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Abstract. We are interested in the normal class of an algebraic hypersurface $\mathcal{Z}$ in the complexified euclidean projective space $\mathbb{P}^n$, that is the number of normal lines to $\mathcal{Z}$ passing through a generic point of $\mathbb{P}^n$. Thanks to the notion of normal polars, we state a formula for the normal class valid for a general hypersurface $\mathcal{Z} \subset \mathbb{P}^n$. We give a generic result and illustrate our formula on examples in $\mathbb{P}^n$. We define the orthogonal incidence variety and compute the Schubert class of the variety of projective normal lines to a surface of $\mathbb{P}^3$ in the Chow ring of $\mathbb{G}(1,3)$. We complete our work with a generalization of Salmon's formula for the normal class of a Plücker curve to any plane curve with any kind of singularity.

Introduction

The notion of normal lines to an hypersurface of an euclidean space is extended here to the complexified euclidean projective space $\mathbb{P}^n$ ($n \geq 2$). In this setting, $\mathcal{H}^\infty$ the hyperplane at infinity is fixed, together with the umbilical at infinity $\mathcal{U}_\infty \subset \mathcal{H}^\infty$, the smooth quadric in $\mathcal{H}^\infty$ corresponding to the intersection of $\mathcal{H}^\infty$ with any hypersphere (see Section 1.1 for details). The aim of the present work is the study the **normal class** $c_\nu(\mathcal{Z})$ of a hypersurface $\mathcal{Z}$ of $\mathbb{P}^n$, that is the number of $m \in \mathcal{Z}$ such that the projective normal line $\mathcal{N}_m(\mathcal{Z})$ to $\mathcal{Z}$ at m passing through a generic $m_1 \in \mathbb{P}^n$ (see Section 1 for details). Our estimates provide upper bounds for the number of normal lines, of a real algebraic surface in an n-dimensional affine euclidean space E_n, passing through a generic point in E_n. Let us consider the **variety $\mathfrak{N}_\mathcal{Z}$ of projective normal lines of** $\mathcal{Z}$ by

$$\mathfrak{N}_\mathcal{Z} := \overline{\{\mathcal{N}_m(\mathcal{Z}); m \in \mathcal{Z}\}} \subset \mathbb{G}(1,n) \subset \mathbb{P}^{\frac{n(n+1)}{2}-1}$$

and its Schubert class $\mathfrak{n}_\mathcal{Z} := [\mathfrak{N}_\mathcal{Z}] \in A^{n-1}(\mathbb{G}(1,n))$ (when $\dim \mathfrak{N}_\mathcal{Z} = n-1$). The fact that $PGL(n,\mathbb{C})$ **does not preserve normal lines** complicates our study compared to the study of tangent hyperplanes. We prove namely the following result valid for a wide family of surfaces of $\mathbb{P}^n$. Let $\mathcal{Z} = V(F)$ be an irreducible hypersurface of $\mathbb{P}^n$. We write $\mathcal{Z}_\infty := \mathcal{Z} \cap \mathcal{H}^\infty$. Note that the singular points of $\mathcal{Z}_\infty$ correspond to the points of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$.

Theorem 1. *Let $\mathcal{Z} \in \mathbb{P}^n$ be a smooth irreducible hypersurface of degree $d_\mathcal{Z} \geq 2$ such that $\mathcal{H}^\infty$ is not tangent to $\mathcal{Z}$ and that at any $m \in \mathcal{Z}_\infty \cap \mathcal{U}_\infty$, the tangent planes to $\mathcal{Z}_\infty$ and to $\mathcal{U}_\infty$ at m are distinct. Then the normal class $c_\nu(\mathcal{Z})$ of $\mathcal{Z}$ is*

$$c_\nu(\mathcal{Z}) = d_\mathcal{Z} \sum_{k=0}^{n-1} (d_\mathcal{Z} - 1)^k.$$

In particular,

- *if $d_\mathcal{Z} = 2$, $c_\nu(\mathcal{Z}) = n$;*
- *if $n = 2$, $c_\nu(\mathcal{Z}) = d_\mathcal{Z}$;*

Date: April 2, 2016.
2000 *Mathematics Subject Classification.* 14J99,14H50,14E05,14N05,14N10.
Key words and phrases. projective normal, class, Plücker, Grassmannian, Puiseux.

- *if* $n = 3$, $c_\nu(\mathcal{Z}) = d_{\mathcal{Z}}^3 - d_{\mathcal{Z}}^2 + d_{\mathcal{Z}}$;
- *if* $n = 4$, $c_\nu(\mathcal{Z}) = d_{\mathcal{Z}}^4 - 2d_{\mathcal{Z}}^3 + 2d_{\mathcal{Z}}^2$;
- *if* $n = 5$, $c_\nu(\mathcal{Z}) = d_{\mathcal{Z}}^5 - 3d_{\mathcal{Z}}^4 + 4d_{\mathcal{Z}}^3 - 2d_{\mathcal{Z}}^2 + d_{\mathcal{Z}}$.

The normal class of an hyperplane $\mathcal{H} \subset \mathbb{P}^n$ *(other than* $\mathcal{H}^\infty$*) is* $c_\nu(\mathcal{H}) = 1$.

Actually we establish a general formula which is valid for a wider family of hypersurfaces of $\mathbb{P}^n$. The notion of normal polars $\mathcal{P}_{A,\mathcal{Z}}$ plays an important role in our study. It is a notion analogous to the notion of polars [2]. Given an irreducible hypersurface $\mathcal{Z} \subset \mathbb{P}^n$ of degree $d_{\mathcal{Z}}$, we extend the definition of the line $\mathcal{N}_m(\mathcal{S})$ to any $m \in \mathbb{P}^n$. We then define a regular map $\alpha_{\mathcal{Z}} : \mathbb{P}^n \setminus \mathcal{B}_{\mathcal{Z}}^{(0)} \to \mathbb{P}^{\frac{n(n+1)}{2}-1}$ corresponding to $m \mapsto \mathcal{N}_m(\mathcal{Z})$ (where $\mathcal{B}_{\mathcal{Z}}^{(0)}$ is the set of base points of $\alpha_{\mathcal{Z}}$). We will see that $\mathcal{B}_{\mathcal{Z}} := \mathcal{B}_{\mathcal{Z}}^{(0)} \cap \mathcal{Z}$ corresponds to the union of the set of singular points of $\mathcal{Z}$, of the set of points of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and of the set of points of tangency of $\mathcal{Z}_\infty$ with $\mathcal{U}_\infty$. For any $A \in \mathbb{P}^n$, we will introduce the notion of **normal polar** $\mathcal{P}_{A,\mathcal{Z}}$ of $\mathcal{Z}$ with respect to A as the set of $m \in \mathbb{P}^n$ such that either $m \in \mathcal{B}_{\mathcal{Z}}^{(0)}$ or $A \in \mathcal{N}_m(\mathcal{Z})$. We will see that, if $\dim \mathcal{B}_{\mathcal{Z}}^{(0)} \le 1$, then, for a generic $A \in \mathbb{P}^n$,

$$\dim \mathcal{P}_{A,\mathcal{Z}} = 1 \quad \text{and} \quad \deg(\mathcal{P}_{A,\mathcal{Z}}) = \sum_{k=0}^{n-1} (d_{\mathcal{Z}} - 1)^k.$$

Theorem 2. *Let* $\mathcal{Z}$ *be an irreducible hypersurface of* $\mathbb{P}^n$ *with isolated singularities, admitting a finite number of points of tangency with* $\mathcal{H}^\infty$ *and such that* $\mathcal{Z}_\infty$ *has a finite number of points of tangency with* $\mathcal{U}_\infty$*. Then the normal class* $c_\nu(\mathcal{Z})$ *of* $\mathcal{Z}$ *is given by*

$$c_\nu(\mathcal{Z}) = d_{\mathcal{Z}}.\sum_{k=0}^{n-1} (d_{\mathcal{Z}} - 1)^k - \sum_{P \in B_{\mathcal{Z}}} i_P(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}),$$

for a generic $A \in \mathbb{P}^n$*, where* $i_P(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}})$ *is the intersection multiplicity of* $\mathcal{Z}$ *with* $\mathcal{P}_{A,\mathcal{Z}}$.

In dimension 3, we obtain the following result.

Theorem 3 (n=3, normal class and Chow ring)**.** *Let* $\mathcal{S}$ *be an irreducible surface of* $\mathbb{P}^3$ *with isolated singularities, admitting a finite number of points of tangency with* $\mathcal{H}^\infty$ *and such that* $\mathcal{S}_\infty$ *has a finite number of (non singular) points of tangency with* $\mathcal{U}_\infty$*. Then*

$$\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}} - 1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3)),$$

where the normal class $c_\nu(\mathcal{S})$ *of* $\mathcal{Z}$ *is equal to* $d_{\mathcal{Z}}.\deg(\mathcal{P}_{A,\mathcal{Z}})$ *(for a generic* $A \in \mathbb{P}^n$*) minus the sum of the intersection multiplicities of* $\mathcal{S}$ *with its generic normal polars* $\mathcal{P}_{A,\mathcal{S}}$ *at points of* $\mathcal{B}_{\mathcal{S}}$.

Corollary 4 (n=3)**.** *For a generic irreducible surface* $\mathcal{S} \subset \mathbb{P}^3$ *of degree* $d \ge 2$*, we have* $c_\nu(\mathcal{S}) = d^3 - d^2 + d$ *and*

$$\mathfrak{n}_{\mathcal{S}} = (d^3 - d^2 + d).\sigma_2 + d(d-1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3)).$$

In the next statement, we consider smooth surfaces $\mathcal{S}$ of $\mathbb{P}^3$ ($\mathbb{P}^3$ being endowed with projective coordinates $[x : y : z : t]$) such that $\mathcal{S}_\infty$ has no worse singularities than ordinary multiple points and ordinary cusps.

Theorem 5 (n=3)**.** *Let* $\mathcal{S} \subset \mathbb{P}^3$ *be a smooth irreducible surface of degree* $d_{\mathcal{S}} \ge 2$ *such that:*

(i) *in* $\mathcal{H}^\infty$*, the curve* $\mathcal{S}_\infty$ *has a finite number of points of tangency with* $\mathcal{U}_\infty$*,*
(ii) *any singular point of* $\mathcal{S}_\infty$ *is either an ordinary multiple point or an ordinary cusp,*
(iii) *at any (non singular) point of tangency of* $\mathcal{S}_\infty$ *with* $\mathcal{U}_\infty$*, the contact is ordinary,*
(iv) *at any singular point of* $\mathcal{S}_\infty$ *contained in* $\mathcal{U}_\infty$*, the tangent line to* $\mathcal{U}_\infty$ *is not contained in the tangent cone to* $\mathcal{S}_\infty$.

Then

$$\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}}-1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3))$$

and the normal class of $\mathcal{S}$ *is*

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}}^3 - d_{\mathcal{S}}^2 + d_{\mathcal{S}} - \sum_{k\geq 2}((k-1)^2 m_\infty^{*(k)} + k(k-1)\tilde{m}_\infty^{(k)}) - 2\kappa_\infty^* - 3\tilde{\kappa}_\infty - c_\infty,$$

where

- $m_\infty^{*(k)}$ *(resp.* $\tilde{m}_\infty^{(k)}$*) is the number of ordinary multiple points of order* k *of* $\mathcal{S}_\infty$ *outside (resp. contained in)* $\mathcal{U}_\infty$*,*
- κ_∞^* *(resp.* $\tilde{\kappa}_\infty$*) is the number of ordinary cusps of* $\mathcal{S}_\infty$ *outside (resp. contained in)* $\mathcal{U}_\infty$*,*
- c_∞ *is the number of ordinary (non singular) points of tangency of* $\mathcal{S}_\infty$ *with* $\mathcal{U}_\infty$*.*

Example 6 (n=3)**.** *The surface* $\mathcal{S} = V(xzt - tx^2 - zt^2 - xz^2 + y^3) \subset \mathbb{P}^3$ *is smooth, its only point of tangency with* $\mathcal{H}_\infty = V(t)$ *is* $P[1:0:0:0]$ *which is an ordinary cusp of* $\mathcal{S}_\infty = V(t, -xz^2+y^3)$*. Moreover* $\mathcal{S}_\infty$ *has no point of tangency with* $\mathcal{U}_\infty$*. Hence the normal class of* $\mathcal{S}$ *is* $27-9+3-2 = 19$*.*

Theorem 1 (resp. 5) is a consequence of Theorem 2 (resp. 3). In a more general setting, when $n=3$, we can replace $\alpha_{\mathcal{S}}$ in $\tilde{\alpha}_{\mathcal{S}} = \frac{\alpha_{\mathcal{S}}}{H}$ (for some homogeneous polynomial H of degree d_H) so that the set $\tilde{\mathcal{B}}_{\mathcal{S}}^{(0)}$ of base points of $\tilde{\alpha}_{\mathcal{S}}$ has dimension at most 1. In this case, we consider a notion of normal polars associated to $\tilde{\alpha}_{\mathcal{S}}$ which have generically dimension 1 and degree $\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1$ (with $\tilde{d}_{\mathcal{S}} = d_{\mathcal{S}} - d_H$).

Theorem 7 (n=3)**.** *Let* $\mathcal{S}$ *be an irreducible surface of* $\mathbb{P}^3$*. If the set* $\tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \cap \mathcal{S}$ *is finite, then* $\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(\tilde{d}_{\mathcal{S}} - 1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3))$ *and the normal class* $c_\nu(\mathcal{S})$ *of* $\mathcal{S}$ *is equal to* $d_{\mathcal{S}}(\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1)$ *minus the intersection multiplicity of* $\mathcal{S}$ *with its generic normal polars* $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ *at points* $m \in \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \cap \mathcal{S}$*.*

When the surface is a "cylinder" or a surface of revolution, its normal class is equal to the normal class of its plane base curve. The normal class of any plane curve is given by the simple formula of Theorem 8 below, that we give for completness. Let us recall that, when $\mathcal{C} = V(F)$ is an irreducible curve of $\mathbb{P}^2$, the evolute of $\mathcal{C}$ is the curve tangent to the family of normal lines to $\mathcal{Z}$ and that the evolute of a line or a circle is reduced to a single point. Hence, except for lines and circles, the normal class of $\mathcal{C}$ is simply the class (with multiplicity) of its evolute. The following result generalizes the result by Salmon [9, p. 137] proved in the case of Plücker curves (plane curves with no worse multiple tangents than ordinary double tangents, no singularities other than ordinary nodes and cusps) to any plane curve (with any type of singularities). We write ℓ_∞ for the line at infinity of $\mathbb{P}^2$. We define the two cyclic points $I[1:i:0]$ and $J[1:-i:0]$ in $\mathbb{P}^2$ (when $n=2$, $\mathcal{U}_\infty = \{I, J\}$).

Theorem 8 (n=2)**.** *Let* $\mathcal{C} = V(F)$ *be an irreducible curve of* $\mathbb{P}^2$ *of degree* $d \geq 2$ *with class* $d^\vee$*. Then its normal class is*

$$c_\nu(\mathcal{C}) = d + d^\vee - \Omega(\mathcal{C}, \ell_\infty) - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C}),$$

where Ω *denotes the sum of the contact numbers between two curves and where* $\mu_P(\mathcal{C})$ *is the multiplicity of* P *on* $\mathcal{C}$*.*

In [4], Fantechi proved that the evolute map is birational from $\mathcal{C}$ to its evolute curve unless if[1] $F_x^2 + F_y^2$ is a square modulo F and that in this latest case the evolute map is $2:1$ (if $\mathcal{C}$ is neither a line nor a circle). Therefore, the normal class $c_\nu(\mathcal{C})$ of a plane curve $\mathcal{C}$ corresponds to the class

[1] We write $[x:y:z]$ for the coordinates of $m \in \mathbb{P}^2$ and F_x, F_y, F_z for the partial derivatives of F.

of its evolute unless $F_x^2 + F_y^2$ is a square modulo F and in this last case, the normal class $c_\nu(\mathcal{C})$ of $\mathcal{C}$ corresponds to the class of its evolute times 2 (if $\mathcal{C}$ is neither a line nor a circle).

The notion of focal loci generalizes the notion of evolute to higher dimension [10, 1]. The normal lines of an hypersurface $\mathcal{Z}$ are tangent to the focal loci hypersurface of $\mathcal{Z}$ but of course the normal class of $\mathcal{Z}$ does not correspond anymore (in general) to the class of its focal loci (the normal lines to $\mathcal{Z}$ are contained in but are not equal to the tangent hyperplanes of its focal loci).

In Section 1, we introduce normal lines, normal class, normal polars in $\mathbb{P}^n$ (see also Appendix B for the link between projective orthogonality and affine orthogonality). In Section 2, we study normal polars and prove Theorems 2 and 1. In Section 3, we introduce the orthogonal incidence variety $\mathcal{I}^\perp$ in $\mathbb{G}(1,n)$, give some recalls on the Schubert classes in the Chow ring of $\mathbb{G}(1,3)$ and prove Theorems 3 and 7. In Section 4, we prove Theorem 5. In Section 5.1, we apply our results on examples in $\mathbb{P}^3$: we compute the normal class of every quadric and of a cubic surface with singularity E_6. In Section 6, we prove Theorem 8. Appendix A on the normal class of "cylinders" and of surfaces of revolution in $\mathbb{P}^n$.

1. Normal lines, normal class and normal polars

1.1. **Definitions and notations.** Let $\mathbf{V}$ be a $\mathbb{C}$-vector space of dimension $n+1$. Given $\mathcal{Z} = V(F) \neq \mathcal{H}^\infty$ an irreducible hypersurface of $\mathbb{P}^n = \mathbb{P}(\mathbf{V})$ (with $F \in Sym(\mathbf{V}^\vee) \cong \mathbb{C}[x_1, ..., x_{n+1}]$), we consider the rational map $n_\mathcal{Z} : \mathbb{P}^n \dashrightarrow \mathcal{H}^\infty$ given by $n_\mathcal{Z} = [F_{x_1} : \cdots : F_{x_n} : 0]$. Note that, for nonsingular $m \in \mathcal{Z}$ such that the tangent hyperplane $\mathcal{T}_m\mathcal{Z}$ to $\mathcal{Z}$ at m is not $\mathcal{H}^\infty$, $n_\mathcal{Z}(m)$ is the pole of the $(n-2)$-variety at infinity $\mathcal{T}_m\mathcal{Z} \cap \mathcal{H}^\infty \subset \mathcal{H}^\infty$ with respect to the **umbilical** $\mathcal{U}_\infty := V(x_1^2 + ... + x_n^2) \cap \mathcal{H}^\infty \subset \mathcal{H}^\infty$. $\mathcal{U}_\infty$ corresponds to the set of **circular points at infinity**.

Definition 9. **The projective normal line** *$\mathcal{N}_m\mathcal{Z}$ to $\mathcal{Z}$ at $m \in \mathcal{Z}$ is the line $(m\, n_\mathcal{Z}(m))$ when $n_\mathcal{Z}(m)$ is well defined in $\mathbb{P}^n$ and not equal to m.*

Remark 10. *This is a generalization of affine normal lines in the euclidean space E_n. Indeed, if F has real coefficients and if $m \in \mathcal{Z} \setminus \mathcal{H}_\infty$ has real coordinates $[x_1^{(0)} : \cdots : x_n^{(0)} : 1]$, then $\mathcal{N}_m\mathcal{Z}$ corresponds to the affine normal line of the affine hypersurface $V(F(x_1, ..., x_n, 1)) \subset E_n$ at the point of coordinates $(x_1^{(0)}, \cdots, x_n^{(0)})$ (see Section B).*

The aim of this work is the study of the notion of normal class.

Definition 11. *Let $\mathcal{Z}$ be an irreducible hypersurface of $\mathbb{P}^n$.* **The normal class** *of $\mathcal{Z}$ is the number $c_\nu(\mathcal{Z})$ of $m \in \mathcal{Z}$ such that $\mathcal{N}_m(\mathcal{Z})$ contains m_1 for a generic $m_1 \in \mathbb{P}^n$.*

Let $\Delta := \{(m_1, m_2) \in \mathbb{P}^n \times \mathbb{P}^n \ : \ m_1 = m_2\}$ be the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. Recall that the **Plücker embedding** $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta \overset{Pl}{\hookrightarrow} \mathbb{P}(\bigwedge^2 \mathbf{V}) \cong \mathbb{P}^{\frac{n(n+1)}{2}-1}$ is defined by

$$Pl(u,v) = \bigwedge^2 (u,v) = [p_{i,j} = u_i v_j - u_j v_i]_{1 \le i < j \le n+1} \in \mathbb{P}^{\frac{n(n+1)}{2}-1},$$

with $p_{i,j} = -p_{j,i}$ the (i,j)-th Plücker coordinate, identifying $\mathbb{P}^{\frac{n(n+1)}{2}-1}$ with the projective space of $n \times n$ antisymmetric matrices. Its image is the Grassmannian $\mathbb{G}(1,n)$ (see [3]) given by

$$\mathbb{G}(1,n) := Pl((\mathbb{P}^n)^2 \setminus \Delta) = \bigcap_{(i,j_1,j_2,j_3) \in \mathcal{I}} V(B_{i,j_1,j_2,j_3}) \subset \mathbb{P}^{\frac{n(n+1)}{2}-1},$$

where $B_{i,j_1,j_2,j_3} := p_{i,j_1} p_{j_2,j_3} - p_{i,j_2} p_{j_1,j_3} + p_{i,j_3} p_{j_1,j_2}$ and where $\mathcal{I}$ is the set of $(i, j_1, j_2, j_3) \in \{1, ..., n+1\}$ such that $j_1 < j_2 < j_3$ and $j_1, j_2, j_3 \neq i$. We recall also that $\dim \mathbb{G}(1,n) = 2n-2$.

Remark 12. *Let $h_{\mathcal{Z}} : \mathbb{P}^n \setminus V(F_{x_1}, \cdots, F_{x_n}) \to \mathbb{P}^n \times \mathbb{P}^n$ be the morphism defined by $j(m) = (m, n_{\mathcal{Z}}(m))$. The variety $\mathfrak{N}_{\mathcal{Z}} \subset \mathbb{G}(1,n)$ of projective normal lines to $\mathcal{Z}$ is the (Zariski closure of the) image of $\mathcal{Z}$ by the regular map $\alpha_{\mathcal{Z}} := Pl \circ h_{\mathcal{Z}} : \mathbb{P}^n \setminus \mathcal{B}_{\mathcal{Z}}^{(0)} \to \mathbb{P}^{\frac{n(n+1)}{2}-1}$, with $\mathcal{B}_{\mathcal{Z}}^{(0)} := V(F_{x_1}, ..., F_{x_n}) \cup j^{-1}(\Delta)$, i.e.*

$$\mathcal{B}_{\mathcal{Z}}^{(0)} := \left\{ m \in \mathbb{P}^n;\ \bigwedge^2 (\mathbf{m}, \mathbf{n}_{\mathcal{Z}}(\mathbf{m})) = \mathbf{0} \text{ in } \bigwedge^2 \mathbf{V} \right\}.$$

Note that the number of normal lines to $\mathcal{Z}$ passing through $A \in \mathbb{P}^n$ corresponds to the number of $m \in \mathcal{Z} \setminus \mathcal{B}_{\mathcal{Z}}$ satisfying the following sets of equations :

$$\bigwedge^3 [\mathbf{m}\ \mathbf{n}_{\mathcal{Z}}(\mathbf{m})\ \mathbf{A}] = \mathbf{0} \quad \text{in } \bigwedge^3 \mathbf{V}. \tag{1}$$

Definition 13. *For any $A \in \mathbb{P}^n$, the set of points $m \in \mathbb{P}^n$ satisfying (1) is called* **normal polar** *$\mathcal{P}_{A,\mathcal{Z}}$ of $\mathcal{Z}$ with respect to A*

1.2. **Projective similitudes.** Recall that, for every field $\mathbb{k}$,

$$GO(n, \mathbb{k}) = \left\{ A \in GL(n, \mathbb{k}); \exists \lambda \in \mathbb{k}^*, A \cdot^t A = \lambda \cdot I_n \right\}$$

is the **orthogonal similitude group** (for the standard products) and that $GOAff(n, \mathbb{k}) = \mathbb{k}^n \rtimes GO(n, \mathbb{k})$ is the **orthogonal similitude affine group**. We have a natural monomorphism of groups $\kappa : Aff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GL(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$ given by

$$\kappa(b, A) = \begin{pmatrix} a_{11} & ... & ... & a_{1n} & b_1 \\ a_{21} & ... & ... & a_{2n} & b_2 \\ & & & & \\ a_{n1} & .. & ... & a_{nn} & b_n \\ 0 & ... & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and, by restriction, $\kappa|_{GOAff(n,\mathbb{R})} : GOAff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GO(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$. Analogously we have a natural monomorphism of groups $\kappa' := (\kappa \otimes 1)|_{GOAff(n,\mathbb{C})} : GOAff(n, \mathbb{C}) = \mathbb{C}^n \rtimes GO(n, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$. Composing with the canonical projection $\pi : GL(n+1, \mathbb{C}) \longrightarrow \mathbb{P}(GL(n+1, \mathbb{C}))$ we obtain the **projective complex similitude Group**:

$$\widehat{Sim_{\mathbb{C}}(n)} := (\pi \circ \kappa')(GOAff(n, \mathbb{C})).$$

which acts naturally on $\mathbb{P}^n$.

Definition 14. *An element of $\mathbb{P}(Gl(\mathbf{V}))$ corresponding to an element of $\widehat{Sim_{\mathbb{C}}(n)}$ with respect to the basis $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ is called a* ***projective similitude of*** *$\mathbb{P}^n$.*

The set of projective similitudes of $\mathbb{P}^n$ is isomorphic to $\widehat{Sim_{\mathbb{C}}(n)}$.

Lemma 15. *The projective similitude preserves the orthogonality structure in $\mathbb{P}^n$. They preserve namely the normal lines and the normal class of surfaces of $\mathbb{P}^n$.*

This lemma has a straightforward proof that is omitted.

2. Proof of Theorem 2

2.1. **Geometric study of $\mathcal{B}_{\mathcal{Z}} := \mathcal{B}_{\mathcal{Z}}^{(0)} \cap \mathcal{Z}$.** We write $\mathcal{Z}_\infty := \mathcal{Z} \cap \mathcal{H}^\infty$. Recall that $\mathcal{U}_\infty := \mathcal{H}^\infty \cap V\left(x_1^2 + ... + x_n^2\right)$.

Proposition 16. *A point of $\mathcal{Z}$ is in $\mathcal{B}_\mathcal{Z}$ if it is a singular point of $\mathcal{Z}$ or a tangential point of $\mathcal{Z}$ at infinity or a tangential point of $\mathcal{Z}_\infty$ to the umbilical, i.e. $\mathcal{B}_\mathcal{Z} = \mathrm{Sing}(\mathcal{Z}) \cup \mathcal{K}_\infty(\mathcal{Z}) \cup \Gamma_\infty(\mathcal{Z})$, where*

- $\mathrm{Sing}(\mathcal{Z})$ *is the set of singular points of $\mathcal{Z}$,*
- $\mathcal{K}_\infty(\mathcal{Z})$ *is the set of points of $\mathcal{Z}$ at which the tangent hyperplane is $\mathcal{H}^\infty$,*
- $\Gamma_\infty(\mathcal{Z})$ *is the set of points of $\mathcal{Z}_\infty \cap \mathcal{U}_\infty$ at which the tangent space to $\mathcal{Z}_\infty$ and to $\mathcal{U}_\infty$ are the same.*

Proof. Let $m \in \mathcal{Z}$. We have

$$\begin{aligned} m \in \mathcal{B}_\mathcal{Z} \quad &\Leftrightarrow \quad \bigwedge^2 (\mathbf{m}, \mathbf{n}_\mathcal{Z}(\mathbf{m})) = 0 \\ &\Leftrightarrow \quad \mathbf{n}_\mathcal{Z}(\mathbf{m}) = \mathbf{0} \text{ or } m = n_\mathcal{Z}(m) \\ &\Leftrightarrow \quad m \in V(F_{x_1}, \cdots, F_{x_n}) \text{ or } m = n_\mathcal{Z}(m). \end{aligned}$$

Now $m \in V(F_{x_1}, \cdots, F_{x_n})$ means either that m is a singular point of $\mathcal{S}$ or that $\mathcal{T}_m\mathcal{Z} = \mathcal{H}^\infty$.

Let $m = [x_1 : \cdots : x_{n+1}] \in \mathcal{Z}$ be such that $m = n_\mathcal{S}(m)$. So $[x_1 : \cdots : x_{n+1}] = [F_{x_1} : \cdots : F_{x_n} : 0]$. In particular $x_{n+1} = 0$. Due to the Euler identity, we have $0 = \sum_{i=1}^{n+1} x_i F_{x_i} = \sum_{i=1}^{n+1} x_i^2$. Hence $m \in \mathcal{U}_\infty$. Note that the $(n-2)$-dimensional tangent space $\mathcal{T}_m\mathcal{U}_\infty$ to $\mathcal{U}_\infty$ at m has equations $X_{n+1} = 0$ and $\langle m, \cdot\rangle$ and that the $(n-2)$-dimensional tangent space $\mathcal{T}_m\mathcal{Z}_\infty$ to $\mathcal{Z}_\infty$ at m has equations $X_{n+1} = 0$ and $\langle (n_\mathcal{S}(m), \cdot\rangle$. We conclude that $\mathcal{T}_m\mathcal{U}_\infty = \mathcal{T}_m\mathcal{Z}_\infty$.

Conversely, if $m = [x_1 : \cdots : x_n : 0]$ is a nonsingular point of $\mathcal{Z}_\infty \cap \mathcal{U}_\infty$ such that $\mathcal{T}_m\mathcal{U}_\infty = \mathcal{T}_m\mathcal{Z}_\infty$, then the linear spaces $Span(\mathbf{m}, \vec{e}_{n+1})$ and $Span(\nabla F, \vec{e}_{n+1})$ are equal which implies that $[x_1 : \cdots : x_n : 0] = [F_{x_1} : \cdots : F_{x_n} : 0]$. □

Recall that the dual variety of $\mathcal{Z}_\infty \subset \mathcal{H}^\infty$ is the variety $\mathcal{Z}_\infty^\vee \subset (\mathcal{H}^\infty)^\vee \cong (\mathbb{P}^{n-1})^\vee$ of tangent hyperplanes to $\mathcal{Z}_\infty$. It corresponds to the (Zariski closure of the) image of $\mathcal{Z}_\infty$ by the rational map $n_\mathcal{Z}$. We write $\mathcal{Z}_\infty^\wedge \subset \mathbb{P}^n$ for this image. With this notation, $\mathcal{B}_\mathcal{Z} = \mathrm{Sing}(\mathcal{Z}) \cup (\mathcal{Z}_\infty \cap \mathcal{Z}_\infty^\wedge)$.

Remark 17. *For a generic hypersurface of $\mathbb{P}^n$, $\mathcal{B}_\mathcal{Z} = \emptyset$ and so $\dim \mathcal{B}_\mathcal{Z}^{(0)} \leq 0$.*

But we will also consider cases for which $\#\mathcal{B}_\mathcal{Z} < \infty$, and so $\dim \mathcal{B}_\mathcal{Z}^{(0)} \leq 1$.

Example 18 (n=3)**.** *For the saddle surface $\mathcal{S}_1 = V(xy - zt)$, the set $\mathcal{B}_{\mathcal{S}_1}$ contains a single point $[0 : 0 : 1 : 0]$ which is a point of tangency at infinity of $\mathcal{S}_1$.*

For the ellipsoid $\mathcal{E}_1 := V(x^2 + 2y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_1}$ is empty.

For the ellipsoid $\mathcal{E}_2 := V(x^2 + 4y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_2}$ has two elements: $[0 : 1 : \pm i : 0]$ which are points of tangency of $\mathcal{E}_2$ with $\mathcal{U}_\infty$.

Example 19. *For the cuartic $\mathcal{Z} := V(x_1^2 + x_2^2 + (x_3 + x_5)x_3 + (2x_3 + x_4)x_4) \subset \mathbb{P}^4$, $Sing(\mathcal{Z}) = \emptyset$, $\mathcal{K}_\infty(\mathcal{Z}) = \{[0 : 0 : 1 : -1 : 0]\}$ and $\Gamma_\infty(\mathcal{Z}) = \{I_1, I_2\}$, with $I_1[1 : i : 0 : 0 : 0]$ and $I_2[1 : -i : 0 : 0 : 0]$.*

2.2. **Normal polars of $\mathcal{Z} \subset \mathbb{P}^n$.** Let $\mathcal{Z} = V(F) \subset \mathbb{P}^n$ (with $F \in Sym(\mathbf{V}^\vee)$) be an irreducible hypersurface. For every $A \in \mathbb{P}^n$, **the normal polar** $\mathcal{P}_{A,\mathcal{Z}}$ of $\mathcal{Z}$ with respect to A is the set of $m \in \mathbb{P}^n$ satisfying the $\begin{pmatrix} n+1 \\ 3 \end{pmatrix}$ equations of (1). For every $m, A \in \mathbb{P}^n$, we have

$$m \in \mathcal{P}_{A,\mathcal{Z}} \Leftrightarrow \ m \in \mathcal{B}_\mathcal{Z}^{(0)} \text{ or } A \in \mathcal{N}_m\mathcal{Z},$$

extending the definition of $\mathcal{N}_m\mathcal{Z}$ from $m \in \mathcal{Z}$ to $m \in \mathbb{P}^n$.

Lemma 20 (The projective similitudes preserve the normal polars)**.** *Let $\mathcal{Z} = V(F) \subset \mathbb{P}^n$ be a hypersurface and φ be any projective similitude, then $\varphi(\mathcal{P}_{A,\mathcal{Z}}) = \mathcal{P}_{\varphi(A),\varphi(\mathcal{Z})}$.*

Proof. Due to Lemma 15, $\varphi(\mathcal{N}_m\mathcal{Z}) = \mathcal{N}_{\varphi(m)}(\varphi(\mathcal{Z}))$ which gives the result. □

Note that

$$\mathcal{P}_{A,\mathcal{Z}} = \mathcal{B}_{\mathcal{Z}}^{(0)} \cup \left(\bigcap_{i<j<k} \alpha_{\mathcal{Z}}^{-1} \mathcal{H}_{A,i,j,k} \right),$$

where $\mathcal{H}_{A,i,j,k}$ is the hyperplane of $\mathbb{P}^{\frac{n(n+1)}{2}-1}$ given by $\mathcal{H}_{A,i,j,k} := V(D_{i,j,k}) \subset \mathbb{P}^{\frac{n(n+1)}{2}-1}$, with $D_{i,j,k} := a_i p_{j,k} - a_j p_{i,k} + a_k p_{i,j}$. On $\mathbb{G}(1,n)$, $p = Pl(u,v) \in \bigcap_{i<j<k} \mathcal{H}_{A,i,j,k}$ means that $\bigwedge^3(\mathbf{A},\mathbf{u},\mathbf{v}) = 0$.

Lemma 21. *For every $A \in \mathbb{P}^n$, the set $\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k}$ is a $(n-1)$-dimensional linear space of $\mathbb{P}^{\frac{n(n+1)}{2}-1}$ contained in $\mathbb{G}(1,n)$.*

Proof. Let $A[a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$. Assume for example $a_{j_0} \neq 0$ (the proof being analogous when $a_j \neq 0$ for symetry reason).

Let $p \in \bigcap_{i<j<k} \mathcal{H}_{A,i,j,k}$. Let us prove that $p \in \mathbb{G}(1,n)$. Let $i, j_1, j_2, j_3 \in \{1, ..., n+1\}$ be distinct indices. Due to $D_{j_1,j_2,j_3} = D_{i,j_1,j_2} = 0$, we have

$$\begin{aligned} a_{j_1} a_{j_2} p_{i,j_1} p_{j_2,j_3} &= a_{j_1} a_{j_2} p_{j_1,j_3} p_{i,j_2} + a_{j_1} p_{j_1,j_2}(-a_{j_3} p_{i,j_2}) \\ &\quad + a_{j_2} p_{j_1,j_2}(-a_i p_{j_1,j_3}) + p_{j_1,j_2}(a_i a_{j_3} p_{j_1,j_2}) \\ &= a_{j_1} a_{j_2} p_{j_1,j_3} p_{i,j_2} - a_{j_1} a_{j_2} p_{i,j_3} p_{j_1,j_2} - a_j p_{j_1,j_2} D_{i,j_2,j_3} + a_i p_{j_1,j_2} D_{j_1,j_2,j_3}. \end{aligned}$$

Hence $a_{j_1} a_{j_2} B_{i,j_1,j_2,j_3} = 0$, for every $i, j_1, j_2, j_3 \in \{1, ..., n+1\}$. So $B_{i,j_1,j_2,j_3} \neq 0$ implies that $a_{j_1} = a_{j_2} = a_{j_3} = 0$ (up to a permutation of (i, j_1, j_2, j_3)) and so $0 = D_{j_0,j_1,j_2} = D_{j_0,j_2,j_3} = D_{j_0,j_1,j_3}$ imply $p_{j_1,j_2} = p_{j_2,j_3} = p_{j_1,j_3} = 0$ which contradicts $B_{i,j_1,j_2,j_3} \neq 0$. Since $\bigwedge^4(\mathbf{A},\mathbf{A},\mathbf{u},\mathbf{v}) = 0$, we get that $a_{j_0} D_{i,j,k} - a_i D_{j_0,j,k} + a_j D_{j_0,i,k} - a_k D_{j_0,i,j} = 0$ for every $1 \leq i < j < k \leq n+1$ such that $i, j, k \neq j_0$. Hence $\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k} = \bigcap_{i<j, i,j \neq j_0} \mathcal{H}_{A,j_0,i,j}$. Since $a_1 \neq 0$, the $\frac{n(n-1)}{2}$ corresponding linear equations are linearly independent and so $\bigcap_{i<j, i,j \neq j_0} \mathcal{H}_{A,j_0,i,j} \subset \mathbb{P}^{\frac{n(n+1)}{2}-1}$ has dimension $\frac{n(n+1)}{2} - 1 - \frac{(n-1)n}{2} = n-1$. □

Proposition 22. *Let $\mathcal{Z} = V(F) \subset \mathbb{P}^n$ be an irreducible hypersurface such that $d_{\mathcal{Z}} := \deg \mathcal{Z} \geq 2$ and $\dim \mathcal{B}_{\mathcal{Z}}^{(0)} \leq 1$. Then, for a generic $A \in \mathbf{V}$, we have $\dim \mathcal{P}_{A,\mathcal{Z}} = 1$ and*

$$\deg \mathcal{P}_{A,\mathcal{Z}} = \sum_{k=0}^{n-1} (d_{\mathcal{Z}} - 1)^k.$$

Proof. Due to the proof of Lemma 21, for every $A \in \mathbb{P}^n$ such that $a_{n+1} \neq 0$, we have $\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k} = \bigcap_{i<j<n+1} \mathcal{H}_{A,n+1,i,j} \subset \mathbb{P}^{\frac{n(n+1)}{2}-1}$ and so

$$\mathcal{P}_{A,\mathcal{Z}} = \bigcap_{1 \leq i<j \leq n} V(E_{A,i,j}) \subset \mathbb{P}^n,$$

with

$$\forall i, j \in \{1, ..., n\}, \quad E_{A,i,j} := L_{A,i} F_{x_j} - L_{A,j} F_{x_i} \quad \text{and} \quad L_{A,i} := a_{n+1} x_i - a_i x_{n+1},$$

i.e. $E_{A,i,j} = a_{n+1}(x_i F_{x_j} - x_j F_{x_i}) + a_j x_{n+1} F_{x_i} - a_i x_{n+1} F_{x_j}$. Note that

$$L_{A,k} E_{A,i,j} - L_{A,j} E_{A,i,k} = L_{A,i} E_{A,k,j} \quad \text{and} \quad F_{x_k} E_{A,i,j} - F_{x_j} E_{A,i,k} = F_{x_i} E_{A,k,j}. \tag{3}$$

Hence

$$\forall i \in \{1,...,n\}, \quad \mathcal{P}_{A,\mathcal{Z}} \setminus V(L_{A,i}, F_{x_i}) = \bigcap_{j\in\{1,...,n\}\setminus\{i\}} V(E_{A,i,j}) \setminus V(L_{A,i}, F_{x_i}), \tag{4}$$

and so $\dim \mathcal{P}_{A,\mathcal{Z}} \geq 1$. Recall that $\mathcal{B}_{\mathcal{Z}}^{(0)} = \bigcap_{i=1}^{n} V(x_{n+1}F_{x_i}) \cap \bigcap_{i,j=1}^{n} V(x_iF_{x_j} - x_jF_{x_i})$ and that we have assumed that $\dim \mathcal{B}_{\mathcal{Z}}^{(0)} \leq 1$. In particular $\mathcal{P}_{A,\mathcal{Z}} \cap \mathcal{H}^\infty = \mathcal{B}_{\mathcal{Z}}^{(0)} \cap \mathcal{H}^\infty$ and $\mathcal{B}_{\mathcal{Z}}^{(0)} \setminus \mathcal{H}^\infty = V(F_{x_1}, ..., F_{x_n}) \setminus \mathcal{H}^\infty$. This combined with (4) and with the expression of $E_{A,i,j}$ leads to $\dim \mathcal{P}_{A,\mathcal{Z}} = 1$.

Now let us compute the degree of $\mathcal{P}_{A,\mathcal{Z}}$. The idea is to prove an induction formula. Assume that $A \notin \mathcal{H}^\infty$ is such that $\dim \mathcal{P}_{A,\mathcal{Z}} = 1$. Let $\mathcal{H} = V(\sum_{i=1}^{n+1} \alpha_i x_i) \subset \mathbb{P}^n$ be an hyperplane such that $\#(\mathcal{H} \cap \mathcal{P}_{A,\mathcal{Z}}) = \deg \mathcal{P}_{A,\mathcal{Z}}$ and $\sum_{i=1}^{n} \alpha_i^2 \neq 0$. We compose by a projective similitude $\phi : \mathbb{P}^n \to \mathbb{P}^n$ so that $\phi(A)$ has projective coordinates $[0 : \cdots : 0 : 1]$ and that $\hat{\mathcal{H}} := \phi(\mathcal{H}) = V(x_1 - bx_{n+1}) \subset \mathbb{P}^n$. Set $\hat{\mathcal{Z}} := \phi(\mathcal{Z}) = V(\hat{F}) \subset \mathbb{P}^n$, with $\hat{F} := F \circ \phi$. Hence $\phi(\mathcal{P}_{A,\mathcal{Z}}) = \mathcal{P}_{\phi(A),\tilde{\mathcal{Z}}}$ is the set of points $m[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n$ such that $\bigwedge^2 \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{F}_{x_1} \\ \vdots \\ \hat{F}_{x_n} \\ 0 \end{pmatrix} \right) = 0$ in $\bigwedge^2 \mathbf{V}$.

We then define $G(x_2, ..., x_{n+1}) := \hat{F}(bx_{n+1}, x_2, ..., x_{n+1}) \in \mathbb{C}[x_2, ..., x_{n+1}]$ and $H(x_3, ..., x_{n+1}) := G(0, x_3, ..., x_{n+1}) \in \mathbb{C}[x_3, ..., x_{n+1}]$. We set $\mathcal{Z}_1 := V(G) \subset \mathbb{P}^{n-1}$, $\mathcal{Z}_2 := V(H) \subset \mathbb{P}^{n-2}$ and $B_k[0 : ... : 0 : 1] \in \mathbb{P}^k$. We then write $\mathcal{P}_{n-k,B_{n-k},\mathcal{Z}_k}$ for the normal polar in $\mathbb{P}^{n-k}$ of $\mathcal{Z}_k \subset \mathbb{P}^{n-k}$ with respect to B_{n-k}, with the conventions $\mathcal{P}_{0,B_0,\mathcal{Z}_k} = \emptyset$ (if $k = n$) and $\mathcal{P}_{1,B_1,\mathcal{Z}_k} = \mathbb{P}^1$ (if $k = n - 1$). We will prove that

$$\deg \mathcal{P}_{A,\mathcal{Z}} = d \times \deg \mathcal{P}_{n-1,B_{n-1},\mathcal{Z}_1} - (d-1) \times \deg \mathcal{P}_{n-2,B_{n-2},\mathcal{Z}_2}.$$

Let $\Pi_1 : \mathbb{P}^n \to \mathbb{P}^{n-1}$ and $\Pi_2 : \mathbb{P}^n \to \mathbb{P}^{n-2}$ be the projections given by $\Pi_1[x_1 : ... : x_{n+1}] = [x_2 : ... : x_{n+1}]$ and $\Pi_2[x_1 : ... : x_{n+1}] = [x_3 : ... : x_{n+1}]$. Due to (3),

$$\hat{\mathcal{H}} \cap V(x_1\hat{F}_{x_2} - x_2\hat{F}_{x_1}) \cap \Pi_1^{-1}(\mathcal{P}_{n-1,B_{n-1},\mathcal{Z}_1}) = (\hat{\mathcal{H}} \cap \mathcal{P}_{\phi(A),\hat{\mathcal{Z}}}) \cup [\hat{\mathcal{H}} \cap V(x_2, \hat{F}_{x_2}) \cap \Pi_2^{-1}(\mathcal{P}_{n-2,B_{n-2},\mathcal{Z}_2})]. \tag{5}$$

For a generic $\mathcal{H}$ and for a good choice of ϕ, the union in the right hand side of (5) is disjoint and

$$\begin{aligned} \deg \mathcal{P}_{A,\mathcal{Z}} &= \#(\mathcal{H} \cap \mathcal{P}_{A,\mathcal{Z}}) \\ &= \#(\hat{\mathcal{H}} \cap \deg \mathcal{P}_{\phi(A),\hat{\mathcal{Z}}}) \\ &= d_{\mathcal{Z}}. \deg \mathcal{P}_{n-1,B_{n-1},\mathcal{Z}_1} - (d_{\mathcal{Z}} - 1).\mathcal{P}_{n-2,B_{n-2},\mathcal{Z}_2}. \end{aligned}$$

Hence $\deg \mathcal{P}_{A,\mathcal{Z}} = d_{\mathcal{Z}}$ if $n = 2$ and $\deg \mathcal{P}_{A,\mathcal{Z}} = d_{\mathcal{Z}}^2 - d_{\mathcal{Z}} + 1$ if $n = 3$. The formula in the general case follows by induction. □

Analogously we have the following.

Proposition 23 (n=3)**.** *If $\mathcal{S}$ is an irreducible algebraic surface of $\mathbb{P}^3$ (with projective coordinates $[x : y : z : t]$) and if $\dim \mathcal{B}_{\mathcal{S}}^{(0)} = 2$, then the two dimensional part of $\mathcal{B}_{\mathcal{S}}^{(0)}$ is $V(H) \subset \mathbb{P}^3$ for some homogeneous polynomial $H \in \mathbb{C}[x, y, z, t]$ of degree d_H. We write $\boldsymbol{\alpha}_{\mathcal{S}} = H \cdot \tilde{\boldsymbol{\alpha}}_{\mathcal{S}}$. Note that the regular map $\tilde{\alpha}_{\mathcal{S}} : \mathbb{P}^3 \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \to \mathbb{P}^3$ (with $\dim \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \leq 1$) associated to $\tilde{\boldsymbol{\alpha}}_{\mathcal{S}}$. We then adapt our study with respect to $\tilde{\alpha}_{\mathcal{S}}$ instead of $\alpha_{\mathcal{S}}$ and define the corresponding polar $\tilde{\mathcal{P}}_{A,\mathcal{S}}$. Then, we have $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = (d_{\mathcal{S}} - d_H)^2 - d_{\mathcal{S}} + d_H + 1$.*

Example 24. *Note that the only irreducible quadrics $\mathcal{S} = V(F) \subset \mathbb{P}^3$ such that $\dim \mathcal{B}_{\mathcal{S}}^{(0)} \geq 2$ are the spheres and cones, i.e. with F of the following form*

$$F(x, y, z, t) = (x - x_0 t)^2 + (y - y_0 t)^2 + (z - z_0 t)^2 + a_0 t^2,$$

where x_0, y_0, z_0, a_0 are complex numbers (it is a sphere if $a_0 \neq 0$ and it is a cone otherwise).

Hence, due to Proposition 22, the degree of a generic normal polar of any irreducible quadric of $\mathbb{P}^3$ which is neither a sphere nor a cone is 3.

Moreover, for a sphere or for a cone, applying Proposition 23 with $H = t$, $\tilde{\mathcal{P}}_{A,S}$ is a line for a generic $A \in \mathbb{P}^3$.

2.3. **Proof of Theorems 1 and 2.**

Proof of Theorem 2. Let $\mathcal{Z}$ be an irreducible surface of $\mathbb{P}^n$ of degree $d_{\mathcal{Z}} \geq 2$ such that $\#\mathcal{B}_{\mathcal{Z}} < \infty$. It remains to prove that

$$c_\nu(\mathcal{S}) = d_{\mathcal{Z}}. \deg \mathcal{P}_{A,\mathcal{Z}} - \sum_{P \in \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right), \tag{6}$$

for a generic $A \in \mathbb{P}^n$. Note that, for a generic $A \in \mathbb{P}^n$, since $\overline{\alpha_{\mathcal{Z}}(\mathcal{Z})}$ is irreducible of dimension at most $n-1$, we have $\#\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k} \cap \overline{\alpha_{\mathcal{Z}}(\mathcal{Z})} < \infty$ and so $\#\mathcal{P}_{A,\mathcal{Z}} \cap \mathcal{Z} < \infty$ (since $\#\mathcal{B}_{\mathcal{Z}} < \infty$). Since $\dim \mathcal{P}_{A,\mathcal{Z}} = 1$ and $\#\mathcal{Z} \cap \mathcal{P}_{A,\mathcal{Z}} < \infty$ for a generic $A \in \mathbb{P}^n$, due to Proposition 22 and to the Bezout formula, we have:

$$\begin{aligned} d_{\mathcal{Z}}. \deg\left(\mathcal{P}_{A,\mathcal{Z}}\right) &= \deg\left(\mathcal{Z} \cap \mathcal{P}_{A,\mathcal{Z}}\right) \\ &= \sum_{P \in \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right) + \sum_{P \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right). \end{aligned}$$

Now let us prove that, for a generic $A \in \mathbb{P}^n$,

$$\sum_{P \in \mathcal{Z} \setminus \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right) = \#((\mathcal{Z} \cap \mathcal{P}_{A,\mathcal{Z}}) \setminus \mathcal{B}_{\mathcal{Z}}). \tag{7}$$

Since $\alpha_{\mathcal{Z}}$ defines a rational map, $\overline{\alpha_{\mathcal{Z}}(\mathcal{Z})}$ is irreducible and its dimension is at most $n-1$.

Assume first that $\dim \overline{\alpha_{\mathcal{Z}}(\mathcal{Z})} < n-1$. For a generic $A \in \mathbb{P}^n$, the plane $\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k}$ does not meet $\overline{\alpha_{\mathcal{Z}}(\mathcal{Z})}$ and so the left and right hand sides of (7) are both zero. So Formula (6) holds true with $c_\nu(\mathcal{Z}) = 0$.

Assume now that $\dim \overline{\alpha_{\mathcal{Z}}(\mathcal{Z})} = n-1$. Then, for a generic $A \in \mathbb{P}^n$, the plane $\bigcap_{i<j<k} \mathcal{H}_{A,i,j,k}$ meets $\overline{\alpha_{\mathcal{Z}}(\mathcal{Z})}$ transversally (with intersection number 1 at every intersection point) and does not meet $\overline{\alpha_{\mathcal{Z}}(\mathcal{Z})} \setminus \alpha_{\mathcal{Z}}(\mathcal{Z})$. This implies that, for a generic $A \in \mathbb{P}^n$, we have $i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right) = 1$ for every $P \in (\mathcal{S} \cap \mathcal{P}_{A,\mathcal{Z}}) \setminus \mathcal{B}_{\mathcal{Z}}$ and so (7) follows. Hence, for a generic $A \in \mathbb{P}^n$, we have

$$\begin{aligned} d_{\mathcal{Z}}. \deg\left(\mathcal{P}_{A,\mathcal{Z}}\right) &= \sum_{P \in \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{Z}, \mathcal{P}_{A,\mathcal{Z}}\right) + \#\{P \in \mathcal{Z} \setminus \mathcal{B}_{\mathcal{Z}} : \ A \in \mathcal{N}_m \mathcal{Z}\} \\ &= \sum_{P \in \mathcal{B}_{\mathcal{Z}}} i_P\left(\mathcal{S}, \mathcal{P}_{A,\mathcal{Z}}\right) + c_\nu(\mathcal{Z}), \end{aligned}$$

which gives (6). □

Proof of Theorem 1. Let $\mathcal{H} = V(\sum_{i=1}^{n+1} a_i x_i)$ be an hyperplane such that $\mathcal{H} \neq \mathcal{H}^\infty$. For every $m \in \mathcal{H}$, $n_{\mathcal{H}}(m)[a_1 : \cdots : a_n : 0] \in \mathbb{P}^n$. Hence every $A \in \mathbb{P}^n$ belong to a single normal line to $\mathcal{H}$ (the line containing A and $[a_1 : \cdots : a_n : 0]$).

The case $d_{\mathcal{Z}} \geq 2$ follows from Theorem 2 and Remark 17. □

3. Orthogonal incidence variety and Schubert classes

3.1. **Orthogonal incidence variety.** Let us write as usual $\mathbb{G}(1,n)$ (resp. $\mathbb{G}(n-1,n)$) for the grassmannian of the lines (resp. of the hyperplanes) of $\mathbb{P}^n$. Let us write $pr_1 : \mathbb{G}(1,n) \times \mathbb{G}(n-1,n) \to \mathbb{G}(1,n)$ and $pr_2 : \mathbb{G}(1,n) \times \mathbb{G}(n-1,n) \to \mathbb{G}(n-1,n)$ for the canonical projections. We define the **orthogonal incidence variety** $\mathcal{I}^\perp$ by

$$\mathcal{I}^\perp := \{(\mathcal{L}_1, \mathcal{H}_1) \in \mathbb{G}(1,n) \times \mathbb{G}(n-1,n) \, : \, \mathcal{L}_1 \perp \mathcal{H}_1\}.$$

Let us write $p_1 : \mathcal{I}^\perp \to \mathbb{G}(1,n)$ and $p_2 : \mathcal{I}^\perp \to \mathbb{G}(n-1,n)$ for the restrictions of pr_1 and pr_2. We want to describe in the Chow ring of $\mathbb{G}(1,n)$ and $\mathbb{G}(n-1,n) \equiv \mathbb{P}^{n\vee}$ the rational equivalence class of $p_2 p_1^{-1}(\mathcal{L})$ and $p_1 p_2^{-1}(\mathcal{H})$.

Lemma 25. *$p_2 \circ p_1^{-1} : \mathbb{G}(1,n) \setminus \{\mathcal{L} \subset \mathcal{H}^\infty\} \to \mathbb{G}(n-1,n)$ is a line projective bundle and $p_1 \circ p_2^{-1} : \mathbb{G}(n-1,n) \setminus \{\mathcal{H}^\infty\} \to \mathbb{G}(1,n)$ and is a plane projective bundle.*

Proof. Let $\mathcal{H} = V(a_1 x_1 + \cdots a_{n+1} x_{n+1})$ be a projective hyperplane of $\mathbb{P}^n$, which is not $\mathcal{H}^\infty$. Then

$$p_1(p_2^{-1}(\mathcal{H})) = \{\mathcal{L} \in \mathbb{G}(1,n), \ (a_1, \cdots, a_n, 0) \in \mathbf{L}\}.$$

Moreover $p_1(p_2^{-1}(\mathcal{H}^\infty)) = \mathbb{G}(1,n)$.
Let $\mathcal{L} \not\subset \mathcal{H}^\infty$ be a line of $\mathbb{P}^3$, let $A_0[a_1 : \cdots : a_n : 0]$ be the only point in $\mathcal{L} \cap \mathcal{H}^\infty$, we have

$$p_2(p_1^{-1}(\mathcal{L})) = \{\mathcal{H} \in \mathbb{G}(n-1,n) \, : \, \exists [a:b] \in \mathbb{P}^1, \ \mathcal{H} = V(aa_1x_1 + \cdots + aa_nx_n + bx_{n+1})\}.$$

Finally, if $\mathcal{L} = \mathbb{P}(\mathbf{L}) \subset \mathcal{H}^\infty$ is a projective line, then we have

$$p_2(p_1^{-1}(\mathcal{L})) = \{\mathcal{H} \in \mathbb{G}(n-1,n) \, : \, \exists a, b \in \mathbb{C}, \ \exists (a_1, \cdots, a_n, 0) \in \mathbf{L}, \ \mathcal{H} = V(aa_1x_1 + \cdots + aa_nx_n + bx_{n+1})\}.$$

□

It follows directly from the proof of this lemma that if $\mathcal{H} \in \mathbb{G}(n-1,n) \setminus \{\mathcal{H}^\infty\}$, the class of $p_1(p_2^{-1}(\mathcal{H}))$ in the Chow ring $A^*(\mathbb{G}(1,n))$ is simply the Schubert class σ_{n-1}.

3.2. **Schubert classes for $\mathbb{G}(1,3)$.** Given a flag $\mathbf{F} = \{\mathbf{V_1} \subset \mathbf{V_2} \subset \mathbf{V_3} \subset \mathbf{V_4} = \mathbf{V}\}$ of $\mathbf{V}$ with $\dim_{\mathbb{C}}(V_i) = i$ for all integer i, we consider its associated projective flag $\mathcal{F}$ of $\mathbb{P}^3$ (image by the canonical projection $\pi : \mathbf{V} \setminus \{\mathbf{0}\} \to \mathbb{P}^3$)

$$\mathcal{F} = \{p \in \mathcal{D} \subset \mathcal{P} \subset \mathbb{P}^3\}.$$

Let $\mathcal{Z}^k$ denote the set of cycles of codimension k in $\mathbb{G}(1,3)$. We recall that the **Schubert cycles** of $\mathbb{G}(1,3)$ associated to $\mathcal{F}$ (or to $\mathbf{F}$) are given by

$$\left\{\begin{array}{c} \Sigma_{0,0} := \mathbb{G}(1,3) \in \mathcal{Z}^0(\mathbb{G}(1,3)) \\ \Sigma_{1,0} := \{\mathcal{L} \in \mathbb{G}(1,3); \mathcal{D} \cap \mathcal{L} \neq \varnothing\} \in \mathcal{Z}^1(\mathbb{G}(1,3)) \\ \Sigma_{2,0} := \{\mathcal{L} \in \mathbb{G}(1,3); p \in \mathcal{L}\} \in \mathcal{Z}^2(\mathbb{G}(1,3)) \\ \Sigma_{1,1} := \{\mathcal{L} \in \mathbb{G}(1,3); \mathcal{L} \subset \mathcal{P}\} \in \mathcal{Z}^2(\mathbb{G}(1,3)) \\ \Sigma_{2,1} := \Sigma_{2,0} \cap \Sigma_{1,1} \in \mathcal{Z}^3(\mathbb{G}(1,3)) \\ \Sigma_{2,2} := \{\mathcal{L} \in \mathbb{G}(1,3); \mathcal{L} = \mathcal{D}\} \in \mathcal{Z}^2(\mathbb{G}(1,3)) \end{array}\right. . \tag{8}$$

We write as usual $A^*(\mathbb{G}(1,3))$ for the Chow ring of $\mathbb{G}(1,3)$ and $\sigma_{i,j} := [\Sigma_{i,j}] \in A^{i+j}(\mathbb{G}(1,3))$ for **Schubert classes**. For commodity we will use the notation $\Sigma_k := \Sigma_{k,0}$ and $\sigma_k := \sigma_{k,0}$. We recall that $A^*(\mathbb{G}(1,3))$ is freely generated as graded $\mathbb{Z}$-module by $\{\sigma_{i,j}; 0 \le j \le i \le 2\}$ with the following multiplicative relations

$$(E)\left\{\begin{array}{c}\sigma_{1,1} = \sigma_1^2 - \sigma_2 \\ \sigma_{1,1}\cdot\sigma_1 = \sigma_1\cdot\sigma_2 = \sigma_{2,1} \\ \sigma_{2,1}\cdot\sigma_1 = \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2} \\ \sigma_{1,1}\cdot\sigma_2 = 0\end{array}\right. .$$

Hence, the Chow ring of the grassmannian is

$$A^*(\mathbb{G}(1,3)) = \frac{\mathbb{Z}[\sigma_1,\sigma_2]}{(2\sigma_1\cdot\sigma_2 - \sigma_1^3, \sigma_1^2\cdot\sigma_2 - \sigma_2^2)}.$$

3.3. **Proofs of Theorems 3 and 7.** Recall that we have defined $\mathfrak{N}_{\mathcal{S}} := \overline{\{\mathcal{N}_m(\mathcal{S}); m \in \mathcal{S}\}} \subset \mathbb{G}(1,3)$ and $\mathfrak{n}_{\mathcal{S}} := [\mathfrak{N}_{\mathcal{S}}] \in A^2(\mathbb{G}(1,3))$.

Proposition 26. *Let $\mathcal{S} \subset \mathbb{P}^3$ be an irreducible surface of degree $d \ge 2$ of $\mathbb{P}^3$.*

- *If $\#\mathcal{B}_{\mathcal{S}} < \infty$, we have*
$$\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}}-1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3)).$$
- *If $\dim \mathcal{B}_{\mathcal{S}}^{(0)} = 2$ with two dimensional part $V(H)$ and $\#\tilde{\mathcal{B}}_{\mathcal{S}} < \infty$ (with the notations of Proposition 23), then we have*
$$\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}} - d_H - 1).\sigma_{1,1} \in A^2(\mathbb{G}(1,3)).$$

Proof. Since $\mathfrak{n}_{\mathcal{S}} \in A^2(\mathbb{G}(1,3))$, we have $\mathfrak{n}_{\mathcal{S}} = a.\sigma_2 + b.\sigma_{1,1}$ for some integers a and b. Morever by Kleiman's transversality theorem (see for example [3, Thm 5.20]), since $\Sigma_{1,1} := \{\mathcal{L} \in \mathbb{G}(1,3); \mathcal{L} \subset \mathcal{P}\} \in \mathcal{Z}^2(\mathbb{G}(1,3))$, we have $\mathfrak{n}_{\mathcal{S}} \cdot \sigma_{1,1} = (a\sigma_2 + b\sigma_{1,1}) \cdot \sigma_{1,1}$ and so, using (8), we obtain

$$\mathfrak{n}_{\mathcal{S}} \cdot \sigma_{1,1} = b.\sigma_{1,1}^2 = b.\sigma_{2,2} = b. \tag{9}$$

Analogously, since $\Sigma_2 := \{\mathcal{L} \in \mathbb{G}(1,3); p \in \mathcal{L}\} \in \mathcal{Z}^2(\mathbb{G}(1,3))$, due to (8), we have

$$\mathfrak{n}_{\mathcal{S}} \cdot \sigma_2 = (a\sigma_2 + b\sigma_{1,1}) \cdot \sigma_2 = a\sigma_2^2 = a\sigma_{2,2} = a. \tag{10}$$

Now it remains to compute $\mathfrak{n}_{\mathcal{S}} \cdot \sigma_2$ and $\mathfrak{n}_{\mathcal{S}} \cdot \sigma_{1,1}$, i.e. to compute the cardinality of the intersection of $\mathfrak{N}_{\mathcal{S}}$ with $\Sigma_{1,1}$ and with Σ_2.
Let us start with the computation of $a = \mathfrak{n}_{\mathcal{S}} \cdot \sigma_2$. If $\#\mathcal{B}_{\mathcal{S}} < \infty$, then, for a generic $P \in \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_2 = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; P \in \mathcal{L}\} = \{\mathcal{N}_m\mathcal{S}; m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}},\ P \in \mathcal{N}_m\mathcal{S}\}$$

and if $\dim \mathcal{B}_{\mathcal{S}}^{(0)} = 2$ and $\#\tilde{\mathcal{B}}_{\mathcal{Z}}^{(0)} \cap \mathcal{S} < \infty$, then, for a generic $P \in \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_2 = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; P \in \mathcal{L}\} = \left\{\mathcal{N}_m\mathcal{S}; m \in \mathcal{S} \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)},\ P \in \mathcal{N}_m\mathcal{S}\right\}.$$

So, in any case, $a = c_\nu(\mathcal{S})$ by definition of the normal class of $\mathcal{S}$.
Now, for b, since $\#\mathcal{B}_{\mathcal{S}} < \infty$, we note that, for a generic projective plane $\mathcal{H} \subset \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_{1,1} = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; \mathcal{L} \subset \mathcal{H}\} = \{\mathcal{N}_m\mathcal{S};\ m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}},\ \mathcal{N}_m\mathcal{S} \subset \mathcal{H}\}.$$

We have $\mathcal{H} = V(a_1X + a_2Y + a_3Z + a_4T) \subset \mathbb{P}^3$ for some complex numbers a_1, a_2, a_3 and a_4. Let $m[x:y:z:t] \in \mathbb{P}^3$. For a generic $\mathcal{H}$, we have

$$\begin{aligned} m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}},\ \mathcal{N}_m\mathcal{S} \subset \mathcal{H} \quad &\Leftrightarrow \quad m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}},\ \ m \in \mathcal{H},\ \ n_{\mathcal{S}}(m) \in \mathcal{H} \\ &\Leftrightarrow \quad \left\{\begin{array}{c} F(x,y,z,t) = 0 \\ a_1F_x + a_2F_y + a_3F_z = 0 \\ a_1x + a_2y + a_3z + a_4t = 0 \end{array}\right. . \end{aligned}$$

Hence $b = d_{\mathcal{S}}(d_{\mathcal{S}} - 1)$. Assume now that $\dim \mathcal{B}_{\mathcal{S}}^{(0)} = 2$ with two dimensional part $V(H)$ and $\#\tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \cap \mathcal{S} < \infty$. For a generic projective plane $\mathcal{H} = V(A^\vee) \subset \mathbb{P}^3$, we have

$$\begin{aligned} \mathfrak{N}_{\mathcal{S}} \cap \Sigma_{1,1} &= \left\{ \mathcal{N}_m\mathcal{S};\ m \in \mathcal{S} \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)},\ \mathcal{N}_m\mathcal{S} \subset \mathcal{H} \right\} \\ &= \{\mathcal{N}_m\mathcal{S};\ m \in \mathcal{S} \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)},\ \ m \in \mathcal{H},\ \ n_{\mathcal{S}}(m) \in \mathcal{H}\}. \end{aligned}$$

Now there are two cases:

- If H divides F_x, F_y and F_z and then $n_{\mathcal{S}} = [\frac{F_x}{H} : \frac{F_y}{H} : \frac{F_z}{H}]$ and $b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H - 1)$.
- Otherwise $H = tH_1$, with H_1 dividing F_x, F_y and F_z and $V(X) \subset V(xF_y - yF_x, xF_z - zF_x, yF_z - zF_y)$. Hence $n_{\mathcal{S}} = [\frac{F_x}{H_1} : \frac{F_y}{H_1} : \frac{F_z}{H_1}]$. We have

$$m \in \mathcal{S} \setminus (\mathcal{H}^\infty \cup \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)}),\ \mathcal{N}_m\mathcal{S} \subset \mathcal{H} \quad \Leftrightarrow \quad \begin{cases} F(x,y,z,t) = 0, \quad t \neq 0 \\ a_1\frac{F_x}{H_1} + a_2\frac{F_y}{H_1} + a_3\frac{F_z}{H_1} = 0 \\ a_1x + a_2y + a_3z + a_4t = 0 \end{cases}$$

and

$$m \in \mathcal{S}_\infty \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)},\ \mathcal{N}_m\mathcal{S} \subset \mathcal{H} \quad \Leftrightarrow \quad \begin{cases} F(x,y,z,t) = 0 \\ t = 0 \\ a_1x + a_2y + a_3z = 0 \end{cases},$$

so

$$b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H) + d_{\mathcal{S}} - \sum_{P \in \mathcal{S} \cap \mathcal{H}^\infty \cap \mathcal{H}} i_P\left(\mathcal{S}, V(a_1\frac{F_x}{H_1} + a_2\frac{F_y}{H_1} + a_3\frac{F_z}{H_1}), \mathcal{H}\right)$$

(due to the Bezout Theorem). Now let $P \in \mathcal{S} \cap \mathcal{H}^\infty \cap \mathcal{H}$, we have $x \neq 0$ or $y \neq 0$ or $z \neq 0$. Assume for example that $x \neq 0$, we have

$$\begin{aligned} & i_P\left(\mathcal{S}, V(a_1\frac{F_x}{H_1} + a_2\frac{F_y}{H_1} + a_3\frac{F_z}{H_1}), \mathcal{H}\right) = \\ = \ & i_P\left(\mathcal{S}, V(t(-a_4\frac{F_x}{H_1} + a_2\frac{xF_y - yF_x}{H} + a_3\frac{xF_z - zF_x}{H}), \mathcal{H}\right) \\ = \ & 1 + i_P\left(\mathcal{S}, V(-a_4\frac{F_x}{H_1} + a_2\frac{xF_y - yF_x}{H} + a_3\frac{xF_z - zF_x}{H}), \mathcal{H}\right) \\ = \ & 2 \end{aligned}$$

for a generic $\mathcal{H}$ and so $b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H + 1)$.

□

Theorem 3 follows from Theorem 2 and Proposition 26.

Proof of Theorem 7. If $\dim \mathcal{B}_{\mathcal{S}}^{(0)} = 2$, we saw in Proposition 23 that we can adapt our study to compute the degree of the reduced normal polar $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ associated to the rational map $\tilde{\alpha}_{\mathcal{S}} : \mathbb{P}^3 \setminus \tilde{\mathcal{B}}_{\mathcal{S}}^{(0)} \to \mathbb{P}^3$ such that $\boldsymbol{\alpha}_{\mathcal{S}} = H \cdot \tilde{\boldsymbol{\alpha}}_{\mathcal{S}}$. Using Proposition 23 and following the proof of Theorem 2, we obtain Theorem 7. □

4. Proof of Theorem 5

We apply Theorem 3. Note that, since $\mathcal{S}$ is smooth, it has only a finite number of points of tangency with $\mathcal{H}_\infty$ (due to Zak's theorem on tangencies [13, corolloray 1.8]). Since the surface is smooth, $\mathcal{B}_{\mathcal{S}}$ consists of points of tangency of $\mathcal{S}$ with $\mathcal{H}_\infty$ and of points of tangency of $\mathcal{S}_\infty$ with

$\mathcal{U}_\infty$. It remains to compute the intersection multiplicity of $\mathcal{S}$ with a generic normal polar at these points. Let us recall that if $A \notin \mathcal{H}^\infty$, then

$$i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \dim_{\mathbb{C}} \left((\mathbb{C}[x,y,z,t]/I)_P \right)$$

where I is the ideal $(F, E_{A,1,2}E_{A,1,3}, E_{A,2,3})$ of $\mathbb{C}[x,y,z,t]$, with the notation $E_{A,i,j}$ introduced in the proof of Proposition 22.

To compute these quantities, it may be useful to make an appropriate change of coordinates with the use of a projective similitude of $\mathbb{P}^3$. Note that:

* The umbilical $\mathcal{U}_\infty$ is stable under the action of the group of projective similitudes of $\mathbb{P}^3$.
* For any $P \in \mathcal{H}_\infty \setminus \mathcal{U}_\infty$, there exists a projective similitude ζ of $\mathbb{P}^3$ mapping $[1:0:0:0]$ to P. [2]
* For any $P \in \mathcal{U}_\infty$, there exists a projective similitude ζ of $\mathbb{P}^3$ mapping $[1:i:0:0]$ to P. [3]

We recall that a multiple point of order k of a plane curve is ordinary if its tangent cone contains k pairwise distinct lines and that an ordinary cusp of a plane curve is a double point with a single tangent line in the tangent cone, this tangent line being non contained in the cubic cone of the curve at this point.

- Let P be a (non singular) point of tangency of $\mathcal{S}$ with $\mathcal{H}_\infty$.

  We prove the following:
  (a) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k^2$ for a generic $A \in \mathbb{P}^3$ if P is an ordinary multiple point of order $k+1$ of $\mathcal{S}_\infty \setminus \mathcal{U}_\infty$.
  (b) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k(k+1)$ for a generic $A \in \mathbb{P}^3$ if P is an ordinary multiple point of order $k+1$ of $\mathcal{S}_\infty$, which belongs to $\mathcal{U}_\infty$ and at which the tangent line to $\mathcal{U}_\infty$ is not contain in the tangent cone of $\mathcal{S}_\infty$.
  (c) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 3$ for a generic $A \in \mathbb{P}^3$ if P is an ordinary cusp of $\mathcal{S}_\infty$, which belongs to $\mathcal{U}_\infty$ and at which the tangent line to $\mathcal{U}_\infty$ is not contain in the tangent cone of $\mathcal{S}_\infty$.
  (d) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 2$ for a generic $A \in \mathbb{P}^3$ if P is an ordinary cusp of $\mathcal{S}_\infty \setminus \mathcal{U}_\infty$.

  Due to Lemma 20, we assume that $P[1:\theta:0:0]$ with $\theta = 0$ (if $P \in \mathcal{H}_\infty \setminus \mathcal{U}_\infty$) or $\theta = i$ (if $P \in \mathcal{U}_\infty$). Since $\mathcal{T}_P\mathcal{S} = \mathcal{H}_\infty$, we suppose that $F_x(P) = F_y(P) = F_z(P) = 0$ and $F_t(P) = 1$ (without any loss of generality). Recall that the Hessian determinant H_F of F satisfies[4]

$$H_F = \frac{(d_{\mathcal{S}} - 1)^2}{x^2} \begin{vmatrix} 0 & F_y & F_z & F_t \\ F_y & F_{yy} & F_{yz} & F_{yt} \\ F_z & F_{yz} & F_{zz} & F_{zt} \\ F_t & F_{yt} & F_{zt} & F_{tt} \end{vmatrix}.$$

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[2]Let $P[x_0 : y_0 : z_0 : 0]$ with $x_0^2 + y_0^2 + z_0^2 = 1$. Assume for example that $z_0^2 \neq 1$ (up to a permutation of the coordinates) and take ζ given by $\kappa'(b, A)$ (for any $b \in \mathbb{C}^n$) with $A = (u\,v\,w)$ where $u = (x_0, y_0, z_0)$ and $v = (x_0^2 + y_0^2)^{-\frac{1}{2}}(y_0, -x_0, 0)$ and $w = (x_0^2 + y_0^2)^{-\frac{1}{2}}(x_0 z_0, y_0 z_0, -x_0^2 - y_0^2)$.

[3]Let $P[x_0 : y_0 : z_0 : 0] \in \mathcal{U}_\infty$. Assume for example that $y_0 \neq 0$ and $x_0^2 + y_0^2 \neq 0$ (up to a composition by a permutation matrix). A suitable ζ is given by $\kappa'(b, A)$ (for any $b \in \mathbb{C}^n$) with $A = \begin{pmatrix} \frac{x_0(y_0^2-1)}{2y_0^2} & -\frac{ix_0(1+y_0^2)}{2y_0^2} & \frac{\sqrt{x_0^2+y_0^2}}{y_0} \\ \frac{1+y_0^2}{2y_0^2} & \frac{i(1-y_0^2)}{y_0} & 0 \\ \frac{i(y_0^4+y_0^2x_0^2-y_0^2-x_0^2)}{y_0^2\sqrt{x_0^2+y_0^2}} & \frac{(1+y_0^2)\sqrt{x_0^2+y_0^2}}{2y_0^2} & \frac{ix_0}{y_0} \end{pmatrix}$.

[4]see for example [8].

Hence $H_F(P)\neq 0 \ \Leftrightarrow\ [F_{yy}F_{zz}-F_{yz}^2](P)\neq 0$. For a generic $A\in\mathbb{P}^3$, we have

$$\left(\frac{\mathbb{C}[x,y,z,t]}{I}\right)_P\cong\left(\frac{\mathbb{C}[x,y,z,t]}{(F,A_2,A_3)}\right)_P.$$

Recall that $A_2=atF_z-ctF_x+d(zF_x-xF_z)$ and $A_3=-atF_y+btF_x+d(xF_y-yF_x)$ (with $A[a:b:c:d]$). Using the Euler identity $xF_x+yF_y+zF_z+tF_t=d_{\mathcal{S}}F$, we obtain that

$$\left(\frac{\mathbb{C}[x,y,z,t]}{I}\right)_P\cong\left(\frac{\mathbb{C}[x,y,z,t]}{(F,A'_2,A'_3)}\right)_P\cong\left(\frac{\mathbb{C}[y,z,t]}{(F_*,A'_{2*},A'_{3*})}\right)_{(0,0,0)}$$

with

$$A'_2:=atxF_z+ct(yF_y+zF_z+tF_t)-d(z(yF_y+zF_z+tF_t)+x^2F_z),$$
$$A'_3:=-atxF_y-bt(yF_y+zF_z+tF_t)+d(x^2F_y+y(yF_y+zF_z+tF_t))$$

and with $G_*(y,z,t):=G(1,\theta+y,z,t)$ for any homogeneous G. In a neighbourhood of $(0,0,0)$, $V(F_*)$ is given by $t=\varphi(y,z)$ with $\varphi(y,z)\in\mathbb{C}[[y,z]]$ and

$$\varphi_y(y,z)=-\frac{F_y(1,\theta+y,z,\varphi(y,z))}{F_t(1,\theta+y,z,\varphi(y,z))}\quad\text{and}\quad\varphi_z(y,z)=-\frac{F_z(1,\theta+y,z,\varphi(y,z))}{F_t(1,\theta+y,z,\varphi(y,z))}.\tag{11}$$

So

$$\left(\frac{\mathbb{C}[x,y,z,t]}{I}\right)_P\cong\frac{\mathbb{C}[[y,z]]}{(A'_{2**},A'_{3**})},$$

with $G_{**}(y,z):=G(1,\theta+y,z,\varphi(y,z))$. Now due to (11), we have

$$H:=A'_{2**}=(F_t)_{**}[-a\varphi\varphi_z+c\varphi(\varphi-(\theta+y)\varphi_y-z\varphi_z)+d(z((\theta+y)\varphi_y+z\varphi_z-\varphi)+\varphi_z)]$$

and

$$K:=A'_{3**}=(F_t)_{**}[a\varphi\varphi_y-b\varphi(\varphi-(\theta+y)\varphi_y-z\varphi_z)-d(\varphi_y+(\theta+y)((\theta+y)\varphi_y+z\varphi_z-\varphi))].$$

Hence

$$i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=i_{(0,0)}(\Gamma_H,\Gamma_K),$$

where Γ_H and Γ_K are the analytic plane curves of respective equations H and K of $\mathbb{C}[[y,z]]$.

Note that $(H_y(0,0),H_z(0:0))=d(\varphi_{yz}(0,0),\varphi_{zz}(0,0))$. Analogously, we obtain $(K_y(0,0),K_z(0,0))=-d(1+\theta^2)(\varphi_{y,y}(0,0),\varphi_{yz}(0,0))$.

(a) If $P\notin\mathcal{U}_\infty$ and if P is an ordinary multiple point of order $k+1$ of $\mathcal{S}_\infty$, with our change of coordinates we have $P[1:0:0:0]$ (i.e. $\theta=0$) and $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.[5] Then the first homogeneous parts of H and K have order k and are $H_k=d(\varphi_{k+1})_z$ and $K_k=-d(\varphi_{k+1})_y$ respectively. Since Γ_{H_k} and Γ_{K_k} have no common lines, we conclude that $i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=k^2$.

(b) Assume now that $P\in\mathcal{U}_\infty$ is an ordinary multiple point of $\mathcal{S}_\infty$, which belongs to $\mathcal{U}_\infty$ and at which the tangent line to $\mathcal{U}_\infty$ is not contain in the tangent cone of $\mathcal{S}_\infty$. With our changes of coordinates, this means that $P[1:i:0:0]$ (i.e. $\theta=i$) that y does divide φ_{k+1} (since $V(y)$ is the tangent line to $\mathcal{U}_\infty$ at P) and that $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.

Note that the first homogeneous parts of H and K have respective orders k and $k+1$ and are respectively $H_k=d(\varphi_{k+1})_z$ and

$$\begin{aligned}K_{k+1}&=-di[2y(\varphi_{k+1})_y+z(\varphi_{k+1})_z-\varphi_{k+1}]\\&=-di\left[\left(2-\frac{1}{k+1}\right)y(\varphi_{k+1})_y+\left(1-\frac{1}{k+1}\right)z(\varphi_{k+1})_z\right],\end{aligned}$$

[5]Recall that the tangent cone $V(\varphi_{k+1})$ of Γ_φ at $(0,0)$ (corresponding to the tangent cone of $V(\mathcal{S}_\infty)$ at P) has pairwise distinct tangent lines if and only if $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.

due to the Euler identity applied to φ_{k+1}. Hence $i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=k(k+1)$ if $V((\varphi_{k+1})_z)$ and $V(y(\varphi_{k+1})_y)$ have no common lines, which is true since $V((\varphi_{k+1})_z)$ and $V((\varphi_{k+1})_y)$ have no common lines and since y does not divide $(\varphi_{k+1})_z$.

(c) Assume that $P\in\mathcal{U}_\infty$ is an ordinary cusp (of order 2) of $\mathcal{S}_\infty$, which belongs to $\mathcal{U}_\infty$ and at which the tangent line to $\mathcal{U}_\infty$ is not contain in the tangent cone of $\mathcal{S}_\infty$. With our changes of coordinates, this means that $P[1:i:0:0]$ (i.e. $\theta=i$), that $H_F(P)=0$ and $F_{zz}(P)\neq 0$ (since $V(y)$ is the tangent line to $\mathcal{U}_\infty$ at P).
Note that $H=-d(F_{yz}(P)y+F_{zz}(P)z)+\dots$. In a neighbourhood of $(0,0)$, $H(y,z)=0 \Leftrightarrow z=h(y)$ with
$h(y)=-\frac{F_{yz}(P)}{F_{zz}(P)}y-\frac{F_{zzz}(P)F_{yz}^2(P)-2F_{yzz}(P)F_{yz}(P)F_{zz}(P)+F_{yyz}(P)F_{zz}^2(P)}{F_{zz}^3(P)}y^2+\dots$
and we obtain that $\mathrm{val}_y\,K(y,h(y))=3$ if $P\notin V(F_{yyy}F_{zz}^3-3F_{yyz}F_{zz}^2F_{yz}+3F_{yzz}F_{zz}F_{yz}^2-F_{zzz}^3F_{yz}^3)$ which means that the line $V(F_{yz}(P)y+F_{zz}(P)z)$ (corresponding to the tangent cone of $V(F(1,y,z,0))$) is not contained in the cubic cone of $V(F(1,y,z,0))$. Hence $i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=3$ if the node P of $\mathcal{S}_\infty$ is ordinary.

(d) Assume now that P is an ordinary cusp (of order 2) of $\mathcal{S}_\infty\setminus\mathcal{U}_\infty$.
With our change of coordinates, this means that $P[1:0:0:0]$ (i.e. $\theta=0$), that $H_F(P)=0$ and $P\notin V(F_{yy},F_{yz},F_{zz})$. This implies that $F_{yy}(P)\neq 0$ or $F_{zz}(P)\neq 0$.
If $F_{yy}(P)\neq 0$, the tangent line to $\mathcal{S}_\infty$ at P is given by $V(t,F_{yy}(P)y+F_{yz}(P)z)$.
If $F_{zz}(P)\neq 0$, the tangent line to $\mathcal{S}_\infty$ at P is given by $V(t,F_{zz}(P)z+F_{yz}(P)y)$.
The fact that the cusp is ordinary implies also that the tangent line is not contained in the cubic cone of $V(F(1,y,z,0))$, i.e. this tangent line is not contained $V(F_{yyy}(P)y^3+3F_{yyz}(P)y^2z+3F_{yzz}(P)yz^2+F_{zzz}(P)z^3](P)$.
Hence, we have either

$$F_{yy}[F_{yyy}F_{yz}^3-3F_{yyz}F_{yz}^2F_{yy}+3F_{yzz}F_{yz}F_{yy}^2-F_{zzz}F_{yy}^3](P)\neq 0$$

or

$$F_{zz}[F_{zzz}F_{yz}^3-3F_{yzz}F_{yz}^2F_{zz}+3F_{yyz}F_{yz}F_{zz}^2-F_{yyy}F_{zz}^3](P)\neq 0.$$

Note that, if $F_{yy}(P)$ and $F_{zz}(P)$ are both non null, these two conditions are equivalent.
Assume for example that the first condition holds. In a neighbourhood of $(0,0)$, $H(y,z)=0\Leftrightarrow y=h(z)$ and $K(y,z)=0\Leftrightarrow y=k(z)$, with

$$h'(z)=-\frac{H_z(h(z),z)}{H_y(h(z),z)}\quad\text{and}\quad k'(z)=-\frac{K_z(h(z),z)}{K_y(h(z),z)}.$$

Hence we have

$$\left(\frac{\mathbb{C}[x,y,z,t]}{I}\right)_P\cong\frac{\mathbb{C}[[y,z]]}{(y-h(z),y-k(z))}\cong\frac{\mathbb{C}[[z]]}{((h-k)(z))}.$$

We have $h'(0)=k'(0)=-\varphi_{yz}(0,0)/\varphi_{yy}(0,0)$ and
$(h''-k'')(0)=\left[\varphi_{yy}^3\varphi_{zy}\left(\varphi_{yyy}\varphi_{yz}^3-3\varphi_{yyz}\varphi_{yz}^2\varphi_{yy}+3\varphi_{yzz}\varphi_{yy}^2\varphi_{yz}-\varphi_{zzz}\varphi_{yy}^3\right)\right](0,0)$

Hence $i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=\dim_{\mathbb{C}}\frac{\mathbb{C}[[z]]}{((h-k)(z))}=2$.

- Let P be a simple (non singular) point of tangency of $\mathcal{S}_\infty$ with $\mathcal{U}_\infty$.

Let us prove that $i_P(\mathcal{S},\mathcal{P}_{A,\mathcal{S}})=1$. Due to Lemma 20, we can assume that $P=[1:i:0:0]$ (i.e. $\theta=i$) and that $F_t(P)=1$. As previously, we note that

$$\left(\frac{\mathbb{C}[x,y,z,t]}{I}\right)_P\cong\frac{\mathbb{C}[[y,z]]}{(A_{1**},A'_{3**})},$$

with

$$A_{1**}(y,z)=(F_t)_{**}[\varphi(b\varphi_z-c\varphi_y)-d((y+i)\varphi_z-z\varphi_y)]$$

and

$$A_{3**} = (F_t)_{**}[a\varphi\varphi_y - b\varphi(\varphi - (\theta + y)\varphi_y - z\varphi_z) - d(\varphi_y + (i + y)((i + y)\varphi_y + z\varphi_z - \varphi))].$$

The fact that P is a simple contact point of $\mathcal{S}_\infty$ with $\mathcal{U}_\infty$ implies that $[F_x(P) : F_y(P) : F_z(P)] = [1 : i : 0]$ and that $\varphi_{zz}(0,0) \neq 1$. Indeed $V(t, F(1, i + y, z, t))$ is given by $t = 0$, $y = g(z)$ with $g(0) = 0$ and $g'(z) = -\varphi_z(g(z), z)/\varphi_y(g(z), z)$ (in particular $g'(0) = 0$), so

$$\frac{\mathbb{C}[[y, z]]}{(y - g(z), 1 + (i + y)^2 + z^2)} \cong \frac{\mathbb{C}[[z]]}{(1 + (i + g(z))^2 + z^2)}$$

and finally

$$i_P(\mathcal{S}_\infty, \mathcal{U}_\infty) = \mathrm{val}_z(1 + (i + g(z))^2 + z^2) = 1 + \mathrm{val}_z((i + g(z))g'(z) + z)$$

which is equal to 2 if and only if $\varphi_{zz}(0,0) \neq 1$.

In a neighbourhood of $(i, 0)$, A_{1**} can be rewritten $\varphi - \kappa$ with $\kappa = d((y + i)\varphi_z - z\varphi_y)/(b\varphi_z - c\varphi_y)$. Since $\varphi_{zz}(0) \neq 1$, $\kappa_z(0,0) \neq 0$ and, in a neighbourhood of 0, $\varphi - \kappa = 0$ corresponds to $y = h(z)$ with $h'(0) \neq 0$ (recall that $\varphi_y(0) = i \neq 0$ and that A is generic) which gives

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I}\right)_P \cong \frac{\mathbb{C}[[y, z]]}{(y - h(z), A'_{3**})}$$

and finally $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \mathrm{val}_z\, A'_{3**}(h(z), z) = 1$.

5. Examples in $\mathbb{P}^3$

5.1. **Normal class of quadrics.** The aim of the present section is the study of the normal class of every irreducible quadric. Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be an irreducible quadric. We recall that, up to a composition by $\varphi \in \widehat{Sim_{\mathbb{C}}}(3)$, one can suppose that F has the one of the following forms:

$$\begin{aligned}
&(a) \quad F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2 + t^2\\
&(b) \quad F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2\\
&(c) \quad F(x, y, z, t) = x^2 + \alpha y^2 - 2tz\\
&(d) \quad F(x, y, z, t) = x^2 + \alpha y^2 + t^2,
\end{aligned}$$

with α, β two non zero complex numbers. Spheres, ellipsoids and hyperboloids are particular cases of (a), paraboloids (including the saddle surface) are particular cases of (c), (b) correspond to cones and (d) to cylinders.

We will see, in Appendix A, that in the case (d) (cylinders) and in the cases (a) and (b) with $\alpha = \beta = 1$, the normal class of the quadric is naturally related to the normal class of a conic.

Proposition 27. *The normal class of a sphere is 2.*

The normal class of a quadric $V(F)$ with F given by (a) is 6 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (a) is 4 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 4 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (b) is 2 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 0 if $\alpha = \beta = 1$.

The normal class of a quadric $V(F)$ with F given by (c) is 5 if $\alpha \neq 1$ and 3 if $\alpha = 1$.

The normal class of a quadric $V(F)$ with F given by (d) is 4 if $\alpha \neq 1$ and 2 if $\alpha = 1$.

Corollary 28. *The normal class of the saddle surface $\mathcal{S}_1 = V(xy - zt)$ is 5.*

The normal class of the ellipsoid $\mathcal{E}_1 = V(x^2 + 2y^2 + 4z^2 - t^2)$ with three different length of axis is 6.

The normal class of the ellipsoid $\mathcal{E}_2 = V(x^2 + 4y^2 + 4z^2 - t^2)$ with two different length of axis is 4.

Proof of Proposition 27. Let $\mathcal{S} = V(F)$ be a quadric with F of the form (a), (b), (c) or (d).

- The easiest cases is (a) with $1, \alpha, \beta$ pairwise distinct since $\mathcal{B}_{\mathcal{S}}$ is empty. In this case, since the generic degree of the normal polar curves is 3 and since $\mathcal{E}_1$ has degree 2, we simply have $c_\nu(\mathcal{E}_1) = 2 \cdot 3 = 6$ (due to Theorem 3).
- The case of a sphere $\mathcal{S}$ is analogous. In this case, $\tilde{\mathcal{B}}^{(0)}_{\mathcal{S}} \cap \mathcal{S} = \emptyset$ and $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = 1$ for a generic $A \in \mathbb{P}^3$ (see Example 24). Hence, we have $c_\nu(\mathcal{E}_1) = 2 \cdot 1 = 2$ (due to Theorem 7).
- In case (a) with $\alpha = 1 \neq \beta$, the set $\mathcal{B}_{\mathcal{S}}$ contains two points $[1 : \pm i : 0 : 0]$. We find the parametrization $\psi(y) = [1 : \pm i + y : 0 : 0]$ of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of $P[1 : \pm i : 0 : 0]$, which gives $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \mathrm{val}_z(1 + (\pm i + y)^2) = 1$ and so $c_\nu(\mathcal{S}) = 2 \cdot 3 - 1 - 1 = 4$.
- In case (b) with α, β and 1 are pairwise distinct, the set $\mathcal{B}_{\mathcal{S}}$ contains a single point $P[0 : 0 : 0 : 1]$ and a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ in a neighbourhood of P is
$$\psi(x) = \left[x : -\frac{bx}{d(\alpha - 1)x - a} : \frac{cx}{a + d(1 - \beta)x} : 1\right]. \tag{12}$$
Hence $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \mathrm{val}_x(F(\psi(x))) = 2$ and so $c_\nu(\mathcal{S}) = 2 \cdot 3 - 2 = 4$.
- In case (b) with $\alpha = 1 \neq \beta$, we have $\mathcal{B}_{\mathcal{S}} = \{P, P'_+, P'_-\}$ with $P[0 : 0 : 0 : 1]$ and $P'_\pm[1 : \pm i : 0 : 0]$. A parametrization of $\mathcal{P}_{A,\mathcal{S}}$ in a neighbourhood of P is given by (12) with $\alpha = 1$ and so $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 2$. A parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of $P'_\pm$ is $\psi(z) = [1 : \pm i + y : 0 : 0]$ and so $i_{P'_\pm}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \cdot 3 - 2 - 1 - 1 = 2$.
- In case (b) with $\alpha = \beta = 1$, for a generic $A \in \mathbb{P}^3$, we have $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = 1$ (see Example 24) but here $\tilde{\mathcal{B}}^{(0)}_{\mathcal{S}} \cap \mathcal{S} = \{[0 : 0 : 0 : 1]\}$. We find the parametrization $\psi(x) = [x : (bx/a) : (cx/a) : 1]$ of $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ at the neighbourhood of $P[0 : 0 : 0 : 1]$. Hence $i_P(\mathcal{S}, \tilde{\mathcal{P}}_{A,\mathcal{S}}) = 2$ and so $c_\nu(\mathcal{S}) = 2 \cdot 1 - 2 = 0$.
- In case (c) with $\alpha \neq 1$, the only point of $\mathcal{B}_{\mathcal{S}}$ is $P_1[0 : 0 : 1 : 0]$ and a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of this point is
$$\psi(t) = \left[\frac{at^2}{d + (d - c)t} : \frac{bt^2}{t(d - c\alpha) + \alpha d} : 1 : t\right], \tag{13}$$
which gives $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \times 3 - 1 = 5$.
- In case (c) with $\alpha = 1$, $\mathcal{B}_{\mathcal{S}}$ is made of three points: $P_1[0 : 0 : 1 : 0]$, $P_{2,\pm}[1 : \pm i : 0 : 0]$. As in the previous case, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of P_1 is given by (13) with $\alpha = 1$ and so $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Now, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of $P_{2,\pm}$ is $\psi(t) = [1 : \pm i + y : 0 : 0]$ and so $i_{P_{2,\pm}} = (\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \mathrm{val}_y(1 + (y \pm i)^2) = 1$.
- For the case (d), due to Proposition 30, $c_\nu(\mathcal{S}) = c_\nu(\mathcal{C})$ with $\mathcal{C} = V(x^2 + \alpha y^2 + z^2) \subset \mathbb{P}^2$ which is a circle if $\alpha = 1$ and an ellipse otherwise. Hence, due to Theorem 8, $c_\nu(\mathcal{C}) = 2 + 2 - 0 - 1 - 1 = 2$ if $\alpha = 1$ and $c_\nu(\mathcal{C}) = 2 + 2 = 4$ otherwise.

□

5.2. **Normal class of a cubic surface with singularity E_6.** Consider $S = V(F) \subset \mathbb{P}^3$ with $F(x, y, z, t) := x^2 z + z^2 t + y^3$. $\mathcal{S}$ is a singular cubic surface with E_6−singularity at $p[0 : 0 : 0 : 1]$. Let a generic $A[a : b : c : d] \in \mathbb{P}^3$. The ideal of the normal polar $\mathcal{P}_{\mathcal{S},A}$ is given by

$I(\mathcal{P}_{\mathcal{S},A}) = \langle H_1, H_2, H_3\rangle \subset \mathbb{C}[x,y,z,t]$ with $H_1 := (y(x^2+2zt) - 3y^2z)d - b(x^2+2zt)t + 3y^2ct$, $H_2 := (x(x^2+2zt) - 2xz^2)d - a(x^2+2zt)t + 2xztc$ and $H_3 := (-2xzy + 3xy^2)d - 3ay^2t + 2xztb$. $\mathcal{B}_{\mathcal{S}}$ is made of two points: p and $q[0:0:1:0]$. Actually q is the point of tangency of $\mathcal{S}$ with $\mathcal{H}_\infty$. This point is an ordinary cusp of $\mathcal{S}_\infty$.

(1) Study at p.
Near p the ideal of the normal polar, in the chart $t=1$, $H_3 = 0$ gives $z = g(x,y) := \frac{3y^2(-xd+a)}{2x(-yd+b)}$. Now $V(A_1(x,y,g(x,y),1))$ corresponds to a quintic with a cusp at the origine (and with tangent cone $V(y^2)$). Its single branch has Puiseux expansion $y^2 = -\frac{b}{3a}x^3 + o(x^3)$, with probranches $y = \varphi_\varepsilon(x)$ with $\varphi_\varepsilon(x) = i\varepsilon\sqrt{\frac{b}{3a}}x^{\frac{3}{2}} + o(x^{\frac{3}{2}})$ for $\varepsilon \in \{\pm 1\}$. Hence, $g(x,\varphi_\varepsilon(x)) = -\frac{x^2}{2} + o(x^2)$. Hence parametrizations of the probranches of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of p are

$$\Gamma_\varepsilon(x) = [x : \varphi_\varepsilon(x) : g(x,\varphi_\varepsilon(x)) : 1]$$

and $F(\Gamma_\varepsilon(x)) = -\frac{x^4}{4} + o(x^4)$. Therefore $i_p(\mathcal{P}_{\mathcal{S},A},\mathcal{S}) = 8$.

(2) Study at q.
Assume that $b=1$. Near $q[0:0:1:0]$, in the chart $z=1$, $H_3 = 0$ gives $t = h(x,y) := \frac{d(-2+3y)xy}{3ay^2-2x}$ and $V(H_2(x,y,1,h(x,y)))$ is a quartic with a (tacnode) double point in $(0,0)$ with vertical tangent and which has Puiseux expansion

$$x = \theta_\varepsilon(y) = \omega_\varepsilon a\, y^2 + o(y^2),$$

with $\omega_\varepsilon = \frac{3-d}{2} + \frac{\varepsilon}{2}\sqrt{d(d-6)}$ for $\varepsilon \in \{\pm 1\}$ and $h(\theta_\varepsilon(y),y) = -\frac{2d\omega_\varepsilon}{3-2\omega_\varepsilon}y + o(y)$. Hence parametrizations of the probranches of $\mathcal{P}_{\mathcal{S},A}$ in a neighbourhood of q are given by

$$\Gamma_\varepsilon(y) := [\theta_\varepsilon(y) : y : 1 : h(\theta_\varepsilon(y),y)]$$

for $\varepsilon \in \{\pm 1\}$ and $F(\Gamma_\varepsilon(y)) = -\frac{2\omega_\varepsilon d}{3-2\omega_\varepsilon}y + o(y)$. Hence $i_q(\mathcal{P}_{\mathcal{S},A},\mathcal{S}) = 2$
We can also apply directly Item (c) of Section 4 to prove that $i_q(\mathcal{P}_{\mathcal{S},A},\mathcal{S}) = 2$.

Therefore, due to Theorem 3, the normal class of $\mathcal{S} = V(x^2z + z^2t + y^3) \subset \mathbb{P}^3(\mathbb{C})$ is

$$c_\nu(\mathcal{S}) = 3\cdot(3^2 - 3 + 1) - 8 - 2 = 11.$$

6. Normal class of plane curves : Proof of Theorem 8

Let $\mathbf{V}$ be a three dimensional complex vector space and set $\mathbb{P}^2 := \mathbb{P}(\mathbf{V})$ with projective coordinates x,y,z. We denote by $\ell_\infty = V(z)$ the line at infinity.

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$. For any nonsingular $m[x:y:z] \in \mathcal{C}$ (with coordinates $\mathbf{m} = (x,y,z) \in \mathbb{C}^3$), we write $\mathcal{T}_m\mathcal{C}$ for the tangent line to $\mathcal{C}$ at m. If $\mathcal{T}_m\mathcal{C} \neq \ell_\infty$, then $n_{\mathcal{C}}(m) = [F_x : F_y : 0]$ is well defined in $\mathbb{P}^2$ and **the projective normal line** $\mathcal{N}_m\mathcal{C}$ to $\mathcal{C}$ at m is the line $(m\, n_{\mathcal{C}}(m))$ if $n_{\mathcal{C}}(m) \neq m$. An equation of this normal line is then given by $\langle \mathbf{N}_{\mathcal{C}}(m), \cdot\rangle$ where $N_{\mathcal{C}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the rational map defined by

$$\mathbf{N}_{\mathcal{C}}(\mathbf{m}) := \mathbf{m} \wedge \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix} = \begin{pmatrix} -zF_y(m) \\ zF_x(m) \\ xF_y(m) - yF_x(m) \end{pmatrix}. \tag{14}$$

Lemma 29. *The base points of $(N_{\mathcal{C}})_{|\mathcal{C}}$ are the singular points of $\mathcal{C}$, the points of tangency with the line at infinity and the points of $\{I,J\} \cap \mathcal{C}$.*

Proof. A point $m \in \mathcal{C}$ is a base point of $N_\mathcal{C}$ if and only if $F_x = F_y = 0$ or $z = xF_y - yF_x = 0$. Hence, singular points of $\mathcal{C}$ are base points of $N_\mathcal{C}$.

Let $m = [x : y : z]$ be a nonsingular point of $\mathcal{C}$. First $F_x = F_y = 0$ is equivalent to $\mathcal{T}_m\mathcal{C} = \ell_\infty$. Assume now that $z = xF_y - yF_x = 0$ and $(F_x, F_y) \neq (0,0)$. Then $m = [x : y : 0] = [F_x : F_y : 0]$ and, due to the Euler formula, we have $0 = -zF_z = xF_x + yF_y$ and so $x^2 + y^2 = 0$, which implies $m = I$ or $m = J$.

Finally note that if $m \in \{I, J\} \cap \mathcal{C}$, then $m = [-y : x : 0]$ and, due to the Euler formula, $0 = -zF_z = xF_x + yF_y = xF_y - yF_x$. □

Since the degree of each non zero coordinate of $N_\mathcal{C}$ is d, we have

$$c_\nu(\mathcal{C}) = d^2 - \sum_{P \in \mathrm{Base}\left((N_\mathcal{C})_{|\mathcal{C}}\right)} i_P(\mathcal{C}, V(\langle L, \mathbf{N}_\mathcal{C}(\cdot)\rangle)), \tag{15}$$

for a generic $L \in \mathbb{P}^2$, where we write $\mathrm{Base}\left((N_\mathcal{C})_{|\mathcal{C}}\right)$ for the set of base points of $(N_\mathcal{C})_{|\mathcal{C}}$. The set $V(\langle L, \mathbf{N}_\mathcal{C}(\cdot)\rangle) \subset \mathbb{P}^2$ is called **the normal polar** of $\mathcal{C}$ with respect to L. It satisfies

$$m \in V(\langle L, \mathbf{N}_\mathcal{C}(\cdot)\rangle) \quad \Leftrightarrow \quad \mathbf{N}_\mathcal{C}(\mathbf{m}) = 0 \text{ or } L \in \mathcal{N}_m(\mathcal{C}).$$

Now, to compute the generic intersection numbers, we use the notion of probranches [5, 11, 12]. See section 4 of [6] for details. Let $P \in \mathcal{C}$ be an indeterminancy point of $N_\mathcal{C}$ and let us write μ_P for the multiplicity of $\mathcal{C}$ at P. Recall that $\mu_P = 1$ means that P is a nonsingular point of $\mathcal{C}$. Let $M \in GL(\mathbf{V})$ be such that $M(\mathbf{O}) = \mathbf{P}$ with $\mathbf{O} = (0,0,1)$ (we set also $O = [0 : 0 : 1]$) and such that $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O. Recall that the equation of this tangent cone is the homogeneous part of lowest degree in (x, y) of $F(x, y, 1) \in \mathbb{C}[x, y]$ and that this lowest degree is μ_P. Using the combination of the Weierstrass preparation theorem and of the Puiseux expansions,

$$F \circ M(x, y, 1) = U(x, y) \prod_{j=1}^{\mu_P} (y - g_j(x)),$$

for some $U(x, y)$ in the ring of convergent series in x, y with $U(0,0) \neq 0$ and where $g_j(x) = \sum_{m \geq 1} a_{j,m} x^{\frac{m}{q_j}}$ for some integer $q_j \neq 0$. The $y = g_j(x)$ correspond to the equations of the probranches of $\mathcal{C}$ at P. Since $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O, the valuation in x of g_j is strictly larger than or equal to 1 and so the probranch $y = g_j(x)$ is tangent to $V(y - xg_j'(0))$. We write $\mathcal{T}_P^{(i)} := M(V(y - xg_j'(0)))$ the associated (eventually singular) tangent line to $\mathcal{C}$ at P ($\mathcal{T}_P^{(i)}$ is the tangent to the branch of $\mathcal{C}$ at P corresponding to this probranch) and we denote by $i_P^{(j)}$ the tangential intersection number of this probranch:

$$i_P^{(j)} = \mathrm{val}_x(g_j(x) - xg_j'(0)) = \mathrm{val}_x(g_j(x) - xg_j'(x)).$$

We recall that for any homogeneous polynomial $H \in \mathbb{C}[x, y, z]$, we have

$$\begin{aligned} i_P(\mathcal{C}, V(H)) &= i_O(V(F \circ M), V(H \circ M)) \\ &= \sum_{j=1}^{\mu_P} \mathrm{val}_x(H(M(G_j(x)))), \end{aligned}$$

where $G_j(x) := (x, g_j(x), 1)$. With these notations and results, we have

$$\Omega(\mathcal{C}, \ell_\infty) = \sum_{P \in \mathcal{C} \cap \ell_\infty} (i_P(\mathcal{C}, \ell_\infty) - \mu_P(\mathcal{C})) = \sum_{P \in \mathcal{C} \cap \ell_\infty} \sum_{j : \mathcal{T}_P^{(j)} = \ell_\infty} (i_P^{(j)} - 1).$$

For a generic $L \in \mathbf{V}^\vee$, we also have

$$\begin{aligned} i_P(\mathcal{C}, V(L \circ N_\mathcal{C})) &= \sum_{j=1}^{\mu_P} \mathrm{val}_x(L(N_\mathcal{C}(M(G_j(x))))) \\ &= \sum_{j=1}^{\mu_P} \min_k \mathrm{val}_x([N_\mathcal{C} \circ M]_k(G_j(x))), \end{aligned}$$

where $[\cdot]_k$ denotes the k-th coordinate. Moreover, due to (14), as seen in Proposition 16 of [7], we have

$$\mathbf{N}_\mathcal{C} \circ M(\mathbf{m}) = Com(M) \cdot (\mathbf{m} \wedge [\Delta_\mathbf{A} G(\mathbf{m}) \cdot \mathbf{A} + \Delta_\mathbf{B} G(\mathbf{m}) \cdot \mathbf{B}]),$$

where $G := F \circ M$, $\mathbf{A} := M^{-1}(1,0,0)$, $\mathbf{B} := M^{-1}(0,1,0)$ and $\Delta_{(x_1,y_1,z_1)} H = x_1 H_x + y_1 H_y + z_1 H_z$. As seen in Lemma 33 of [6], we have

$$\Delta_{(x_1,y_1,z_1)} G(x, g_j(x), 1) = R_j(x) W_{(x_1,y_1,z_1),j}(x),$$

where $R_j(x) = U(x, g_j(x)) \prod_{j' \neq j}(g_{j'}(x) - g_j(x))$ and $W_{(x_1,y_1,z_1),j}(x) := y_1 - x_1 g_j'(x) + z_1(x g_j'(x) - g_j(x))$. Therefore, for a generic $L \in \mathbf{V}^\vee$, we have

$$i_P(\mathcal{C}, V(L \circ N_\mathcal{C})) = V_P + \sum_{j=1}^{\mu_P} \min_k \mathrm{val}_x([G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})]_k)$$

where $V_P := \sum_{j=1}^{\mu_P} \sum_{j' \neq j} \mathrm{val}(g_{j'} - g_j)$. Now, we write $h_P^{(j)} := \min_k \mathrm{val}_x([G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})]_k)$ and $h_P := \sum_{j=1}^{\mu_P} h_P^{(j)}$. Note that $V(P) = 0$ if P is a nonsingular point of $\mathcal{C}$. We recall that, due to Corollary 31 of [6], we have

$$\sum_{P \in \mathcal{C} \cap \mathrm{Base}(N_\mathcal{C})} V_P = d(d-1) - d^\vee$$

and so, due to (15), we obtain

$$c_\nu(\mathcal{C}) = d + d^\vee - \sum_{P \in \mathcal{C} \cap \mathrm{Base}(N_\mathcal{C})} h_P. \tag{16}$$

Now we have to compute the contribution $h_P^{(j)}$ of each probranch of each $P \in \mathcal{C} \cap \mathrm{Base}(N_\mathcal{C})$. We have seen, in Proposition 29 of [6], that we can adapt our choice of M to each probranch (or, to be more precise, to each branch corresponding to the probranch). This fact will be useful in the sequel. In particular, for each probranch, we take M such that $g_j'(0) = 0$ so $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ can be rewritten:

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_A y_A - (x_A^2 + x_B^2) g_j'(x) + x_B y_B + (z_A x_A + z_B x_B)(x g_j'(x) - g_j(x)) \\ y_A^2 + y_B^2 - (x_A y_A + x_B y_B) g_j'(x) + (z_A y_A + z_B y_B)(x g_j'(x) - g_j(x)) \\ y_A z_A + y_B z_B - (x_A z_A + x_B z_B) g_j'(x) + (z_A^2 + z_B^2)(x g_j'(x) - g_j(x)) \end{pmatrix}. \tag{17}$$

- Assume first that P is a point of $\mathcal{C}$ outside ℓ_∞. Then for M as above and such that $z_A = z_B = 0$, we have

$$G_j(0) \wedge (W_{\mathbf{A},j}(0) \cdot \mathbf{A} + W_{\mathbf{B},j}(0) \cdot \mathbf{B}) = \begin{pmatrix} -y_A^2 - y_B^2 \\ x_A y_A + x_B y_B \\ 0 \end{pmatrix}$$

  which is non null since $(y_A, y_B) \neq (0,0)$ and since $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. So $h_P^{(j)} = 0$.

- Assume now that $P \in \mathcal{C}\cap\ell_\infty\setminus\{I,J\}$ and $\mathcal{T}_P^{(j)} \neq \ell_\infty$. Then $y_A+iy_B\neq 0$ and $y_A-iy_B\neq 0$ (since $I,J\notin\mathcal{T}_P^{(j)}$) and so $y_A^2+y_B^2\neq 0$ which together with (17) implies that $h_P^{(j)}=0$ as in the previous case.
- Assume that $P\in\mathcal{C}\cap\ell_\infty\setminus\{I,J\}$ and $\mathcal{T}_P^{(i)}=\ell_\infty$. Assume that $M(1,0,0)=(1,i,0)$. Hence $\mathbf{A}+i\mathbf{B}=(1,0,0)$. Then $y_A=y_B=0$, $x_A+ix_B=1$, $z_A+iz_B=0$. So $z_A^2+z_B^2=0$ and $z_Ax_A+z_Bx_B=z_A\neq 0$ (since $z_B=iz_A$ and $x_B=i(x_A-1)$). Note that $P\neq J$ implies also that $x_A-ix_B\neq 0$. So that $x_A^2+x_B^2\neq 0$. Hence, due to (17), $G_j(x)\wedge(W_{\mathbf{A},j}(x)\cdot\mathbf{A}+W_{\mathbf{B},j}(x)\cdot\mathbf{B})$ is equal to
$$\begin{pmatrix} x\\ g_j(x)\\ 1\end{pmatrix}\wedge\begin{pmatrix}(x_A^2+x_B^2)g_j'(x)+z_A(xg_j'(x)-g_j(x))\\ 0\\ z_Ag_j'(x)\end{pmatrix}.$$
Therefore we have $h_P^{(j)}=\mathrm{val}_x((x_A^2+x_B^2)g_j'(x))=i_P^{(j)}-1$.
- Assume that $P=I$ and that $\mathcal{T}_P^{(j)}=\ell_\infty$. Take M such that $M(\mathbf{O})=(1,i,0)$, $\mathbf{B}=(1,0,0)$ and so $\mathbf{A}=(-i,0,1)$. Due to (17), $G_j(x)\wedge(W_{\mathbf{A},j}(x)\cdot\mathbf{A}+W_{\mathbf{B},j}(x)\cdot\mathbf{B})$ is equal to
$$\begin{pmatrix} x\\ g_j(x)\\ 1\end{pmatrix}\wedge\begin{pmatrix}-i(xg_j'(x)-g_j(x))\\ 0\\ ig_j'(x)+(xg_j'(x)-g_j(x))\end{pmatrix}. \tag{18}$$
Note that each coordinate has valuation at least equal to $i_P^{(j)}=\mathrm{val}\, g_j$ and that the term of degree $i_P^{(j)}$ of the second coordinate is the term of degree $i_P^{(j)}$ of
$$-i(xg_j'(x)-g_j(x))+xig_j'(x)=ig_j(x)\neq 0$$
which is non null. Therefore $h_P^{(j)}=i_P^{(j)}$.
- Assume finally that $P=I$ and that $\mathcal{T}_P^{(j)}\neq\ell_\infty$. Take M such that $M(\mathbf{O})=(1,i,0)$, $\mathbf{B}=(0,1,0)$ and so $\mathbf{A}=(0,-i,1)$. Due to (17), $G_j(x)\wedge(W_{\mathbf{A},j}(x)\cdot\mathbf{A}+W_{\mathbf{B},j}(x)\cdot\mathbf{B})$ is equal to
$$\begin{pmatrix} x\\ g_j(x)\\ 1\end{pmatrix}\wedge\begin{pmatrix}0\\ -i(xg_j'(x)-g_j(x))\\ -i+(xg_j'(x)-g_j(x))\end{pmatrix}. \tag{19}$$
Note that each coordinate has valuation at least equal to 1 and that the term of degree 1 of the second coordinate is $ix\neq 0$. Hence $i_P^{(j)}=1$.

Note that the case $P=J$ can be treated in the same way than the case $P=I$.

Theorem 8 follows from (16) and from the previous computation of h_P.

Appendix A. Dimension decrease

Here, we consider two particular cases of hypersurfaces the normal class of which is equal to the normal class of a hypersurface of lower dimension: cylinders (i.e. cones at a point at infinity) and revolution hypersurfaces (circles fibers).

Let $n\geq 3$. Let $\tilde F\in\mathbb{C}[u_1,...,u_n]$ be homogeneous. We call **cylinder of base** $\tilde{\mathcal{Z}}=V(\tilde F)\subset\mathbb{P}^n$ **and of axis** $V(x_2,...,x_n)\subset\mathbb{P}^n$ the hypersurface $V(F)\subset\mathbb{P}^n$, with $F(x_1,\dots,x_{n+1}):=\tilde F(x_2,\dots,x_{n+1})$.

Proposition 30. *Let $n\geq 3$ and $d\geq 2$. Let $\mathcal{Z}=V(F)\subset\mathbb{P}^n$ be the cylinder of axis $V(x_2,...,x_n)\subset\mathbb{P}^n$ and of base $\tilde{\mathcal{Z}}=V(\tilde F)\subset\mathbb{P}^{n-1}$. Then $c_\nu(\mathcal{Z})=c_\nu(\tilde{\mathcal{Z}})$.*

Proof. Note that $\mathcal{Z}\cap V(x_2,...,x_{n+1})\subset \mathrm{Sing}(\mathcal{Z})\subset \mathcal{B}_{\mathcal{Z}}$. Let $m[x_1^{(1)}:\dots:x_{n+1}^{(1)}]\in\mathcal{Z}\setminus V(x_2,...,x_{n+1})$ and $P[x_1^{(0)}:\dots:x_{n+1}^{(0)}]\in\mathbb{P}^n\setminus V(x_2,...,x_{n+1})$. Set $\tilde{m}[x_2^{(1)}:\dots:x_{n+1}^{(1)}]\in\tilde{\mathcal{Z}}$ and $\tilde{P}[x_2^{(0)}:\dots:x_{n+1}^{(0)}]\in\mathbb{P}^{n-1}$. Note that $n_{\mathcal{Z}}(m)[0:\tilde{F}_{u_1}(\tilde{\mathbf{m}}):\dots:\tilde{F}_{u_{n-1}}(\tilde{\mathbf{m}}):0]\in\mathbb{P}^n$.

- Let $\mathcal{H}=V(\alpha x_1+\beta x_{n+1})\subset\mathbb{P}^n$ be a hyperplane orthogonal to $V(x_2,...,x_n)$ such that $\mathcal{H}\neq\mathcal{H}^\infty$ (i.e. $\alpha\neq 0$). Assume $m\in\mathcal{H}$. Then $m\in\mathcal{B}_{\mathcal{Z}}\Leftrightarrow\tilde{m}\in\mathcal{B}_{\tilde{\mathcal{Z}}}$. If $m\in\mathcal{H}\cap\mathcal{Z}\setminus\mathcal{B}_{\mathcal{Z}}$, then $\mathcal{N}_m(\mathcal{Z})\subset\mathcal{H}$.
- Assume $P\in\mathbb{P}^n\setminus V(x_1,x_{n+1})$. Then $\mathcal{H}:=V(x_1^{(0)}x_{n+1}-x_{n+1}^{(0)}x_1)$ is the unique hyperplane orthogonal to $V(x_2,...,x_n)$ containing P and

$$P\in\mathcal{N}_m(\mathcal{Z}),\ m\in\mathcal{Z}\setminus\mathcal{B}_{\mathcal{Z}}\quad\Leftrightarrow\quad m\in\mathcal{H},\ \tilde{m}\in\tilde{\mathcal{Z}}\setminus\mathcal{B}_{\tilde{\mathcal{Z}}},\ \tilde{P}\in\mathcal{N}_{\tilde{m}}(\tilde{\mathcal{Z}}).$$

Hence $c_\nu(\mathcal{Z})=c_\nu(\tilde{\mathcal{Z}})$ □

Let $\tilde{F}\in\mathbb{C}[u_1,...,u_n]$ be a homogeneous polynomial of the form $\tilde{F}(u_1,...,u_n)=G(u_1^2,...,u_n)$ for some $G\in\mathbb{C}[u_1,...,u_n]$. Let $\tilde{\mathcal{Z}}:=V(\tilde{F})\subset\mathbb{P}^{n-1}$.

We call **algebraic hypersurface of revolution of $\tilde{\mathcal{Z}}$ around the subspace** $V(x_1,x_2)$ the hypersurface $\mathcal{Z}=V(F)\subset\mathbb{P}^n$ with $F(x_1,....,x_{n+1}):=G(x_1^2+x_2^2,x_3,...,x_{n+1})$.

Note that if $m[x_1^{(1)}:\dots:x_{n+1}^{(1)}]\in\mathcal{Z}\setminus\mathcal{H}^\infty$ with $x_{n+1}^{(1)}=1$, then the "circle" $V(x_1^2+x_2^2-(x_1^{(1)})^2-(x_2^{(1)})^2)\cap\bigcap_{i=3}^n V(x_i-x_i^{(1)}x_{n+1})$ of center $[0:0:x_3^{(1)}:\dots:x_{n+1}^{(1)}]$ that passes through m is contained in $\mathcal{Z}$.

Proposition 31. *Let $n\geq 3$ and $d\geq 2$. Let $\mathcal{Z}=V(F)\subset\mathbb{P}^n$ be the algebraic hypersurface of revolution of $\tilde{\mathcal{Z}}=V(\tilde{F})\subset\mathbb{P}^{n-1}$ (with $\tilde{F}\in\mathbb{C}[u_1,...,u_n]$ as above) around the subspace $V(x_1,x_2)$, then $c_\nu(\mathcal{Z})=c_\nu(\tilde{\mathcal{Z}})$.*

Proof. Let $m[x_1^{(1)}:\dots:x_{n+1}^{(1)}]\in\mathcal{Z}$ and $P[x_1^{(0)}:\dots:x_{n+1}^{(0)}]\in\mathbb{P}^n$. Then

$$n_{\mathcal{Z}}(m)[2x_1^{(1)}G_{u_1}(\mathbf{m}_1):2x_2^{(1)}G_{u_1}(\mathbf{m}_1):G_{u_2}(\mathbf{m}_1):\dots:G_{u_{n-1}}(\tilde{\mathbf{m}}_1):0]\in\mathbb{P}^n,$$

with $\mathbf{m}_1((x_1^{(1)})^2+(x_2^{(1)})^2,x_3^{(1)},...,x_{n+1}^{(1)})\in\mathbb{C}^n$. Hence if $m\in\mathcal{Z}\cap V(x_1^2+x_2^2)\setminus\mathcal{B}_{\mathcal{Z}}$, then $\mathcal{N}_m\mathcal{Z}\subset V(x_1^2+x_2^2)$. Assume from now on that $m\in\mathcal{Z}\setminus V(x_1^2+x_2^2)$ and that $P\in\mathbb{P}^n\setminus(V(x_1^2+x_2^2)\cup V(x_1))$.

Let $\tilde{m}[y_1^{(1)}:x_3^{(1)}:\dots:x_{n+1}^{(1)}]\in\mathbb{P}^{n-1}$ and $\tilde{P}[y_1^{(0)}:x_3^{(0)}:\dots:x_{n+1}^{(0)}]\in\mathbb{P}^{n-1}$ with $(y_1^{(i)})^2=(x_1^{(i)})^2+(x_2^{(i)})^2$. Note that $\tilde{m}\in\tilde{\mathcal{Z}}$. Then

$$n_{\mathcal{Z}}(m)[x_1^{(1)}\tilde{F}_{u_1}(\tilde{\mathbf{m}})/y_1^{(1)}:x_2^{(1)}\tilde{F}_{u_1}(\tilde{\mathbf{m}})/y_1^{(1)}:\tilde{F}_{u_2}(\tilde{\mathbf{m}}):\dots:\tilde{F}_{u_{n-1}}(\tilde{\mathbf{m}}):0]\in\mathbb{P}^n.$$

- Note that $m\in\mathcal{B}_{\mathcal{Z}}\Leftrightarrow\tilde{m}\in\mathcal{B}_{\tilde{\mathcal{Z}}}$ (since $x_1^{(1)}$ and $x_1^{(1)}$ are not both null).
- Let $\mathcal{H}=V(\alpha x_1+\beta x_2)\subset\mathbb{P}^n$ be a hyperplane that contains $V(x_1,x_2)$ but not contained in $V(x_1^2+x_2^2)$ (i.e. $\alpha^2+\beta^2\neq 0$). If $m\in\mathcal{H}\cap\mathcal{Z}\setminus\mathcal{B}_{\mathcal{Z}}$, then $\mathcal{N}_m\mathcal{Z}\subset\mathcal{H}$.
- Let $\mathcal{H}:=V(x_1^{(0)}x_2-x_2^{(0)}x_1)$ be the unique hyperplane that contains $V(x_1,x_2)$ and P. Then

  $$P\in\mathcal{N}_m(\mathcal{Z}),\ m\in\mathcal{Z}\setminus\mathcal{B}_{\mathcal{Z}}\quad\Leftrightarrow\quad m\in\mathcal{H},\ \tilde{m}\in\tilde{\mathcal{Z}}\setminus\mathcal{B}_{\tilde{\mathcal{Z}}},\ \tilde{P}\in\mathcal{N}_{\tilde{m}}(\tilde{\mathcal{Z}}),$$

  by choosing $y_1^{(1)}:=y_1^{(0)}x_1^{(1)}/x_1^{(0)}$.

Hence $c_\nu(\mathcal{Z})=c_\nu(\tilde{\mathcal{Z}})$ □

Appendix B. Projective orthogonality in $\mathbb{P}^n$

B.1. **From affine orthogonality to projective orthogonality.** Let E_n be an euclidean affine n-space of direction the n-vector space $\mathbf{E}_n$ (endowed with some fix basis). Let $\mathbf{V} := (\mathbf{E}_n \oplus \mathbb{R}) \otimes \mathbb{C}$ (endowed with the induced basis $\mathbf{e}_1, ..., \mathbf{e}_{n+1}$). We consider the complex projective space $\mathbb{P}^n := \mathbb{P}(\mathbf{V})$ with projective coordinates $x_1, ..., x_{n+1}$. Let us write $\pi : \mathbf{V} \setminus \{0\} \to \mathbb{P}^3$ for the canonical projection. We denote by $\mathcal{H}^\infty := V(x_{n+1}) \subset \mathbb{P}^n$ the hyperplane at infinity. We consider the affine space $A^n := \mathbb{P}^n \setminus \mathcal{H}^\infty$ endowed with the vector space $\overrightarrow{\mathbf{E}} := Span(\mathbf{e}_1, \cdots, \mathbf{e}_n) \subset \mathbf{V}$ (with the affine structure $m + \overrightarrow{\mathbf{v}} = \pi(\mathbf{m} + \overrightarrow{\mathbf{v}})$ if $\overrightarrow{\mathbf{v}} \in \overrightarrow{\mathbf{E}}$ and $m = \pi(\mathbf{m}) \in A^n$ with $\mathbf{m}(x_1, \cdots, x_n, 1)$).

Let us consider $\mathcal{W}_1 = \mathbb{P}(\mathbf{W}_1) \subset \mathbb{P}^n$ and $\mathcal{W}_2 = \mathbb{P}(\mathbf{W}_2) \subset \mathbb{P}^n$ where $\mathbf{W}_1$ and $\mathbf{W}_2$ are two vector subspaces of $\mathbf{V}$ not contained in $\overrightarrow{\mathbf{E}}$ such that $\dim \mathbf{W}_1 + \dim \mathbf{W}_2 = n + 2$. Since $\mathcal{W}_i$ is not contained in $\mathcal{H}^\infty$, $W_i := \mathcal{W}_i \setminus \mathcal{H}^\infty$ is an affine subspace of A^n with vector space $\overrightarrow{\mathbf{W}_i} := \mathbf{W}_i \cap \overrightarrow{\mathbf{E}}$, that is to say that there exists m_i such that $W_i = m_i + \overrightarrow{\mathbf{W}_i}$ in A^n. Consider the usual bilinear symmetric form $\langle u, v \rangle = \sum_{i=0}^3 u_i v_i$ on $\mathbf{V}$, the associated orthogonality on $\mathbf{V}$ is written $\perp$.

Definition 32. *Let us consider $\mathcal{W}_1 = \mathbb{P}(\mathbf{W}_1) \subset \mathbb{P}^n$ and $\mathcal{W}_2 = \mathbb{P}(\mathbf{W}_2) \subset \mathbb{P}^n$ where $\mathbf{W}_1$ and $\mathbf{W}_2$ are two vector subspaces of $\mathbf{V}$ not contained in $\overrightarrow{\mathbf{E}}$ and such that $\dim \mathbf{W}_1 + \dim \mathbf{W}_2 = n + 2$. With the above notations, we say that $\mathcal{W}_1$ and $\mathcal{W}_2$ are orthogonal in $\mathbb{P}^3$ if $\overrightarrow{\mathbf{W}}_1 \perp \overrightarrow{\mathbf{W}}_2$. We then write $\mathcal{W}_1 \perp \mathcal{W}_2$.*

Note that if $\mathcal{H} \subset \mathbb{P}^n$ and $\mathcal{L} \subset \mathbb{P}^n$ are respectively an hyperplane and a line in $\mathbb{P}^n$ not contained in $\mathcal{H}^\infty$, then $\mathcal{H} \perp \mathcal{L}$ if and only if the point at infinity of $\mathcal{L}$ is the pole in $\mathcal{H}^\infty$ of the line $\mathcal{H} \cap \mathcal{H}^\infty \subset \mathcal{H}^\infty$ with respect to the **umbilical** $\mathcal{U}_\infty := V(x_1^2 + ... + x_n^2) \cap \mathcal{H}^\infty \subset \mathcal{H}^\infty$. This leads us to the following generalization of normal lines to an hyperplane.

Definition 33. *We say that a projective hyperplane $\mathcal{H} = V(a_1 x_1 + \cdots + a_{n+1} x_{n+1}) \subset \mathbb{P}^n$ and a projective line $\mathcal{L} = \mathbb{P}(\mathbf{L}) \subset \mathbb{P}^n$ are orthogonal in $\mathbb{P}^n$ if $(a_1, \cdots, a_n, 0) \in \mathbf{L}$. We then write $\mathcal{L} \perp \mathcal{H}$.*

It is worthful to note that, with this definition, an orthogonal line to an hyperplane $\mathcal{H}$ may be included in $\mathcal{H}$.

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