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Absence of traveling wave solutions of conductivity type for the Novikov-Veselov equation at zero energy

A.V. Kazeykina

Abstract. We prove that the Novikov-Veselov equation (an analog of KdV in dimension 2 + 1) at zero energy does not have sufficiently localized soliton solutions of conductivity type.

1 Introduction

In this note we are concerned with the Novikov-Veselov equation at zero energy

\[ \begin{align*}
\partial_t v &= 4\text{Re}(4\partial_x^3 v + \partial_x(vw)), \\
\partial_z w &= -3\partial_x v, \quad v = \tilde{v}, \\
v &= v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},
\end{align*} \tag{1} \]

where

\[ \partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \partial_x = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \]

Definition 1. A pair \((v, w)\) is a sufficiently localized solution of equation (1) if

- \(v, w \in C(\mathbb{R}^2 \times \mathbb{R}), v(\cdot, t) \in C^3(\mathbb{R}^3),\)
- \(|\partial^j_x v(x,t)| \leq \frac{q(t)}{(1 + |x|)^{2+\varepsilon}}, \quad |j| \leq 3, \text{ for some } \varepsilon > 0, \quad w(x,t) \to 0, |x| \to \infty,\)
- \((v, w)\) satisfies (1).

Definition 2. A solution \((v, w)\) of (1) is a soliton (a traveling wave) if \(v(x,t) = V(x-ct),\) \(c \in \mathbb{R}^2.\)

Equation (1) is an analog of the classic KdV equation. When \(v = v(x_1, t), w = w(x_1, t),\) then equation (1) is reduced to KdV. Besides, equation (1) is integrable via the scattering transform for the 2-dimensional Schrödinger equation

\[ \begin{align*}
L \psi &= 0, \\
L &= -\Delta + v(x,t), \quad \Delta = 4\partial_x \partial_{\bar{z}}, \quad x \in \mathbb{R}^2.
\end{align*} \tag{2} \]

Equation (1) is contained implicitly in [M] as an equation possessing the following representation

\[ \frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{3} \]

where \(L\) is defined in (2), \(A\) and \(B\) are suitable differential operators of the third and zero order respectively and \([\cdot, \cdot]\) denotes the commutator. In the explicit form equation (1) was written in [NV1], [NV2], where it was also studied in the periodic setting. For the rapidly decaying potentials the studies of equation (1) and the scattering problem for (2) were carried out in [BLMP], [GN] [T], [LMS]. In [LMS] the relation with the Calderón conductivity problem was discussed in detail.
Definition 3. A potential $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, is of conductivity type if $v = \gamma^{-1/2} \Delta \gamma^{1/2}$ for some real-valued positive $\gamma \in L^\infty(\mathbb{R}^2)$, such that $\gamma \geq \delta_0 > 0$ and $\nabla \gamma^{1/2} \in L^p(\mathbb{R}^2)$.

The potentials of conductivity type arise naturally when the Calderón conductivity problem is studied in the setting of the boundary value problem for the 2-dimensional Schrödinger equation at zero energy (see [Nov1], [N], [LMS]); in addition, in [N] it was shown that for this type of potentials the scattering data for (2) are well-defined everywhere.

The main result of the present note consists in the following: there are no solitons of conductivity type for equation (1). The proof is based on the ideas proposed in [Nov2].

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

2 Scattering data for the 2-dimensional Schrödinger equation at zero energy with a potential of conductivity type

Consider the Schrödinger equation (2) on the plane with the potential $v(z) = x_1 + ix_2$, satisfying

$$v(z) = \overline{v(z)}, \quad v(z) \in L^\infty(\mathbb{C}),$$

$$|v(z)| < q(1 + |z|)^{-2-\varepsilon} \text{ for some } q > 0, \varepsilon > 0.$$ \hfill (4)

For $k \in \mathbb{C}$ we consider solutions $\psi(z, k)$ of (2) having the following asymptotics

$$\psi(z, k) = e^{ikz} \mu(z, k), \quad \mu(z, k) = 1 + o(1), \text{ as } |z| \to \infty,$$ \hfill (5)

i.e. Faddeev’s exponentially growing solutions for the two-dimensional Schrödinger equation (2) at zero energy, see [F], [GN], [Nov1].

It was shown that if $v$ satisfies (4) and is of conductivity type, then $\forall k \in \mathbb{C}\setminus0$ there exists a unique continuous solution of (2) satisfying (5) (see [N]). Thus the scattering data $b$ for the potential $v$ of conductivity type are well-defined and continuous:

$$b(k) = \int_{\mathbb{C}} e^{iky+\bar{k}\bar{y}} v(y) \mu(y, k) dRey dImy, \quad k \in \mathbb{C}\setminus0.$$ \hfill (6)

In addition (see [N]), the function $\mu(z, k)$ from (5) satisfies the following $\bar{\partial}$-equation

$$\frac{\partial \mu(z, k)}{\partial k} = \frac{1}{4\pi k} e^{-i(kz+k\bar{z})} b(k) \mu(z, k), \quad z \in \mathbb{C}, \quad k \in \mathbb{C}\setminus0$$ \hfill (7)

and the following limit properties:

$$\mu(z, k) \to 1, \text{ as } |k| \to \infty,$$ \hfill (8)

$$\mu(z, k) \text{ is bounded in the neighborhood of } k = 0.$$ \hfill (9)

The following lemma describes the scattering data corresponding to a shifted potential.

Lemma 1. Let $v(z)$ be a potential satisfying (4) with the scattering data $b(k)$. The scattering data $b_y(k)$ for the potential $v_y(z) = v(z - y)$ are related to $b(k)$ by the following formula

$$b_y(k) = e^{i(ky+\bar{k}\bar{y})} b(k), \quad k \in \mathbb{C}\setminus0, \quad y \in \mathbb{C}.$$ \hfill (10)
Proof. We note that $\psi(z - y, k)$ satisfies (2) with $v_y(z)$ and has the asymptotics $\psi(z - y, k) = e^{ik(z-y)}(1 + o(1))$ as $|z| \to \infty$. Thus $\psi_y(z, k) = e^{iky}\psi(z - y, k)$ and $\mu_y(z, k) = \mu(z - y, k)$. Finally, we have

$$b_y(k) = \int\int_C e^{i(k\zeta + \bar{k}\bar{\zeta})} v_y(\zeta) \mu_y(\zeta, k) d\text{Re}\zeta d\text{Im}\zeta =$$

$$= \int\int_C e^{i(k\zeta + \bar{k}\bar{\zeta})} v(\zeta - y) \mu(\zeta - y, k) d\text{Re}\zeta d\text{Im}\zeta = e^{i(ky + \bar{k}\bar{y})} b(k).$$

As for the time dynamics of the scattering data, in [BLMP], [GN] it was shown that if the solution $(v, w)$ of (1) exists and the scattering data for this solution are well-defined, then the time evolution of these scattering data is described as follows:

$$b(k, t) = e^{i(k^2 + \bar{k}^2)t} b(k, 0), \quad k \in \mathbb{C} \setminus 0, \quad t \in \mathbb{R}. \quad (11)$$

3 Absence of solitons of conductivity type

Theorem 1. Let $(v, w)$ be a sufficiently localized traveling wave solution of (1) of conductivity type. Then $v \equiv 0$, $w \equiv 0$.

Scheme of proof. From (10), (11), continuity of $b(k)$ on $\mathbb{C} \setminus 0$ and the fact that the functions $k$, $\bar{k}$, $k^3$, $\bar{k}^3$, 1 are linearly independent in the neighborhood of any point, it follows that $b \equiv 0$. Equation (7) implies that in this case the function $\mu(z, k)$ is holomorphic on $k$, $k \in \mathbb{C} \setminus 0$. Using properties (8) and (9) we apply Liouville theorem to obtain that $\mu \equiv 1$. Then $\psi(z, k) = e^{ikz}$ and from (2) it follows that $v \equiv 0$. \qed

References


