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HAL Id: inria-00593408
https://hal.inria.fr/inria-00593408
Submitted on 16 May 2011

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Parameters estimation of a noisy sinusoidal signal with time-varying amplitude

Da-yan Liu, Olivier Gibaru and Wilfrid Perruquetti

Abstract— In this paper, we give estimators of the frequency, amplitude and phase of a noisy sinusoidal signal with time-varying amplitude by using the algebraic parametric techniques introduced by Fliess and Sira-Ramirez. We apply a similar strategy to estimate these parameters by using modulating functions method. The convergence of the noise error part due to a large class of noises is studied to show the robustness and the stability of these methods. We also show that the estimators obtained by modulating functions method are robust to "large" sampling period and to non zero-mean noises.

I. INTRODUCTION

Recent algebraic parametric estimation techniques for linear systems [1], [2], [3] have been extended to various problems in signal processing (see, e.g., [4], [5], [6], [7], [8]). In [9], [10], [11], these methods are devoted to estimate the frequency, amplitude and phase of a noisy sinusoidal signal with time-invariant amplitude. Let us emphasize that these methods, which are algebraic and non-asymptotic, exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties (see [12], [13] for more theoretical details). We have shown in [14] that the differentiation estimators proposed by algebraic parametric techniques can cope with a large class of noises for which the mean and covariance are polynomials in time. The robustness properties have already been confirmed by numerous computer simulations and several laboratory experiments. In [15], [9], modulating functions methods are used to estimate unknown parameters of noisy sinusoidal signals. These methods have similar advantages than algebraic parametric techniques especially concerning the robustness of estimations to corrupting noises. The aim of this paper is to estimate the frequency, amplitude and phase of a noisy sinusoidal signal with time-varying amplitude by using respectively algebraic parametric techniques and modulating functions method. In Section III, the estimators are given in discrete case. Then, we study the influence of sampling period on the associated noise error part due to a class of noises. In Section IV, inspired by [15] a recursive algorithm for the frequency estimator is given, then some numerical simulations are given to show the efficiency and stability of our estimators.

II. NOTATIONS PRELIMINARIES

Let us denote by $D_T := \{ T \in \mathbb{R}^+_0 : [0, T] \subset \Omega \}$, and $w_{\mu, k}(\tau) = (1 - \tau)^{\mu} e^{\tau}$ for any $\tau \in [0, 1]$ with $\mu, k \in [-1, +\infty]$. By using the Rodrigues formula (see [16] p.67), we have

$$
\frac{d^i}{d\tau^i} \left\{ w_{\mu, k}(\tau) \right\} = (-1)^i i! w_{\mu - i, k - i}(\tau) R^i_{\mu - i, k - i}(\tau),
$$

where $R^i_{\mu - i, k - i}$, $\min(\mu, k) \geq i \in \mathbb{N}$, is the $i^{th}$ order Jacobi polynomial defined on $[0, 1]$ (see [16]): $\forall \tau \in [0, 1],

$$
P^i_{\mu - i, k - i}(\tau) = \sum_{s=0}^{i} (-1)^{i-s} \binom{i}{s} \binom{\mu}{k} w_{s-i, s-i}(\tau).$$

Then, we have the following lema.

Lemma 1: Let $f$ be a $\mathcal{C}^{n+1}(\Omega)$-continuous function ($n \in \mathbb{N}$) and $\Pi^n_{k, \mu}$ be a differential operator defined as follows

$$
\Pi^n_{k, \mu} = \frac{1}{s^{n+1} + \mu} \frac{d^{n+k}}{ds^{n+k}} f^s,
$$

where $s$ is the Laplace variable, $k \in \mathbb{N}$ and $-1 < \mu \in \mathbb{R}$. Then, the inverse Laplace transform of $\Pi^n_{k, \mu} f$ where $f$ is the Laplace transformation of $f$ is given by

$$
\mathcal{L}^{-1}\left\{ \Pi^n_{k, \mu} \hat{f}(s) \right\}(T) = T^{n+1+k} c_{\mu+n, k} \int_0^1 \left( \frac{\tau^k}{\Gamma(\mu+n+1)} \right)^{n+1} f(T \tau)d\tau,
$$

where $T \in D_T$ and $c_{\mu+n, k} = \left( \frac{\Gamma(\mu+n+1)}{\Gamma(\mu+n)} \right)$. In order to prove this lemma, let us recall that the $\alpha$-order ($\alpha \in \mathbb{R}^+_0$) Riemann-Liouville integral (see [17]) of a real function $g$ ($\mathbb{R} \to \mathbb{R}$) is defined by

$$
J^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} g(\tau)d\tau.
$$

The associated Laplace transform is given by

$$
\mathcal{L}\{ J^\alpha g(t) \}(s) = s^{-\alpha} \hat{g}(s),
$$

where $\hat{g}$ denotes the Laplace transform of $g$.

Proof. Let us denote $W_{\mu+n, k+n}(t) = (T - t)^{\mu+n} e^{(k+n)\tau}$ for any $t \in [0, T]$. Then, by applying the Laplace transform to
the following Riemann-Liouville integral and doing some classical operational calculations, we obtain

\[
\mathcal{L} \left\{ \sum_{n=0}^{\infty} \left( \int_{0}^{T} W_{k+n}(\tau) f(n)(\tau) d\tau \right) \right\} = \mathcal{L} \left\{ (\tau - \frac{1}{2})(\tau + \frac{1}{2}) \right\} = \sum_{n=0}^{\infty} \left( \int_{0}^{T} W_{k+n}(\tau) f(n)(\tau) d\tau \right)
\]

Then, by substituting \( \tau = T \tau \) we have

\[
\mathcal{L} \left\{ \sum_{n=0}^{\infty} \left( \int_{0}^{T} W_{k+n}(\tau) f(n)(\tau) d\tau \right) \right\} = \sum_{n=0}^{\infty} \left( \int_{0}^{T} W_{k+n}(\tau) f(n)(\tau) d\tau \right)
\]

By using \( \mathcal{L} \), we obtain \( \sum_{n=0}^{\infty} \left( \int_{0}^{T} W_{k+n}(\tau) f(n)(\tau) d\tau \right) = 0 \) for \( i = 0, \ldots, n - 1 \). Finally, this proof can be completed by applying \( n \) times integration by parts to \( \mathcal{L} \).

III. ALGEBRAIC PARAMETRIC TECHNIQUES

Let \( y = x + \sigma \) be a noisy observation on a finite time interval \( \Omega \subset \mathbb{R}^+ \) of a real valued signal \( x \), where \( \sigma \) is an additive corrupting noise and

\[
\forall t \in \Omega, \ x(t) = (A_0 + A_1 t) \sin(\omega t + \phi)
\]

with \( A_0 \in \mathbb{R}^+ \), \( A_1 \in \mathbb{R}^+ \), \( \omega \in \mathbb{R}^+ \), and \(-\pi \leq \phi < \pi \). Observe that \( x \) is a time-varying variant sinusoidal signal, which is a solution to the harmonic oscillator equation

\[
\forall t \in \Omega, \ x(\dot{t}) + 2 \omega^2 \dot{x}(t) + \omega^4 x(t) = 0.
\]

Then, we can estimate the parameters \( \omega, A_0 \) and \( \phi \) by applying algebraic parametric techniques to \( \mathcal{L} \).

Proposition 1: Let \( k \in \mathbb{N}, \ -1 < \mu \in \mathbb{R} \) and \( T \in D_T \) such that \( A_1 \int_{0}^{T} \mu_{k+\mu+k+4}(\tau) \sin(\omega_2 t + \phi) d\tau \leq 0 \), then the parameter \( \omega \) is estimated from the noisy observation \( y \) by

\[
\hat{\omega} = \left( \frac{-\hat{B}_2 + \sqrt{\hat{B}_2^2 - 4\hat{A}_2 \hat{C}_2}}{2\hat{A}_2} \right) \frac{1}{2},
\]

where \( \hat{A}_2 = T^2 \int_{0}^{T} \mu_{k+\mu+k+4}(\tau) x(T) d\tau, \hat{B}_2 = 2T^2 \int_{0}^{T} \mu_{k+\mu+k+4}(\tau) y(T) d\tau, \hat{C}_2 = \int_{0}^{T} \mu_{k+\mu+k+4}(\tau) y(T) d\tau, \hat{\omega}(\mu_{k+\mu+k+4}(\tau)) \) is given by \( \mathcal{L} \) with \( i = 1, 2, 4 \).

Proof. By applying the Laplace transform to \( \hat{y}(s) \), we get

\[
\hat{y}(s) + 2 \omega^2 \dot{\hat{y}}(s) + \omega^4 \hat{y}(s) = 0.
\]

Let us apply \( k+4(k \in \mathbb{N}) \) times derivations to both sides of \( \hat{y}(s) \) with respect to \( s \). By multiplying the resulting equation by \( s^{5-\mu} \) with \(-1 < \mu \in \mathbb{R} \), we get

\[
\Pi_{k+\mu, \mu}^4 \hat{y}(s) + 2 \omega^2 \Pi_{k+2, \mu+2}^4 \hat{y}(s) + \omega^4 \Pi_{k+4, \mu+4}^4 \hat{y}(s) = 0.
\]

Let us apply the inverse Laplace transform to \( \hat{y}(s) \), then by using Lemma \( \mathcal{L} \), we obtain

\[
\int_{0}^{T} \left( w_{k+\mu+k+4, j}(\tau) + 2(\omega_T)^2 w_{k+\mu+k+4, j}(\tau) \right) x(T) d\tau = 0.
\]

According to \( \mathcal{L} \), we have \( \int_{0}^{T} w_{k+\mu+k+4, j}(\tau) x(T) d\tau = 0 \) for \( i = 0, \ldots, 3 \). Then by applying integration by parts, we get

\[
\omega^2 \int_{0}^{T} w_{k+\mu+k+4, j}(\tau) x(T) d\tau + 2 \omega^2 w_{k+\mu+k+4, j}(\tau) x(T) d\tau = 0.
\]

Thus, \( \omega^2 \) is obtained by

\[
\omega^2 = A \frac{B}{2} = 2A \frac{B}{2} + 4 \frac{C}{2} = \frac{1}{2} \left( \hat{B}_2 + \sqrt{\hat{B}_2^2 - 4\hat{A}_2 \hat{C}_2} \right) \frac{1}{2}.
\]

Finally, this proof can be completed by applying integration by parts and substituting \( x \) by \( y \) in the last equation.

By observing that \( x_0 = x(0) = A_0 \sin \phi, x_0 = x(\hat{y}(s)) = A_0 \cos \phi + A_1 \sin \phi \) and \( x_0^{(3)} = x(\hat{y}(s)) = \hat{y}(s) \), then we can obtain \( A_0 \cos \phi = \frac{A_0}{2} \left( x_0^{(3)} + 3 \omega^2 x_0 \right) \). Hence, if \(-\frac{\pi}{2} < \phi < \frac{\pi}{2} \), then we have

\[
A_0 = \left( x_0^2 + \frac{x_0^{(3)} + 3 \omega^2 x_0}{4 \omega^6} \right)^{\frac{1}{2}},
\]

Thus, we need to estimate \( x_0, x_0^{(3)} \) as so to obtain the estimations of \( A_0 \) and \( \phi \).

Proposition 2: Let \(-1 < \mu \in \mathbb{R} \) and \( T \in D_T \), then the parameters \( A_0 \) and \( \phi \) are estimated from the noisy observation
y and the estimated value of \( \omega \) given in (14):

\[
\hat{A}_0 = \left( \frac{x_0^2}{4} + 3\hat{\omega}^2 \hat{x}_0 \right)^{\frac{1}{2}},
\]

\[
\hat{\phi} = \arctan \left( \frac{2\hat{\omega}^2 x_0}{3\hat{\omega}^2 \hat{x}_0 + \hat{\omega}^2 x_0} \right),
\]

where

\[
\hat{x}_0 = \int_0^T P_0^0(y(T) dt, \quad \hat{x}_0 = \frac{1}{T} \int_0^T P_0^0(y(T) dt,
\]

\[
\hat{x}_0^{(3)} = \frac{3}{T^5} \int_0^T P_0^0(y(T) dt - \tau^2 \hat{x}_0,
\]

\[
\hat{y}_0 = \int_0^T \frac{6}{\Gamma(\mu+5)} P_0^0(\tau) y(T) d\tau, \quad \hat{y}_0 = \frac{1}{T} \int_0^T P_0^0(\tau) y(T) d\tau, \quad \hat{y}_0^{(3)} = \frac{3}{T^5} \int_0^T P_0^0(\tau) y(T) dt - \tau^2 \hat{y}_0.
\]

Hence, by substituting \( \tau \) by \( T \), \( x \) by \( y \) and taking the estimation of \( \omega \) given in Proposition 1, we obtain an estimate for \( x_0 \). Similarly, we apply the operator \( \Pi_2 = \frac{1}{\sin \phi} \frac{d^3}{ds^3} \frac{1}{s} \) to (11) to compute an estimate for \( x_0^{(3)} \) (resp. \( \Pi_3 = \frac{d^3}{ds^3} \frac{1}{s} \)) by using the estimations of \( x_0, x_0^{(3)} \) and \( \omega \).

IV. MODULATING FUNCTIONS METHOD

\begin{proposition}
Let \( f \) be a function belonging to \( C^4([0,1]) \) which satisfies the following conditions \( f(i) = f(i+1) \) for \( i = 0, \ldots, 3 \). Assume that \( A_1 f_0(T) \sin(\omega T + \phi) dt \leq 0 \) with \( T \in D_T \), then the parameter \( \omega \) is estimated from the noisy observation \( y \) by

\[
\hat{\omega} = - \frac{B_2 + \sqrt{B_2^2 - 4A_1C_1}}{2A_1},
\]

where \( A_4 = T^4 \int_0^T f(T) \sin(\omega T + \phi) dt, \quad B_4 = T^2 \int_0^T f(T) \sin(\omega T + \phi) dt, \quad C_4 = \int_0^T f(T) \sin(\omega T + \phi) dt \).

\begin{proof}
Recall that \( x^{(i)}(T) + 2\omega^2 x^{(2)}(T) + \omega^4 x(T) = 0 \) for any \( \tau \in [0,1] \). As \( f \) is continuous on \([0,1]\), then we have

\[
\int_0^T f(T) x^{(i)}(T) dt + 2\omega^2 \int_0^T f(T) x^{(2)}(T) dt + \omega^4 \int_0^T f(T) x(T) dt = 0.
\]

Then, this proof can be completed similarly to the one of Proposition 1.
\end{proof}

\begin{proposition}
Let \( f_i \) for \( i = 1, \ldots, 4 \) be four continuous functions defined on \([0,1]\). Assume that there exists \( T \in D_T \) such that the determinant of the matrix \( M_{vi} = (M_{v,i})_{1 \leq i \leq 4} \) is different to zero, where for \( i = 1, \ldots, 4 \)

\[
M_{vi} = \int_0^T f_i(T) \sin(\omega T + \phi) dt, \quad M_{v,1} = \int_0^T f_i(T) \cos(\omega T + \phi) dt.
\]

Then, for any \( \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), the estimations of \( A_0, A_1 \) and \( \phi \) are given by

\[
\hat{A}_i = \left( A_i \cos \phi \right)^2 + \left( A_i \sin \phi \right)^2, \quad \hat{\phi} = \arctan \left( \frac{A_0 \sin \phi}{A_0 \cos \phi} \right),
\]

where the estimates of \( A_i \cos \phi \) and \( A_i \sin \phi \) for \( i = 0, 1 \) are obtained by solving the following linear system

\[
M_{v0} \begin{pmatrix} A_0 \cos \phi \\ A_0 \sin \phi \\ A_1 \cos \phi \\ A_0 \sin \phi \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix},
\]

where \( f_i = \int_0^T f_i(T) \sin(\omega T + \phi) dt \) for \( i = 1, \ldots, 4 \), and \( \phi \) is the estimate of \( \phi \) given by Proposition 3.
Proof. Let us take an expansion of \( x \)
\[
x(T \tau) = A_0 \cos \phi \sin (\omega T \tau) + A_0 \sin \phi \cos (\omega T \tau)
\]
+ \( A_1 \cos \phi T \tau \sin (\omega T \tau) + A_1 \sin \phi T \tau \cos (\omega T \tau) \),
where \( \tau \in [0, 1], \ T \in D. \) By multiplying both sides of the last equation by the continuous functions \( f_i \) for \( i = 1, \ldots, 4 \) and by integrating the resulting equations between 0 and 1, we obtain
\[
I_{fi}^T = A_0 \cos \phi M_{i,1}^\phi + A_0 \sin \phi M_{i,1}^\phi + A_1 \cos \phi M_{i,3}^\phi + A_1 \sin \phi M_{i,4}^\phi.
\]
Then, it yields the following linear system
\[
M = \left( \begin{array}{c}
\begin{array}{c}
A_0 \cos \phi \\
A_0 \sin \phi \\
A_1 \cos \phi \\
A_1 \sin \phi 
\end{array}
\end{array} \right) = \left( \begin{array}{c}
I_{f1}^T \\
I_{f2}^T \\
I_{f3}^T \\
I_{f4}^T
\end{array} \right).
\]
Since \( \det(M_{i,0}) \neq 0 \), we obtain \( A_0 \cos \phi \) and \( A_1 \sin \phi \) for \( i \) in \( 0, 1, \ldots, 4 \). Finally, the proof can be completed by substituting \( x \) by \( y \) in the so obtained formulae of \( A_0 \cos \phi \) and \( A_1 \sin \phi \).

From now on, we choose functions \( w_{n,0}^{\mu+n,0} \) with \( n \in \mathbb{N} \), \( \mu, \kappa \in \mathbb{C} \). Since the previous modulating functions, we obtain an equidistant sampling period \( T \). Let \( \{ \varphi(\tau), \tau \geq 0 \} \) be a continuous stochastic process satisfying conditions \( (C_1) \) and \( (C_3) \). Assume that \( q \in \mathbb{L}^2([0, 1]) \), then we have
\[
\lim_{m \to +\infty} E \left[ e_{q,m} \right] = 0.
\]
Proof. Since \( \varphi(t) \) is a sequence of independent random variables \( \{ \tau \} \), then by using the properties of mean value and variance functions we have
\[
E \left[ e_{q,m} \right] = \frac{1}{m} \sum_{i=0}^{m} a_i q(\tau) \varphi(\tau),
\]
\[
Var \left[ e_{q,m} \right] = \frac{1}{m} \sum_{i=0}^{m} a_i^2 q^2(\tau) \varphi(\tau).
\]
According to \( (C_1) \), the variance function of \( \varphi(\tau) \) is bounded. Then we have
\[
0 \leq \frac{1}{m^2} \sum_{i=0}^{m} a_i^2 q^2(\tau) \varphi(\tau) \leq U a(\mu) \frac{1}{m} \sum_{i=0}^{m} a_i q^2(\tau),
\]
where \( a(\mu) = \max_{0 \leq \mu \leq m} \sup_{0 \leq \tau \leq 1} \varphi(\tau) \) and \( U = \sup_{0 \leq \tau \leq 1} \varphi(\tau) \).

Moreover, the variance function of \( \varphi(\tau) \) is integrable \( (C_2) \), then we have
\[
\lim_{m \to +\infty} E \left[ e_{q,m} \right] = \frac{1}{m} \sum_{i=0}^{m} a_i q^2(\tau) \varphi(\tau),
\]
\[
\lim_{m \to +\infty} \sum_{i=0}^{m} a_i^2 q^2(\tau) \varphi(\tau) = \frac{1}{m} \sum_{i=0}^{m} a_i^2 q^2(\tau) \varphi(\tau) < +\infty.
\]
As all \( a_i \) are bounded, we have \( \frac{1}{m} \sum_{i=0}^{m} a_i^2 q^2(\tau) = 0 \). This proof is completed.

\textbf{Theorem 1:} With the same conditions given in Lemma 3, we have the following convergence
\[
e_{q,m}^{\mathbb{L}^2([0,1])} \to \left( \begin{array}{c}
\varphi(\tau) \end{array} \right) E \varphi(\tau) d\tau, \quad \text{when } T \to 0.
\]
Moreover, if noise \( \varphi(\tau) \) satisfies the following conditions
\[
(C_4) \quad E(\varphi(\tau)) = \sum_{i=0}^{n-1} v_i \tau^i \quad \text{with } n \in \mathbb{N} \quad \text{and } v_i \in \mathbb{R},
\]
and \( q \equiv w_{n,0}^{\mu+n,0} \) with \( \mu, \kappa \in \mathbb{C} \), then we have
\[
\lim_{m \to +\infty} E \left[ e_{q,m} \right] = 0,
\]
and
\[
e_{q,m}^{\mathbb{L}^2([0,1])} \to 0, \quad \text{when } T \to 0.
\]
Proof. Recall that $E[(Y_n - c)^2] = Var[Y_n] + (E[Y_n] - c)^2$ for any sequence of random variables $Y_n$ with $c \in \mathbb{R}$, then by using Lemma 2, $e(q^{\alpha,m})$ converges in mean square to $\int_0^1 q(\tau)E[\sigma(T\tau)]d\tau$ when $r \to 0$. Hence, if $E[\sigma(\tau)] = \sum_{i=0}^{n-1} \nu_i$, and $\mu, \kappa \in [-\frac{1}{2}, +\infty]$, then by using the Rodrigues formula given by (4) we obtain $w_{\nu,m}^{\mu,n}(\tau) \in L^2([-1, 1])$ and $\int_0^1 w_{\nu,m}^{\mu,n}(\tau)E[\sigma(T\tau)]d\tau = 0$. Hence, this proof is completed.

VI. NUMERICAL IMPLEMENTATIONS

In our identification procedure, we use a moving integration window. Hence, the estimate of $\omega$ at $t_i$ is given by Proposition 3 as follows

$$\forall t_i \in \Omega, \quad \omega^2(t_i) = -\frac{B_{\nu_i}}{2A_{\nu_i}} + \frac{A_{\nu_i}}{2A_{\nu_i}}, \quad i = 0, 1, \ldots, \quad (27)$$

where $A_{\nu_i} = \sqrt{B_{\nu_i}^2 - 4A_{\nu_i}C_{\nu_i}}$, $A_{\nu_i} = T^2P_{\nu,m}^{-1}$, $B_{\nu_i} = 2T^2P_{\nu,m}^{-1}$, $C_{\nu_i} = \nu_i P_{\nu,m}^{-1} + \nu_i^{\nu_i}$ with $\nu_i \equiv y(T + t_i)$. Note that if $A_{\nu_i} = 0$, then there is a singular value in (27). If we denote by $\theta_i = \frac{n_{\nu_i}}{D_{\nu_i}}$ where $D_{\nu_i} = -B_{\nu_i}$ or $D_{\nu_i} = A_{\nu_i}$, then we can apply the following criterion (see [15]) to improve the estimation of $\omega$

$$\min_{\theta_i \in \mathbb{R}} J(\theta_i) = \frac{1}{2} \sum_{j=0}^{n-1} \nu_{i+j} A_{\nu_i} \theta_i^2,$$

where $i = 0, 1, \ldots, \nu \in [0, 1]$. The parameter $\nu$ represents a forgetting factor to exponentially discard the “old” data in the recursive schema. The value of $\theta_i$, which minimizes the criterion (28), is obtained by seeking the value which cancels $\frac{J(\theta)}{\sigma(\theta)}$. Thus, we get

$$\theta_i = -\frac{\sum_{j=0}^{n-1} \nu_{i+j} A_{\nu_i} \theta_i}{\sum_{j=0}^{n-1} \nu_{i+j} A_{\nu_i}} \quad \text{(29)}$$

Similarly to [15], we can get the following recursive algorithm for (29)

$$\theta_{i+1} = \frac{\nu}{\alpha_{i+1}} \left( \alpha_i \theta_i + D_{\nu_i} A_{\nu_i} \right), \quad i = 0, 1, \ldots, \quad (30)$$

where $\alpha_i = \sum_{j=0}^{i-1} \nu_{i+j} A_{\nu_i}$ Moreover, $\alpha_{i+1}$ can be recursively calculated as follows $\alpha_{i+1} = \nu \left( \alpha_i + A_{\nu_i} \right)^2$.

Example 1: According to Section III we can reduce the noise error part in our estimations by decreasing the sampling period. Hence, let $(y(t_i) = x(t_i) + c\sigma(t_i))_{i \geq 0}$ be a generated noise data set with a small sampling period $T_s = 5\pi \times 10^{-4}$ in the interval $[0, 3\pi]$ (see Fig. 1) where

$$x(t_i) = \begin{cases} \sin(10t_i + \frac{\pi}{4}), & \text{if } 0 \leq t_i \leq \pi, \\ \frac{1}{2} \sin(10t_i + \frac{\pi}{4}), & \text{if } \pi < t_i \leq 2\pi, \\ 2\sin(10t_i + \frac{\pi}{4}), & \text{if } 2\pi < t_i \leq 3\pi, \end{cases} \quad (31)$$

and noise $c\sigma(x_i)$ is simulated from a zero-mean white Gaussian iid sequence with $c = 0.1$. Hence, the signal-to-noise ratio $SNR = 10\log_{10} \left( \frac{\sum_{i=0}^{n-1} \nu_i}{\sum_{n=1}^{m-1} \nu_n} \right)^2$ is equal to $SNR = 20.8$ dB. In order to estimate the frequency, by applying the previous recursive algorithm we use Proposition 3 with $\kappa = \mu = 0$, $m = 450$ and $\nu = 1$. The relating estimation error is shown in Fig. 2. By using the estimated frequency value, we estimate the amplitude and phase of the signal by applying Proposition 4 with $\mu = 0$, $m = 500$ and Proposition 5 with $m = 500$, $f_1 \equiv w_{1,2}$, $f_2 \equiv w_{2,3}$, $f_3 \equiv w_{3,4}$ and $f_4 \equiv w_{4,3}$. The relating estimation errors are shown in Fig. 3 and Fig. 4. We can observe that with small value of $T_s$ the relating estimation errors are also small.

Example 2: In this example, we increase the value of $T_s$ to $T_s = 2\pi \times 10^{-2}$ and reduce the noise level to $c = 0.01$. Moreover, we add a bias term perturbation $\xi = 0.25$ in (51) when $t_i \in [2\pi, 3\pi]$. The estimations of $\omega$ are obtained by Proposition 3 with $\kappa = \mu = 0$, $m = 12$ and $\nu = 1$. The estimations of the amplitude and phase are given by applying Proposition 5 with $\mu = 0$, $m = 12$ and Proposition 6 with $m = 15$, $f_1 \equiv w_{1,2}$, $f_2 \equiv w_{2,3}$, $f_3 \equiv w_{3,4}$ and $f_4 \equiv w_{4,3}$. The relating estimation errors are shown in Fig. 5 and Fig. 6. We can observe that the estimators obtained by modulating functions method are more robust to the sampling period and to the non zero-mean noise than the ones obtained by algebraic parametric techniques.
In this paper, two methods are given to estimate the frequency, amplitude and phase of a noisy sinusoidal signal with time-varying amplitude, where the estimates are obtained by using integrals. There are two types of errors for these estimates: the numerical error and the noise error part. Then, the convergence in mean square of the noise error part is studied. A recursive algorithm for frequency estimator is given. In numerical examples, we show some comparisons between the two proposed methods. Moreover, these methods can also be used to estimate the frequencies, the amplitudes and the phases of two sinusoidal signals from their noisy sum (see [11]). The analysis for colored noises will be done in a future work.

REFERENCES