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A note on single-machine scheduling problems with position-dependent processing times

Julien Moncel*, Gerd Finke†, Vincent Jost‡

Abstract

The purpose of this note is two-fold. First, it answers an open problem about a single-machine scheduling problem with exponential position-dependent processing times defined in [V. S. Gordon, C. N. Potts, V. A. Strusevich, J. D. Whitehead, Single machine scheduling models with deterioration and learning: Handling precedence constraints via priority generation, Journal of Scheduling 11 (2008), 357–370]. In this problem, the processing time of job \(i\) when scheduled in rank \(r\) is equal to \(p_i(r) = p_i \gamma^{r-1}\), with \(\gamma\) a positive constant. Gordon et al. show in the above-mentioned paper with priority-generating techniques that the problem of minimizing the total flow-time on one machine admits an \(O(n \log n)\) algorithm when \(2^{-1} < \gamma < 2\), and leave the case \(\gamma \in [1, 2]\) open. We show that the problem admits an \(O(n \log n)\) algorithm also for \(\gamma \in [1, 2]\). The second purpose of this note is to provide a simple and general insight on why and when position-dependent scheduling problems on one machine can be solved in time \(O(n \log n)\).

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1 Scheduling with position-dependent processing times

There is a growing literature dealing with scheduling problems where the actual processing time of a job depends on its position in the schedule, and/or its starting processing time (see for instance the recent monography [4] for a survey on time-dependent scheduling). This enables in particular one to model the so-called learning and deteriorating effects. Practical applications include operators becoming more efficient while getting used to a new procedure (learning effect), and forest fires that take longer to extinct as time flows (deteriorating effect).

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In this paper we focus on position-dependent processing times, where the processing time of job $i$ when scheduled in rank $r$ is equal to $p(i, r)$. Hence processing times of jobs are defined by a function $p : i, r \mapsto p(i, r)$. For a general function $p$ it is known that problems $1 \mid p(i, r) \mid C_{\text{max}}$ and $1 \mid p(i, r) \mid \sum C_i$ can be modelled as assignment problems and thus admit $O(n^3)$ algorithms [1, 2] ($n$ being the number of jobs).

But for most practical purposes we do not need processing times in such a general form as $p(i, r)$, and the problem can be solved with simpler methods than solving an assignment problem. Let us assume that the function $p(i, r)$ can be written as $p(i, r) = f(r)p_i$. In this case, we say that the position-dependent scheduling times are decomposable, $p_i$ being the normal processing time of job $i$, and $p(i, r)$ its actual processing time if scheduled in position $r$. This is the case for many scheduling problems of the literature, such as the model of Biskup [2] $p(i, r) = p_ir^a$ (with $a < 0$ a constant “learning index”), the model of Wang and Xia [11] $p(i, r) = p_i(a - br)$ (with $a \geq 0$ integer, $b \geq 0$ rational, and $a - (n + 1)b > 0$), or the model of Gordon et al [5] $p(i, r) = p_i\gamma^{r-1}$ (with $\gamma > 0$).

For this last model, it is shown with priority-generating techniques in [5] that the problem $1 \mid p(i, r) = p_i\gamma^{r-1} \mid \sum C_i$ can be solved in time $O(n \log n)$ for $\gamma \geq 0$. This case $\gamma \in [1, 2]$ being left open. In the next section we prove that this last problem admits an $O(n \log n)$ algorithm for every $\gamma > 0$.

We get this result as a consequence of a more general result on scheduling jobs on one machine with position-dependent processing times. Let us say that an objective function $\gamma$ is decomposable if it can be written as $\gamma = \sum \nu_r p_r$, where $p_r$ denotes the actual processing time of job scheduled in position $r$, and $\nu_1, \ldots, \nu_n$ are parameters that depend on the number of jobs of the problem but not on the processing time of the jobs. Many classical objective functions are decomposable. Trivially, $C_{\text{max}} = \sum p_r$ is decomposable (we have $\nu_r = 1$ for all $r$). Similarly, since $\sum C_i$ can be rewritten as $\sum (n + 1 - r)p_r$, then it is a decomposable objective function with $\nu_r = (n + 1 - r)$ for all $r$. Other functions are decomposable, such as for instance the total absolute difference in completion times (TADC), defined as TADC = $\sum_{i<j} |C_i - C_j|$. Indeed, it is easy to see that TADC can be rewritten as $\sum \nu_r p_r$ with $\nu_r = \sum_{j \geq r} (2j - (n + 1))$ for all $r$.

In the next section, we show that, if both the objective function $\gamma$ and the scheduling times $p(i, r)$ are decomposable, then the single-machine scheduling problem $1 \mid p(i, r) \mid \gamma$ admits an $O(n \log n)$ algorithm, that consists essentially in sorting two series of numbers. Some consequences of this result are discussed in Section 3. We then provide in Section 4 a characterization of the processing times for which an optimal schedule can be found by a sorting algorithm.

2 A general result on decomposable objective functions and position-dependent processing-times

We start by a well-known lemma of Hardy et al [6] on minimizing the scalar product of the permutation of two sequences of numbers.

Lemma 1 (Hardy et al) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sequences of numbers, and let us assume that $x_1 \leq x_2 \leq \ldots \leq x_n$. Let $\pi$ denote a per-
mutation on \{1, \ldots, n\}. Then the the minimum of \(\sum x_i y_{\pi(i)}\), taken over all permutations of \{1, \ldots, n\}, is attained for any \(\pi^*\) satisfying \(y_{\pi^*(1)} \geq y_{\pi^*(2)} \geq \ldots \geq y_{\pi^*(n)}\).

The proof of this lemma is easy, since it suffices to notice that if \(x_1 \leq x_2\) and \(y_1 \leq y_2\), then \(x_1 y_2 + x_2 y_1 \leq x_1 y_1 + x_2 y_2\). The next theorems are consequences of this result.

**Theorem 1** Let \(\gamma\) be a decomposable objective function. Then any single-machine scheduling problem \(1 \mid p(i, r) \mid \gamma\) can be modelled by an assignment problem, and thus solved in time \(O(n^3)\), where \(n\) is the number of jobs.

**Proof:** The result is immediate. Indeed, by definition, if \(\gamma\) is decomposable, then it can be written as \(\gamma = \sum r \nu_r p_r\), where \(p_r\) denotes the (actual) processing time of job scheduled in position \(r\). This can be seen as an assignment problem, where the weight from job \(i\) to position \(r\) is \(\nu_r p_r\).

If the processing times are also decomposable then we get the following stronger result.

**Theorem 2** Let \(\gamma\) be a decomposable objective function, and let us assume that the position-dependent processing times of jobs are also decomposable. Then any single-machine scheduling problem \(1 \mid p(i, r) = f(r) p_i \mid \gamma\) can be solved in time \(O(n \log n)\), where \(n\) is the number of jobs.

**Proof:** By definition, if \(\gamma\) is decomposable, then it can be written as \(\gamma = \sum r \nu_r p_r\), where \(p_r\) denotes the (actual) processing time of job scheduled in position \(r\). Let us assume that the schedule is described by a permutation \(\pi\), such that \(\pi(r) = i\) if and only if job \(i\) is scheduled in position \(r\). Now, we clearly have \(p_r = p(\pi(r), r) = f(r) p_{\pi(r)}\), such that \(\gamma = \sum \nu_r p_r = \sum \nu_r f(r) p_{\pi(r)}\). By Lemma 1, it suffices to sort the parameters \(\nu_r f(r)\) in non-decreasing order, and sort the jobs in non-increasing order of their normal processing times in order to minimize the objective function \(\gamma\). To terminate the proof it then suffices to notice that sorting two sequences of \(n\) numbers can be made in time \(O(n \log n)\).

This last theorem shows that, if both the objective function and the processing times are decomposable, an optimal schedule can be found by running a sorting algorithm. Indeed, assuming that the jobs are sorted in an SPT order (that is to say \(p_1 \leq p_2 \leq \ldots \leq p_n\)), there exists a fixed permutation \(\pi\) (that depends only on the function \(f\) and on \(\gamma\)) such that the schedule defined by “\(i\) is scheduled at rank \(r\) if and only if \(\pi(r) = i\)” is optimal. This permutation \(\pi\) is defined by \(\nu_{\pi^{-1}(1)} f(\pi^{-1}(1)) \geq \nu_{\pi^{-1}(2)} f(\pi^{-1}(2)) \geq \ldots \geq \nu_{\pi^{-1}(n)} f(\pi^{-1}(n))\).

**3 Some consequences of the general result in the decomposable case**

Theorem 2 generalizes and unifies in a single framework many results of the literature, including those described in Table 1.
Theorem 2. These results are sorted chronologically. Most of them use an interchange argument, and some explicitly use Lemma 1 (for instance [12] and [9]).

Table 1: Sample of existing results of the literature that are generalized by Theorem 2. These results are sorted chronologically. Most of them use an interchange argument, and some explicitly use Lemma 1 (for instance [12] and [9]).

<table>
<thead>
<tr>
<th>Reference</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2]</td>
<td>1 (p(i, r) = p_ir^a \mid \sum C_i ) (with (a &lt; 0))</td>
</tr>
<tr>
<td>[7]</td>
<td>1 (p(i, r) = p_ir^a \mid C_{\text{max}}) (with (a &lt; 0))</td>
</tr>
<tr>
<td>[8]</td>
<td>1 (p(i, r) = p_ir^a \mid C_{\text{max}}) (with (a &gt; 0))</td>
</tr>
<tr>
<td>[10]</td>
<td>1 (p(i, r) = p_i(M + (1 - M)r^a) \mid C_{\text{max}}) (with (a \leq 0) and (M \in [0, 1]))</td>
</tr>
<tr>
<td>[11]</td>
<td>1 (p(i, r) = p_i(a - br) \mid \sum C_i) and 1 (p(i, r) = p_i(a - br) \mid C_{\text{max}}) (with (a \geq 0) integer, (b \geq 0) rational, and (a - (n + 1)b &gt; 0))</td>
</tr>
<tr>
<td>[5]</td>
<td>1 (p(i, r) = p_i\gamma^{r-1} \mid \sum C_i) (with (\gamma \in [0, 1]\cup[2, +\infty]))</td>
</tr>
<tr>
<td>[12]</td>
<td>1 (p(i, r) = p_ir^a \mid \text{TADC}) (with (a &lt; 0))</td>
</tr>
<tr>
<td>[9]</td>
<td>1 (p(i, r) = f(r)p_i \mid C_{\text{max}}) (with (f) increasing or decreasing)</td>
</tr>
</tbody>
</table>

Mosheiov shows in [8] that there always exists a so-called “V-shaped” optimal schedule for problem 1 \(p(i, r) = p_ir^a \mid \sum C_i\) (with \(a > 0\)). Recall that a schedule is said “V-shaped” if it consists of a subset of jobs arranged in a non-increasing order of processing times, followed by the remaining jobs in non-decreasing order of their processing times. This result can also be seen as a consequence of Theorem 2. Indeed, in this case we have \(\sum C_i = \sum(n + 1 - r)p_ir^a = \sum(n + 1 - r)^a p_ir^a\). The result follows from the fact that \(g : r \mapsto g(r) = (n + 1 - r)^a\) is \(\wedge\)-shaped (that is to say it is impossible to have simultaneously \(g(r) < g(r - 1)\) and \(g(r) < g(r + 1)\) for \(1 < r < n\)).

Whereas most of the results listed in Table 1 use an interchange argument, the result of Gordon et al [5] is based on priority-generating techniques and is rather involved. This in particular explains why they get a proof only for \(\gamma \in [0, 1]\cup[2, +\infty]\). Indeed, they show that for \(\gamma \in [1, 2]\) there does not exist 1-priority functions for the problem (we refer the interested reader to [5] for the definition of priority functions). As an immediate consequence of Theorem 2, we get the following.

**Corollary 1** The problem 1 \(p(i, r) = p_i\gamma^{r-1} \mid \sum C_i\) (with \(\gamma > 0\)) admits an \(O(n \log n)\) algorithm. \(\square\)

### 4 Characterisation of sortable processing times

Let us consider a decomposable objective function \(\gamma = \sum \nu_j p_j^a\). We now show that decomposability of processing times is not necessary for the existence of a fixed permutation yielding an optimal schedule provided the processing times are sorted. Let \(\mathcal{P} \subseteq \mathbb{R}_+\) be the set of all the possible normal processing times of the jobs, and let us assume now that \(p(i, r)\) can be seen as \(f_r(p_i)\). In this framework, each \(f_r\) is a function \(f_r : \mathcal{P} \rightarrow \mathbb{R}_+\). For all \(r\), set \(g_r = \nu_r f_r\), and define \(\mathcal{G}\) as the set \(\{g_1, \ldots, g_n\}\). We say that \(g_r \succeq g_s\) if

\[
g_r(p) - g_r(q) \geq g_s(p) - g_s(q) \quad \forall (p,q) \in \mathcal{P}^2 \text{ with } p \geq q
\]

(1)

Clearly, \(\succeq\) defines a preorder on \(\mathcal{G}\) (that is to say, \(\succeq\) is reflexive and transitive). We say that \(g_r\) and \(g_s\) are comparable if either \(g_r \succeq g_s\) or \(g_s \succeq g_r\).
The justification of this definition of comparability of processing times is two-fold. One is Theorem 3 below stating somehow that comparability is the essential property for the double-sorting algorithm of Theorem 2 to work. The other is the variety of examples of comparable processing times, including for instance if \( \gamma = C_{\max}^r \\
\).

- \( f_r(p_i) := k_r p_i \) for \( \mathcal{P} \subseteq \mathbb{R}_+ \) and \( k_r \in \mathbb{R}_+ \)
- \( f_r(p_i) := p_i^{k_r} \) for \( \mathcal{P} \subseteq [1, +\infty] \) and \( k_r \in \mathbb{R}_+ \)
- \( f_r(p_i) := k_r p_i^{\gamma} \) for \( \mathcal{P} \subseteq \mathbb{R}_+ \) and \( k_r \in \mathbb{R}_+ \)

Note also that \( f_r \geq f_r' \) and \( f_r' \geq f_r'' \) implies \( f_r + f_r' \geq f_r + f_r'' \).

The following theorem states that, provided the processing times are sorted of \( n \) admissible processing times, and \( f_1, \ldots, f_n \) be any number of jobs,

**Theorem 3** Let \( \gamma = \sum f_r p_i \) be a decomposable objective function. Let \( n \) be any number of jobs, \( \mathcal{P} \) be any set of admissible processing times, and \( f_1, \ldots, f_n \) be \( n \) functions from \( \mathcal{P} \) to \( \mathbb{R}_+ \). Let \( \mathcal{G} = \{ g_r \mid 1 \leq r \leq n \} \), where for all \( r \) the function \( g_r \) is defined as \( f_r f_r \). Then \( (\mathcal{G}, \preceq) \) is a totally preordered set if and only if there exists a permutation \( \pi \) on \( \{1, \ldots, n\} \) such that, for any instance \( (p_1, \ldots, p_n) \in \mathcal{P}^n \) of \( f_r(p_i) \mid \gamma \) such that \( p_1 \leq p_2 \leq \ldots \leq p_n \), assigning job \( i \) to rank \( r \) if and only if \( \pi(r) = i \) leads to an optimal schedule.

**Proof:** Let us first assume that \( (\mathcal{G}, \preceq) \) is a totally preordered set, that is to say the functions \( g_r \) are all pairwise comparable. In this case, there exists a permutation \( \pi \) such that \( g_{\pi^{-1}(1)} \geq g_{\pi^{-1}(2)} \geq \ldots \geq g_{\pi^{-1}(n)} \). Consider now any instance \( (p_1, \ldots, p_n) \in \mathcal{P}^n \) of \( f_r(p_i) \mid \gamma \) such that \( p_1 \leq p_2 \leq \ldots \leq p_n \). Then an interchange argument shows that assigning job \( i \) to rank \( r \) if and only if \( \pi(r) = i \) leads to an optimal schedule. Indeed, if two jobs \( i \) and \( j \) such that \( p_i \leq p_j \) are scheduled \( i \) at a rank \( r \), and \( j \) at a rank \( s \), with \( g_i \geq g_r \), then exchanging jobs \( i \) and \( j \) can only improve \( \gamma \).

Now let us assume that there exist two functions \( g_r \) and \( g_s \) that are not comparable. This implies that there exist \( p \geq q \) and \( p' \geq q' \) such that \( g_r(p) - g_r(q) > g_s(p) - g_s(q) \) and \( g_r(p') - g_r(q') < g_s(p') - g_s(q') \). As a consequence, there can not exist a fixed permutation \( \pi \) on \( \{1, \ldots, n\} \) such that, for any instance \( (p_1, \ldots, p_n) \in \mathcal{P}^n \) of \( f_r(p_i) \mid \gamma \) such that \( p_1 \leq p_2 \leq \ldots \leq p_n \), assigning job \( i \) to rank \( r \) if and only if \( \pi(r) = i \) leads to an optimal schedule. Indeed, let us consider one instance such that \( p_1 = q \) and \( p_2 = p \), and another instance such that \( p_1 = q' \) and \( p_2 = p' \). Since \( g_r(p) - g_r(q) > g_s(p) - g_s(q) \) and \( g_r(p') - g_r(q') < g_s(p') - g_s(q') \), then for one of these instances job 1 is scheduled before job 2 in all optimal schedules, and for the other one job 2 is scheduled before job 1 in all optimal schedules.

\[ \square \]

## 5 Conclusion

In this note we presented general results on decomposable objective functions and position-dependent processing-times. These results cover in particular the
classical objective functions $C_{\text{max}}$, $\sum C_i$, and TADC. They also generalize several existing results of the literature. In particular, Theorem 2 states that any problem of the form $1 \mid p(i, r) = f(r)p_i \mid \gamma$ can be optimally solved by a sorting algorithm if $\gamma$ is decomposable. This theorem simplifies a result of Gordon et al on a single-machine scheduling problem with exponential position-dependent processing times [5], and enables one to extend this result.

Furthermore, Theorem 3 provides a characterization of processing times for which there exists a sorting algorithm that optimally solves any problem of the form $1 \mid p(i, r) = f_r(p_i) \mid \gamma$ in the case where $\gamma$ is decomposable. This result uses a notion of comparability between functions, which in some sense is an extension of so-called Monge properties for matrices (see e.g. the survey [3]).

References


