} w_{ij} \min\ \{\alpha_c^a c_{ij}^a, \alpha_c^g c_{ij}^g\} \leq \sum_{i\in N}\sum_{j\in N} w_{ij}\bar{C}_{ij}$.

Since all the terms in *LB* proved to have lower values than the corresponding terms in $\bar{z}$, we conclude that $LB \leq \bar{z}$. As $\bar{z}$ was chosen arbitrarily, *LB* provides a lower bound to the objective function value of any feasible solution in set F.□

We tested the quality and the performance of our lower bound on the Turkish network data set. Our lower bound turned out to be superior to both the LP relaxation values of the mathematical model and the initial lower bounds reported by CPLEX. The detailed computational results are provided in Section 6.4.

6.3.4 Upper Bound

In this section we derive an upper bound for our problem. The aim in providing this upper bound is to decrease the CPU time requirements for the model using the commercial solver CPLEX. The calculation of the upper bound is based on establishing complete hub network solutions.

First, let's introduce a new problem named *multimodal hub location problem with complete hub networks*. The only difference of this new problem from the multimodal hub location and hub network design problem is that the hub network to be established is forced to be complete in this new problem.

By using the previously defined parameters, decision variables, and constraints, the linear integer programming formulation for the multimodal hub location problem with complete hub networks is:

Minimize (6.1)

subject to

(2.3)–(2.6), (6.2)–(6.8), (6.12), (6.14), (6.16*)–(6.21),

$$\sum_{i \in H} \sum_{j \in H: j > i} Z_{ij}^{a} + Z_{ij}^{g} = \frac{p(p-1)}{2} \quad (6.22)$$

$$C_{kl} \geq \alpha_c^a c_{kl}^a (Z_{kl}^a + Z_{kl}^a) + \alpha_c^g c_{kl}^g (Z_{kl}^g + Z_{kl}^g) \quad \forall\, k, l \in H \quad (6.23)$$

$$R_j^1 + \alpha_t^a t_{jk}^a (Z_{jk}^a + Z_{kj}^a) + \alpha_t^g t_{jk}^g (Z_{jk}^g + Z_{kj}^g) + R_k^1 \leq \beta_1 \quad \forall\, j, k \in H \quad (6.24)$$

$$R_j^2 + \alpha_t^a t_{jk}^a (Z_{jk}^a + Z_{kj}^a) + \alpha_t^g t_{jk}^g (Z_{jk}^g + Z_{kj}^g) + R_k^2 \leq \beta_2 \quad \forall\, j, k \in H \quad (6.25)$$

For the complete hub network problem, we need neither the *Y* variables associated with the spanning trees nor the *T* variables to calculate travel time between pairs of hub nodes. It is assumed that each pair of hub nodes sends their flow directly through the hub link that is established between them. Both the discounted travel time and the discounted unit transportation costs between hub nodes are calculated in the model based on this assumption.

In a complete network with n nodes, the number of links is calculated as $\frac{n(n-1)}{2}$. So, in a complete hub network with p hubs the number of hub links to be established is $\frac{p(p-1)}{2}$, which is ensured by Constraint (6.22). In fact, Constraint (6.22) guarantees that a hub link is to be established between each pair of hub nodes, however, the model decides on which type of hub link to establish.

Constraint (6.23) calculates the unit discounted transportation costs between two hub nodes.

Constraint (6.24) is analogous to Constraint (6.13) from the incomplete version of the problem. Since there is no need for T_{jk} variables, they are replaced with $\alpha_t^a t_{jk}^a\left(Z_{jk}^a + Z_{kj}^a\right) + \alpha_t^g t_{jk}^g\left(Z_{jk}^g + Z_{kj}^g\right)$, which is the discounted travel time between hubs j and k. Similarly, Constraint (6.25) is analogous to Constraint (6.15).

The mathematical formulation presented for the multimodal hub location problem with complete hub networks has O(n^2) variables and a single O(n^3) constraint (Constraint (6.16*)). Since the number of variables is decreased by a factor of n and there are fewer constraints, the complete version is expected to be solved more efficiently.

Let F^c be the feasible set for the multimodal hub location problem with complete hub networks. As previously defined, F is the feasible set for the multimodal hub location and hub network design problem.

Theorem 6.2: F^c ⊆ F, *in the sense that any feasible solution to the multimodal hub location problem with complete hub networks is a feasible solution to the multimodal hub location and hub network design problem.*

Proof Let $(\bar{X}, \bar{H}, \bar{Z}, \bar{C}, \bar{R})$ be a feasible solution to the problem with complete hub networks. Let us show that it is also feasible to the incomplete version. As all the constraints (2.3)–(2.6), (6.2)–(6.8), (6.12), (6.14), (6.16*)–(6.21) are common to both, it suffices to show that $(\bar{X}, \bar{H}, \bar{Z}, \bar{C}, \bar{R})$ is feasible to rest of the constraints, (5.2), (5.3), (5.9), (6.9)–(6.11), (6.13), (6.15), (6.19), of the multimodal hub location and hub network design problem.

First, let's construct a solution from $(\bar{X}, \bar{H}, \bar{Z}, \bar{C}, \bar{R})$ to the incomplete problem and then prove its feasibility to the remaining constraints. Set,

$$\bar{T}_{jk} = \alpha_t^a t_{jk}^a \left(\bar{Z}_{jk}^a + \bar{Z}_{kj}^a\right) + \alpha_t^g t_{jk}^g \left(\bar{Z}_{jk}^g + \bar{Z}_{kj}^g\right) \quad \forall\, j, k \in H$$

$$\bar{Y}_{kjk} = 1 \text{ for } \bar{X}_{jj} = \bar{X}_{kk} = 1$$

$$\bar{Y}_{ijk} = 0 \quad \forall\, i, j, k \in H : i \neq k$$

Clearly, $\bar{T}_{jk} \geq 0 \quad \forall\, j, k \in H$, and $\bar{Y}_{ijk} \in \{0,1\} \quad \forall\, i, j, k \in H$. Thus, Constraints (5.9) and (6.19) are satisfied.

Since $\bar{Y}_{ijk}$ can take on the value 1 only for $i = k$ and for $\bar{X}_{jj} = \bar{X}_{kk} = 1$, then $\sum_{i \in H : i \neq j} \bar{Y}_{ijk} \geq \bar{X}_{kk} + \bar{X}_{jj} - 1$ and $\sum_{i \in H : i \neq j} \bar{Y}_{ijk} \leq \bar{X}_{kk} \quad \forall\, j, k \in H : j \neq k$. Hence, Constraints (5.2) and (5.3) are satisfied.

Since we are building a complete hub network by Constraint (6.22), $\bar{Z}_{jk}^a + \bar{Z}_{jk}^g = 1$ for $\bar{X}_{jj} = \bar{X}_{kk} = 1$ when $j < k$. Thus, $(Y_{kjk} = 1) \leq \left(\bar{Z}_{jk}^a + \bar{Z}_{jk}^g = 1\right)$ for $\bar{X}_{jj} = \bar{X}_{kk} = 1 : j < k$. Thus, Constraint (6.9) is also satisfied.

With $\bar{Y}$ variables, Constraint (6.10) reduces to $\bar{T}_{kj} \geq \bar{T}_{kk} + \alpha_t^a t_{kj}^a \left(\bar{Z}_{kj}^a + \bar{Z}_{jk}^a\right) + \alpha_t^g t_{kj}^g \left(\bar{Z}_{kj}^g + \bar{Z}_{jk}^g\right)$ for $\bar{X}_{jj} = \bar{X}_{kk} = 1$. Since $\bar{T}_{kk} = 0$ and $\bar{T}_{kj} =$

$\alpha_t^a t_{kj}^a(\bar{Z}_{jk}^a + \bar{Z}_{kj}^a) + \alpha_t^g t_{kj}^g(\bar{Z}_{jk}^g + \bar{Z}_{kj}^g)$, Constraint (6.10) is feasible. Similarly, Constraint (6.11) reduces to $\bar{C}_{kj} \geq \bar{C}_{kk} + \alpha_c^a c_{kj}^a(\bar{Z}_{kj}^a + \bar{Z}_{jk}^a) + \alpha_c^g c_{kj}^g(\bar{Z}_{kj}^g + \bar{Z}_{jk}^g)$ for $\bar{X}_{jj} = \bar{X}_{kk} = 1$. Since, when $\bar{X}_{jj} = \bar{X}_{kk} = 1$, either $(\bar{Z}_{kj}^a + \bar{Z}_{jk}^a)$ or $(\bar{Z}_{kj}^g + \bar{Z}_{jk}^g)$ takes on the value 1, $\bar{C}_{kj} = \alpha_c^a c_{kj}^a(\bar{Z}_{kj}^a + \bar{Z}_{jk}^a) + \alpha_c^g c_{kj}^g(\bar{Z}_{kj}^g + \bar{Z}_{jk}^g)$ by Constraint (6.23) in the complete hub network problem and $\bar{C}_{kk} = 0$, Constraint (6.11) is also feasible.

When, $\bar{T}_{jk}$ is set to $\alpha_t^a t_{jk}^a(\bar{Z}_{jk}^a + \bar{Z}_{kj}^a) + \alpha_t^g t_{jk}^g(\bar{Z}_{jk}^g + \bar{Z}_{kj}^g)$ $\forall j,k \in H$, Constraints (6.13) and (6.15) reduces to the Constraints (6.24) and (6.25). Hence, they are feasible by the feasibility of Constraints (6.24) and (6.25) with $\bar{R}$.

So, we conclude that the solution $(\bar{X}, \bar{H}, \bar{Z}, \bar{C}, \bar{R}, \bar{T}, \bar{Y})$ satisfies all of the remaining Constraints (5.2), (5.3), (5.9), (6.9)–(6.11), (6.13), (6.15), and (6.19). Hence, it is a feasible solution to the multimodal hub location and hub network design problem. Since the feasible solution from F^c was chosen arbitrarily, $\mathrm{F}^c \subseteq \mathrm{F}$.□

By Theorem 6.2 any feasible solution to the multimodal hub location problem with complete hub networks is a feasible solution to the multimodal hub location and hub network design problem. Thus, the objective function value of any feasible solution to the complete version provides an upper bound to the objective function of the multimodal hub location and hub network design problem.

Note that the converse of the Theorem 6.2 does not hold. This is because we force the complete model to use the direct hub link established between two hubs. That is, the complete model does not allow traversing two or more hub links when travelling between two hub nodes. However, the travel time between two hub nodes of traversing two hub links with air transportation may be less than the travel time of using the direct hub link in-between with ground transportation. Thus, for the instances when the complete problem does not have a feasible solution, the incomplete version may have one.

We obtained upper bound values by solving the presented integer programming formulation of the multimodal hub location problem with complete hub networks optimally using CPLEX. Then, we use the optimum objective function values of the complete problem as an upper bound for the objective function value of the incomplete version, while solving the incomplete problem using CPLEX. We also tested the effect of providing an initial feasible solution to the incomplete problem by using the optimum solutions obtained from the complete version of the problem. During our preliminary analysis, we observed that neither providing upper bounds for the objective function nor an initial feasible solution for the multimodal hub location and hub network design problem has any positive effect on the solution times using CPLEX.

On the other hand, we observed that the solutions obtained from the multimodal hub location problem with complete hub networks, provided good feasible solutions in reasonable amounts of CPU times. Thus, may be considered as a heuristic for the problem rather than as an upper bounding technique. For the test instances, we calculated the gap of the complete hub network solutions from the optimal solutions of the multimodal hub location and hub network design problem. The detailed results of our computational analysis are provided in the next section.

6.4 Computational Analysis

We tested our multimodal hub location and hub network design model on the Turkish network data set. Unfortunately, since we do not have enough data for the required parameters on the CAB data set, we could not test it for this problem.

For the Turkish network, the values of the parameters w_{ij}, c_{ij}^g, FH_j^g, FL_{ij}^g, t_{ij}^g, and α_t^g comply with the values presented in Beasley (1990) which are also used in the previous chapters of this thesis. To determine the values of the parameters associated with air transportation, several interviews are held with various cargo companies operating in Turkey. Based on these interviews, we observed that the cost values associated with air hub and hub link usage is higher than that of ground hub and hub link usage, as expected. For the sake of simplicity, we took the cost values associated with air transportation as 10 times the corresponding value of using ground transportation. The travel times using air transportation is estimated by using the information from the Turkish Airlines. It is observed that the economies of scale time discount factor of using air transportation (α_t^a) is negligible, thus it is taken as one.

We took the service time bounds, as $\beta_1 = 12$ and $\beta_2 = 24$ hours, where the first one corresponds to VIP service and the second to next day delivery, respectively. The set of origin–destination pairs requiring service within 12 hours are taken as the same as the VIP service set of a well-known cargo company operating in Turkey. Based on their service promises, the demand centers Adana (1), Ankara (6), Antalya (7), Bursa (16), Erzurum (25), İstanbul (34), and İzmir (35) can mutually use the VIP service. So, all of these demand centers are origins and destinations of each other in our set S_{β_1}. All of the remaining pairs from 81 demand centers are included in the set S_{β_2}.

Then, we varied the values of the remaining parameters p, q, α_c^a, and α_c^g on the Turkish network data set. As customarily done in the literature, we varied α_c^a and α_c^g values between 0.2 and 0.8 with increments of 0.2. While varying α_c^a and α_c^g, we assumed that the economies of scale discount factor of using air transportation to be lower than using ground transportation; that is, $\alpha_c^a \leq \alpha_c^g$.

We took our runs on a server with 2.66 GHz Intel Xeon processor and 8GB of RAM and we used the optimization software CPLEX version 11.2.

First, we tested the methods proposed for enhancing our multimodal hub location and hub network design model. For this analysis, we generated a smaller data set from the Turkish network. We took 25 nodes from 81 demand centers and chose 8 candidate hub locations from 16 presented in the Turkish network (|N|=25, |H|=8). For the |N|=25, |H|=8 Turkish data set we ranged the values of p, q, α_c^a and α_c^g. Table 6.1 presents the values of the test parameters. The other parameters required for the model comply with the parameters of the Turkish network data set.

First, we tested and compared the three sets of valid inequalities, (A), (B) and (C), introduced in Section 6.3.2, with the linear integer programming formulation of the multimodal hub location and hub network design problem on the instances presented in Table 6.1. We put a time limit of one hour (3600 seconds) on CPLEX.

Table 6.1 Test bed for |N|=25, |H|=8.

Instance number	p	q	α_c^a	α_c^g
1	3	2	0.6	0.8
2	3	3	0.6	0.6
3	4	2	0.4	0.8
4	4	3	0.2	0.6
5	4	4	0.2	0.6
6	5	2	0.2	0.8
7	5	3	0.2	0.4
8	5	4	0.4	0.6
9	5	5	0.4	0.4

We compared the initial lower bound values reported by CPLEX, the CPU time requirements, and the number of nodes in the branch and bound tree. We provide the results in Table 6.2.

The first column in Table 6.2 reports the instance number corresponding to the instances listed in Table 6.1. For each instance, the first row lists the initial lower bound value reported by CPLEX (denoted by 'lb'), the second row lists the CPU time requirement in seconds by CPLEX, and if the problem could not be solved within one hour the gap reported by CPLEX (denoted by 'cpu (gap)'), the third row lists the number of nodes in the branch and bound tree (denoted by 'nodes'), and the last row lists the actual gap of the final solution reported by CPLEX from the optimum solution. The actual gap is calculated as $\frac{Obj_{CPLEX}-Obj_{Optimum}}{Obj_{Optimum}} \times 100$. The last four rows of Table 6.2 list the average of these values for each column. The highlighted values in each row correspond to the best values in each category.

Table 6.2 The effect of valid inequalities.

		No v.i's	**A**	**B**	**C**	**AB**	**AC**	**BC**	**ABC**
1	**lb**	6264.6485	6484.0554	6530.1959	6264.6485	**6530.2952**	6484.0554	6530.1959	**6530.2952**
	cpu (gap)	2.94	2.84	2.58	3.32	3.44	**2.21**	2.61	4.58
	nodes	508	543	**479**	1165	506	500	497	490
	actual gap	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**
2	**lb**	8773.1415	8971.5761	9007.7685	8773.1415	**9008.1126**	8971.5761	9007.7685	**9008.1126**
	cpu (gap)	**1.56**	1.77	1.58	1.97	1.60	1.60	1.80	1.90
	nodes	510	593	**473**	704	506	493	541	490
	actual gap	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**
3	**lb**	3176.1052	3431.8707	3483.3042	3176.1052	**3483.9731**	3431.8707	3483.3042	**3483.9731**
	cpu (gap)	3600 (19.58)	2724.92	3600 (3.00)	3600 (3.03)	1627.00	3600 (2.11)	3600 (3.23)	**594.64**
	nodes	711691	525756	632438	607046	264948	700589	806897	**77516**
	actual gap	1.7%	**0%**	0.15%	0.002%	**0%**	**0%**	0.11%	**0%**
4	**lb**	4080.9902	4313.8083	4346.4299	4080.9902	**4348.6642**	4313.8083	4346.4299	**4348.6642**
	cpu (gap)	2387.86	2422.28	1250.84	2961.97	3600 (1.22)	411.13	2263.60	**301.51**
	nodes	558233	509062	253414	528940	659316	82296	504070	**53697**
	actual gap	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**	**0%**
5	**lb**	4985.8972	5219.8742	5250.0074	4985.8972	**5252.4697**	5219.8742	5250.0074	**5252.4697**
	cpu (gap)	3600 (1.76)	3600 (0.64)	**1472.36**	3600 (1.76)	1625.37	3600 (0.83)	1640.97	2345.17
	nodes	456063	525544	**259051**	503980	266604	556981	274169	323698
	actual gap	**0%**	**0%**	**0%**	0.03%	**0%**	**0%**	**0%**	**0%**
6	**lb**	3459.4816	3708.2446	3755.6168	3459.4816	**3755.8178**	3708.2446	3755.6168	**3755.8178**
	cpu (gap)	3600 (15.49)	3600 (15.79)	3600 (16.57)	3600 (17.79)	**3600 (7.49)**	3600 (16.25)	3600 (15.77)	3600 (14.72)
	nodes	442493	**297171**	458167	464342	485927	389443	372297	355295
	actual gap	7.51%	10.09%	10.08%	10.31%	**2.06%**	10.08%	12.01%	10.23%
7	**lb**	4357.8580	4547.2003	4572.0537	4357.8580	**4573.5534**	4547.2003	4572.0537	**4573.5534**
	cpu (gap)	3600 (9.03)	3600 (5.76)	3600 (7.08)	3600 (4.35)	3600 (15.53)	3600 (7.66)	3600 (12.13)	**3600 (3.04)**
	nodes	642614	620520	453093	665058	405696	534149	623720	**336688**
	actual gap	0.83%	**0%**	0.45%	0.70%	10.48%	**0%**	6.71%	**0%**
8	**lb**	5256.2343	5477.1654	5509.4325	5256.2343	**5510.3771**	5477.1654	5509.4325	**5510.3771**
	cpu (gap)	3600 (13.61)	3600 (12.18)	**3600 (2.63)**	3600 (9.49)	3600 (2.99)	3600 (3.31)	3600 (4.11)	3600 (8.55)
	nodes	627951	**523743**	554993	620253	439547	653143	572903	554891
	actual gap	10.47%	3.35%	0.58%	6.05%	0.70%	1.48%	1.68%	**0%**
9	**lb**	6154.6517	6343.2091	6365.9006	6154.6517	**6367.6962**	6343.2091	6365.9006	**6367.6962**
	cpu (gap)	3600 (11.89)	3600 (2.36)	**3600 (1.81)**	3600 (18.10)	3600 (2.36)	3600 (19.12)	3600 (4.79)	3600 (5.54)
	nodes	538228	487796	549802	583608	548914	547758	562940	**396501**
	actual gap	8.47%	0.02%	**0%**	7.93%	0.03%	10.99%	1.76%	2.75%
Average	**lb**	5167.6676	5388.5560	5424.5233	5167.6676	**5425.6621**	5388.5560	5424.5233	**5425.6621**
	cpu (gap)	2665.82 (7.93)	2572.42 (4.08)	**1903.04** (3.45)	2729.7 (6.06)	2361.93 **(3.29)**	2446.1 (5.48)	2434.33 (4.45)	1960.87 (3.54)
	nodes	442032.33	387858.67	351323.33	441677.33	341329.33	385039.11	413114.89	**233251.78**
	actual gap	3.22%	1.50%	**1.25%**	2.78%	1.47%	2.51%	2.47%	1.44%

Columns three to ten correspond to all possible combinations of the three sets of valid inequalities. In particular, the column indicated by 'No v.i.'s', correspond to the linear integer formulation of the problem without any of the valid inequalities, and the last column includes all the three sets of valid inequalities in our formulation.

When we compare the initial lower bounds, observe that inequality set (C) does not have any effect on the lower bounds. This is because these sets of inequalities are related to travel times, which does not have an associated cost in the objective function. The inequality set (B) always resulted in higher lower bounds when compared with set (A). Observe from Table 6.2 that the highest initial lower bounds at every instance are obtained with inequality sets (AB) and (ABC).

Instances 1,2, 4, and 5 (with one exception) are all solved to optimality with all sets of inequalities. Instances 1 and 2 are solved within 3 seconds, thus the CPU times for these instances are not comparable. In the average CPU times reported in Table 6.2, inequality set (B) has the lowest average, whereas the second lowest is the set (ABC). Inequality set (B) has the lowest CPU time requirement or the best gap reported by CPLEX at the instances 5, 8 and 9, whereas instances 3, 4, and 7 solved more effectively with the set (ABC).

In terms of number of nodes in the branch and bound tree, on the average, set (ABC) showed the best behavior. Whereas, for the actual gaps from the optimum solution set (B) has the lowest average with 1.25%, followed by the set (ABC) having 1.44%.

In light of these observations, since the behavior of (ABC) is satisfactory at each instance and in the average values, we decided to use (ABC) and include all of the valid inequalities to our mathematical formulation.

We tested the effect of our lower bounding technique and the performance of the solutions obtained with the multimodal hub location problem with complete hub networks on the same test instances. The results are provided in Table 6.3.

First, we compared the model results with (ABC) and (ABC) with the inclusion of the lower bound. Observe from Table 6.3 that except at instance number two, our lower bounds turned out to be superior to the initial lower bounds reported by CPLEX. The CPU times at the instances 2, 3, 4, and 5 with the inclusion of the lower bound turned out to be higher. As expected, at the instances that could not be solved optimally in one hour, the gaps reported by CPLEX with our lower bound are lower, with one exception.

When we look at the averages, both the average CPU time and the average number of nodes in the branch and bound tree are higher with the lower bound, whereas the average actual gaps are lower.

In Table 6.3, we also compared the effect of the solutions obtained with the complete version of the problem, denoted by ‘Complete solution’. In order to obtain the ‘Complete solution’ we solved the integer programming formulation of the multimodal hub location problem with complete hub networks using CPLEX 11.2 on the same server.

Table 6.3 The effect of the lower bound and the performance of the solution with complete hub network.

Instance		ABC	ABC with *LB*	Complete solution
1	**lb**	6530.2952	7939.8720	6584.7057
	cpu (gap)	4.58	4.50	0.32
	nodes	490	497	6
	actual gap	0%	0%	0.52%
2	**lb**	9008.1126	7888.7808	9102.8287
	cpu (gap)	1.90	2.23	0.70
	nodes	490	568	254
	actual gap	0%	0%	0%
3	**lb**	3483.9731	8250.3298	3488.7547
	cpu (gap)	594.64	1349.08	4.09
	nodes	77516	153937	125
	actual gap	0%	0%	7.92%
4	**lb**	4348.6642	8199.2188	4350.6507
	cpu (gap)	301.51	639.87	13.11
	nodes	53697	91772	1246
	actual gap	0%	0%	3.02%
5	**lb**	5252.4697	10993.5221	5255.0442
	cpu (gap)	2345.17	3600 (0.12)	15.02
	nodes	323698	515732	1370
	actual gap	0%	0%	6.27%
6	**lb**	3755.8178	8576.8269	3763.1394
	cpu (gap)	3600 (14.72)	3600 (0.18)	14.35
	nodes	355295	346865	553
	actual gap	10.23%	0%	Infeasible
7	**lb**	4573.5534	8474.4258	4577.8692
	cpu (gap)	3600 (3.04)	3600 (2.96)	11.26
	nodes	336688	649661	575
	actual gap	0%	0%	12.88%
8	**lb**	5510.3771	11319.9830	5313.2748
	cpu (gap)	3600 (8.55)	3600 (2.25)	45.08
	nodes	554891	383991	3235
	actual gap	0%	0.21%	10.65%
9	**lb**	6367.6962	11269.0789	6220.4070
	cpu (gap)	3600 (5.54)	3600 (5.73)	178.2
	nodes	396501	177269	39341
	actual gap	2.75%	0%	5.59%
Average	**lb**	5425.6621	9212.4487	5611.6919
	cpu (gap)	1960.87 (3.54)	2221.74 (1.25)	33.47
	nodes	233251.78	257810.22	5769
	actual gap	1.44%	0.02%	5.86%

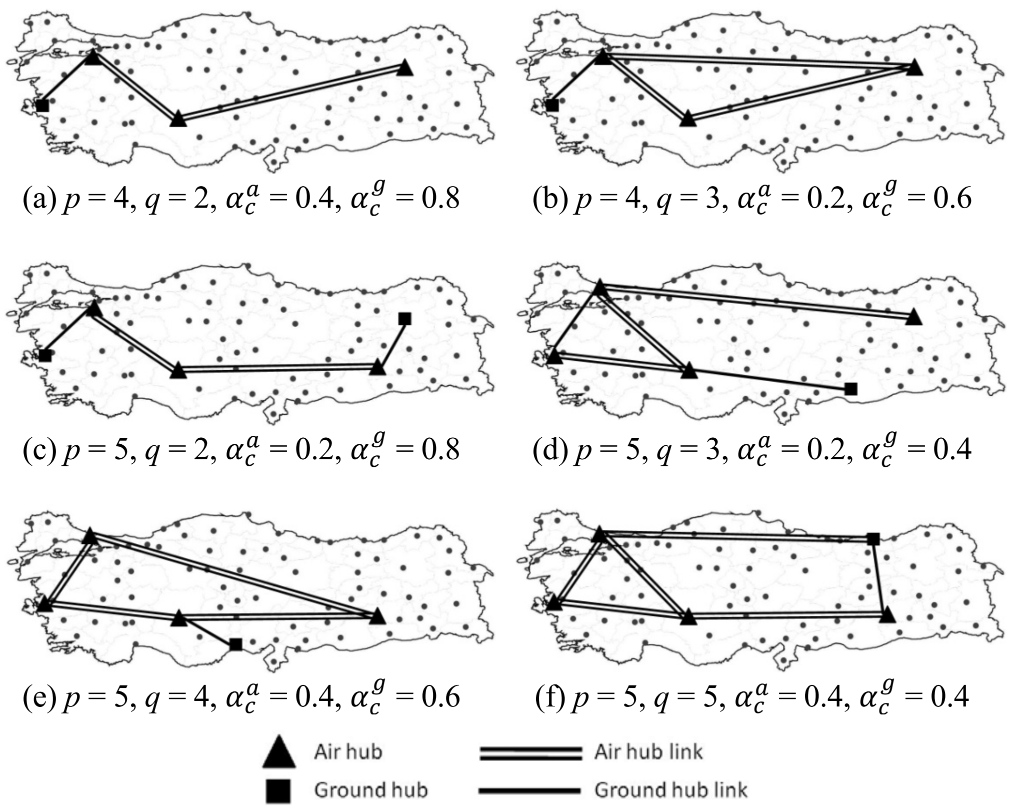
Except at instance six, the multimodal hub location model with complete hub networks was able to find a feasible solution. The CPU time requirements to solve the complete version of the problem ranged from 0.32 seconds to 2.97 minutes. On the average, the model is solved in less than 34 seconds.

The gaps of the complete solution from the optimal solution are very promising. They ranged from 0% to less than 13%. In instance two, since the model opens 3 hubs and 3 air hub links the resulting hub network is forced to be complete, thus the complete version of the problem outputs the same optimal solution. The average gaps of the complete solution reported in Table 6.3 is 5.86%, proving the efficiency of the feasible solutions obtained by solving the multimodal hub location problem with complete hub networks.

As a conclusion, the computational results proved the effect of all of our valid inequalities proposed in Section 6.3.2. The lower bound that we presented in Section 6.3.3 is very easy to calculate and outperforms the initial lower bound values reported by CPLEX. However, the inclusion of lower bounds does not proved to have a positive effect on the solution times using CPLEX. The computational results also showed that the multimodal hub location problem with complete hub networks, introduced in Section 6.3.4, provide good feasible solutions in reasonable amounts of time

We then tested the multimodal hub location and hub network design problem on the Turkish data set with 81 demand centers and 16 candidate hub locations. We included all sets of valid inequalities, (A), (B), and (C), as they were proven to be effective. We put a time limit of 24 hours (86400 seconds) on CPLEX.

The multimodal hub location and hub network design problem is very difficult to solve. Unfortunately, we were not able to solve all test instances on the Turkish data set with 81 demand nodes and 16 candidate hub locations to optimality within 24 hours. Figure 6.1a–f depicts some examples of resulting hub network solutions from the best solutions found by CPLEX.



(a) $p = 4$, $q = 2$, $\alpha_c^a = 0.4$, $\alpha_c^g = 0.8$ (b) $p = 4$, $q = 3$, $\alpha_c^a = 0.2$, $\alpha_c^g = 0.6$

(c) $p = 5$, $q = 2$, $\alpha_c^a = 0.2$, $\alpha_c^g = 0.8$ (d) $p = 5$, $q = 3$, $\alpha_c^a = 0.2$, $\alpha_c^g = 0.4$

(e) $p = 5$, $q = 4$, $\alpha_c^a = 0.4$, $\alpha_c^g = 0.6$ (f) $p = 5$, $q = 5$, $\alpha_c^a = 0.4$, $\alpha_c^g = 0.4$

Figure 6.1 Resulting hub networks.

Air and ground hubs and hub links are indicated in the Figure 6.1. In our results with the Turkish data set, there is a hub that is usually located in the eastern part or Turkey which is connected to the hub network by an air hub link. Most of the demand centers in the eastern part of the region are allocated to this hub. Konya (42) is almost always present in the hub set. Most of the demand centers from the southern parts and the Middle Anatolian Region are allocated to Konya. In the solutions, there is a hub commonly located near or at the demand center Istanbul (34), since Istanbul generates the highest amount of flow. Again, there is usually a hub located near or at Izmir (35), which serves the western part of the region.

When we look at the best solutions reported by CPLEX, all solutions resulted in the establishment of incomplete hub networks. Thus, the experimental results on the multimodal hub location and hub network design problem also suggest the impracticality of building complete hub networks in hub location problems.

We also analyzed the effect of α_c^a and α_c^g values on the hub network to be established. Thus, we fixed p and q, to four and three, respectively, and tested the possible combinations of α_c^a and α_c^g values. With 0.2 increments between 0.2 and 0.8 and with $\alpha_c^a \leq \alpha_c^g$ there are a total of ten instances for fixed p and q values. We observed that the established hubs and hub links do not change with the values chosen for α_c^a and α_c^g. All the possible combinations of α_c^a and α_c^g values resulted in the hub network depicted in Figure 6.1b. The resulting hub networks turned out to be the same since the service time bounds are tight. With tight service time bounds, there may not be many feasible solutions satisfying the service time limits with the given p and q values.

6.5 Conclusions

In this chapter, we introduced the multimodal hub location and hub network design problem to the literature. The model includes various observations from real-life hub networks. We relaxed the assumption of building complete hub networks, considered transportation costs and travel times simultaneously in the model, offered different types of service time promises between origin–

destination pairs, and considered the choice of different modes of transportation for hubs and hub links. To the best of our knowledge, there is not any study in the hub location literature including all of these stated real-life observations from hub networks.

We provided a linear integer programming formulation of the multimodal hub location and hub network design problem, with two types of hubs and hub links, accounting for different transportation modes, and two different service time parameters, with $O(n^3)$ variables and constraints.

Various methods are suggested for enhancing the presented mathematical formulation. Some efficient valid inequalities are proposed, and effective lower and upper bounding techniques are presented. Detailed computational analysis is presented using the Turkish network data set.

As it can be seen from the computational analysis, the problem turned out to be very difficult. As a future research, one may develop optimization algorithms such as a branch-and-cut algorithm or an efficient heuristic for the multimodal hub location and hub network design problem.

Chapter 7

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

In this thesis, we studied hub location and hub network design problems. We observed from many real-life applications of hub location problems that the basic assumption in the hub location literature of building complete hub networks is not always valid. Thus, we relaxed the complete hub network assumption in hub location problems and focused on designing hub networks that are not necessarily complete. In addition to the location and allocation decisions in hub location problems, we incorporated the decision on how the hub networks must be designed. We defined new hub location problems with incomplete hub networks to the literature by justifying with real world examples. We focused on the single allocation versions of the problems, which are computationally harder than their multiple allocation counterparts.

First, we defined the 3-stop hub covering network design problem. The problem is motivated by observations from a specific cargo delivery company operating in Turkey. The aim of the 3-stop hub covering network design problem is to locate hubs and design hub networks so that all origin–destination pairs receive service by visiting at most three hubs on a route, within a given service time limit. The problem also includes the decisions on the synchronization of trucks.

We then included hub network design decisions in classical hub location problems presented in the literature. We introduced and studied four new problems: the single allocation incomplete *p*-hub median, hub location with fixed costs, hub covering, and *p*-hub center problems. Very efficient novel integer programming formulations are proposed for these problems that can solve various instances from well-known data sets within minutes.

In the extensive literature on hub location, different service levels for different origin–destination pairs are not considered. In addition to that, hub location problems proposed in the literature focus on using only a single transportation mode. Especially, with different service time promises multimodal transportation is a necessity. Thus, in this thesis, we proposed the multimodal hub location and hub network design problem. We included the possibility of using different hub links and allowed for different transportation modes, and for different types of service time promises between origin–destination pairs, while designing the hub network in the multimodal problem. In this problem, we also considered transportation costs and travel times simultaneously, which are studied separately in hub location problems presented in the literature. We proposed an integer programming formulation, with two types of hubs and transportation modes and two different service time parameters, for the multimodal hub location and hub network design problem. We also proposed very efficient valid inequalities and effective lower and upper bounding techniques.

All of the problems studied in this thesis are among the first single allocation hub location and hub network design problems. We were able to solve various instances from the commonly used CAB and Turkish network data sets in reasonable times.

Our test results demonstrated the importance of hub network design decisions on hub locations. The hub location and hub network design problems should be addressed simultaneously, since the hub networks to be designed effects the locations of the hub nodes.

An extension to the hub location and hub network design problems introduced in this thesis is to include capacity restrictions. All of the proposed models are readily extendible to include capacity constraints. The capacity constraints can be on the amount of flow on the hub links as well as on the available number of trucks or airplanes to employ on each hub link to be established.

For future research, one may develop efficient formulations for the multiple allocation versions of the hub location and hub network design problems introduced in this thesis. Cleary, the multiple allocation hub location problems provide a lower bound for the single allocation versions. By using this property, multiple allocation problems can be solved to obtain lower bounds for the single allocation versions. These lower bounds can be utilized in solving the single allocation problems with exact solution algorithms such as a branch and bound.

Another future research direction is to include real-life operational requirements into the mathematical models. Operational decisions such as the synchronization of airplanes and trucks are important especially for the multimodal hub location and hub network design problem. The scheduling of airplanes can be considered together with hub location and hub network design decisions to maximize the utilization of airplanes. The inclusion of such decisions into the mathematical models may result in the establishment of different hubs and hub links. With more restrictions the problems will become harder to solve. Thus, one may need to develop efficient heuristics for hub location and hub network design problems in the future.

The economy of scale discount factor is in fact dependent on the flow on the hub links. To the best of our knowledge, there is no study in the literature modeling the relation between the amount of flow and the discount factor in designing incomplete hub networks. Thus, the relation between the amount of flow on the hub links and the economies of scale discount factor should also be addressed in future.

Lastly, another area for future research is to incorporate stochasticity in hub location and hub network design. Both the travel times and the amount of flow generated between demand points have stochastic nature. The inclusion of stochasticity in the models may lead to more robust hub network solutions.

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APPENDIX

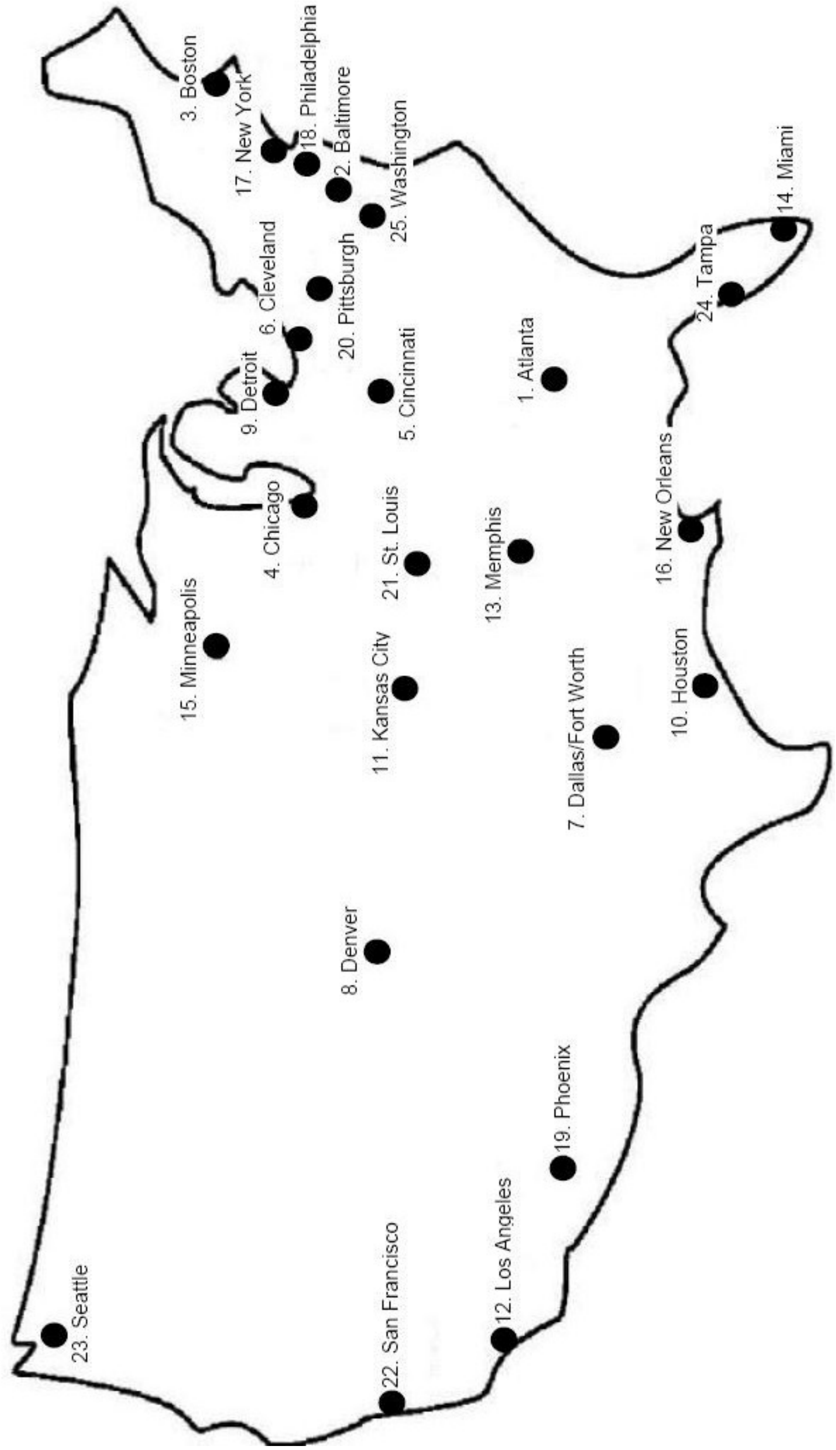

Figure A.1 Names and geographical locations of the cities in the CAB data set.

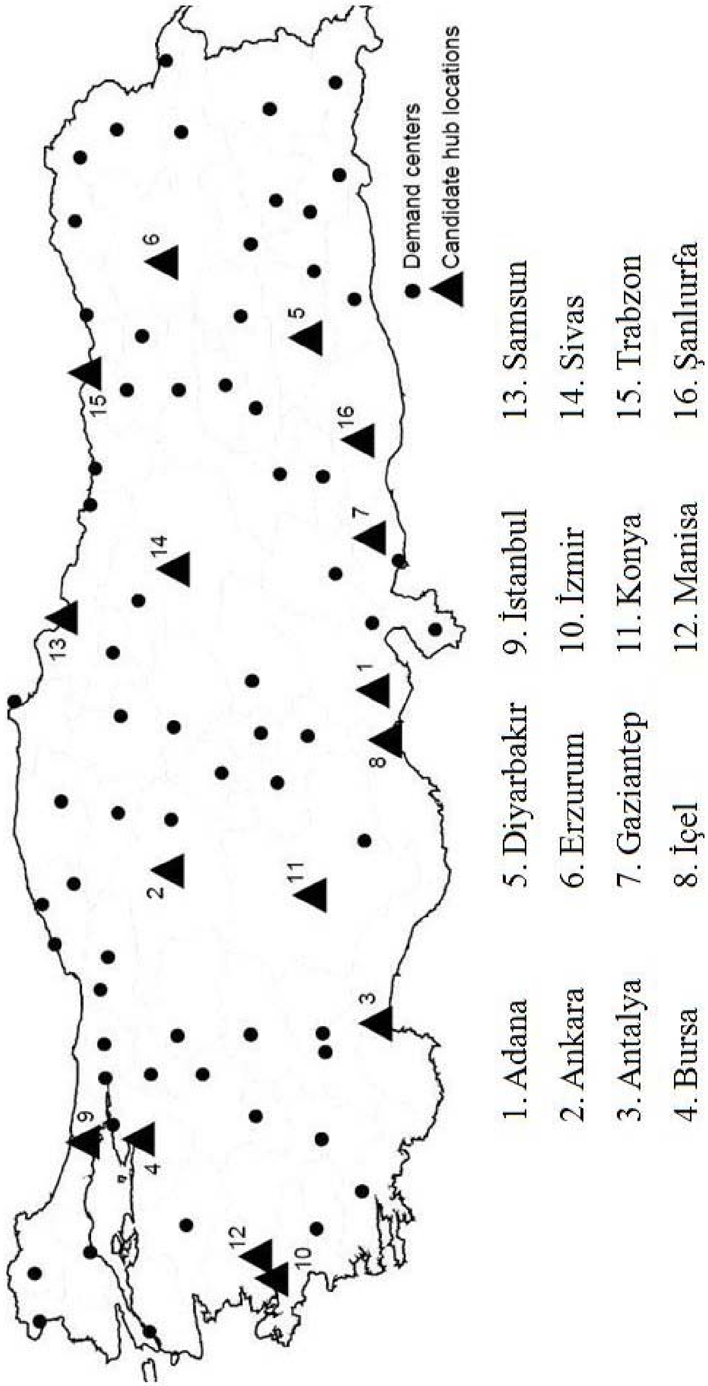

Figure A.2 Geographical locations of the 81 demand centers and names of the 16 candidate hub locations on the Turkish network.