Invited Review

Network hub location problems: The state of the art

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Abstract

Hubs are special facilities that serve as switching, transshipment and sorting points in many-to-many distribution systems. The hub location problem is concerned with locating hub facilities and allocating demand nodes to hubs in order to route the traffic between origin–destination pairs. In this paper we classify and survey network hub location models. We also include some recent trends on hub location and provide a synthesis of the literature.
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1. Introduction

Hubs are special facilities that serve as switching, transshipment and sorting points in many-to-many distribution systems. Instead of serving each origin–destination pair directly, hub facilities concentrate flows in order to take advantage of economies of scale. Flows from the same origin with different destinations are consolidated on their route to the hub and are combined with flows that have different origins but the same destination. The consolidation is on the route from the origin to the hub and from the hub to the destination as well as between hubs.

The *hub location problem* is concerned with locating hub facilities and allocating demand nodes to hubs in order to route the traffic between origin–destination pairs. There are two basic types of hub networks – *single allocation* and *multiple allocation*. They differ in how non-hub nodes are allocated to hubs. In single allocation, all the incoming and outgoing traffic of every demand center is routed through a single hub; in multiple allocation, each demand center can receive and send flow through more than one hub. Some papers are concerned only with the allocation aspect of the problem. But since optimal allocations are affected by hub locations and optimal hub locations are affected by allocation decisions, location and allocation problems must be considered together in designing hub networks.

Studies on the hub location problem often assume three things: that the hub network is complete with a link between every hub pair; that there is economies of scale incorporated by a discount factor (α) for using the

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inter-hub connections; and that no direct service (between two non-hub nodes) is allowed. Although these assumptions are relaxed in some studies, this paper assumes that these three assumptions are satisfied unless otherwise stated.

This paper classifies and surveys network hub location models. In network hub location problems there is a given network with n nodes on which the set of origins, destinations and potential hub locations are identified. The flow between origin–destination pairs, an attribute of interest associated with flows on links in the network (cost, time, distance, etc.) and the hub-to-hub transportation discount factor α are known.

This review does not include any studies on the continuous hub location problem. This problem is concerned with locating hub facilities on a plane rather than on the nodes of a network. One may refer to O'Kelly (1986a, 1992b), Aykin (1988, 1995b), Campbell (1990), O'Kelly and Miller (1991), and Aykin and Brown (1992) for such studies.

The problem of hub location has attracted many researchers: this paper cites more than 100 papers related to the hub location problem. Fig. 1 shows the distribution of these papers over the years.

The research on hub location began with the pioneering work of O'Kelly (1986a,b, 1987). As Fig. 1 shows, there was a steep increase in the number of publications after the year 2000. Clearly interest in the hub location area is still strong and there are currently a number of researchers working on this topic with several papers pending; in fact the previous reviews are out of date. In this paper, we summarize the work that has been done and provide a synthesis of the existing literature. We also highlight some deficiencies of the literature and present some directions for future research.

Perhaps, Goldman (1969) is the first paper addressing the network hub location problem. However, O'Kelly (1987) presented the first recognized mathematical formulation for a hub location problem by studying airline passenger networks. His formulation is referred to the *single allocation p-hub median* problem. Given n demand nodes, flow between origin–destination pairs and the required number of hubs (p), the objective is to minimize the total transportation cost (time, distance, etc.) to serve the given set of flows.

Let W_{ij} be the flow between nodes i and j and C_{ij} be the transportation cost of a unit of flow between i and j. Define X_{ik} as 1 if node i is allocated to hub at k, and 0 otherwise; X_{kk} takes on the value 1 if node k is a hub and it is 0 otherwise. The integer programming formulation of the single allocation p-hub median problem given by O'Kelly (1987) is

$$(\text{O'Kelly, 1987}) \quad \text{Min} \quad \sum_i \sum_j W_{ij} \left(\sum_k X_{ik} C_{ik} + \sum_m X_{jm} C_{jm} + \alpha \sum_k \sum_m X_{ik} X_{jm} C_{km} \right) \tag{1}$$

$$\text{s.t.} \quad (n - p + 1) X_{kk} - \sum_i X_{ik} \geqslant 0 \quad \text{for all } k, \tag{2}$$

$$\sum_k X_{ik} = 1 \quad \text{for all } i, \tag{3}$$

$$\sum_k X_{kk} = p, \tag{4}$$
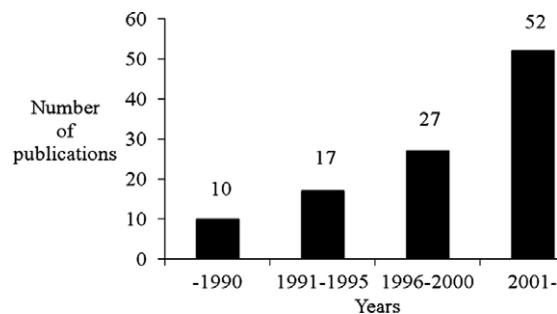
$$X_{ik} \in \{0, 1\} \quad \text{for all } i, k. \tag{5}$$

Fig. 1. The number of hub-location studies in the literature according to year.

The objective function, (1), calculates the cost of flow. α in the third term is the economies of scale factor; the cost of flow between the hub facilities must be smaller than the original costs since hub facilities concentrate flow, so $0 \leqslant \alpha < 1$. Note that this objective function is quadratic due to the fact that the hub-to-hub discount is a product of the allocation decisions. Constraint (2) ensures that no node is assigned to a location unless a hub is opened at that site. This constraint can be replaced with:

$$X_{ij} \leqslant X_{jj} \quad \text{for all } i, j. \tag{6}$$

Constraint (3) and (5) ensure that each node is assigned to exactly one hub, and constraint (4) states that the number of hubs to be located is p.

The nearest allocation strategy – assigning each demand node to its nearest hub – does not necessarily give optimal solutions for the hub location problem. Thus, Aykin (1990) formulated the difference in the objective function if node i is assigned to hub k instead of hub t and defined a procedure to find the optimal allocation of demand points to a given set of hubs.

O'Kelly (1987) introduced a data set based on the airline passenger interactions between 25 US cities in 1970 evaluated by the Civil Aeronautics Board (CAB). Later, this data set has been used by almost all of the hub location researchers and will be referred to as the CAB data set. Another commonly used data set is the Australia Post (AP) data set (first used in Ernst and Krishnamoorthy (1996)). AP data set is based on a postal delivery in Sydney, Australia and consists of 200 nodes representing postal districts. The main difference of the AP data set from the CAB data set other than the number of nodes is that the flow matrix of the AP data set is not symmetrical.

Hub location has various application areas in transportation (air passenger, cargo) and telecommunication network design. Hall (1989) analyzed the impact of overnight restrictions and time zones on the configuration of an air freight network. Later, O'Kelly and Lao (1991) presented a zero-one linear programming model to decide the mode choice in the hub network discussed by Hall (1989). In their model the location of a master and a mini hub are assumed to be known and the model is used to determine which cities may be served by truck rather than air. Iyer and Ratliff (1990) tried to locate accumulation points (hubs) to service the origin–destination pairs within a guaranteed time. Powell and Sheffi (1983) studied the load planning problem of less-than-truckload (LTL) motor carriers. The load plan specifies how a shipment is to be routed beginning from the origin terminal and consisting of a sequence of one or more consolidation terminals before reaching the destination terminal. There are various studies considering LTL structures in the literature. One may refer to Campbell (2005) for a survey on strategic network design for motor carriers. Jaillet et al. (1996) presented models for designing capacitated airline networks. They did not assume a priori hub-network structure. The resulting network may suggest the presence of hubs, if cost efficient. The authors proposed different integer programming formulations and a heuristic algorithm for their problem. Kuby and Gray (1993) explored the tradeoffs and savings involved with stopovers and feeders in package delivery systems, and developed a mixed integer program to design the least cost single-hub air network assuming that the hub location is already identified. Their model answers the question of how the network should be configured to achieve the greatest operational efficiency once the hub site is chosen. A recent paper by Cetiner et al. (2006) studied a combined hubbing and routing problem in postal delivery systems. They proposed an iterative solution procedure for a case study using the Turkish postal delivery system data.

The hub location problem is also studied in telecommunication network design (also called backbone/tributary network design). The reader may refer to Klincewicz (1998) for an extensive review on hub location in network design, telecommunication and computer systems. The hub location problem in network design differs somewhat from the classical hub location literature. For example, in addition to locating hub facilities and allocating nodes, Carello et al. (2004) considered the cost of installing on each edge the capacity needed to route the traffic on the edge itself. Yaman (2005) studied a similar problem which she named the *uncapacitated hub location problem with modular arc capacities*. Integer amounts of capacity units are installed on the arcs while minimizing the costs of installing hubs and capacity units on the arcs. The capacitated version of this problem is studied in Yaman and Carello (2005) where the capacity of a hub is defined as the amount of traffic passing through the hub.

Almost all of the hub location models defined in the literature have analogous location versions. Our review of the literature follows this classification. The next four sections of this paper are devoted in turn to the *p*-hub

median problem, the hub location problem with fixed costs, the p-hub center problem and hub covering problems. In the sixth section we present some studies discussing the discount factor α. In the seventh section of the paper we present some other hub location studies that do not fit into previous sections of the paper and the last section synthesizes the existing literature and suggests future research directions.

2. The p-hub median problem

The objective of the p-hub median problem is to minimize the total transportation cost (time, distance, etc.) needed to serve the given set of flows, given n demand nodes, flow between origin–destination pairs and the number of hubs to locate (p). The studies considering the p-hub median problem are analyzed here in two different subsections: single allocation and multiple allocation.

2.1. Single allocation

Campbell (1994b) produced the first linear integer programming formulation for the single allocation p-hub median problem. His formulation has $(n^4 + n^2 + n)$ variables of which $(n^2 + n)$ are binary and it has $(n^4 + 2n^2 + n + 1)$ linear constraints. Campbell (1994b) also formulated the problem with flow thresholds which he defined as the minimum flow value needed to allow service on a link. When flow thresholds are set to their maximum values, each demand node is assigned to a single hub and the formulation reduces to the single allocation p-hub median problem.

Skorin-Kapov et al. (1996) stated that the LP relaxation of Campbell (1994b) formulation resulted in highly fractional solutions. They proposed a new mixed integer formulation for the single allocation p-hub median problem. Define,

X_{ijkm} = Fraction of flow from node i to node j that is routed via hubs at locations k and m
in that order

and let $C_{ijkm} = C_{ik} + C_{mj} + \alpha C_{km}$.

(Skorin-Kapov et al., 1996)   Min $\displaystyle\sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} C_{ijkm}$ (7)

s.t.   (3)–(6),

$$\sum_m X_{ijkm} = X_{ik} \quad \text{for all } i, j, k,$$ (8)

$$\sum_k X_{ijkm} = X_{jm} \quad \text{for all } i, j, m,$$ (9)

$$X_{ijkm} \geqslant 0 \quad \text{for all } i, j, k, m.$$ (10)

This resulting formulation has $(n^4 + n^2)$ variables of which n^2 are binary and it has $(2n^3 + n^2 + n + 1)$ linear constraints. The authors showed that the linear relaxation of this formulation is tight as it almost always yields integral solutions with the CAB data set. For those instances with non-integral LP solutions, the LP relaxation resulted in an objective function value less than 1% below the optimal objective function value. They obtained the optimal values by using CPLEX. To the best of our knowledge Skorin-Kapov et al. (1996) presented the first attempt at optimally solving the single allocation p-hub median problem.

O'Kelly et al. (1996) presented a formulation that assumed a symmetric flow data, thus further reducing the size of the problem. This reduced formulation still finds integer solutions to the LP relaxation most of the time. An important aspect of O'Kelly et al. (1996) is its discussion of the sensitivity of the solutions to the inter-hub discount factor α. Sohn and Park (1998) formulation presents a further reduction in the number of variables and constraints for the case when the unit flow cost is symmetric and proportional to the distance.

Ernst and Krishnamoorthy (1996) propose a different linear integer programming formulation which requires fewer variables and constraints in an attempt to solve larger problems. They treated the inter-hub transfers as a multicommodity flow problem where each commodity represents the traffic flow originating from a particular node. The authors observed and modeled how Australia Post uses different discount factors

for collection and distribution. Let χ be the discount factor for collection (non-hub to hub) and δ be the discount factor for distribution (hub to non-hub). Define Y_{kl}^i as the total amount of flow of commodity i (i.e., traffic emanating from node i) that is routed between hubs k and l. Let $O_i = \sum_j W_{ij}$ be the total amount of flow originating at node i and $D_i = \sum_i W_{ij}$ be the total amount of flow destined to node i. Using previously defined decision variables and parameters their formulation is

$$\text{(Ernst and Krishnamoorthy, 1996)} \quad \text{Min} \quad \sum_i \sum_k C_{ik} X_{ik} (\chi O_i + \delta D_i) + \sum_i \sum_k \sum_l \alpha C_{kl} Y_{kl}^i \tag{11}$$

$$\text{s.t.} \quad (3)\text{--}(6),$$

$$\sum_l Y_{kl}^i - \sum_l Y_{lk}^i = O_i X_{ik} - \sum_j W_{ij} X_{jk} \quad \text{for all } i, k, \tag{12}$$

$$Y_{kl}^i \geqslant 0 \quad \text{for all } i, k, l. \tag{13}$$

Eq. (12) is the flow balance equation (divergence equation) for commodity i at node k where the demand and supply at the node is determined by the allocations X_{ik}.

This formulation has $(n^3 + n^2)$ variables of which n^2 are binary and it requires $(2n^2 + n + 1)$ linear constraints. Note that, the problem size from the previous formulation (Skorin-Kapov et al., 1996) is reduced, both in terms of variables and constraints, by a factor of approximately n.

Ebery (2001) presented another formulation for the single allocation p-hub median problem that requires $O(n^2)$ variables and $O(n^2)$ constraints. This formulation uses fewer variables than all of the other models previously presented in the literature. However, in practice, the computational time required to solve this new formulation was greater than that required to solve the (Ernst and Krishnamoorthy, 1996) formulation.

The p-hub median problem is NP-hard. Moreover, even if the locations of the hubs are fixed, the allocation part of the problem remains NP-hard. (Kara, 1999).

The first two heuristics for the single allocation p-hub median problem were proposed by O'Kelly (1987). Both of the proposed heuristics enumerate all possible choices of p hub locations. In the first heuristic (HEUR1) demand nodes are assigned to its nearest hub and in the second heuristic (HEUR2) the better, in terms of the objective function value, of the first and second nearest hubs is selected. The heuristics are used to solve the CAB data set. Klincewicz (1991, 1992) developed various heuristics for the single allocation p-hub median problem. Klincewicz (1991) developed an exchange heuristic based on local improvement considering both the single and double exchange procedures. His comparison showed that these heuristics are superior to a clustering heuristic and to the heuristics proposed in O'Kelly (1987). Then, Klincewicz (1992) presented a tabu search and a GRASP (greedy randomized search procedure) heuristic; in both of these heuristics demand nodes are allocated to their nearest hubs. Both papers (Klincewicz, 1991, 1992) used the CAB data set and a larger data set with 52 demand points and up to 10 hubs to test the performance of the heuristics.

Another tabu search heuristic for the single allocation p-hub median problem was developed by Skorin-Kapov and Skorin-Kapov (1994). Using the CAB data set they compared their results with the heuristics of O'Kelly (1987) (HEUR1 and HEUR2) and the tabu search of Klincewicz (1992). Their results are superior but CPU time requirement was greater due to more emphasis on the allocation phase of the problem.

O'Kelly et al. (1995) presented a lower bounding technique for the single allocation p-hub median problem based on the linearization of the quadratic objective function where distances are assumed to satisfy the triangle inequality. Using their method, the authors showed that the tabu search method of Skorin-Kapov and Skorin-Kapov (1994) was within an average gap of 3.3% for smaller problems (10–15 nodes) and an average gap of 5.9% for the 20 and 25 node problems. Later, with the optimal solutions to the CAB data set Skorin-Kapov et al. (1996) were able to validate the optimality of the tabu search solutions obtained in Skorin-Kapov and Skorin-Kapov (1994).

Clearly, the multiple allocation p-hub median solutions provide a lower bound on the optimal solution of the single allocation p-hub median problem (Campbell, 1996). Using this idea, Campbell (1996) proposed two new heuristics for the single allocation p-hub median problem. These two heuristics, MAXFLO and ALLFLO, derive solutions to the single allocation p-hub median problem from the solution to the multiple allocation p-hub median problem. In these heuristics, the allocations are done according to different rules but location decisions are the same.

Ernst and Krishnamoorthy (1996) developed a simulated annealing heuristic and showed that it is comparable, in both solution quality and computational time, with the tabu search heuristic of Skorin-Kapov and Skorin-Kapov (1994). They used this upper bound obtained from the simulated annealing heuristic to develop an LP-based branch-and-bound solution method. They tested both their heuristic and the branch-and-bound algorithm on the CAB and AP data sets, but they were unable to solve any problem greater than $n = 50$. Later, Ernst and Krishnamoorthy (1998b) proposed another branch-and-bound algorithm which solves shortest-path problems to obtain lower bounds. Unlike the traditional branch-and-bound algorithms, their algorithm does not start with a single root node, but with a set of root nodes. They tested the effectiveness of this algorithm by comparing its performance with the results provided in Ernst and Krishnamoorthy (1996) on the CAB and AP data sets. They stated that this new algorithm is significantly faster for small values of p and it requires less memory than the LP based branch-and-bound algorithm presented in Ernst and Krishnamoorthy (1996). The largest single allocation problems to date have been solved to optimality with this algorithm. The authors solved problems with 100 nodes and with $p = 2$ and 3 in approximately 228 and 2629 seconds respectively. However, they were still unable to solve problems with 100 nodes when $p > 3$ in a reasonable amount of computational time. Ebery (2001) presented a formulation for the single allocation p-hub median problem with two or three hubs. The results indicate that for large problems with $p = 2$ or $p = 3$ using CPLEX with this formulation is better than the shortest-path based approach presented in Ernst and Krishnamoorthy (1998b).

Pirkul and Schilling (1998) developed an efficient lagrangean relaxation method which finds tight upper and lower bounds in a reasonable amount of CPU time. They used subgradient optimization on the lagrangean relaxation of the model and they also provided a cut constraint for one of the subproblems. In computational experiments on the CAB data set, they stated that the average gaps of this heuristic are 0.048% and even the maximal gaps are under 1% – the tightest bounds of any heuristic up to that date.

Smith et al. (1996) mapped the single allocation p-hub median problem onto a modified Hopfield neural network. They used the quadratic integer programming formulation of O'Kelly (1987) because this formulation has a reduced number of variables and constraints. They compared their results on the CAB data set with the simulated annealing heuristic of Ernst and Krishnamoorthy (1996) and the commercial package GAMS with the solver MINOS-5. They found the performance of GAMS/MINOS-5 considerably poorer than the other approaches since it is designed to minimize convex functions; they also found that the Hopfield neural network approach is able to compete effectively with simulated annealing. Another simulated annealing heuristic for the single allocation p-hub median problem is proposed by Abdinnour-Helm (2001). However, Ernst and Krishnamoorthy (1996) obtained better results than Abdinnour-Helm (2001).

Sohn and Park (1997) studied the single allocation two-hub median problem. They showed that this problem can be solved in polynomial time when hub locations are fixed. They provided a linear programming formulation for the single allocation problem with fixed hub locations and showed that the problem can be transformed into the minimum cut problem. Since there are $O(n^2)$ ways to choose the hub locations, the two-hub location problem can be solved in polynomial time. In a subsequent study, Sohn and Park (1998) presented methods to find optimal solutions for the allocation problems with fixed hub locations. They presented a mixed integer formulation for a model with fixed hub locations where fixed costs for opening links are also considered. Another study by the same authors (Sohn and Park, 2000) focuses on the single allocation problem on a three-hub network with fixed hub locations. They provided a mixed integer formulation and studied its polyhedral properties. Although the single allocation problem in a two-hub system has a polynomial time algorithm, the authors showed that it is NP-hard as soon as the number of hubs is three. Ebery (2001) presented a new mixed integer formulation for the p-hub single allocation problem where hub locations are fixed. His computational results indicated that this new formulation is more effective than the formulations presented in Sohn and Park (1997, 2000) for $p = 2$ and $p = 3$.

Elhedhli and Hu (2005) considered the congestion at the hubs and proposed a non-linear convex cost function for the objective function of the single allocation p-hub median model. They linearized this model by using piecewise linear functions, and then applied Lagrangean relaxation. Via comparison with the non-congestion problem on the CAB data set, the authors stated that the congestion model results in a more balanced distribution of flows through hubs.

Table 1 summarizes the studies on the single allocation p-hub median problem. In terms of required number of variables and constraints, Ebery (2001) provides the best mathematical formulation. However, the best

Table 1
Single allocation *p*-hub median literature

Year	Authors	Notes
1987	O'Kelly	Quadratic integer program, HEUR1, HEUR2
1990	Aykin	Procedure to find optimal allocations
1991	Klincewicz	Exchange heuristic
1992	Klincewicz	Tabu search and GRASP heuristics
1994b	Campbell	First linear integer formulation
1994	Skorin-Kapov and Skorin-Kapov	Tabu search heuristic
1995	O'Kelly, Skorin-Kapov and Skorin-Kapov	Lower bounding technique
1996	Campbell	MAXFLO and ALLFLO heuristics
1996	Ernst and Krishnamoorthy	New formulation, simulated annealing heuristic, B&B method
1996	O'Kelly, Bryan, Skorin-Kapov and Skorin-Kapov	New formulation for symmetric flow data
1996	Skorin-Kapov, Skorin-Kapov and O'Kelly	New mathematical formulation leading to tight LP relaxation
1996	Smith, Krishnamoorthy and Palaniswami	Modified Hopfield neural network heuristic
1997	Sohn and Park	Two-hub location problem
1998b	Ernst and Krishnamoorthy	Shortest path based B&B algorithm
1998	Pirkul and Schilling	Lagrangean relaxation heuristic
1998	Sohn and Park	New formulation for symmetric cost, and allocation problem
2000	Sohn and Park	Three-hub allocation problem
2001	Abdinnour-Helm	Simulated annealing heuristic
2001	Ebery	New formulations for $p = 2$ and 3
2005	Elhedhli and Hu	Minimized congestion at hubs

mathematical formulation in terms of computation time requirement is that of Ernst and Krishnamoorthy (1996). The most efficient exact solution procedure is the shortest-path based branch-and-bound algorithm presented in Ernst and Krishnamoorthy (1998b). Up to now the largest set of problems that has been solved to optimality has 100 nodes. The most effective heuristic is the lagrangean relaxation based heuristic presented in Pirkul and Schilling (1998). And among the best metaheuristics are the tabu search heuristic presented in Skorin-Kapov and Skorin-Kapov (1994) and the simulated annealing heuristic presented in Ernst and Krishnamoorthy (1996).

2.2. Multiple allocation

Recall that in the multiple allocation problem each demand center can receive and send flow through more than one hub; that is, each demand center can be allocated to more than one hub.

Campbell (1992) was the first to formulate the multiple allocation *p*-hub median problem as a linear integer program.

$$\text{(Campbell, 1992)} \quad \text{Min} \quad (7)$$

$$\text{s.t.} \quad (4), (5) \text{ and } (10)$$

$$\sum_k \sum_m X_{ijkm} = 1 \quad \text{for all } i, j, \tag{14}$$

$$X_{ijkm} \leqslant X_{kk} \quad \text{for all } i, j, k, m, \tag{15}$$

$$X_{ijkm} \leqslant X_{mm} \quad \text{for all } i, j, k, m. \tag{16}$$

Campbell (1994b) stated that in the absence of capacity constraints on the links, there is an optimal solution where all X_{ijkm} variables are set to zero or one since the total flow for each origin–destination pair should be routed via the least-cost hub pair. Thus, there is no need to restrict X_{ijkm} variables to integers. The author formulated the multiple allocation *p*-hub median problem also with flow thresholds and fixed costs as a linear integer program.

Skorin-Kapov et al. (1996) proposed a new mixed integer formulation, where constraints (15) and (16) are replaced with their aggregate forms. This modified formulation has $(n^4 + n)$ variables of which *n* are binary and it requires $(2n^3 + n^2 + 1)$ linear constraints. This formulation resulted in tighter LP relaxations and produced integral results in almost all instances using the CAB data set. For the cases when LP relaxation

did not yield an integer solution, the authors employed an implicit enumeration search tree to obtain optimal solutions. This search tree normally involved very few tree nodes.

Ernst and Krishnamoorthy (1998a) proposed a new formulation for the multiple allocation p-hub median problem based on the idea that they have proposed for the single allocation version in Ernst and Krishnamoorthy (1996). Define Z_{ik} as the flow from node i to hub k, X^i_{lj} as the flow of commodity i flowing from hub l to node j, H_k as the binary variable (which is one if node k is a hub and zero otherwise); all the other decision variables and parameters are defined as before

$$\text{(Ernst and Krishnamoorthy, 1998a)} \quad \text{Min} \quad \sum_i \left[\sum_k \chi C_{ik} Z_{ik} + \sum_k \sum_l \alpha C_{kl} Y^i_{kl} + \sum_l \sum_j \delta C_{lj} X^i_{lj} \right] \quad (17)$$

$$\text{s.t.} \quad \sum_k H_k = p, \quad (18)$$

$$\sum_k Z_{ik} = O_i \quad \text{for all } i, \quad (19)$$

$$\sum_l X^i_{lj} = W_{ij} \quad \text{for all } i,j, \quad (20)$$

$$\sum_l Y^i_{kl} - \sum_j X^i_{kj} - \sum_l Y^i_{lk} - Z_{ik} = 0 \quad \text{for all } i,k, \quad (21)$$

$$Z_{ik} \leqslant O_i H_k \quad \text{for all } i,k, \quad (22)$$

$$\sum_i X^i_{lj} \leqslant D_j H_i \quad \text{for all } l,j, \quad (23)$$

$$H_k \in \{0,1\} \quad \text{for all } i,k, \quad (24)$$

$$Y^i_{kl}, X^i_{lj}, Z_{ik} \geqslant 0 \quad \text{for all } i,j,k,l. \quad (25)$$

This new formulation has $(2n^3 + n^2 + n)$ variables of which n are binary and it requires $(4n^2 + n + 1)$ linear constraints. Ernst and Krishnamoorthy (1998a) showed that this formulation is more effective than the formulation of Skorin-Kapov et al. (1996).

Campbell (1996) proposed a greedy-interchange heuristic for the multiple allocation p-hub median problem. To obtain exact solutions, Ernst and Krishnamoorthy (1998a) presented an LP based branch-and-bound method. They strengthened the lower bound by identifying violated inequalities and adding them to the LP. They also proposed two heuristics. The first one is a shortest path based heuristic and the second one is an explicit enumeration heuristic. Note that if the hub locations are fixed, the allocation decision is straightforward: each pair of nodes sends flow from their shortest paths via the given hubs. Both of the heuristics employ this idea. The authors presented computational results for both the CAB and AP data sets. That year, the same authors presented another paper (Ernst and Krishnamoorthy, 1998b) in which they developed another but more effective (in terms of CPU time requirement) branch-and-bound algorithm. This time they obtained lower bounds by solving the shortest path problems rather than solving the LP relaxation. This new branch-and-bound algorithm consistently outperformed the LP-based branch-and-bound algorithm of Ernst and Krishnamoorthy (1998a); it runs about 500 times faster and requires significantly less memory. With this new algorithm they were able to provide exact solutions to problems larger than anyone attempted in the literature. They were able to obtain exact solutions even for problems of size $n = 200$ with $p = 3$ in approximately 632 seconds. However, they were unable to solve the AP data set problems where $n = 100$, $p > 5$ and where $n = 200$, $p > 3$ in a reasonable amount of computational time.

Boland et al. (2004) suggested that even though the formulation in Ernst and Krishnamoorthy (1998a) results in faster computational times and requires less memory, it still suffers from weak lower bounds. In order to overcome this deficiency, the authors identified some characteristics of optimal solutions to develop preprocessing techniques and tightening constraints. When they applied these to the multiple allocation p-hub median problem, the results indicate that tightening does significantly improve some of the results.

Sasaki et al. (1999) considered a special case of the problem where each route in the network is allowed to use only one hub. They called this the 1-stop multiple allocation p-hub median problem. They presented a

Table 2
Multiple allocation *p*-hub median literature

Year	Authors	Notes
1992	Campbell	First linear integer program
1994b	Campbell	New formulations, flow thresholds, fixed costs
1996	Campbell	Greedy-interchange heuristic
1996	Skorin-Kapov, Skorin-Kapov and O'Kelly	New mathematical formulation leading to tight LP relaxation
1998a	Ernst and Krishnamoorthy	New formulation, B&B method, two heuristics
1998b	Ernst and Krishnamoorthy	Shortest path based B&B algorithm
1999	Sasaki, Suzuki and Drezner	1-stop problem
2004	Boland, Krishnamoorthy, Ernst and Ebery	Preprocessing and tightening constraints

mixed integer formulation of this model, which can be further transformed into the *p*-median problem. They proposed a branch-and-bound algorithm and a greedy-type heuristic; they tested the performance of their algorithm on the CAB data set.

Table 2 summarizes the studies on the multiple allocation *p*-hub median problem.

3. The hub location problem with fixed costs

In the *p*-hub median problem, the fixed costs of opening facilities are ignored. O'Kelly (1992a) introduced the single allocation hub location problem with fixed costs making the number of hubs a decision variable. He formulated this problem as a quadratic integer program as:

$$\text{(O'Kelly, 1992)} \quad \text{Min} \quad \sum_i \sum_k X_{ik} C_{ik} (O_i + D_i) + \sum_j \sum_m X_{jm} \sum_i \sum_k X_{ik} (\alpha W_{ij} C_{km}) + \sum_j X_{jj} F_j \tag{26}$$
$$\text{s.t.} \quad (3), (5) \text{ and } (6),$$

where F_j is the fixed cost of opening a hub at node j and all the other decision variables and parameters are defined as in the previous section.

In addition to having single/multiple allocation versions, since the number of hubs is not fixed it is possible to have uncapacitated/capacitated hub location problems with fixed costs. Campbell (1994b) presented the first linear programming formulations for multiple/single allocation uncapacitated/capacitated hub location problems. The multiple allocation uncapacitated hub location problem with previously defined decision variables and parameters is

$$\text{(Campbell, 1994)} \quad \text{Min} \quad \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} C_{ijkm} + \sum_k F_k X_{kk} \tag{27}$$
$$\text{s.t.} \quad (5), (14)\text{--}(16),$$
$$0 \leqslant X_{ijkm} \leqslant 1 \quad \text{for all } i, j, k, m. \tag{28}$$

Like the *p*-hub median problem, in the absence of capacity constraints on the links there is always an optimal solution where all X_{ijkm} variables are set to zero or one.

Several studies have looked at the single allocation uncapacitated hub location problem. Abdinnour-Helm and Venkataramanan (1998) presented a new quadratic integer formulation based on the idea of multicommodity flows in networks. They then solved this formulation with a branch-and-bound procedure that used the underlying network structure of the problem to obtain the lower bounds. Since the branch-and-bound procedure requires a considerable amount of time, they also proposed a genetic algorithm to find solutions quickly and efficiently. Abdinnour-Helm (1998) proposed a new heuristic method based on a hybrid of genetic algorithms and tabu search. Firstly, the genetic algorithm is used to determine the number and the location of hubs and then each demand point is assigned to its closest hub to form a starting solution for the tabu search heuristic which finds the optimal allocations. She compared her results with the genetic algorithm of

Abdinnour-Helm and Venkataramanan (1998) on the CAB data set and found that using tabu search in combination with a genetic algorithm leads to much better solutions than using the genetic algorithms alone.

Topcuoglu et al. (2005) proposed another genetic algorithm for the uncapacitated single allocation hub location problem. Their heuristic outperformed the hybrid heuristic proposed in Abdinnour-Helm (1998) with respect to both solution quality and required computational time over the CAB and AP data sets. Unaware of the work by Topcuoglu et al. (2005), Cunha and Silva (2007) proposed another genetic algorithm combined with a simulated annealing heuristic. This new hybrid heuristic outperformed the genetic algorithms of both Abdinnour-Helm (1998) and Abdinnour-Helm and Venkataramanan (1998). Another heuristic for this problem is proposed in Chen (2007). His hybrid heuristic is based on the simulated annealing method, tabu list and improvement procedures. This heuristic outperformed the heuristic presented in Topcuoglu et al. (2005) both in solution time and quality. Since there is not yet a published comparison of the heuristics provided in Cunha and Silva (2007) and Chen (2007), we conclude that these two are the best heuristics proposed for the single allocation hub location problem up to now.

For the single allocation uncapacitated hub location problem Labbé and Yaman (2004) derived a family of valid inequalities that generalizes the facet-defining inequalities and that can be separated in polynomial time.

For the multiple allocation uncapacitated hub location problem, Klincewicz (1996) presented an algorithm based on dual-ascent and dual adjustment techniques within a branch-and-bound scheme. Hubs are chosen from a predetermined set of potential hubs and the algorithm is tested on the CAB data set.

Mayer and Wagner (2002) developed a new branch-and-bound method, the Hublocater, for the uncapacitated multiple allocation hub location problem. The main advantage of Hublocater is in obtaining lower bounds. The lower bounds are tighter and reduce the computational effort required within the branch-and-bound algorithm. They compared Hublocater with the algorithm presented in Klincewicz (1996) and with CPLEX on the CAB and AP data sets. To compare their algorithm with CPLEX they used a mathematical formulation based on the multicommodity flow modeling approach developed by Ernst and Krishnamoorthy (1998a) for the p-hub median problem. Even though their algorithm is superior to the one presented in Klincewicz (1996), it was not always able to outperform CPLEX. Later, Cánovas et al. (2007) presented a new heuristic based again on a dual-ascent technique. They then implemented this heuristic within a branch-and-bound algorithm. Through computational analysis using CAB and AP data sets they were able to solve instances up to 120 nodes. These are the best computational results for the uncapacitated multiple allocation hub location problem up to now.

Hamacher et al. (2004) present a polyhedral study of the multiple allocation uncapacitated hub location problem. The authors determined the dimension and derived some classes of facets for this polyhedron. They developed a general rule about lifting facets from the uncapacitated facility location problem to the multiple allocation uncapacitated hub location problem. They developed a new formulation whose constraints are all facet-defining. Marín (2005b) presented some facet-defining valid inequalities for the uncapacitated hub location problem with the costs satisfying triangle inequality. He used previous knowledge about the polyhedron associated with the set-packing problem and applied it to the uncapacitated hub location problem. He solved the problem by a relax-and-cut algorithm. Marín et al. (2006) presented a new formulation which is a generalization of the earlier formulations and relaxes the assumption of having a cost structure satisfying triangle inequality. By using some polyhedral results they were able to tighten and reduce the number of constraints. Their formulation outperformed all of the previous formulations.

Aykin (1994) presented the capacitated version of the hub location problem with fixed costs where hubs have limited capacities. He formulated the problem such that direct connections (between non-hub nodes) are also allowed. He proposed a branch-and-bound algorithm where the lower bounds are obtained by lagrangean relaxation which is solved by subgradient optimization. Aykin (1995a) analyzed a similar problem with fixed costs and a given number of hubs to locate. He compared two hubbing policies which he named as strict and non-strict (direct connections are allowed). He proposed an enumeration algorithm and a simulated annealing-based greedy interchange heuristic.

Ernst and Krishnamoorthy (1999) presented two new formulations for the capacitated single allocation hub location problem. Their formulations are a modified version of the previous mixed integer formulations developed for the p-hub median problem. The better (in terms of required number of variables and constraints) among these formulations is as follows:

(Ernst and Krishnamoorthy, 1999) $\text{Min} \quad \sum_i \sum_k C_{ik} X_{ik} (\chi O_i + \delta D_i)$

$$+ \sum_i \sum_k \sum_l \alpha C_{kl} Y_{kl}^i + \sum_k F_k X_{kk} \tag{29}$$

$$\text{s.t.} \quad (3), (5), (6), (12), (13)$$

$$\sum_i O_i X_{ik} \leqslant \Gamma_k X_{kk} \quad \text{for all } k, \tag{30}$$

where Γ_k is the capacity of hub k and all the other decision variables and parameters are as defined before. Note that the capacity restrictions are only applied to the traffic arriving at the hub directly from non-hub nodes. This capacity definition is usually used in postal service applications in order to represent the sorting capacity of hubs.

Ernst and Krishnamoorthy (1999) proposed two heuristics. The first is based on simulated annealing and the other is based on random descent. They obtained optimal solutions by using an LP-based branch-and-bound method with the initial upper bound provided by these heuristics. They also proposed some preprocessing steps to improve the performance of the branch-and-bound algorithm. They tested the algorithm on the AP data set since the CAB data set does not include fixed costs and capacities.

Labbé et al. (2005) studied the single allocation capacitated hub location problem where each hub has a fixed capacity in terms of the traffic that passes through it. They investigated some polyhedral properties of this problem and developed a branch-and-cut algorithm based on these results.

Costa et al. (2007) suggested a different approach to the capacitated single allocation hub location problem. Instead of using capacity constraints on the amount of flow processed in the hubs the authors introduced a second objective function into their mathematical model, which minimizes the time hubs take to process flows. They considered two different bi-criteria problems. In addition to minimizing total cost in both of the problems, in the first one they minimized the total time of processing the flow (service time) at the hubs and in the second one they minimized the maximum service time on the hubs. They proposed an iterative approach which is used to calculate non-dominated solutions.

Ebery et al. (2000) considered the multiple allocation version of the capacitated hub location problem. Their formulation is similar to the one proposed in Ernst and Krishnamoorthy (1998a) for the multiple allocation p-hub median problem, except that there is no restriction on the number of hubs to be located and the capacity constraint is similar to (30). They presented an efficient heuristic algorithm based on shortest paths and incorporated the upper bound obtained from this heuristic in an LP-based branch-and-bound solution procedure. Boland et al. (2004) outlined some properties of the optimal solutions for both the uncapacitated and capacitated multiple allocation hub location problems. Based on these results they developed preprocessing procedures and tightening constraints to improve the linear programming relaxations for existing mixed integer linear programming formulations. They also employed flow-cover constraints for the capacitated version to improve computation times. These formulations led to an overall reduction in the CPU time required using CPLEX compared to the existing formulations. Marín (2005a) presented a new formulation for the multiple allocation capacitated hub location problem based on the same idea used in Ebery et al. (2000) but exploiting some of the ideas used in Marín et al. (2006) to reduce the size.

Sasaki and Fukushima (2003) presented a model for the capacitated 1-stop multiple allocation hub location problem. Their model involves capacity constraints both on hubs and arcs. They then solved this model by a branch-and-bound algorithm with lagrangean relaxation bounding strategy and tested the performance on the CAB data set.

The literature on the hub location problem with fixed costs is classified in Table 3.

4. The p-hub center problem

The p-hub center problem is a minimax type problem which is analogous to the p-center problem. Campbell (1994b) was first to formulate and discuss the p-hub center problem in the hub literature. He defined three different types of p-hub center problems:

Table 3
The literature on the hub location problem with fixed costs

Uncapacitated hub location problem	Single Allocation	O'Kelly (1992a)	Quadratic integer program
		Campbell (1994b)	First linear integer formulation
		Abdinnour-Helm and Venkataramanan (1998)	New quadratic formulation, genetic algorithm
		Abdinnour-Helm (1998)	Hybrid heuristic algorithm
		Labbé and Yaman (2004)	Valid and facet defining inequalities
		Topcuoglu et al. (2005)	Genetic algorithm
		Cunha and Silva (2007)	Hybrid genetic algorithm
		Chen (2007)	Hybrid heuristic algorithm
	Multiple Allocation	Campbell (1994b)	First linear integer formulation
		Klincewicz (1996)	B&B algorithm
		Mayer and Wagner (2002)	B&B algorithm (HubLocater)
		Boland et al. (2004)	Preprocessing procedures, tightening constraints
		Hamacher et al. (2004)	Polyhedral study, new formulation
		Marín (2005b)	Valid inequalities, relax-and-cut algorithm
		Marín et al. (2006)	New formulation
		Cánovas et al. (2007)	Heuristic based on dual-ascent technique, best B&B algorithm
Capacitated hub location problem	Single Allocation	Campbell (1994b)	First linear integer formulation
		Aykin (1994)	New formulation allowing direct connections
		Aykin (1995a)	Formulation with given number of hubs to locate, allows direct connections
		Ernst and Krishnamoorthy (1999)	New formulation, two heuristics, B&B algorithm
		Labbé et al. (2005)	B&B algorithm
		Costa et al. (2007)	New bi-criteria problems minimizing total cost and service time
	Multiple Allocation	Campbell (1994b)	First linear integer formulation
		Ebery et al. (2000)	New formulation, a heuristic, B&B algorithm
		Sasaki and Fukushima (2003)	Formulation for the 1-stop hub location problem, B&B algorithm
		Boland et al. (2004)	Preprocessing procedures, tightening constraints
		Marín (2005a)	New formulation

(I) The maximum cost for any origin–destination pair is minimized.
(II) The maximum cost for movement on any single link (origin-to-hub, hub-to-hub and hub-to-destination) is minimized.
(III) The maximum cost of movement between a hub and an origin/destination is minimized (vertex center).

According to Campbell (1994b), the first type of hub center problem is important for a hub system involving perishable or time sensitive items in which cost refers to time. An example of the second type of p-hub center problem is items that require some preserving/processing such as heating or cooling which is available at the hub locations; another example is the vehicle drivers that are subject to a time limit on continuous service. For the third type, similar examples to the second type can be given considering that hub-to-hub links may have some special attributes. Campbell (1994b) presented formulations for both single and multiple allocation versions for all three types of p-hub center problem.

Kara and Tansel (2000) provided various linear formulations for the single allocation p-hub center problem. They provided three different linearizations of the Campbell (1994b) type I model together with a new formulation that they proposed. Through computational analysis using CPLEX, their new formulation is superior to all of the three linearizations. Their new formulation has $(n^2 + 1)$ variables of which n^2 are binary and it has $(n^3 + n^2 + n + 1)$ linear constraints.

Kara and Tansel (2000) also provided a combinatorial formulation of the single allocation p-hub center problem and proved that it is NP-complete by a reduction from the dominating set problem.

Ernst et al. (2002a) developed a new formulation for the single allocation p-hub center problem. They defined a new variable r_k as the maximum collection/distribution cost between hub k and the nodes

that are allocated to hub k. Using previously defined parameters and decision variables, their new formulation is

(Ernst et al., 2002a)   Min Z

s.t.   (3)–(6)

$$r_k \geqslant C_{ik}X_{ik} \quad \text{for all } i, k, \tag{31}$$

$$Z \geqslant r_k + r_m + \alpha C_{km} \quad \text{for all } k, m, \tag{32}$$

$$r_k \geqslant 0 \quad \text{for all } k. \tag{33}$$

This formulation has $(n^2 + n + 1)$ variables of which n^2 are binary and it has $(3n^2 + n + 1)$ linear constraints. Even though this model has n more continuous variables than the model proposed in Kara and Tansel (2000), it has fewer constraints. Computational analysis using CPLEX on the CAB and AP data sets showed that the Ernst et al. (2002a) formulation is better in terms of CPU time requirements.

Baumgartner (2003) investigated the polyhedral properties of the Ernst et al. (2002a) formulation. She identified some facet-defining inequalities and defined separation procedures. She proposed a branch-and-cut algorithm. Hamacher and Meyer (2006) proposed solving hub covering problems with binary search for the solution of the p-hub center problem.

The first heuristic for the single allocation p-hub center problem is presented in Pamuk and Sepil (2001). They proposed a single-relocation heuristic for generating location-allocation decisions in a reasonable time and they superimposed tabu search on this underlying algorithm so as to decrease the possibility of being trapped by local optima.

Ernst et al. (2002a) also studied the multiple allocation p-hub center problem. They proposed two new formulations and proved that the problem is *NP*-hard. They presented a heuristic method for both the single and multiple allocation p-hub center problems. For the multiple allocation version they also proposed a shortest path based branch-and-bound algorithm which is very similar to the algorithm developed for the multiple allocation p-hub median problem presented in Ernst and Krishnamoorthy (1998b).

Ernst et al. (2002b) studied the allocation subproblem of the single allocation p-hub center problem when hub locations are fixed. They proved the NP-hardness of this problem and presented linear programming formulations. They proposed five heuristic algorithms and analyzed their worst case performances. Campbell et al. (2007) also studied the allocation subproblem. They presented various complexity results and provided integer programming formulations for both uncapacitated and capacitated cases. They established some special uncapacitated cases that are polynomially solvable, such as when $\alpha = 0$, $p = 2$ and when the hub network is a tree or path.

A working paper by Gavriliouk and Hamacher (2006) applied aggregation to various hub location models (single and multiple allocation center, median and fixed cost). They proposed some error measurements and developed error bounds for these models.

Table 4 summarizes the literature on the p-hub center problem.

Table 4
p-Hub center literature

Year	Authors	Notes
1994b	Campbell	Different types of p-hub center formulations
2000	Kara and Tansel	Various linear formulations for single allocation
2001	Pamuk and Sepil	Heuristic for the single allocation problem
2002a	Ernst et al.	New formulations for both single and multiple allocation, heuristic and a B&B algorithm
2002b	Ernst et al.	Heuristic algorithms for the allocation subproblem
2003	Baumgartner	Polyhedral properties, valid inequalities and branch-and-cut algorithm
2006	Hamacher and Meyer	Solving hub covering problems combined with binary search
2006	Gavriliouk and Hamacher	Applied aggregation and proposed error measurements
2007	Campbell, Lowe and Zhang	Complexity results and formulations for the allocation subproblem

5. Hub covering problems

In facility covering problems, demand nodes are considered to be covered if they are within a specified distance of a facility that can serve their demand. As in the p-hub center problem, Campbell (1994b) defined three coverage criteria for hubs. The origin destination pair (i,j) is covered by hubs k and m if

  (I) the cost from i to j via k and m does not exceed a specified value,
  (II) the cost for each link in the path from i to j via k and m does not exceed a specified value,
(III) each of the origin-hub and hub-destination links meets separate specified values.

The hub set-covering problem is to locate hubs to cover all demand such that the cost of opening hub facilities is minimized. The maximal hub-covering problem on the other hand maximizes the demand covered with a given number of hubs to locate. Campbell (1994b) presented the first mixed integer formulations for both of these problems.

Kara and Tansel (2003) studied the single allocation hub set-covering problem and proved that it is *NP*-hard. The authors presented and compared three different linearizations of the original quadratic model as well as presenting a new linear model. The new model's performance turned out to be superior to all of the other presented linear models.

Wagner (2004b) proposed new formulations for both single and multiple allocation hub covering problems. By his proposed preprocessing techniques he rules out some hub assignments and thus the formulations require less number of variables and constraints than that of Kara and Tansel (2003) formulation. He further improved these formulations with a procedure for aggregating some constraints.

Later, Ernst et al. (2005) presented a new formulation for the single allocation hub set covering problem similar to the one that is proposed in Ernst et al. (2002a) for the p-hub center problem. The new formulation is

$$\text{(Ernst et al., 2005)} \quad \text{Min} \quad \sum_k X_{kk} \tag{34}$$

$$\text{s.t.} \quad (3), (5), (6), (31), (33)$$

$$r_k + r_m + \alpha C_{km} \leqslant \beta \quad \text{for all } k, m, \tag{35}$$

where β is the cover radius.

In order to compare this new model with the previous formulation presented in Kara and Tansel (2003) the authors strengthened Kara and Tansel (2003) formulation by replacing a constraint with its aggregate form. The Ernst et al. (2005) formulation performs better in terms of CPU time requirement than the strengthened Kara and Tansel (2003) formulation.

Ernst et al. (2005) also studied the multiple allocation hub set-covering problem. They proposed two new formulations and an implicit enumerative method for this problem.

Hamacher and Meyer (2006) compared various formulations of the hub covering problem. They analyzed the feasibility polyhedron and identified some facet-defining valid inequalities. They solved the hub set-covering problem for a given cover radius β and then iteratively reduced β to obtain the optimum solution of the p-hub center problem.

Table 5 summarizes the literature on the hub covering problem.

Table 5
Hub covering literature

Year	Authors	Notes
1994b	Campbell	Different types of hub-covering formulations
2003	Kara and Tansel	Various linear formulations for the single allocation hub set covering problem
2004b	Wagner	Improved model formulations for both single and multiple allocation hub covering problems
2005	Ernst et al.	New formulations for both single and multiple allocation, implicit enumerative solution method
2006	Hamacher and Meyer	Compared formulations, identified facet defining valid inequalities

6. The discount factor α

In classical hub location problems, the hub-to-hub arcs are typically discounted by a fixed discount factor α, such that $0 \leqslant \alpha < 1$. However, the number and location of hubs may be seriously affected by the value chosen for α. Most hub location models have assumed that this inter-hub discount factor is not dependent on the amount of flow using the links. O'Kelly and Bryan (1998) pointed out that "the assumption of flow-independent costs not only miscalculates total network cost but may also erroneously select optimal hub locations and allocations". They proposed a non-linear cost function which allows costs to increase at a decreasing rate as flows increase. They then approximated this non-linear cost function by a piecewise-linear concave function and incorporated it into the multiple allocation hub location problem. The authors presented an illustrative example showing that the optimal solution differs by using this new cost function. Bryan (1998) presented some variations and extensions to the formulation presented in O'Kelly and Bryan (1998). She considered capacities and minimum flows on inter-hub links, and flow-dependent costs in all network links. Later, Klincewicz (2002) showed that for a fixed set of hubs, the concave cost model presented in O'Kelly and Bryan (1998) can be converted to the classic *Uncapacitated Facility Location Problem* for which effective solution procedures exist. He proposed an enumeration procedure, and some heuristics based on tabu search and greedy random adaptive search procedures. He have tested the algorithm on the CAB data set and showed that the optimal set of hubs does change for different cost functions.

Another non-linear cost function is proposed by Horner and O'Kelly (2001). The authors claimed that discounts could be earned along any portion of a route that had sufficient volume. Thus, like Bryan (1998), they applied this non-linear concave cost function, which rewards economies of scale, on all network links in a GIS environment and compared solutions for different assumptions about network costs. Kimms (2005) also assumed that economies of scale can occur on all kinds of connections, not just on inter-hub connections. He proposed a formulation with a piecewise linear cost function which incurs a fixed cost for using a link. In Wagner (2004b) the author defined α to be a non-increasing quantity-dependent function in his single allocation hub covering formulations.

Racunicam and Wynter (2005) presented an uncapacitated hub location model for determining the optimal location of intermodal freight hubs. They used a non-linear concave cost function to represent the economies of scale generated in both inter-hub and hub-to-destination links. Their function is such that the inter-hub costs are higher than the linear cost up to a threshold value and less costly thereafter. When compared to O'Kelly (1998) non-linear function, theirs acts directly on flow on the link while O'Kelly (1998) acts on the ratio of inter-hub link flow to total network flow. They approximated this function with a piecewise-linear function and presented some polyhedral properties of the new linear model. They developed two variable-reduction heuristics and provided a case study on the Alpine freight network.

Cunha and Silva (2007) considered the problem of configuring a hub network for a less-than-truckload trucking company in Brazil. Rather than taking a constant hub-to-hub discount factor, they allowed a variable discount factor, which varies according to the total amount of freight between hubs, in their model.

7. Other studies

Considering that the standard hub location models were developed mainly for airline applications, some more cargo-specific models have been developed recently. Kara and Tansel (2001) observed that the time spent at hubs for unloading, loading and sorting operations (transient times) may constitute a significant portion of the total delivery time for cargo delivery systems. They proposed new models, called *the latest arrival hub location problem*, for systems where the transient times are incorporated. Several versions of the latest arrival hub location problem are possible: single or multiple allocation minimax, covering and minisum versions.

The focus in Kara and Tansel (2001) was on the single allocation minimax version. The objective is to minimize the arrival time of the last arrived item while taking into account both the flight times and the transient times. The authors showed that this minimax version is a special case of the *p*-center problem hence it is *NP*-hard. The authors presented two linear mixed integer formulations of this problem. Tan and Kara (2007) studied the latest arrival hub covering problem on an application for the cargo delivery sector in Turkey. They

presented integer programming formulations for the latest arrival hub covering problem to accommodate the varying requirements of the cargo delivery sector. Another study is presented in Yaman et al. (2005). They proposed a latest arrival hub center model which incorporates multiple stopovers and vehicle routes. They further improved this formulation by using valid inequalities and lifting. They then tested the formulation on the CAB data set and on the Turkish highway network.

A note by Wagner (2004a) showed that if the objective function depends only on maximum travel time, the critical path of the minimax latest arrival hub location problem is the same as the critical path of the p-hub center model that ignores transient times. The author pointed out that this result is also valid for the covering version of the problem but the minisum version is a genuinely different problem. Note that the transient times for the origin–destination pairs on the critical path will be zero. Thus, the critical paths of the p-hub center problem and the minimax latest arrival hub location problem will be the same. However, the latest arrival hub location problem provides a different modeling approach which is handy in implementing additional sector-specific real life requirements; it also provides more insights. Besides, the minimax latest arrival hub location formulation using CPLEX was superior to the p-hub center formulation in terms of CPU time requirement.

Nickel et al. (2001) presented new hub location models applicable to urban public transportation networks. They considered the hub location problem as a network design problem and incurred a fixed cost for locating hub arcs. Podnar et al. (2002) considered a new network design problem where they do not locate hubs but they decide on the links with reduced unit transportation costs. In their model the cost of flow is reduced according to a prescribed discount factor α, if the flow through that link is larger than a given threshold value.

Similarly, Campbell et al. (2005a) introduced a new model called the *hub arc location* model which assumes neither a fully interconnected hub network nor that the flow on every hub-to-hub arc is discounted. Rather than locating hub facilities, their model locates hubs arcs which have reduced unit costs. They examined in detail four special cases of the general hub arc location model, one being the p-hub median problem. They gave the optimal solutions of these special cases on the CAB data set and compared the results with the p-hub median model solutions. A companion paper, Campbell et al. (2005b), provided integer programming formulations for these four special cases and two optimal solution algorithms for these new hub-arc problems. It also provides details and computation times for these algorithms. Campbell et al. (2003) implemented the enumeration-based algorithm presented in Campbell et al. (2005b) in a parallel environment in an attempt to optimally solve larger hub arc location problems. They have tested this parallel implementation on the CAB and AP data sets.

Sung and Jin (2001) presented a new hub network design problem where the nodes of the network are partitioned into clusters and one node in each cluster is selected as a hub. The clusters are assumed to be fixed and each node in a cluster is allocated to the hub of that cluster. The authors also allowed direct link services between non-hub nodes within a cluster in their model and proposed a dual-based heuristic algorithm. Later, Wagner (2007) showed that this problem is NP-hard and proposed a new mixed-integer programming formulation requiring fewer variables. He proposed preprocessing techniques for using MIP solvers and a constraint programming approach.

All of the classical hub location problems discussed above are *NP*-hard (except for some special cases). Thus the exact solution potential for these problems is limited. Recently, some studies have explored the polyhedral properties of hub location problems with fixed costs (Boland et al. (2004), Hamacher et al. (2004), Labbé and Yaman (2004), Labbé et al. (2005)). Others have studied p-hub center problem (Baumgartner, 2003) and hub covering problems (Hamacher and Meyer, 2006). Using facet-defining valid inequalities increases the exact solution potential.

In addition to the hub applications in airline transportation and postal delivery networks, various studies investigated the use of hub networks in marine and railway transportation as well. For example, Aversa et al. (2005) proposed a model for locating a hub port in South America. A study by Konings (2005) investigates the effects of using hub networks for container-on-barge transportation. There are also various studies in railway transportation as well. A recent study by Jeong et al. (2007) investigates a hub network problem for European freight railway system. The difference in railway applications is that the main focus is on routing and scheduling of the trains rather than the location of the hubs. One may refer to Crainic and Laporte (1997) and Cordeau et al. (1998) for recent reviews related to railway transportation.

The earliest reviews on hub location are by O'Kelly and Miller (1994) and Campbell (1994a). O'Kelly and Miller (1994) provided real world examples which violated some of the assumptions of the standard hub location model. They proposed eight classes of hub location problems corresponding to different decisions on allocation, hub interconnection and non-hub routes; they included references and examples. Campbell (1994a) presented an extensive survey on the network hub location problem that included both the transportation and computer-communication oriented models. Klincewicz (1998) offered another extensive review involving facility location, network design, telecommunication, computer systems and transportation aspects in hub location. O'Kelly (1998) reviewed some distinctive features of hub networks with special attention paid to the contrast between air passenger and air express freight applications. Later, Bryan and O'Kelly (1999) presented an analytical review of the studies on discrete hub networks for passenger airlines and package delivery systems. The most recent review of hub location problems is a book chapter by Campbell et al. (2002). Also a tutorial on recent studies on hub location is presented by Horst Hamacher (Baumgartner et al., 2005) at the ISOLDE X meeting.

8. Synthesis of the existing literature and future research directions

In this paper we have reviewed over 100 papers dealing with or related to the network hub location problem. Looking at the existing literature, it is obvious that the hub location literature is highly influenced by the location literature. Almost all of the problems identified for hub location have analogous location versions.

Fig. 2 shows the total number of the publications among the presented models and Fig. 3 shows their distribution in years.
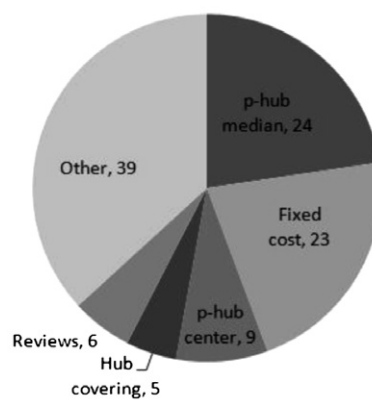

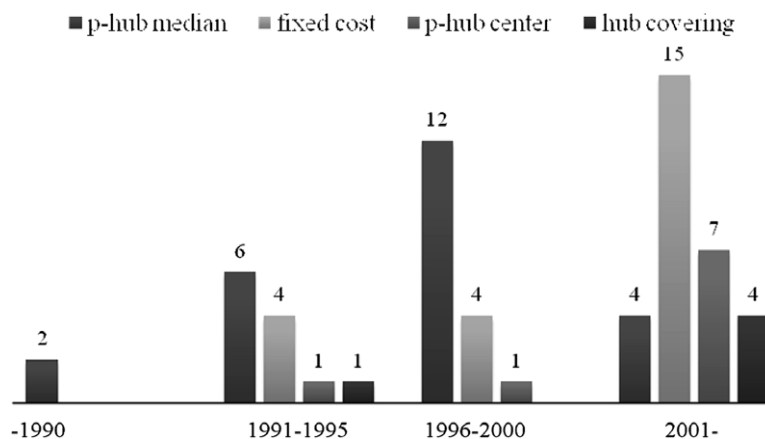Fig. 2. The total number of publications among presented models.

Fig. 3. The number of publications among presented models in years.

Before the year 2000, hub location research is more focused on defining and formulating new problems. These new problems were mainly *p*-hub median variants due the first mathematical formulation (O'Kelly, 1987). After the year 2000, the focus is twisted towards investigating different solution methodologies for these problems. Considering that the *p*-hub median models are very similar in structure to and a special case of the hub location models with fixed costs, there are more studies on solving the fixed cost problem (both heuristic and exact). As it can be observed from Figs. 2 and 3, after the year 2000, there are 4 out of 24 studies on the *p*-hub median problem whereas 15 out of 23 studies on the hub location problem with fixed costs. For single and multiple allocation versions, different integer programming models, branch-and-bound algorithms and heuristics (both construction heuristics and metaheuristics) have been developed. Observe from Fig. 2 that these two problems in total are the most frequently addressed hub location problems.

On the other hand, as Fig. 2 shows, the total number of papers on the *p*-hub center or hub covering type problems is very few compared to other models. The main reason is that these problems are proposed in 1994 (Campbell, 1994b) and remain untouched since the year 2000. These problems are a fairly new research area and there is still a lot of ground to cover; there is a need to develop more exact solution procedures and heuristic algorithms for these problems. Campbell (1994b) defined three versions of both center and covering type hub location problems. The research focused on only one type for each of the problems. In the *p*-hub center problem the maximum cost for any origin–destination pair is minimized; in the hub covering problems only the hub set covering problem is studied. Nor is there any study of the capacitated versions of the center and covering type problems.

There are not enough studies in the literature that considers more than one, possibly conflicting, objectives of the hub location problem. To the best of our knowledge there is only one multiobjective study in the hub location literature: Costa et al. (2007) considered objectives related to both cost and time. Their study is a theoretic study rather than being an application oriented one. We have also noted that both time and cost are major concerns especially for cargo applications. However, the literature lacks such realistic studies. So, modeling the multiobjective nature of the hub location problem and developing solution procedures for such models is a possible future research direction.

Even though recently there are more studies that model real life aspects of the hub location problem, we are merely at the beginning. Observing real-life situations will introduce many new requirements; the models will become more complicated making them even harder to solve (like Yaman et al. (2005)). Developing new mathematical models and incorporating real life aspects is another line of research that is definitely worth pursuing.

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