Wavelet Packets of fractional Brownian motion: Asymptotic Analysis and Spectrum Estimation

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Abstract

This work provides asymptotic properties of the autocorrelation functions of the wavelet packet coefficients of a fractional Brownian motion. It also discusses the convergence speed to the limit autocorrelation function, when the input random process is either a fractional Brownian motion or a wide-sense stationary second-order random process. The analysis concerns some families of wavelet paraunitary filters that converge almost everywhere to the Shannon paraunitary filters. From this analysis, we derive wavelet packet based spectrum estimation for fractional Brownian motions and wide-sense stationary random processes. Experimental tests show good results for estimating the spectrum of $1/f$ processes.

Index Terms

Wavelet packet transforms, Fractional Brownian motion, Gray code, Spectral analysis.

I. Introduction

Wavelet and wavelet packet analysis of stochastic processes have gained much interest in the last two decades, since the earlier works of [1], [2], [3], [4], [5]. Concerning the correlation structure of the wavelet coefficients, and according to the nature of the input random process, one can distinguish,

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first, some results [6], [7], [8], [9], [10], [11], [12], [13], [14] dedicated to the wavelet transform of certain non-stationary processes such as processes with stationary increments and fractionally differenced processes. These references highlight that wavelet coefficients tend to be decorrelated provided that the decomposition level tends to infinity and the decomposition filters satisfy suitable properties. Second, results of the same order holds true for stationary random processes as shown in [15], [16], [17].

In [17], one can find an attempt for the generalization of the decorrelation properties to the case of the wavelet packet transform, when the input random process is stationary. On the basis of the framework of [17], [18] proposes an extension to the case of the dual-tree wavelet packet transform. The results stated in [17] and [18] stipulate that for stationary random processes, the limit autocorrelation functions of the wavelet packet coefficients do not depend on the wavelet packet path and the decomposition filters considered.

However, by using certain families of wavelet filters, it is shown in [19] that the limit autocorrelation functions of the wavelet packet coefficients of band-limited wide-sense stationary random process still depend on the path followed in the wavelet packet decomposition tree. The decomposition considered in [19] is performed by using certain paraunitary filters that converge almost everywhere to the Shannon filters (Daubechies and Battle-Lemarié filters are examples of such families of filters). In fact, the dependency of the decorrelation process and the wavelet filters has been highlighted earlier by [20] and this dependency also appears in [14] which discusses the decorrelation rate for the standard wavelet packet decomposition, when the Daubechies filters are used.

More precisely, [21] shows that the results presented in [17] and [18] concern only one path of the wavelet packet decomposition tree, that is the approximation path of the standard wavelet transform. The analysis of the limit autocorrelation functions cannot be performed independently of the type of the decomposition filters or, equivalently, on the type of mother wavelet used because for the wavelet packet decomposition, the shift parameter depends on the decomposition level and cannot be upper-bounded, so that convergence criterion such as the Lebesgue's dominated convergence theorem cannot easily apply (see [21]).

This paper first extends the results of [19] when the input random process for the wavelet packet decomposition is not constrained to be band-limited. The paper also provides, as a main contribution, the asymptotic autocorrelation functions of the wavelet packet coefficients for fractional Brownian motions. We use the same formalism as that of [19]. The results obtained complete those of [6], [7], [8], [9], [12] which are dedicated to the standard wavelet transform of a fractional Brownian motion.

The paper is organized as follows. In Section III the asymptotic properties of the autocorrelation functions of the wavelet packet coefficients of stationary random processes and fractional Brownian motions are discussed. Section IV addresses the convergence speed of the decorrelation process in order to evaluate how well we can approach the limit autocorrelation function of the wavelet packet coefficients. This convergence speed informs us whether we can obtain, in practice, a good convergence rate at finite decomposition levels. As a consequence of the theoretical results obtained in Sections III and IV, Section V discusses wavelet packet based spectrum estimation, by using suitable decomposition filters. Finally, Section VI concludes this work. The next section provides definitions and basic material used in the paper (see [19], [22], [23] for further details).

II. Basics on wavelet packets

Let $\Phi \in L^2(\mathbb{R})$ and $\mathbf{U}$ be closure of the space spanned by the translated versions of Φ:

$$\mathbf{U} = \text{Closure}\langle \tau_k \Phi : k \in \mathbb{Z} \rangle.$$

The wavelet packet decomposition of $\mathbf{U}$ is obtained by recursively splitting the space $\mathbf{U}$ into orthogonal subspaces, $\mathbf{U} = \mathbf{W}_{1,0} \oplus \mathbf{W}_{1,1}$ and $\mathbf{W}_{j,n} = \mathbf{W}_{j+1,2n} \oplus \mathbf{W}_{j+1,2n+1}$, where $\mathbf{W}_{j,n} \subset \mathbf{U}$ is defined by

$$\mathbf{W}_{j,n} = \text{Closure}\langle W_{j,n,k} : k \in \mathbb{Z} \rangle,$$

and $\{W_{j,n,k} : k \in \mathbb{Z}\}$ is the orthonormal set of the *wavelet packet functions*. In this decomposition, any $W_{j,n,k}$ is defined by

$$\begin{aligned} W_{j,n,k}(t) &= \tau_{2^j k} W_{j,n}(t) \\ &= \tau_{2^j k} \left(2^{-j/2} W_n(2^{-j} t) \right) \\ &= 2^{-j/2} W_n(2^{-j} t - k), \end{aligned} \tag{1}$$

and the sequence $(W_n)_{n \geqslant 0}$ is computed recursively from Φ and some *paraunitary filters* $(H_\epsilon)_{\epsilon=0,1}$ with impulse responses $(h_\epsilon)_{\epsilon=0,1}$ (see [19], [23] for details).

In this paper, we assume that Φ is the *scaling function* associated with the low-pass filter H_0 so that $W_0 = \Phi$ ([22], [23]). The decomposition space $\mathbf{U}$ is then the space generated by the translated versions of the scaling function. The recursive splitting of $\mathbf{U}$ yields a *wavelet packet tree* composed of the subspaces $\mathbf{W}_{j,n}$, where j is the decomposition (or resolution) level and n is the shift parameter. For a given path $\mathcal{P} = (\mathbf{U}, \{\mathbf{W}_{j,n}\}_{j \in \mathbb{N}})$ in the wavelet packet decomposition tree, the shift parameter $n = n_{\mathcal{P}}(j) \in \{0, \ldots, 2^j - 1\}$ is such that $n_{\mathcal{P}}(0) = 0$ and

$$n_{\mathcal{P}}(j) = 2 n_{\mathcal{P}}(j-1) + \epsilon_j = \sum_{\ell=1}^{j} \epsilon_\ell 2^{j-\ell}, \tag{2}$$

where $\epsilon_\ell \in \{0,1\}$, ϵ_ℓ indicates that filter H_{ϵ_ℓ} is used at the decomposition level ℓ, with $\ell \geqslant 1$ (see [19] for details on paths and shift parameter characterization).

Consider a real-valued centered second-order random process X assumed to be continuous in quadratic mean. The projection of X on a *wavelet packet space* $\mathbf{W}_{j,n}$ yields coefficients that define a discrete random process $c_{j,n} = (c_{j,n}[k])_{k\in\mathbb{Z}}$. We have, with convergence in the quadratic mean sense :

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t) W_{j,n,k}(t) dt. \tag{3}$$

In what follows, we are concerned by a family of scaling functions $(\Phi^{[r]})_r$ that satisfy almost everywhere (a.e.) the following property

$$\lim_{r\to\infty} \mathcal{F}\Phi^{[r]} = \mathcal{F}\Phi^{\mathsf{S}} \quad \text{(a.e.)}, \tag{4}$$

where $\Phi^{\mathsf{S}}(t) = \sin(\pi t)/\pi t$ is the Shannon scaling function. The Fourier transform of Φ^{S} is

$$\mathcal{F}\Phi^{\mathsf{S}} = 1\!\!1_{[-\pi,\pi]}, \tag{5}$$

where $1\!\!1_\Delta$ denotes the indicator function of a given set Δ ($1\!\!1_\Delta(x) = 1$ if $x \in \Delta$ and $1\!\!1_\Delta(x) = 0$, otherwise).

The Daubechies and spline Battle-Lemarié scaling functions satisfy Eq. (4). The parameter r, hereafter called *order*, is the number of vanishing moments of the wavelet function for the Daubechies functions [24] and this parameter is the order of the spline scaling function for the Battle-Lemarié functions [25], [26]. The decomposition filters $(H_\epsilon^{[r]})_{\epsilon\in\{0,1\}}$ associated with these functions satisfy (see [24], [25], [26]):

$$\lim_{r\to\infty} H_\epsilon^{[r]} = H_\epsilon^{\mathsf{S}} \quad \text{(a.e.)}. \tag{6}$$

where $(H_\epsilon^{\mathsf{S}})_{\epsilon\in\{0,1\}}$ are the ideal low-pass and high-pass Shannon filters. In the rest of the paper, we assume that $H_\epsilon^{[r]}$ for $\epsilon \in \{0,1\}$ are with finite impulse responses. This holds true for the Daubechies and Battle-Lemarié paraunitary filters. It the follows that:

Remark 1: The wavelet packet function $W_{j,n,k}^{[r]}$ is obtained by a recursive decomposition involving the wavelet function $W_1^{[r]}$: $W_{j,n,k}^{[r]}(t) = 2^{-j/2} W_n^{[r]}(2^{-j}t - k)$ where $W_n^{[r]}$ is defined for $\epsilon = 0,1$ by $W_{2n+\epsilon}^{[r]}(t) = \sqrt{2}\sum_{\ell\in\mathcal{I}} h_\epsilon^{[r]}[\ell] W_n^{[r]}(2t-\ell)$ for every $n \geqslant 1$, $\mathcal{I}$ being a set of finite cardinality (because we assume that the wavelet paraunitary filters with finite impulse responses).

The remark above will prove useful in the sequel. When the Shannon paraunitary ideal filters H_0^{S} (low-pass) and H_1^{S} (high-pass) are used, then the Fourier transform of a wavelet packet function $W_{j,n}^{\mathsf{S}}$ is (see [23], among others)

$$\mathcal{F}W_{j,n}^{\mathsf{S}} = 2^{j/2} 1\!\!1_{\Delta_{j,G(n)}}. \tag{7}$$

The set $\Delta_{j,G(n)}$ is such that $\Delta_{j,G(n)} = \Delta^-_{j,G(n)} \cup \Delta^+_{j,G(n)}$, where $\Delta^-_{j,G(n)}$ and $\Delta^+_{j,G(n)}$ are symmetrical with respect to the origin, and (see [19], [23], [27])

$$\Delta^+_{j,G(n)} = \left[\frac{G(n)\pi}{2^j}, \frac{(G(n)+1)\pi}{2^j}\right], \tag{8}$$

with

$$G(2\ell+\epsilon) = \begin{cases} 2G(\ell)+\epsilon & \text{if} \quad G(\ell) \quad \text{is even,} \\ 2G(\ell)-\epsilon+1 & \text{if} \quad G(\ell) \quad \text{is odd.} \end{cases} \tag{9}$$

The decomposition space $\mathbf{U} = \mathbf{U}^{\mathsf{S}}$ is then the π-band-limited *Paley-Wiener space*, that is the space generated by the translated versions of the Shannon scaling function Φ^{S}. The Shannon wavelet packet tree and the frequency re-ordering induced by the permutation G are represented in figure 1.

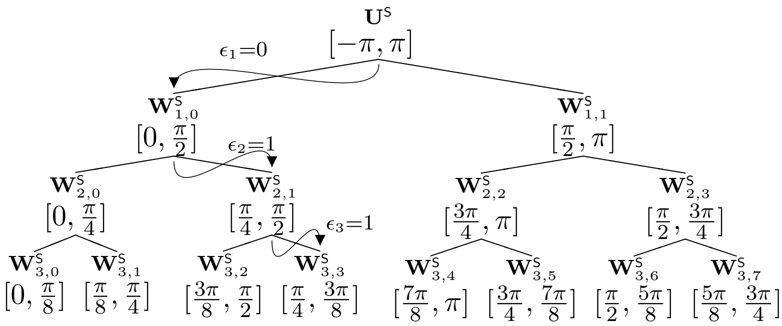

Fig. 1. Shannon wavelet packet decomposition tree. The positive part of the support of $\mathcal{F}W^{\mathsf{S}}_{j,n}$ is indicated below each node $\mathbf{W}^{\mathsf{S}}_{j,n}$. The wavelet packets associated with the sequence $(\epsilon_1, \epsilon_2, \epsilon_3) = (0, 1, 1)$ define a path $(\mathbf{U}^{\mathsf{S}}, \mathbf{W}^{\mathsf{S}}_{j,n})_{j=1,2,3}$. We have $\epsilon_j = 0$ (resp. $\epsilon_j = 1$) if the low-pass (resp. high-pass) filter is used for computing the wavelet packets of decomposition level j. The wavelet packet $\mathbf{W}^{\mathsf{S}}_{3,n(3)}$ of this path is such that $n(3) = \epsilon_3 2^0 + \epsilon_2 2^1 + \epsilon_1 2^2 = 3$ and the positive part of the support of $\mathbf{W}^{\mathsf{S}}_{3,n(3)}$ is $\Delta^+_{j,G(n(3))}$ with $G(n(3)) = 4$.

From now on, an upper index S (resp. $[r]$) will be used, when necessary, to emphasize that the decomposition is achieved by using filters $(H^{\mathsf{S}}_\epsilon)_{\epsilon\in\{0,1\}}$ (resp. $(H^{[r]}_\epsilon)_{\epsilon\in\{0,1\}}$).

III. Asymptotic analysis

A. Asymptotic analysis of the autocorrelation functions

Let $\mathcal{P}$ be a path of the wavelet packet decomposition tree. From the description given in Section II, $\mathcal{P}$ is characterized by a sequence of nodes $(j,n)_{j\geqslant 1}$, where $n = n_{\mathcal{P}}(j)$ is given by Eq. (2) at every decomposition level j. Let $\omega_{\mathcal{P}}$, $0 \leqslant \omega_{\mathcal{P}} \leqslant \pi$, be the value such that (see [19] for the existence of this limit)

$$\omega_{\mathcal{P}} = \lim_{j\to+\infty} \frac{G(n_{\mathcal{P}}(j))\pi}{2^j}. \tag{10}$$

Assume that the input second-order random process X is a wide-sense stationary with spectrum (power spectral density) $\gamma \in L^{\infty}(\mathbb{R})$. Then, the discrete random process $c_{j,n}$ defined by Eq. (3) is wide-sense stationary and its autocorrelation function is (see [17], [19])

$$R_{j,n}[m] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\omega) |\mathcal{F}W_{j,n}(\omega)|^2 e^{i2^j m\omega} d\omega. \tag{11}$$

When j increases, the behavior of the autocorrelation function $R_{j,n}$ depends on the wavelet packet path and the paraunitary filters used to decompose X. More precisely, we have:

Theorem 1: Consider a real-valued centered second-order random process X assumed to be continuous in quadratic mean. Assume that X is wide-sense stationary with spectrum $\gamma \in L^{\infty}(\mathbb{R})$. We have

(i) The autocorrelation function $R^{\mathsf{S}}_{j,n}$ is

$$R^{\mathsf{S}}_{j,n}[m] = \frac{2^j}{\pi} \int_{\Delta^{+}_{j,G(n)}} \gamma(\omega) \cos(2^j m\omega) d\omega. \tag{12}$$

(ii) If γ is continuous at $\omega_{\mathcal{P}}$ given by Eq. (10), then we have, uniformly in $m \in \mathbb{Z}$

$$\lim_{j \to +\infty} R^{\mathsf{S}}_{j,n}[m] = \gamma(\omega_{\mathcal{P}}) \delta[m], \tag{13}$$

where $\delta[\cdot]$ is the Kronecker symbol defined for every integer $k \in \mathbb{Z}$ by

$$\delta[k] = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

(iii) The autocorrelation function $R^{[r]}_{j,n}$ satisfies

$$\lim_{r \to +\infty} R^{[r]}_{j,n}[m] = R^{\mathsf{S}}_{j,n}[m]. \tag{14}$$

Proof: Easy extension of [19, Theorem 1]. In this reference, the decomposition space is the π-band-limited *Paley-Wiener space* and the spectrum γ of X is assumed to be supported in $[-\pi, \pi]$. These assumptions are relaxed here by considering the projection of X on the space generated by the translated versions of the scaling function associated with the decomposition filters used. ■

Now, assume that X is a centered fractional Brownian motion with Hurst parameter α. We assume that $0 < \alpha < 1$, and that the path considered in the wavelet packet tree is $\mathcal{P} \neq \mathcal{P}_0$, where $\mathcal{P}_0$ is the path located at the far left hand side of the wavelet packet tree. Path $\mathcal{P}_0$ corresponds to the standard wavelet approximation path since the low-pass filter is used at every resolution level. For path $\mathcal{P}_0$, there is no convergence for the limit integrals involved in the computation of the wavelet packet coefficients, with respect to the wavelet packet functions considered in this work. In addition, the cases $\alpha = 0$ and $\alpha = 1$

are irrelevant here because $\alpha = 0$ corresponds to a white Gaussian process and the spectral densities of the wavelet packet coefficients are not $L^1(\mathbb{R})$ for $\alpha = 1$.

Let $R(t, s)$ stands for the autocorrelation function of X. We have

$$R(t,s) = \mathbb{E}[X(t)X(s)] = \frac{\sigma^2}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right). \quad (15)$$

Theorem 2 below requires assumptions (**A1-A3**) used in [12] to prove the existence of the spectral density of the wavelet transform of a fractional Brownian motion.

Theorem 2: Assume that the wavelet paraunitary filters $(H_0^{[r]}, H_1^{[r]})$ are with finite impulse responses and that there exists some finite order r_0 such that for every $r \geqslant r_0$, the *wavelet function* $W_1^{[r]}$ satisfy the following assumptions:

(**A1**) $(1+t^2)W_1^{[r]}(t) \in L^1(\mathbb{R})$,

(**A2**) $\int_{\mathbb{R}} W_1^{[r]}(t) = 0$,

(**A3**) $\sup_{|\omega|\leqslant\eta} \left|\mathcal{F}W_1^{[r]}(\omega)/\omega\right| < \infty$ for some $\eta > 0$.

Then, the discrete random process $c_{j,n}^{[r]}$, $n \geqslant 1$, obtained from the projection of the fractional Brownian motion X on the wavelet packet $\mathbf{W}_{j,n}^{[r]}$ is wide-sense stationary and its autocorrelation function is

$$R_{j,n}^{[r]}[m] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma_\alpha(\omega)|\mathcal{F}W_{j,n}^{[r]}(\omega)|^2 e^{i2^j m\omega} d\omega, \quad (16)$$

with

$$\gamma_\alpha(\omega) = \frac{\sigma^2\Gamma(2\alpha+1)\sin(\pi\alpha)}{|\omega|^{2\alpha+1}}, \quad (17)$$

where $\Delta_{j,G(n)}^{+}$ is given by Eq. (8) and Γ is the standard Gamma function.

Proof: Theorem 2 is a consequence of [12, Theorem 1]. In order to apply [12, Theorem 1] for the wavelet packet functions, we need to show that every $W_{j,n,k}^{[r]}$, $j \geqslant 1$ and $n \in \{1, 2, \ldots, 2^j - 1\}$, satisfy assumptions (**A1**), (**A2**) and (**A3**); which simply follows from remark 1. Appendix A summarizes the steps involved in the proof. ■

Remark 2: Under assumption (**A3**), the integral in Eq. (16) is absolutely convergent for every pair (j, n) with $n \neq 0$. Thus, from the Bochner's theorem, we derive that, for a given $j \geqslant 1$ and $n \in \{1, 2, \ldots, 2^j-1\}$, the spectral density of the wavelet packet coefficients $c_{j,n}^{[r]}$ of the fractional Brownian motion X is:

$$\gamma_{j,n}^{[r]}(\omega) = \frac{1}{2\pi}\gamma_\alpha(\omega)|\mathcal{F}W_{j,n}^{[r]}(\omega)|^2.$$

By taking the Fourier transform of Eq. (1), we have $\mathcal{F}W_{j,n}^{[r]}(\omega) = 2^{j/2}\mathcal{F}W_n^{[r]}(2^j\omega)$. Thus, we have

$$\gamma_{j,n}^{[r]}(\omega) = \frac{2^{j-1}}{\pi}\gamma_\alpha(\omega)|\mathcal{F}W_n^{[r]}(2^j\omega)|^2, \tag{18}$$

where (see [19, Lemma 1])

$$\mathcal{F}W_n^{[r]}(\omega) = \left[\prod_{\ell=1}^{j} H_{\epsilon_\ell}^{[r]}(\frac{\omega}{2^{j+1-\ell}})\right]\mathcal{F}\Phi^{[r]}(\frac{\omega}{2^j}), \tag{19}$$

the sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j)$ being the binary sequence associated with the shift parameter n, with n of the form Eq. (2).

Remark 3: Note that assumption **(A1)** is not satisfied for the Shannon wavelet $W_1^{\mathsf{S}}(t)$ defined by

$$W_1^{\mathsf{S}}(t) = 2W_0^{\mathsf{S}}(2t) - W_0^{\mathsf{S}}(t), \tag{20}$$

where $W_0^{\mathsf{S}}(t) = \Phi^{\mathsf{S}}(t) = \sin(\pi t)/\pi t$. Thus, Theorem 2 does apply in order to obtain the analytic form of the spectral density of the Shannon wavelet packet coefficients of X.

Theorem 3: With the same assumptions as in Theorem 2 above, and under assumption:

(A4) there exists some positive function $g \in L^1(\mathbb{R})$ that dominates the sequence $(|\mathcal{F}W_1^{[r]}|^2)_r$ and satisfy: $\sup_{|\omega|\leqslant\eta} g(\omega)/|\omega|^2 < \infty$ for some $\eta > 0$.

The autocorrelation functions of the wavelet packet coefficients of the fractional Brownian motion X satisfy

(i)

$$\lim_{r\to+\infty} R_{j,n}^{[r]}[m] = \frac{2^j}{\pi}\int_{\Delta_{j,G(n)}^+}\gamma_\alpha(\omega)\cos(2^j m\omega)d\omega \triangleq R_{j,n}^{\mathsf{S}}[m] \tag{21}$$

where $\Delta_{j,G(n)}^+$ is given by Eq. (8).

(ii)

$$\lim_{j\to+\infty} R_{j,n}^{S}[m] = \gamma_\alpha(\omega_{\mathcal{P}})\delta[m], \tag{22}$$

where $R_{j,n}^S$ is defined by Eq. (21) with γ_α given by Eq. (17).

Remark 4: As highlighted by remark 3, Theorem 2 does not apply in order to obtain the analytic form of the autocorrelation function $R_{j,n}^S$, $n \neq 0$, for the wavelet packet coefficients of a fractional Brownian motion. The above definition of $R_{j,n}^S$ (second equality in Eq. (21)) shows that results similar to those of

Theorem 2 still hold for the Shannon wavelet packets so that, from Eq. (21), we can define the spectral density of the Shannon wavelet packet coefficients of a fractional Brownian motion as

$$\begin{aligned}\gamma^{\mathsf{S}}_{j,n}(\omega) &= \frac{2^{j-1}}{\pi}\gamma_\alpha(\omega)\mathbb{1}_{\Delta_{j,G(n)}}(2^j\omega), \\ &= \frac{1}{2\pi}\gamma_\alpha(\omega)|\mathcal{F}W^{\mathsf{S}}_{j,n}(\omega)|^2, \end{aligned} \quad (23)$$

where $\mathcal{F}W^{\mathsf{S}}_{j,n}(\omega)$ is given by Eq. (7); with $\gamma^{\mathsf{S}}_{j,n}(0) = 0$ since 0 does not belong to $\Delta_{j,G(n)}$ when $n \neq 0$.

Proof: (of Theorem 3).

***Proof of statement* (i):**

By taking into account [19, Lemma 1], and if $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j)$ is the binary sequence associated with the shift parameter n; that is: if n is of the form Eq. (2), then we have $\mathcal{F}W^{[r]}_{j,n}(\omega) = 2^{j/2}\mathcal{F}W^{[r]}_n(2^j\omega)$, with $\mathcal{F}W^{[r]}_n$ given by Eq. (19). Thus, by taking into account Eqs. (4) and (6), we have that $|\mathcal{F}W^{[r]}_{j,n}|^2$ converges almost everywhere to $|\mathcal{F}W^{S}_{j,n}|^2$ when r tends to infinity.

Since $|H^{[r]}_{\epsilon_\ell}(\omega)| \leqslant 1$ for all $\ell = 1, 2, \ldots, j$, and because we assume $n \neq 0$, we have also from Eq. (19) that $|\mathcal{F}W^{[r]}_{j,n}(\omega)| \leqslant 2^{j/2}|\mathcal{F}W^{[r]}_1(2\omega)|$. Thus, we have

$$\gamma_\alpha(\omega)|\mathcal{F}W^{[r]}_{j,n}(\omega)|^2 \leqslant 2^j\gamma_\alpha(\omega)|\mathcal{F}W^{[r]}_1(2\omega)|^2,$$

and by taking into account assumption (**A4**), we have that $\gamma_\alpha(\omega)|\mathcal{F}W^{[r]}_{j,n}(\omega)|^2$ is dominated by the function $f(\omega) = 2^j\gamma_\alpha(\omega)g(2\omega)$ which does not depends on r. Moreover, the function f is integrable: indeed, by setting $K_1 = 2^j\sigma^2\Gamma(2\alpha+1)\sin(\pi\alpha)$, we have

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(\omega)}{K_1}d\omega &= \int_{\mathbb{R}} \frac{g(2\omega)}{|\omega|^{2\alpha+1}}d\omega \\ &\leqslant \int_{|\omega|\leqslant\eta} \frac{K_2}{|\omega|^{2\alpha-1}}d\omega + \frac{1}{\eta^{2\alpha+1}}\int_{|\omega|\geqslant\eta} g(2\omega)d\omega \\ &< \infty \end{aligned} \quad (24)$$

for every α, $0 < \alpha < 1$, and where K_2 is a constant such that $\sup_{|\omega|\leqslant\eta}\left(g(2\omega)/|\omega|^2\right) < K_2$; the existence of K_2 and η being guaranteed by the assumption (**A4**).

It follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}&\lim_{r\to+\infty} R^{[r]}_{j,n}[m] \\ &= \lim_{r\to+\infty}\left(\frac{1}{2\pi}\int_{\mathbb{R}}\gamma_\alpha(\omega)|\mathcal{F}W^{[r]}_{j,n}(\omega)|^2 e^{i2^j(k-\ell)\omega}d\omega\right) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\gamma_\alpha(\omega)|\mathcal{F}W^{S}_{j,n}(\omega)|^2 e^{i2^j(k-\ell)\omega}d\omega. \end{aligned} \quad (25)$$

Statement (i) derives from Eq. (25), after some straightforward calculations by taking into account that $\mathcal{F}W_{j,n}^{S}$ is given by Eq. (7). One can easily check that integral in Eq. (25) is absolutely convergent for every pair (j, n) with $n \neq 0$, because $|\mathcal{F}W_{j,n}^{\mathsf{S}}(\omega)|$ is compactly supported and 0 does not belong to its support (see Eq. (7)).

***Proof of* (ii):** Statement (ii) simply derives from Lemma 2 given in appendix B: if $\mathcal{P} \neq \mathcal{P}_0$, then $\omega_{\mathcal{P}} \neq 0$, $0 \notin \Delta_{j,G(n)}^{+}$ (which moreover is a closed set), and the function $1/|\omega|^{2\alpha+1}$ is integrable on $\Delta_{j,G(n)}^{+}$ and is continuous at $\omega_{\mathcal{P}}$. ■

From Theorems 2 and 3, we have that $c_{j,n}^{[r]}$ is wide-sense stationary and tend to be decorrelated when both r and j tend to infinity, with variance $\gamma_\alpha(\omega_{\mathcal{P}})$ in path $\mathcal{P} \neq \mathcal{P}_0$ of the wavelet packet decomposition tree. The following highlights that the Daubechies and the spline Battle-Lemarié wavelet families satisfy assumptions of Theorems 2 and 3.

The Fourier transform of a Daubechies or a Battle-Lemarié wavelet $W_1^{[r]}$ of order r has the following form.

$$\mathcal{F}W_1^{[r]}(\omega) = H_1^{[r]}(\omega/2)\mathcal{F}\Phi^{[r]}(\omega/2), \tag{26}$$

where $\Phi^{[r]}$ denotes a scaling function and $H_1^{[r]}$ the associated wavelet filter.

B. Properties of the Daubechies and the spline Battle-Lemarié functions

The following proves that the Daubechies and spline Battle-Lemarié functions satisfy assumptions (**A1**-**A4**) of Theorems 2 and 3. Note that all the Daubechies and Battle-Lemarié wavelet functions satisfy assumption (**A2**) by construction (null moments condition, see [22], [23]). In addition, since the Daubechies wavelet functions are bounded with compact support [22], they satisfy assumption (**A1**). The Battle-Lemarié wavelet functions satisfy assumption (**A1**) as well because these functions are bounded and have exponential decays [22, Corollary 5.4.2]. Since assumption (**A4**) implies (**A3**), it suffices now to check that assumption (**A4**) holds true for the sequences of Daubechies and Battle-Lemarié wavelet functions.

1) The family of Daubechies wavelet functions satisfies assumption (A4): More precisely, we have

Proposition 1: The Daubechies wavelet functions $(W_1^{[r]})_r$ are such that

$$|\mathcal{F}W_1^{[r]}(2\omega)|^2 \leqslant K\left(\left|\sin\frac{\omega}{4}\right|^2 \mathbb{1}_{\{|\omega|\leqslant\eta\}} + \frac{1}{|\omega|^2}\mathbb{1}_{\{|\omega|>\eta\}}\right) \tag{27}$$

for any η such $0 < \eta \leqslant 2\pi/3$, where $K > 0$ is a constant independent of r.

Proof: The Fourier transform of Daubechies wavelet function $W_1^{[r]}$ of order r is of the form Eq. (26).

We have from [22, Lemmas 7.1.7 and 7.1.8] that:

$$|\mathcal{F}\Phi^{[r]}(\omega)| \leqslant \frac{C}{(1+|\omega|)^{r-r\frac{\log(3)}{\log(2)}+\frac{\log(3)}{\log(2)}}}, \tag{28}$$

for every $r = 1, 2, \ldots$, and thus, we derive

$$|\mathcal{F}\Phi^{[r]}(\omega)|^2 \leqslant \frac{C^2}{(1+|\omega|)^2}. \tag{29}$$

On the other hand, the Daubechies wavelet filter $H_1^{[r]}$ is defined by

$$H_1^{[r]}(\omega) = e^{-i\omega/2}\left(\frac{1-e^{i\omega/2}}{2}\right)^r P_r(\omega), \tag{30}$$

where P_r is a trigonometric polynomial (see [22], [23] for more details). From [22, Lemmas 7.1.3 and 7.1.4], we have that $\sup_\omega |P_r(\omega)| \leqslant 2^{r-1}$. Thus, we get

$$|H_1^{[r]}(\omega)| \leqslant \frac{\left|1-e^{i\omega/2}\right|^r}{2} \leqslant 2^{r-1}\left|\sin\frac{\omega}{4}\right|^r. \tag{31}$$

It follows that $|H_1^{[r]}(\omega)| \leqslant |\sin(\omega/4)|$ for $|\omega| \leqslant 2\pi/3$ and the result derives by taking into account Eqs. (26) and (29), with $K = C^2$.

■

*2) The family of Battle-Lemarié wavelet functions satisfies assumption (**A4**):* The Battle-Lemarié scaling and wavelet functions are computed from the normalized central B-spline of order r. The Fourier transform of its associated wavelet function is of the form Eq. (26) with (see [23], [28], [29])

$$H_1^{[r]}(\omega) = e^{-i\omega/2}|\sin(\omega/2)|^r\sqrt{\frac{\Theta_r(\omega+\pi)}{\Theta_r(2\omega)}} \tag{32}$$

and

$$|\mathcal{F}\Phi^{[r]}(\omega)| = \frac{1}{|\omega|^r}\frac{1}{\sqrt{\sum_{k\in\mathbb{Z}}\frac{1}{(\omega+2k\pi)^{2r}}}}, \tag{33}$$

or, equivalently,

$$|\mathcal{F}\Phi^{[r]}(\omega)| = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^r \Big/ \sqrt{\Theta_r(\omega)}, \tag{34}$$

where

$$\Theta_r(\omega) = \sum_{k\in\mathbb{Z}}\left|\frac{\sin(\omega/2+k\pi)}{\omega/2+k\pi}\right|^{2r} \tag{35}$$

$$= \left(\cos\frac{\omega}{4}\right)^{2r}\Theta_r(\frac{\omega}{2}) + \left(\sin\frac{\omega}{4}\right)^{2r}\Theta_r(\frac{\omega}{2}+\pi). \tag{36}$$

Lemma 1: For every $r = 1, 2, \ldots$, the function $H_1^{[r]}$ defined by (32) satisfy

$$\sup_{|\omega| \leqslant \pi/2} |H_1^{[r]}(\omega)/\omega| \leqslant 1/\sqrt{2}. \tag{37}$$

Proof: If $|\omega| \leqslant \pi/2$, then (see [25]) we have $\Theta_r(\omega + \pi) \leqslant \Theta_r(\omega)$, and thus

$$\begin{aligned} \frac{\Theta_r(\omega+\pi)}{\Theta_r(2\omega)} &= \frac{1}{(\sin(\omega/2))^{2r} + (\cos(\omega/2))^{2r} \frac{\Theta_r(\omega)}{\Theta_r(\omega+\pi)}} \\ &\leqslant \frac{1}{(\sin(\omega/2))^{2r} + (\cos(\omega/2))^{2r}}, \end{aligned} \tag{38}$$

and since we assume $|\omega/2| \leqslant \pi/4$, then we obtain

$$\frac{\Theta_r(\omega+\pi)}{\Theta_r(2\omega)} \leqslant 2^r,$$

and the result follows:

$$\begin{aligned} \left|\frac{H_1^{[r]}(\omega)}{\omega}\right| &\leqslant 2^{r/2} \frac{|\sin(\omega/2)|^r}{|\omega|} \\ &= 2^{r/2-1} |\sin(\omega/2)|^{r-1} \frac{|\sin(\omega/2)|}{|\omega/2|} \end{aligned} \tag{39}$$

and for $|\omega/2| \leqslant \pi/4$, we have $|\sin(\omega/2)|^{r-1} \leqslant 2^{-(r+1)/2}$ and $|\sin(\omega/2)|/|\omega/2| \leqslant 1$. ■

Proposition 2: The Battle-Lemarié scaling functions satisfy

$$|\Phi^{[r]}(\omega)|^2 \leqslant \mathbb{1}_{\{|\omega| \leqslant 2\pi\}} + \frac{2\pi}{\omega^2} \times \mathbb{1}_{\{|\omega| > 2\pi\}}, \tag{40}$$

for every $r = 1, 2, \ldots$.

Proof: For every $r = 1, 2, \ldots$, we have from Eq. (33) that $|\mathcal{F}\Phi^{[r]}(\omega)| \leqslant 1$ for every $\omega \in \mathbb{R}$. This result follows from that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(\omega + 2k\pi)^{2r}} = \frac{1}{\omega^{2r}} + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{1}{(\omega + 2k\pi)^{2r}} \geqslant \frac{1}{\omega^{2r}}.$$

On the other hand, for every $\omega \in \mathbb{R}$, there exists some $k_0 \in \mathbb{Z}$ such that $0 \leqslant \omega + 2k_0\pi < 2\pi$. Thus,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \frac{1}{(\omega + 2k\pi)^{2r}} &= \frac{1}{(\omega + k_0\pi)^{2r}} + \sum_{\substack{k \in \mathbb{Z} \\ k \neq k_0}} \frac{1}{(\omega + 2k\pi)^{2r}} \\ &\geqslant \frac{1}{(2\pi)^{2r}}, \end{aligned} \tag{41}$$

so that $|\mathcal{F}\Phi^{[r]}(\omega)|^2 \leqslant (2\pi/\omega)^{2r} = (2\pi/\omega)^2 \times (2\pi/\omega)^{2r-2}$. When $|\omega| \geqslant 2\pi$, we have $(2\pi/\omega)^{2r-2} \leqslant 1$ for every $r = 1, 2, \ldots$. It follows that $|\mathcal{F}\Phi^{[r]}(\omega)|^2 \leqslant (2\pi/\omega)^2$ for $|\omega| \geqslant 2\pi$. ■

Finally, we have that the family of Battle-Lemarié wavelet functions satisfies assumption (**A4**) since from Eqs. (26), (37) and (40), we obtain

$$|\mathcal{F}W_1^{[r]}(2\omega)|^2 \leqslant \frac{\omega^2}{2} \times \mathbb{1}_{\{|\omega|\leqslant\frac{\pi}{2}\}} + \mathbb{1}_{\{\frac{\pi}{2}<|\omega|\leqslant 2\pi\}} + \frac{2\pi}{\omega^2} \times \mathbb{1}_{\{|\omega|>2\pi\}} \tag{42}$$

Theorems 1 and 3 specify the asymptotic behavior of the wavelet packet coefficients when using some families of paraunitary filters that converge almost everywhere to the Shannon filters. The following discusses consequences of Theorems 1 and 3. Due to the complexity of the convergence involved, the key point is the convergence speed to the limit autocorrelation and distributions. In fact, if the convergence speed is fast, we can expect reasonable decorrelation of the wavelet packet coefficients for finite j and r.

IV. On the convergence speed of the decorrelation process

Consider a family of paraunitary filters satisfying Eqs. (6) and a second order centered random process X being either fractional Brownian motion or wide-sense stationary with spectrum γ. The convergence speed to the limit autocorrelation for the wavelet packet coefficients of X depends on two factors:

A. The convergence speed involved in Eq. (6), that is, the speed of the convergence to the Shannon filters.

B. The convergence speed to the limit autocorrelation in the case where the decomposition used is achieved by the Shannon filters.

A. Convergence of paraunitary filters to the Shannon filters

Theorems 1 and 3 concern some paraunitary filters that approximate the Shannon filters in the sense given by Eq. (6). According to these theorems, we can expect that using paraunitary wavelet filters that are close to the Shannon filters will approximately lead to the same behavior as that obtained by using the Shannon filters. In this respect, the following illustrates how close standard Daubechies, Symlets and Coiflets paraunitary filters can be to the Shannon filters. These standard filters are derived from the Daubechies polynomial

$$H_0^{[r]}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^r Q(e^{-i\omega}),$$

so that r describes the flatness of $H_0^{[r]}$ at $\omega = 0$ and $\omega = \pi$ [30]. Figure 2 illustrates the convergence speed for the scaling filters depending on their orders.

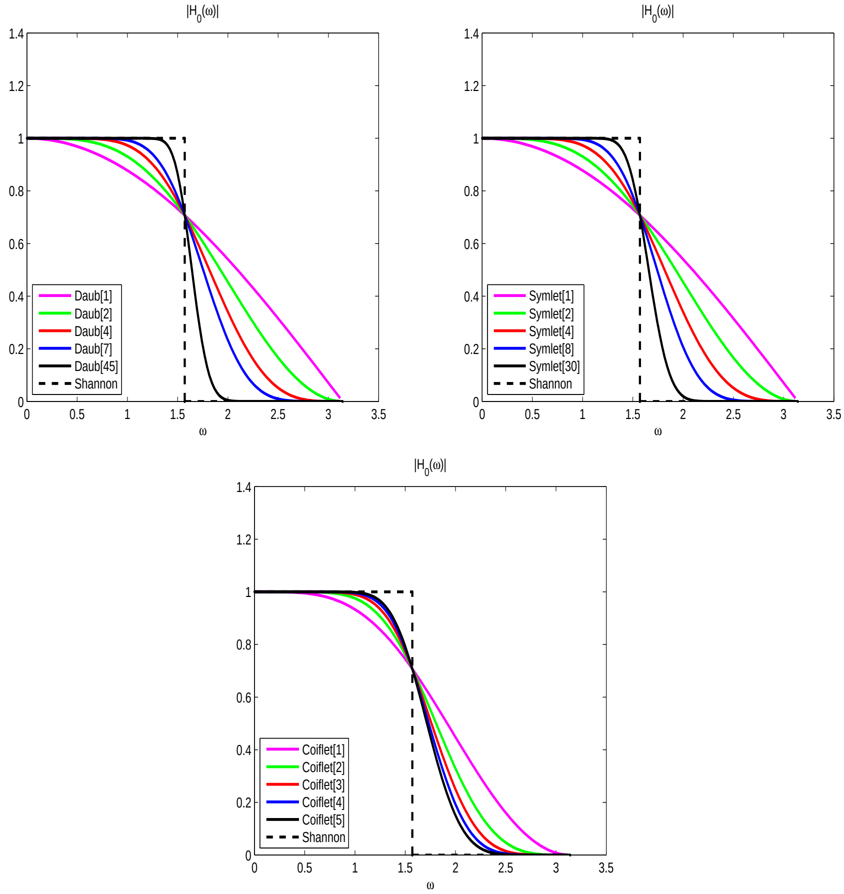

Fig. 2. Graphs of $|H_0^{[r]}|$ for the Daubechies, Symlets and Coiflets scaling filters. "FilterName$[r]$" denotes the filter type and its order, r.

The Meyer paraunitary filters are also close to the Shannon filters in the sense that these filters match the Shannon filters in the interval $[-\pi, -2\pi/3] \cup [-\pi/3, \pi/3] \cup [2\pi/3, \pi]$. The magnitude response of the Meyer scaling filter (normalized by $1/\sqrt{2}$) is given in figure 3.

$$H_0(\omega) = \begin{cases} \sqrt{2} & \text{if} \quad \omega \in [-\frac{\pi}{3}, \frac{\pi}{3}], \\ 0 & \text{if} \quad \omega \in [-\pi, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \pi]. \end{cases} \tag{43}$$

It follows from figures 2 and 3 that we can approach the flatness of the Shannon filters with finite impulse response paraunitary filters. The following now addresses the convergence speed when the wavelet decomposition filters are the Shannon filters.

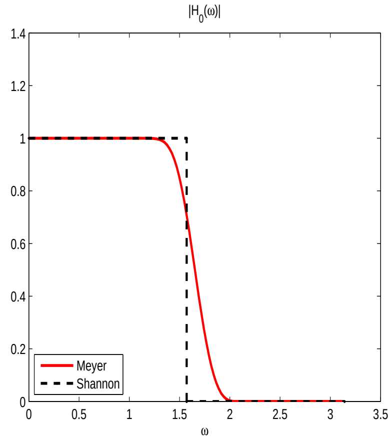

Fig. 3. Magnitude response of Meyer scaling filter normalized by the factor $1/\sqrt{2}$.

B. Convergence speed for the Shannon paraunitary filters

Consider a path $\mathcal{P}$ associated with nodes (subbands) $(j, n)_{j \in \mathbb{N}}$. The speed of the decorrelation process in path $\mathcal{P}$ depends on the shape of spectrum γ of X in the sequence of nested intervals $(\Delta_{j,G(n)})_{j \in \mathbb{N}}$.

First, if γ is constant in $\Delta_{j_0,G(n(j_0))}$ for some $j_0 \geqslant 0$, that is, if $\gamma(\omega) = \gamma(\pi G(n(j_0))/2^{j_0})$ in $\Delta_{j_0,G(n(j_0))}$, then it follows from Eq. (12) that for any $j \geqslant j_0$

$$R^{\mathsf{S}}_{j,n}[m] = \gamma(\frac{\pi G(n(j_0))}{2^{j_0}})\delta[m], \tag{44}$$

and the wavelet packet coefficients are decorrelated in any subband (j, n) of path $\mathcal{P}$, for every $j \geqslant j_0$.

Now, assume that γ is approximately linear, $\gamma(\omega) = a\omega + b$ in $\Delta_{j_0,G(n(j_0))}$, then it follows from Eq. (12) that, in path $\mathcal{P}$ and for every $j \geqslant j_0$,

$$\begin{aligned} R^{\mathsf{S}}_{j,n}[m] &= \gamma(\frac{\pi G(n)}{2^j})\delta[m] \\ &+ \begin{cases} \frac{\pi a}{2^{j+1}} & \text{if} \quad m = 0, \\ \\ \frac{(-1)^{mG(n)}((-1)^m - 1)a}{\pi m^2 2^j} & \text{if} \quad m \neq 0. \end{cases} \end{aligned} \tag{45}$$

Note that $\Delta_{j,G(n)}$ is a tight interval when j is large. For $j = 6$, the diameter of $\Delta_{j,G(n)}$ is $\pi/2^6 \approx 0.05$. It follows that the assumption "γ is constant or linear in $\Delta_{j,G(n)}$" is reasonable for approximating (piecewise linear approximation of a function) the shape of the spectrum γ for large values of the decomposition level, for fractional Brownian motions and for wide-sense stationary processes with regular or piecewise regular spectra.

Eq. (45) has two consequences. First, the convergence speed is very high since the sequence $1/2^j$ decay very fast when j increases. Second, let X^1, X^2 be two processes having spectra with linear shapes a_1 and a_2 in $\Delta_{j,G(n)}$. If $0 < a_1 \ll a_2$, then we can expect that decorrelating process X^1 will be sensibly easier in the paths associated with $\Delta_{j,G(n)}$ than decorrelating process X^2.

C. Decorrelation speed, in practice

We first consider a random process with spectrum $\gamma(\omega) = 1/\omega^\beta$, $0 < \beta < 2$. The spectrum of such a process is very sharp near $\omega = 0$ and becomes less and less sharp when ω increases. Section IV-B thus tells us that the decorrelation speed will be very slow in any path characterized by a sequence of nested intervals $(\Delta_{j,G(n)})_{j\in\mathbb{N}}$ for which the limit value $\omega_{\mathcal{P}}$ close to zero.

More precisely, figure 4 illustrates the decorrelation speed for path $\mathcal{P}_{\pi/4}$ (denoted $\mathcal{P}_{\pi/4}$ because $n(j) = 2^{j-3}$ so that the limit autocorrelation function is $\gamma(\pi/4)\delta[m]$), in comparison with the autocorrelation function obtained in path $\mathcal{P}_0$ (for which, there is no convergence of the integrals involved for computing the autocorrelation functions). It follows that decorrelation can be considered to be attained with reasonable values for decomposition level $j \geqslant 6$ and filter order $r \geqslant 7$ for path $\mathcal{P}_{\pi/4}$ whereas coefficients of path $\mathcal{P}_0$ remain strongly correlated. Note that for a spectrum γ with the form $1/\omega^\beta$, $\gamma(0) = \infty$ and Theorem 3 does not apply for path $\mathcal{P}_0$.

Now, we consider a stationary random process (generated by filtering white noise with an autoregressive filter) with spectrum γ defined for $0 < \mu < 1$, by

$$\gamma(\omega) = (1-\mu)^2/|1-\mu e^{-i\omega}|^2.$$

For such a process, Theorem 1 applies even for path $\mathcal{P}_0$ and the decorrelation speed thus depends on the shape of the spectrum in this path. Figure 6 shows that the decorrelation in $\mathcal{P}_0$ is faster when the spectrum shape is parameterized by μ_1 than when it is parameterized by μ_2 with $\mu_1 < \mu_2$: that is when the shape of the spectrum is less sharp. This confirms the role played by the spectrum shape in the decorrelation speed, as highlighted by Eq. (45). Spectra are plotted in figure 5 for $\mu_1 = 0.5$ and $\mu_2 = 0.9$.

V. WAVELET PACKET BASED SPECTRUM ESTIMATION

We now address wavelet packet based spectrum estimation, on the basis of Theorems 1 and 3. These theorems provide a general non-parametric method for estimating the spectrum of X assumed to be fractional Brownian motion or wide-sense stationary with spectrum γ. The principle of the method is detailed below. Its advantages and limitations are discussed in the Section V-C.

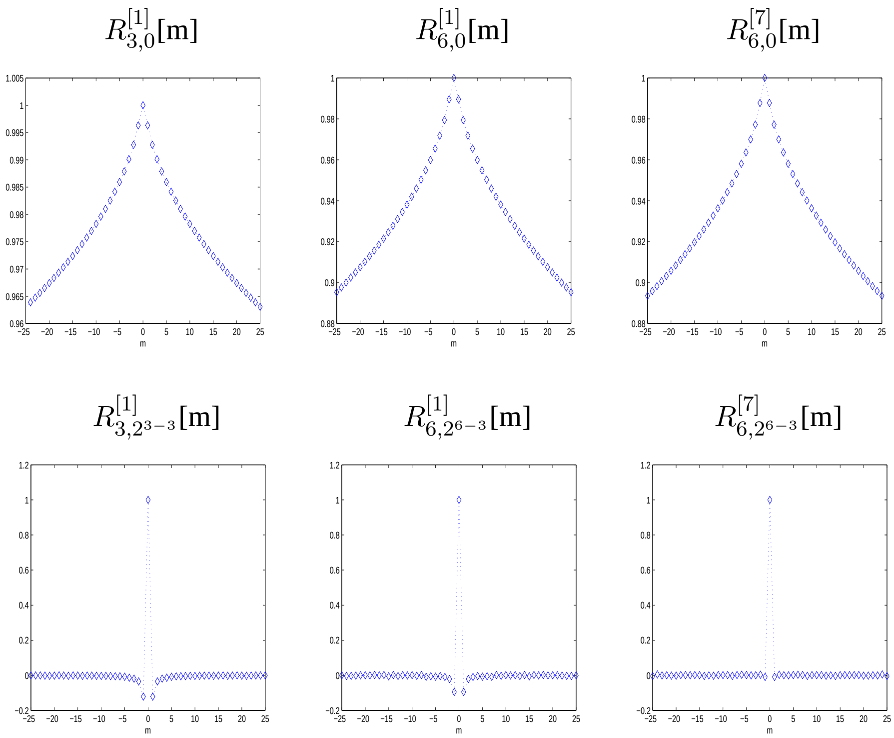

Fig. 4. Normalized autocorrelation functions of the wavelet packet coefficients ($j = 3, 6$, $r = 1, 7$ and $\beta = 1.5$) of a process with spectrum $1/\omega^{\beta}$. The approximation path $\mathcal{P}_0$ and the path $\mathcal{P}_{\pi/4}$ ($n(1) = n(2) = 0$ and $n(j) = 2^{j-3}$ for every $j \geqslant 3$) are considered. Daubechies filters with order $r = 1, 7$ are used.

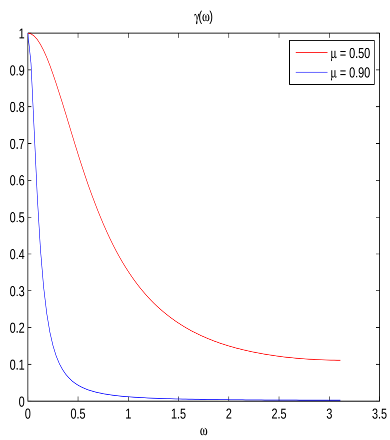

Fig. 5. Spectrum γ for process X_1 (resp. X_2) with parameter $\mu_1 = 0.5$ (resp. $\mu_2 = 0.9$).

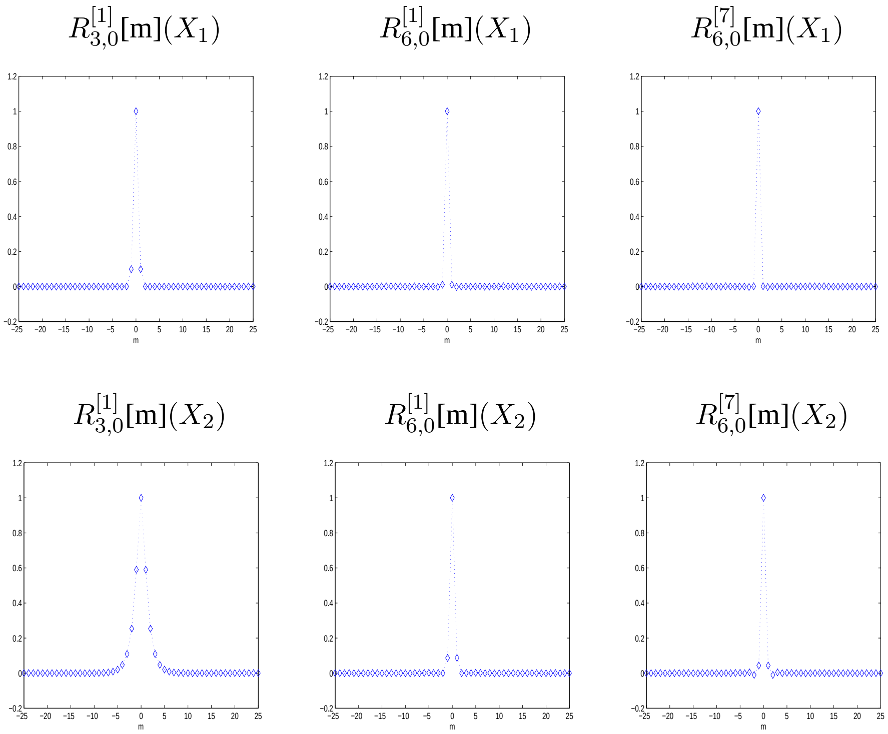

Fig. 6. Normalized autocorrelation functions of the wavelet packet coefficients ($j = 3, 6$, $r = 1, 7$) of processes X_1 and X_2 with parameters $\mu_1 = 0.5, \mu_2 = 0.9$, the spectra of these processes are given by figure 5. The approximation path is considered. For every set of parameters j, n, r considered, the correlation is stronger for process $c_{j,n}^{[r]}(X_2)$ than for process $c_{j,n}^{[r]}(X_1)$. The decorrelation process is fast: Process X_2 spectrum is very sharp around the null frequency, however, the coefficients of this process in the approximation path are sensibly decorrelated by using standard paraunitary filters (Daubechies filters with order $r = 7$ are used).

A. Wavelet packet based spectrum estimation

From Theorems 1 and 3, we have that $R_{j,n}^{[r]}[0]$ is close to $\gamma(\pi G(n)/2^j)$ with a good precision when j and r are large enough since the absolute value of the difference between the two quantities can be made arbitrary small: for every fixed $\eta > 0$, there exist some $j_0 = j_0(\epsilon)$ and $r_0 = r(j_0, \epsilon)$ so that for every $j \geqslant j_0$ and every $r \geqslant r_0$, $|R_{j,n}^{[r]}[0] - \gamma(\pi G(n)/2^j)| < \eta$. Thus the set of the variances of the wavelet packet coefficients at decomposition level j_0, $\{R_{j_0,n}^{[r_0]}[0], n = 0, 1, 2, \ldots, 2^{j_0} - 1\}$, can be described as a set of 2^{j_0} estimates for the spectrum values $\{\gamma(\pi G(n)/2^{j_0}), n = 0, 1, 2, \ldots, 2^{j_0} - 1\}$.

Now, if the spectrum γ is not very singular and if we choose j_0 sufficiently large, then we can assume that γ is approximately constant in $\Delta_{j_0,G(n)}$ (this is reasonable because the diameter $1/2^{j_0}$ of $\Delta_{j_0,G(n)}$ decay very fast to zero when j_0 increases). It follows that for any frequency $\omega_0 \in [0, \pi]$, the value $\gamma(\omega_0)$ can be estimated by the variance $R_{j_0,n}^{[r_0]}[0]$ of the wavelet packet coefficients located at node (j_0, n), where

n is such that $\pi G(n)/2^{j_0} \leqslant \omega_0 < \pi\left(G(n)+1\right)/2^{j_0}$.

Summarizing, assume that we identify sufficiently large values for j and r. We can thus sample uniformly or non-uniformly the spectrum of X with respect to the values $(\omega_\ell)_\ell$ chosen in $[0,\pi]$. For an arbitrary $\omega_\ell \in [0,\pi]$, the estimation is performed along the following steps.

1) Compute the largest integer p so that $\omega_\ell \geqslant p\pi/2^j$, that is

$$p = \left\lfloor \frac{2^j \omega_\ell}{\pi} \right\rfloor.$$

2) Compute the shift parameter n by using the inverse of the permutation G:

$$n = G^{-1}(p),$$

G^{-1} being obtained from the Gray code (see [23]) of p : if $p = \sum_{\ell=1}^{j} \epsilon_\ell 2^{j-\ell}$, with $\epsilon_\ell \in \{0,1\}$, then

$$G^{-1}(p) = \sum_{\ell=1}^{j} \left(\epsilon_\ell \oplus \epsilon_{\ell-1}\right) 2^{j-\ell} \tag{46}$$

with the convention $\epsilon_0 = 0$ and where $\oplus$ denotes the bitwise exclusive-or.

3) Set $\widehat{\gamma(\omega_\ell)} = R^r_{j,n}[0]$, where $R^r_{j,n}[0]$ is the variance of the wavelet packet coefficients located at node (j,n) (projection of X on $\mathbf{W}^r_{j,n}$).

B. *Experimental results*

The experimental tests concern 2^{20} samples of a (simulated) discrete random process X with spectrum $\gamma(\omega) \propto 1/\omega^\beta$. We consider the following wavelet filters for the decomposition of the input process: Daubechies filters with order 7 and 45, Symlet filters with order 8 and 30, Coiflet filters of order 5 and Meyer filters (see figures 2 and 3). The results presented are obtained at decomposition levels 7 and 9. The Welch's averaged modified periodogram method [31] with window size $2^{J+1}-1$, $J = 7, 9$ is also used. The Welch averaged modified periodogram is one of the most efficient methods for estimating spectrum of long data [32]. We choose the window size equal to $2^{J+1}-1$ in order to get the same number of samples of the estimated spectrum as for the wavelet packet method (at level J, we have 2^J subbands and thus, 2^J-1 spectrum samples because the approximation path is not concerned by Theorem 3). The reader can find in [19, Table 1], some complementary tests for the estimates of the values $\widehat{\gamma(0)}, \widehat{\gamma(\pi/4)}, \widehat{\gamma(\pi/2)}, \widehat{\gamma(\pi)}$ as well as their 95% confidence intervals for 100 realizations of the process with spectrum parameterized by $\mu = \mu_2 = 0.9$ (see figure 5).

For a single test, a simple estimate $\hat{\beta}$ of β is obtained by averaging over all the possible combinations of the form $\hat{\beta}(\omega_1,\omega_2) = -\log(\frac{\gamma(\omega_2)}{\gamma(\omega_1)})/\log(\frac{\omega_2}{\omega_1})$, with $\omega_2 > \omega_1 > 0$. This (non-parametric) approach takes

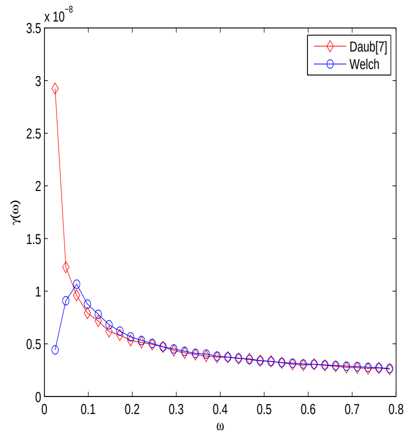

Fig. 7. Spectrum estimated *via* the Wavelet and Fourier-Welch method.

into account the errors made at every sample estimate and thus, reflects more precisely, the estimation errors than extracting β by a parametric method. The empirical mean of the estimate $\hat{\beta}$, the estimation error and the empirical variance of $\hat{\beta}$ are given in table I. These values are those obtained over 25 tests based on different realizations of the random process X. This table shows good performance of the wavelet packet based spectrum estimation, in comparison of the Fourier-Welch method. Note that, surprisingly, the best results for the wavelet packet methods are not those achieved by filters with long impulse responses (filters that are much closer to the Shannon filters): this is due to the fact that the computation of filters with very very long impulse responses[1] and thus, the computation of the wavelet packet coefficients by using such filters, are subject to numerical instabilities [23].

Figure 7 gives an estimate of the spectrum computed from one realization of X, in comparison with the spectrum obtained with the Fourier-Welch method. This figure highlights the good behavior of the wavelet packet method when ω is close to the null frequency, in contrast to the Fourier-Welch method.

C. Discussion

The main limitation of the method seems to be the number of samples required to decompose the input random process up to 6, 7 levels (or more). However, note that if the spectrum shape is not very sharp around certain frequency points, it is not necessary to decompose up to 6 decomposition levels. As an example, if we consider a random process whose spectrum is that of figure 5 for $\mu = 0.9$, then by using the Daubechies filters with order 7, we get (see [19, Figure 5]) a good approximation of

[1]We have 102 (resp. 90) coefficients for the Meyer (resp. Daub[45]) low-pass filter.

TABLE I

EMPIRICAL MEANS, ERRORS, AND VARIANCES, OF THE ESTIMATION OF α OVER 25 NOISE REALIZATIONS, BY USING A FOURIER-WELCH AND WAVELET PACKET BASED METHOD. THE BEST PERFORMANCE OF THE WAVELET PACKET METHOD ARE IN BOLD, IN THE TABLE. THE WELCH'S AVERAGED MODIFIED PERIODOGRAM METHOD WITH WINDOW SIZE $2^{J+1}-1$, $J=7,9$ IS USED AT DECOMPOSITION LEVEL J.

Method		**Fourier**	**Wavelet**					
		'Welch'	'Daub[7]'	'Daub[45]'	'Symlet[8]'	'Symlet[30]'	'Coiflet[5]'	'Meyer'
				$J=7$.				
α=0.25	Mean($\hat{\alpha}$)	0.2563	**0.2520**	0.2534	0.2531	0.2546	0.2531	0.2548
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0063	0.0020	0.0034	0.0031	0.0046	0.0031	0.0048
	$10^4\times$ Var($\hat{\alpha}$)	0.0526	0.0080	0.0271	0.0048	0.0710	0.0084	0.2290
α=0.50	Mean($\hat{\alpha}$)	0.5126	**0.5049**	0.5062	0.5061	0.5075	0.5060	0.5060
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0126	0.0049	0.0062	0.0061	0.0075	0.0060	0.0060
	$10^5\times$ Var($\hat{\alpha}$)	0.6865	0.1967	0.3849	0.0474	0.3276	0.0894	0.3280
α=0.75	Mean($\hat{\alpha}$)	0.7712	**0.7590**	0.7612	0.7607	0.7612	0.7602	0.7624
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0212	0.0090	0.0112	0.0107	0.0112	0.0102	0.0124
	$10^5\times$ Var($\hat{\alpha}$)	0.7520	0.2357	0.6134	0.0298	0.6650	0.1980	0.3396
α=1.00	Mean($\hat{\alpha}$)	1.0297	**1.0135**	1.0138	1.0142	1.0147	1.0146	1.0142
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0297	0.0135	0.0138	0.0142	0.0147	0.0146	0.0142
	$10^4\times$ Var($\hat{\alpha}$)	0.0603	0.0085	0.0773	0.0104	0.0587	0.0168	0.1643
				$J=9$.				
α=0.25	Mean($\hat{\alpha}$)	0.2520	0.2476	0.2490	0.2492	**0.2504**	0.2484	0.2520
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0020	0.0024	0.0010	0.0008	0.0004	0.0016	0.0020
	$10^3\times$ Var($\hat{\alpha}$)	0.0032	0.0085	0.0214	0.0211	0.1027	0.0237	0.1392
α=0.50	Mean($\hat{\alpha}$)	0.5033	0.4976	0.4992	**0.5003**	0.5040	0.4995	0.5027
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0033	0.0024	0.0008	0.0003	0.0040	0.0005	0.0027
	$10^3\times$ Var($\hat{\alpha}$)	0.0100	0.0130	0.0210	0.0068	0.0308	0.0155	0.1185
α=0.75	Mean($\hat{\alpha}$)	0.7569	0.7486	0.7518	**0.7505**	0.7525	0.7511	0.7531
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0069	0.0014	0.0018	0.0005	0.0025	0.0011	0.0031
	$10^4\times$ Var($\hat{\alpha}$)	0.1496	0.0806	0.1958	0.1564	0.4050	0.0845	0.3587
α=1.00	Mean($\hat{\alpha}$)	1.0089	**0.9993**	1.0009	1.0031	1.0099	1.0036	1.0122
	$\lvert\alpha-$ Mean($\hat{\alpha}$)$\rvert$	0.0089	0.0007	0.0009	0.0031	0.0099	0.0036	0.0122
	$10^4\times$ Var($\hat{\alpha}$)	0.0931	0.1154	0.3161	0.1976	0.6106	0.1117	0.2733

- $\gamma(0)$ at decomposition levels $\geqslant 7$,
- $\gamma(\pi/4)$ at decomposition levels $\geqslant 5$,
- $\gamma(\pi/2)$ at decomposition levels $\geqslant 3$,
- $\gamma(\pi)$ at decomposition levels $\geqslant 2$.

Around the null frequency, γ is very sharp and 7 decompositions are necessary; otherwise, less decomposition levels are sufficient because the spectrum is rather flat.

The first advantage of the wavelet packet based method is the simplicity of the spectrum estimation *via* the technique described in Section V-A. Statistical properties of the autocorrelation and the convergence speed to the limit autocorrelation functions ensure that we can expect good performance of the method by using standard Daubechies or Symlets filters with order larger than or equal to 7. The second advantage of the method is that it is non-parametric: in practice, it can be used in many applications with no *a priori* on the spectrum shape. When *a priori* information is available, the method could also be improved with existing techniques. As a matter of fact, if the spectrum of interest has *a priori* exactly the form $1/\omega^{\beta}$, then we can estimate β by maximum-likelihood estimate as done for the wavelet based method in [33], [34] or by techniques such as [35] if the observation is corrupted by additive white noise.

VI. Conclusion

The asymptotic autocorrelation functions of wavelet packet coefficients of fractional Brownian motions have been computed for some paraunitary filters that approximate the Shannon paraunitary filters.

The paper also characterizes the convergence speed to the limit autocorrelation and show that approximate decorrelation can be achieved at finite decomposition levels even by using non-ideal paraunitary filters.

The ideal subband coding yielded by the Shannon wavelet packet decomposition, the convergence of some standard wavelet filters to the Shannon filters, and the asymptotic properties of the wavelet packet autocorrelation allow for defining wavelet packet based spectrum estimation. This spectrum estimation has been tested in the framework of fractional Brownian motion, but also applies to wide-sense stationary random processes.

The new wavelet packet based spectrum estimation presented in the paper derives from theoretical results (those stated in Theorems 1 and 3), has very low complexity and outperforms the standard non-parametric Fourier-Welch based spectrum estimation. The discussion of Section V-C highlights the limitations and the advantages of the new method. It also presents some perspectives on how to improve the wavelet packet based spectrum estimation.

In future work, we plan to investigate the contributions of some of the proposed techniques, among others, the exploitation of redundancy in the signal domain (Hilbert transform) or in the wavelet domain (averaging several ϵ-decimate orthogonal wavelets, using complex wavelets or multiwavelets).

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APPENDIX A
PROOF OF THEOREM 2

By taking into account remark 1 and under assumption (**A1**), the discrete random process $c_{j,n}^{[r]}$ representing the wavelet packet coefficients of the fractional Brownian motion X is defined by

$$c_{j,n}^{[r]}[k]= \int_{\mathbb{R}} X(t) W_{j,n,k}[r](t)dt, \tag{47}$$

with convergence in quadratic mean sense and its autocorrelation function is

$$R_{j,n}^{[r]}[k,\ell] = \iint_{\mathbb{R}^2} R(t,s) W_{j,n,k}^{[r]}(t) W_{j,n,\ell}^{[r]}(s) dt ds. \tag{48}$$

with $R(t,s)$ given by Eq. (15).

By considering again remark 1 and under assumption (**A2**), we have that

$$\iint_{\mathbb{R}} |t|^{2\alpha} W_{j,n,k}^{[r]}(t) dt = 0, \tag{49}$$

and thus

$$\iint_{\mathbb{R}^2} |t|^{2\alpha} W_{j,n,k}^{[r]}(t) W_{j,n,\ell}^{[r]}(s) dt ds = 0. \tag{50}$$

By mimicking the proof of [12, Theorem 1] we get

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} |t-s|^{2\alpha} W^{[r]}_{j,n,k}(t) W^{[r]}_{j,n,\ell}(s) dt ds \\
&\overset{2}{=} \iint_{\mathbb{R}^2} dt ds |t|^{2\alpha} W^{[r]}_{j,n,k}(t+s) W^{[r]}_{j,n,\ell}(s), \\
&\overset{3}{=} \frac{\Gamma(2\alpha+1)\sin(\pi\alpha)}{\pi} \times \\
&\quad \iint_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{|\omega|^{2\alpha+1}} d\omega \right) W^{[r]}_{j,n,k}(t+s) W^{[r]}_{j,n,\ell}(s) dt ds, \\
&\overset{4}{=} \frac{1}{\pi\sigma^2} \int_{\mathbb{R}} d\omega \gamma_\alpha(\omega) \times \\
&\quad \iint_{\mathbb{R}^2} dt ds \left(1-\cos(t\omega)\right) W^{[r]}_{j,n,k}(t+s) W^{[r]}_{j,n,\ell}(s), \\
&\overset{5}{=} -\frac{1}{\pi\sigma^2} \int_{\mathbb{R}} d\omega \gamma_\alpha(\omega) \times \\
&\quad \iint_{\mathbb{R}^2} dt ds \cos(t\omega) W^{[r]}_{j,n,k}(t+s) W^{[r]}_{j,n,\ell}(s), \\
&\overset{6}{=} -\frac{1}{\pi\sigma^2} \int_{\mathbb{R}} \gamma_\alpha(\omega) |\mathcal{F}W^{[r]}_{j,n}(\omega)|^2 e^{i2^j(k-\ell)\omega} d\omega.
\end{aligned} \tag{51}
$$

Thus, from Eqs. (48), (50) and (51), we obtain

$$
R^{[r]}_{j,n}[k,\ell] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma_\alpha(\omega) |\mathcal{F}W^{[r]}_{j,n}(\omega)|^2 e^{i2^j(k-\ell)\omega} d\omega. \tag{52}
$$

One can check that under assumption (**A3**), the integral in Eq. (52) is absolutely convergent for every pair (j,n) with $n \neq 0$. From Eq. (52) we have that $c^{[r]}_{j,n}$ is a wide-sense stationary random process for every $(j,n) \in \mathbb{N} \times \mathbb{N}$. With the standard abuse of language, we denote $R^{[r]}_{j,n}[k,\ell] \equiv R^{[r]}_{j,n}[k-\ell] = R^{[r]}_{j,n}[m]$, with $m = k - \ell$ and Eq. (16) follows.

APPENDIX B

Lemma 2: Let f be a real valued function. Consider the sequence of nested intervals $\left(\Delta^+_{j,G(n_{\mathcal{P}}(j))}\right)_{j\geqslant 1}$ defined by Eq. (8) and associated with a wavelet packet path $\mathcal{P}$. Assume that f is locally integrable on $\mathbb{R}$. If f is continuous at $\omega_{\mathcal{P}}$ given by Eq. (10), then we have uniformly in $k \in \mathbb{Z}$

$$
\lim_{j\to+\infty} \frac{2^j}{\pi} \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega) \cos(2^j k\omega) d\omega = f(\omega_{\mathcal{P}})\delta[k]. \tag{53}
$$

[2]Change of variables.

[3] Bahr and Essen representation of $|t|^{2\alpha}$, see [36].

[4] Fubini's theorem, the integrand is absolutely integrable.

[5]Taking into account Eq. (49).

[6]Write $\cos(t\omega) = (e^{-it\omega} + e^{-it\omega})/2$ to obtain Fourier integrals of $W^{[r]}_{j,n,k}$ and $W^{[r]}_{j,n,\ell}$.

Proof:

Since f is continuous at $\omega_{\mathcal{P}}$, then for every real number $\eta > 0$, there exists a real number $\nu > 0$ such that, for every $\omega \in [\omega_{\mathcal{P}} - \nu, \omega_{\mathcal{P}} + \nu]$, we have $|f(\omega) - f(\omega_{\mathcal{P}})| < \eta$. In addition, since

$$\lim_{j\to+\infty} \frac{G(n_{\mathcal{P}}(j))\pi}{2^j} = \lim_{j\to+\infty} \frac{(G(n_{\mathcal{P}}(j)) + 1)\pi}{2^j} = \omega_{\mathcal{P}},$$

there exists an integer $j_0 = j_0(\nu)$, such that, for every natural number $j \geqslant j_0$, the values $G(n_{\mathcal{P}}(j))\pi/2^j$ and $(G(n_{\mathcal{P}}(j))+1)\pi/2^j$ are within the interval $[\omega_{\mathcal{P}} - \nu, \omega_{\mathcal{P}} + \nu]$. It follows that, for every natural number $j \geqslant j_0$ and every $\omega \in \Delta^+_{j,G(n_{\mathcal{P}}(j))}$,

$$|f(\omega) - f(\omega_{\mathcal{P}})| < \eta.$$

Therefore, for any natural number $j \geqslant j_0$

$$\begin{aligned} &\frac{2^j}{\pi} \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} |f(\omega) - f(\omega_{\mathcal{P}})|\, d\omega \\ &\qquad < \eta \frac{M^j}{\pi} \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} d\omega = \eta. \end{aligned} \tag{54}$$

On the other hand, for any natural number $j \geqslant j_0$ and every integer k,

$$\begin{aligned} &\Big| \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega) \cos(2^j k\omega) d\omega \\ &\qquad - \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega_{\mathcal{P}}) \cos(2^j k\omega) d\omega \Big| \\ &\qquad = \Big| \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} (f(\omega) - f(\omega_{\mathcal{P}})) \cos(2^j k\omega) d\omega \Big| \\ &\qquad \leqslant \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} |f(\omega) - f(\omega_{\mathcal{P}})|\, d\omega. \end{aligned} \tag{55}$$

Hence, we derive from Eqs. (54) and (55) that, for every natural number $j \geqslant j_0$,

$$\begin{aligned} &\frac{2^j}{\pi} \Big| \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega) \cos(2^j k\omega) d\omega \\ &\qquad - \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega_{\mathcal{P}}) \quad \cos(2^j k\omega) d\omega \Big| < \eta \end{aligned}$$

uniformly in $k \in \mathbb{Z}$. Since

$$\frac{2^j}{\pi} \int_{\Delta^+_{j,G(n_{\mathcal{P}}(j))}} f(\omega_{\mathcal{P}}) \cos(2^j k\omega) d\omega = f(\omega_{\mathcal{P}}) \delta[k],$$

we conclude that, for every natural number $j \geqslant j_0$,

$$\left| \frac{2^j}{\pi} \int_{\Delta^+_{j,G(n)}} f(\omega) \cos(2^j k\omega) d\omega - f(\omega_{\mathcal{P}}) \delta[k] \right| < \eta$$

uniformly in $k \in \mathbb{Z}$. ■