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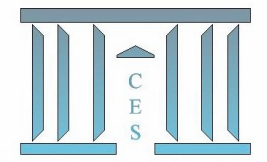

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The no-trade interval of Dow and Werlang: some clarifications

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Abstract

The aim of this paper is two-fold: first, to emphasize that the seminal result of Dow and Werlang [9] remains valid under weaker conditions and this even if non-positive prices are considered, or equally that the no-trade interval result is robust when considering assets which can yield non-positive outcomes, second to make precise the weak uncertainty aversion behavior characteristic of the existence of such an interval.

Keywords: Choquet expected utility, no-trade interval, perfect hedging, comonotone diversification, capacity.

JEL Classification Number: D81

Domain: Decision Theory

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1 Introduction

In a seminal paper [9], Dow and Werlang dealt with the basic portfolio problem under uncertainty. They proved that, given a convex capacity v and a C^2 and concave non-decreasing utility function u for a Choquet expected utility decision maker (CEU DM), there exists an interval of prices within which an uncertainty averse agent neither buys nor sells an asset X. The hightest price at which the agent will buy the asset is the expected value of the asset under v (i.e. the Choquet integral of X, $I(X)$) whereas the lowest price at which the DM sells the asset is the expected value of selling the asset short (i.e. $-I(-X)$). Such a behavior is intuitively plausible and compatible with observed investment comportment. It contrasts with the prediction of expected utility theory under risk (see von Neumann and Morgenstern [19]), according to which a strongly risk averse agent (i.e. with a concave utility function) or equivalently a weakly risk averse agent (result proved by Rothschild and Stiglitz [16]) will invest in an asset X if and only if the expected value of this asset exceeds the price, and will wish to sell the asset short if and only if the expected value is lower than its price and consequently will have no position in the asset if and only if the price is exactly $E(X)$ (see Arrow [3]).
In this paper, we first generalize Dow and Werlang's result by allowing for negative prices or equally we prove that the no-trade interval result is robust when considering assets which can yield non-positive outcomes. We also prove that this result remains valid under weaker hypotheses, requiring only that the capacity satisfies super-additivity at certainty that is, $v(A) + v(A^c) \leq 1$ for any event A instead of being convex and that u is C^1 intead of being C^2. We also obtain the converse implication and so the equivalence between the no-trade interval $[I(X),\ -I(-X)]$ and the following two conditions: 1) v is super-additive at certainty and 2) u is concave.
We furthermore make precise the weak uncertainty aversion behavior of the agent characterictic of the existence of such an interval by proving that the previous conditions 1) and 2) are actually equivalent to: 3) attraction by perfect hedging and 4) preference for comonotone diversification.
Let us note that Chateauneuf and Tallon [6] showed that for a Choquet expected utility decision maker, preference for comonotone diversification is equivalent to concavity of the utility function that is that conditions 2) and 4) are equivalent. The proof 1) and 2) $\Rightarrow$ 3) and 3)$\Rightarrow$ 1) is inspired by a paper of Abouda and Chateauneuf [2] where the same result is proved but under risk for a RDEU agent.
Finally, we show that for a CEU DM endowed with a concave non-decreasing utility function u, super-additivity at certainty of v is equivalent to being averse to some specific increase of uncertainty.

2 Definitions and notations

2.1 Elementary definitions of decision making under uncertainty

The distinction between risk (situations where there exists an objective probability distribution, known by the decision maker) and uncertainty (situations where there is no objective probability distribution, or it is unknown for the decision maker) is due to Knight [12].

Under (non-probabilized) uncertainty, a decision is a mapping, called act, from the set of states (of nature) Ω, which models the lack of information of the decision maker, into a set of outcomes $\mathbb{R}$.

Exactly one state is the "true state", the other states are not true. A decision maker is uncertain about which state of nature is true and has not any influence on the truth of the states.

Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{B}_\infty$ be the set of $\mathcal{A}$-measurable bounded mappings from Ω to $\mathbb{R}$ corresponding to all possible decisions and $X \in \mathcal{B}_\infty$.

By $\succsim$, we denote the preference relation of the decision maker on the set of all acts $\mathcal{B}_\infty$.

For any pair of acts $X,\ Y,\ X \succsim Y$ will read X is (weakly) preferred to Y by the DM, $X \succ Y$ means that X is strictly preferred to Y, and $X \sim Y$ means that X and Y are considered as equivalent by the DM.

As usual, for all $X,\ Y \in \mathcal{B}_\infty$, we write: $X \succ Y$ if $X \succsim Y$ and not $Y \succsim X$; $X \precsim Y$ if $Y \succsim X$; $X \prec Y$ if $Y \succ X$; $X \sim Y$ if $X \succsim Y$ and $Y \succsim X$.

Definition 2.1 *A function* $V :\ \mathcal{B}_\infty \to \mathbb{R}$ *represents* $\succsim$ *if*

$$X \succsim Y \Leftrightarrow V(X) \geq V(Y) \ \textit{for all}\ X,\ Y \in \mathcal{B}_\infty.$$

2.2 The Choquet integral and CEU

For the preferences representation, it will be necessary to define a class of set functions (the capacities) and to give some of their properties that will be used in the remainder of the article.

Definition 2.2 *v is a (normalized) capacity on $(\Omega, \mathcal{A})$ if $v :\ \mathcal{A} \to [0,1]$ is such that $v(\emptyset) = 0,\ v(\Omega) = 1$ and $\forall A,\ B \in \mathcal{A},\ A \subset B \Rightarrow v(A) \leq v(B)$.*

Definition 2.3 *Let v be a capacity on $(\Omega, \mathcal{A})$, v is convex if $\forall A,\ B \in \mathcal{A}$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.*

Definition 2.4 *The Choquet integral of $X \in \mathcal{B}_\infty$ with respect to the capacity v is $I(X) = \int X dv$ where $\int X dv = \int_{-\infty}^{0} (v(X \geq t) - 1)\, dt + \int_{0}^{+\infty} (v(X \geq t))\, dt$.*

Definition 2.5 *$X,\ Y \in \mathcal{B}_\infty$ are comonotone if $\forall s,\ t \in \Omega$, $(X(s) - X(t))\,(Y(s) - Y(t)) \geq 0$. (i.e. X and Y vary in the same direction).*

Definition 2.6 *The core of a capacity v is defined by $C(v) = \{measure\ P :\ P(A) \geq v(A)\ \forall A \in \mathcal{A},\ P(\Omega) = v(\Omega)\}$.*

Definition 2.7 *We say that Choquet expected utility applies if there exist a utility function $u :\ \mathbb{R} \to \mathbb{R}$ and a capacity v on $(\Omega, \mathcal{A})$ so that*

$$I \circ u :\ X \in \mathcal{B}_\infty \mapsto \int_\Omega (u \circ X)\, dv \ \in \mathbb{R}\ represents\ \succsim .$$

In the Choquet expected utility model, preferences depend, on the one hand, on a utility function (which reflects the perception of wealth) and on the other hand on a capacity (reflecting the perception of the occurrence of events). This preferences representation is attractive for at least two reasons: it better represents real choices and allows for a separation between attitude towards uncertainty and attitude towards wealth.
Note that the two attitudes are mixed in the expected utility model, where they are both represented by the utility function.
Note also that, under risk, similar models are proposed by Kahneman and Tversky [11], Quiggin [15], and Yaari [23]. These models are known under the denomination of rank-dependent expected utility (RDEU).

3 Models and results

The study is focused on the case of uncertainty, that is, on non-probabilized risk Choquet expected utility model (Schmeidler [18]). Preferences are then represented by the Choquet integral of a utility function u with respect to a capacity v.

3.1 The result of Dow and Werlang

Let us first present the model and the result of Dow and Werlang [9].

Let $(\Omega, \mathcal{A})$ be a measurable space, v be a convex capacity on $\mathcal{A}$ and u be a utility function. We suppose that u is C^2, $u' > 0$ and $u'' \leq 0$.

The result of Dow and Werlang about the behavior of the "risk averse" or "risk neutral" agent under uncertainty aversion, is the following:

Theorem 3.1 (Dow and Werlang:) *A risk averse (resp. risk neutral) investor with certain wealth $W > 0$, who is faced with an asset which yields a present value X per unit, whose price is $p > 0$ per unit, will buy the asset if $p < I(X)$ (resp. $p \leq I(X)$). He will sell the asset if $p > -I(-X)$ (resp. $p \geq -I(-X)$).*

Remark 3.2 *It may seem strange to call the DM "risk averse" (resp. "risk neutral") since the framework is that of uncertainty and not risk. However, this just means that the utility function u is concave (resp. affine).*

This result is very intuitive and offers an appealing interpretation of the uncertainty aversion in terms of pessimism since, according to a well-known theorem of Schmeidler [17] which says that v is convex if and only if $C(v) \neq \emptyset$ and $I(X) = Min_{P \in C(v)} E_P[X]$, the agent views as possible the set of probabilities above the convex capacity v and will evaluate all assets X by $Min_{P \in C(v)} E_P[X]$.

So, when the price p is less that $I(X)$, the DM will buy a strictly positive amount of X since he considers that the price is lower than the worst expected value of X.

Conversely, when the price p is greater than $-I(-X)$, the DM will sell short a strictly positive amount of X since he considers that the price is greater than $Max_{P \in C(v)} E_P[X]$ i.e. than the best expected value of X.

Thus, he will have no position on the asset X if and only if its price p is between $I(X)$ and $-I(-X)$.

The intuition behind this finding may be grasped in the following example given by Mukerji and Tallon [14]:

Consider an asset that pays off 1 in state L and 3 in state H and assume that the DM is of the CEU type with capacity $v(L) = 0.3$ and $v(H) = 0.4$ and linear utility function. The expected payoff (that is, the Choquet integral) of buying a unit of the risky asset is given by $I(X) = 1 + (3-1)v(H) = 1 + 2.0,4 = 1,8$. The payoff from going short on a unit of the risky asset is $I(-X) = -3 + (3-2)v(L) = -3 + 2.0,3 = -2,4$. Hence, if the price of the asset X lies in the interval $[1,8; 2,4]$, then the investor would strictly prefer a zero position to either going short or buying.

3.2 Generalization and extension of the result of Dow and Werlang

Let $(\Omega, \mathcal{A})$ be a measurable space such that $\mathcal{A}$ contains at least one non-trivial events. Let $\mathcal{B}_\infty$ be the set of bounded $\mathcal{A}$-measurable mappings from Ω to $\mathbb{R}$. We consider a CEU DM with $u : \mathbb{R} \to \mathbb{R}$ a C^1 utility function such that $u' > 0$ and a capacity v on $\mathcal{A}$, non trivial in the sense that there exists at least one event $A \in \mathcal{A}$ such that $0 < v(A) < 1$.

Remark 3.3 *Note that Dow and Werlang only considered assets for which natural reservation prices $I(X)$ and $-I(-X)$ are positive, so limit their study to the natural case where the price p by unit is positive; furthermore since they also assume that the investor is endowed with an initial deterministic positive wealth, any infinitesimal buying transaction is feasible, indeed any short selling position is also a priori feasible. In this paper, we want to study the robustness of Dow and Werlang's result if we relax the above restrictions on X and W. For instance, if $I(X) < 0$, which would occur in case of an asset offering monetary outcomes smaller than a negative one, we wonder if considering a price p smaller than $I(X)$ hence negative, the DM will buy the asset. Similarly, we want to show that if the initial wealth W is negative, then selling short an asset with positive outcomes will occur as soon as the price p is greater than $-I(-X)$.*
Consequently from now on, we will assume that the DM is endowed with an initial wealth W not necessarily positive. To avoid additional intricacy linked with the interest rates, borrowing any amount of money is excluded. Accordingly the sole impossible trading situation for the DM is when he is endowed at time 0 with a non-positive initial wealth, when furthermore the unit price p of X is positive and he intends buying some shares of the asset.
We will call any situation other than $W \leq 0$ and $p > 0$ a situation of feasible trade. We therefore obtain the following generalization of Theorem 3.1.

Theorem 3.4 *The following two propositions (a) and (b) are equivalent:*

(a) For any $X \in \mathcal{B}_\infty$, $I(X) \leq -I(-X)$. For any situation of feasible trade, the DM has no position in the asset X on the range of prices $[I(X), -I(-X)]$, furthermore he holds a positive amount of the asset X at prices below $I(X)$, and holds a short position at prices higher than $-I(-X)$.

$$(b) \begin{cases} (1)\ v(A) + v(A^c) \leq 1 \text{ for all } A \text{ in } \mathcal{A} \\ (2)\ u \text{ is concave} \end{cases}$$

Note that in contrast with Theorem 3.1, the result we obtain is an equivalence and that it remains valid under weaker conditions (we only require v to be super-additive at certainty i.e. $v(A) + v(A^c) \leq 1$ for all A in $\mathcal{A}$ instead of being convex and u to be C^1 instead of being C^2) and this even if non-positive prices are considered.

Theorem 3.4 can be illustrated by the case of a company which has debts and is near bankruptcy. A buyer will purchase the asset X although he values it negatively ($I(X) < 0$) as soon as the negative price p is smaller than $I(X)$ since he thinks the amount of money he will receive per share ($|p|$) is more important than what it will cost him ($|I(X)|$).

Let us now make precise the weak uncertainty aversion behavior characteristic of the existence of such an interval.

Theorem 3.5 *A CEU DM will exhibit the no-transaction interval of Dow and Werlang if and only if:*

(3) He is attracted by perfect hedging
(i.e. $[X, Y \in \mathcal{B}_\infty,\ X \succsim Y,\ \alpha \in [0,1], \alpha X + (1-\alpha)Y = a1_\Omega\ a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$*).*

and

(4) He exhibits preference for comonotone diversification
(i.e. $X, Y \in \mathcal{B}_\infty,\ X$ *and* Y *comonotone,* $X \sim Y \Rightarrow \alpha X + (1-\alpha)Y \succsim Y\ \forall \alpha \in [0,1]$*).*

Let us note that, in the following proposition, Abouda [1] gave three equivalent definitions of preference for perfect hedging.

Proposition 3.6 *(Abouda [1]): The following assertions are equivalent:*
(i) $[X, Y \in \mathcal{B}_\infty,\ \alpha \in [0,1],\ \alpha X + (1-\alpha)Y = a1_\Omega,\ a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim X$ *or* $a1_\Omega \succsim Y$.
(ii) $[X, Y \in \mathcal{B}_\infty,\ X \succsim Y,\ \alpha \in [0,1],\ \alpha X + (1-\alpha)Y = a1_\Omega,\ a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$.
(iii) $[X, Y \in \mathcal{B}_\infty,\ X \sim Y,\ \alpha \in [0,1],\ \alpha X + (1-\alpha)Y = a1_\Omega,\ a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$.

Note that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious while to prove that $(iii) \Rightarrow (i)$, we use the natural fact that for all $X,\ Y \in \mathcal{B}_\infty,\ \lambda \geq 0$ if $X \succsim Y$ then $X + \lambda \succsim Y$.

Remark 3.7 *1) Preference for perfect hedging means that if the decision maker can attain certainty by a convex combination of two assets, then he prefers certainty to one of these assets, which is one of the mildest requirements for uncertainty aversion, so we can also call it attraction for certainty.*

2) Comonotone diversification is nothing but convexity of preferences restricted to comonotone random variables (see Schmeidler [18]), it is therefore a kind of uncertainty aversion. Note that any hedging (in the sense of Wakker [22]) is prohibited in this diversification operation. This type of diversification turns out to be equivalent, in the CEU model, to the concavity of u.

3.3 Relating uncertainty aversion behavior

Definition 3.8 *A DM is symmetrical monotone uncertainty averse (SMUA) if for all* $X,\ Y \in \mathcal{B}_\infty,\ X \succsim_{SM} Y \Rightarrow X \succsim Y$ *where* $X \succsim_{SM} Y$ *means that there exists* $Z \in \mathcal{B}_\infty$, Z *comonotone with* X *such that* $I(Z) = I(-Z)$ *and* $Y = X + Z$.

Y represents a monotone symmetrical increase of uncertainty in relation to X. So, a DM is symmetrical monotone uncertainty averse if he doesn't like the monotone symmetrical increase of uncertainty i.e. if he always prefers X to Y.

A similar notion of monotone symmetrical risk aversion was already defined by Abouda and Chateauneuf [2] for a RDEU agent.

A RDEU agent is said to be symmetrical monotone risk averse (SMRA) if for all $X,\ Y \in \mathcal{B}_\infty,\ X \succsim_{SM} Y \Rightarrow X \succsim Y$ where $X \succsim_{SM} Y$ means that there exists $Z \in \mathcal{B}_\infty$, Z comonotone with X such that $E(Z) = 0,\ Z =_d -Z$ and $Y =_d X + Z$.

Note that in the previous definition, the condition $I(Z) = I(-Z)$ is equivalent to $E(Z) = 0$ when v is a probability measure.

Theorem 3.9 *For a CEU DM with $u : \mathbb{R} \to \mathbb{R}$ a C^1 non-decreasing non-constant concave utility mapping and a non trivial capacity v on $\mathcal{A}$, the following two assertions are equivalent:*

(1) $v(A) + v(A^c) \leq 1$ for all A in $\mathcal{A}$.

(5) The DM is SMUA

4 Concluding comments

After recalling the pioneering result of Dow and Werlang on the no-trade interval of a CEU DM, we generalize this result by allowing for negative prices. We also prove that a DM will exhibit this no-trade interval if and only if he is attracted by perfect hedging and has preference for comotone diversification or equivalently if he presents some kind of uncertainty aversion and satifies the super-additivity at certainty. Finally, we show that for a CEU DM endowed with a concave non-decreasing utility function, super-additivity at certainty is equivalent to being SMUA. While Dow and Werlang only considered positive initial wealth and prices, we generalize their result by allowing for non-positivity. Our goal was achieved under the assumption that borrowing was excluded. We intend in the future to study the robustness of our results if this restriction is removed.

5 Appendix

We now give two technical lemmas which will be useful in the sequel:

Lemma 5.1 *Let $Y \in \mathcal{B}_\infty,\ u : \mathbb{R} \to \mathbb{R}$ be C^1 non-decreasing and let $g :\ \alpha \in \mathbb{R} \to g(\alpha) = I(u(W + \alpha Y))\ \ \forall W \in \mathbb{R}$. Then, $g'_+(0)$ exists and*

$$g'_+(0) = u'(W)I(Y)$$

Proof :

Let $\alpha > 0$, from the mean value theorem, there exists $t \in [0,1]^\Omega$ such that

$$\frac{u(W + \alpha Y) - u(W)}{\alpha} = u'(W + t\alpha Y)Y.$$

It follows from positive homogeneity and constant additivity of the Choquet integral that

$$\frac{g(\alpha)-g(0)}{\alpha}=\int_{\Omega} u'(W+t\alpha Y)Y dv.$$

Note that $W+t\alpha Y$ converges to W in $\mathcal{B}_\infty$ when α goes to zero. Hence since u' is continuous and Y fixed, $u'(W+t\alpha Y)Y$ converges to $u'(W)Y$ in $\mathcal{B}_\infty$ when $\alpha \downarrow 0$. Since I is norm-continuous on $\mathcal{B}_\infty$, $\int_\Omega u'(W+t\alpha Y)Y dv$ converges to $\int_\Omega u'(W)Y dv$ when $\alpha \downarrow 0$.

And since $u'(W)\geq 0$ and $\int_\Omega u'(W)Y dv = u'(W)\int_\Omega Y dv$, $g'_+(0)=u'(W)I(Y)$. □

Lemma 5.2 *Let $Y\in\mathcal{B}_\infty$, and $u:\mathbb{R}\to\mathbb{R}$ be a non-decreasing C^1 concave utility mapping and W in $\mathbb{R}$. If $\alpha\geq 0$ and $I(Y)\leq 0$ then $W+\alpha Y\lesssim W$.*

Proof :

Let us suppose that $\alpha>0$ and $I(Y)\leq 0$, then $\alpha I(Y)\leq 0$ and so $W+\alpha I(Y)\leq W$.

By comonotonicity, $I(W+\alpha Y)\leq W$.

Since u is non-decreasing, $u(I(W+\alpha Y))\leq u(W)$.

But, since u is concave and non-decreasing, we can use Jensen's inequality for capacities proved by Asano [4] Theorem 4 p. 231 to obtain:

$$I(u(W+\alpha Y))\leq u(I(W+\alpha Y))$$

and so

$$I(u(W+\alpha Y))\leq u(W)=I(u(W)) \text{ i.e. } W+\alpha Y\lesssim W.$$

□

Proof of Theorem 3.4:

We first prove that (b) implies (a):

$\star$ Let us prove that (b) (1) implies that for all $X\in\mathcal{B}_\infty$, $I(X)\leq -I(-X)$.

Let $X\in\mathcal{B}_\infty$, then

$$I(X)=\int_{-\infty}^{0}\left(v(X\geq t)-1\right)dt+\int_0^{+\infty} v(X\geq t)\,dt.$$

Note that, since $t\mapsto v(X\leq t)$ is non-decreasing, the set of its discontinuities is at most countable.

Therefore, for $a<b$, $\int_a^b v(X\leq t)dt=\int_a^b v(X<t)dt$.

Thus,

$$\begin{aligned} I(-X) &= \int_{-\infty}^{0} (v\,(X \leq -t) - 1)\,dt + \int_{0}^{+\infty} v\,(X \leq -t)\,dt \\ &= \int_{-\infty}^{0} (v\,(X < -t) - 1)\,dt + \int_{0}^{+\infty} v\,(X < -t)\,dt \\ &= -\int_{+\infty}^{0} (v\,(X < t) - 1)\,dt - \int_{0}^{-\infty} v\,(X < t)\,dt \\ &= \int_{-\infty}^{0} v\,(X < t)\,dt + \int_{0}^{+\infty} (v\,(X < t) - 1)\,dt \end{aligned}$$

Thus,

$$\begin{aligned} I(X) + I(-X) &= \int_{\mathbb{R}} (v\,(X < t) + v\,(X \geq t) - 1)\,dt \\ &\leq 0 \textit{ since by hypothesis } v(A) + v(A^c) \leq 1 \textit{ for all } A \textit{ in } \mathcal{A} \end{aligned}$$

i.e. $I(X) \leq -I(-X)$.

$\star$ We now turn to the no-trade interval result.

Indeed all the proofs below make sense only in case of feasible trades, and consequently are valid as stated in Theorem 3.4, only in these cases.

Let us first prove that if $p \geq I(X)$, the investor is at least as well off not holding the asset, as buying any positive amount α.

Note that for any p, buying any positive amount α of the asset leads to the uncertain future wealth $W(\alpha) = W + \alpha(X - p)$.

The formula is clearly true if $p = 0$.

If $p > 0$, buying a positive amount α of the asset at price p requires an amount of money $\alpha p > 0$, hence $W(\alpha) = W - \alpha p + \alpha X = W + \alpha(X - p)$.

If $p < 0$, buying αX yields a gain equal to $-\alpha p$, so $W(\alpha) = W - \alpha p + \alpha X$, hence $W(\alpha) = W + \alpha(X - p)$.

In order to show that the investor prefers not to buy, it is then enough to see that $W + \alpha(X - p) \lesssim W$ if $\alpha > 0$, but this results directly from Lemma 5.2, since $I(X - p) = I(X) - p \leq 0$.

Let us now prove that if $p \leq -I(-X)$, the investor is at least as well off not selling short the asset, as selling short any positive amount α.

Note that for any p, selling short any positive amount α of the asset leads to the uncertain future wealth $\widetilde{W}(\alpha) = W + \alpha(-X + p)$.

The formula is clearly true if $p = 0$.

If $p > 0$, selling short αX yields a gain equal to αp, so $\widetilde{W}(\alpha) = W + \alpha p + \alpha(-X)$, hence $\widetilde{W}(\alpha) = W + \alpha(-X + p)$.

If $p < 0$, selling short a positive amount α of the asset at price p requires an amount of money $-\alpha p > 0$, hence $\widetilde{W}(\alpha) = W + \alpha p - \alpha X = W + \alpha(p - X)$.

In order to show that the investor prefers not to sell short, it is then enough to see that $W + \alpha(p - X) \lesssim W$ if $\alpha > 0$, but this results from Lemma 5.2, by setting $Y = p - X$; actually $I(Y) = p + I(-X) \leq 0$.

It remains to prove that if $p < I(X)$ the DM will hold a positive amount α of the asset and that if $p > -I(-X)$, he will hold a short position $\alpha > 0$.

Assume now that $p < I(X)$. We only need to show that $W(\alpha) \succ W(0) = W$ for some $\alpha > 0$ or equally that $g(\alpha) = I\left(u\left(W + \alpha\left(X - p\right)\right)\right) > g(0)$ for some $\alpha > 0$.

From Lemma 5.1, $g'_+(0) = u'(W)I(X - p)$ hence $g'_+(0) > 0$, which completes the proof.

Finally let $p > -I(-X)$. We only need to show that $\widetilde{W}(\alpha) \succ \widetilde{W}(0) = W$ for some $\alpha > 0$ or equally that $g(\alpha) = I\left(u\left(W + \alpha\left(p - X\right)\right)\right) > g(0)$ for some $\alpha > 0$.

From Lemma 5.1, $g'_+(0) = u'(W)I(p - X)$ hence $g'_+(0) > 0$, which completes the proof.

We now prove that (a) implies (b):

$\star$ Let us prove that if $I(X) + I(-X) \leq 0$ for all X in $\mathcal{B}_\infty$ then $v(A) + v(A^c) \leq 1$ for all A in $\mathcal{A}$:

Let $A \in \mathcal{A}$, and let $X = 1_A$. We have:

$$I(X) = \int_\Omega X dv = v(X = 1) = v(A).$$

$$I(-X) = -1 + v(X = 0) = -1 + v(A^c).$$

But since $I(X) + I(-X) \leq 0$ by hypothesis, $v(A) + v(A^c) \leq 1$.

$\star$ Let us prove that u is concave:

Let $A \in \mathcal{A}$ such that $0 < v(A) < 1,\ x,\ y \in \mathbb{R}$ such that $y < x,\ X = x1_A + y1_{A^c}$ and $t = v(A)$.

We have

$$I(X) = y + (x - y)t = tx + (1 - t)y.$$

Let $W = p = I(X) = tx + (1 - t)y$ and $\alpha = 1$, note that since $W = p$, we are in a situation of feasible trade, so according to (a), $p = I(X)$ implies $W + \alpha(X - p) \lesssim W$

$$i.e.\ I\left(u\left(W + X - p\right)\right) \leq u(W).$$

Consequently

$$I(u(X)) \leq u(tx + (1 - t)y).$$

On the other hand, since u is non-decreasing,

$$I(u(X)) = u(y) + (u(x) - u(y))t.$$

Therefore,

$$u(y) + (u(x) - u(y))t \leq u(tx + (1-t)y)$$

$$i.e.\ tu(x) + (1-t)u(y) \leq u(tx + (1-t)y).$$

From this, we conclude that u is concave by a result due to Hardy, Littlewood and Polya (see Wakker [20]) which states that, for a continuous function, it is enough to satisfy the concavity inequality for one $t \in (0,1)$ in order to be concave. □

Proof of Theorem 3.5:

By theorem 3.4, it is enough to prove that (1) and (2) is equivalent to (3) and (4).

Chateauneuf and Tallon [6] showed that for a Choquet expected utility decision maker, preference for comonotone diversification is equivalent to the concavity of the utility function i.e. that conditions 2) and 4) are equivalent. The proof that 1) and 2) $\Rightarrow$ 3) and 3)$\Rightarrow$ 1) is inspired by a paper of Abouda and Chateauneuf [2] where the same result is proved but under risk for a RDEU agent.

$\star$ Let us prove that (1) and (2) implies (3) (cf. Abouda and Chateauneuf [2] theorem 3.8 (iii) $\Rightarrow$ (v)):

Let $X, Y \in \mathcal{B}_\infty,\ X \succsim Y$, and $\alpha \in [0,1]$, such that $\alpha X + (1-\alpha)Y = a1_\Omega,\ a \in \mathbb{R}$.

We want to prove that $a1_\Omega \succsim Y$ i.e. $u(a) \geq I(u(Y))$.

Since u is concave and non-decreasing, Jensen's inequality implies hat

$$I(u(X)) \leq u(I(X))$$

and

$$I(u(Y)) \leq u(I(Y)).$$

Furthermore, since $X \succsim Y$, $I(u(Y)) \leq I(u(X))$.

So, $I(u(Y)) \leq Min(u(I(X)), u(I(Y)))$.

Furthermore, if $I(X) \geq I(Y)$ (resp. $I(Y) \geq I(X)$) then $I(a1_\Omega) \geq I(Y)$ (resp. $I(a1_\Omega) \geq I(X)$).

Indeed: if $I(X) \geq I(Y)$ then

$$
\begin{aligned}
I(Y) &= \alpha I(Y) + (1-\alpha) I(Y) \\
&\leq I(\alpha X) + I((1-\alpha)Y)) \text{ since } I(X) \geq I(Y) \\
&\leq -I(-\alpha X) + I((1-\alpha)Y)) \text{ since } v(A) + v(A^c) \leq 1 \ \forall A \text{ implies } I(X) \leq -I(-X) \ \forall X \\
&\leq -I((1-\alpha)Y - a1_\Omega) + I((1-\alpha)Y)) \text{ since } \alpha X + (1-\alpha)Y = a1_\Omega \\
&= I(a1_\Omega).
\end{aligned}
$$

So,

$$Min(I(X), I(Y)) \leq I(a1_\Omega) = a.$$

Since u is non-decreasing,

$$Min(u(I(X)), u(I(Y))) = u(Min(I(X), I(Y))) \leq u(a)$$

and so

$$I(u(Y)) \leq u(a) \text{ } i.e. \text{ } a1_\Omega \succsim Y.$$

$\star$ Let us prove that (3) implies (1) (cf. Abouda and Chateauneuf [2] theorem 3.8 (v) $\Rightarrow$ (iii)):

By contradiction: Suppose that there exists A in $\mathcal{A}$ such that $v(A) + v(A^c) > 1$.

Note that this implies $v(A) > 0$ and $v(A^c) > 0$.

Let $a \in \mathbb{R}, \ \epsilon > 0$,

$$X_\epsilon = (a - \epsilon v(A^c))1_A + (a + \epsilon v(A))1_{A^c}$$

and

$$Y_\epsilon = (a + \epsilon v(A^c))1_A + (a - \epsilon v(A))1_{A^c}.$$

Thus,

$$\frac{1}{2}(X_\epsilon + Y_\epsilon) = a(1_A + 1_{A^c}) = a1_\Omega.$$

and since u is non-decreasing,

$$
\begin{aligned}
&I(u(X_\epsilon)) = u(a - \epsilon v(A^c)) + (u(a + \epsilon v(A)) - u(a - \epsilon v(A^c)))\, v(A^c) \\
&i.e. \ \ I(u(X_\epsilon)) = (u(a - \epsilon v(A^c)))(1 - v(A^c)) + u(a + \epsilon v(A))\, v(A^c).
\end{aligned}
$$

Furthermore,

$$I(u(a)1_\Omega) = u(a) = u(a)(1 - v(A^c)) + u(a)v(A^c).$$

So,

$$I(u(X_\epsilon)) - I(u(a)1_\Omega) = (1 - v(A^c))(u(a - \epsilon v(A^c)) - u(a)) + v(A^c)(u(a + \epsilon v(A)) - u(a)).$$

By a Taylor expansion of order 1, we obtain:

$$u\left(a-\epsilon v(A^c)\right)-u(a)=-\epsilon v(A^c)u'(a)+\circ_1(\epsilon)$$
$$u\left(a+\epsilon v(A)\right)-u(a)=\epsilon v(A)u'(a)+\circ_2(\epsilon)$$

So,

$$\begin{aligned} I\left(u\left(X_\epsilon\right)\right)-I(u(a)1_\Omega) &= \left(1-v(A^c)\right)\left(-\epsilon v(A^c)u'(a)+\circ_1(\epsilon)\right)+v(A^c)\left(\epsilon v(A)u'(a)+\circ_2(\epsilon)\right) \\ &= \epsilon\left[v(A^c)u'(a)\left(v(A)+v(A^c)-1\right)+\tfrac{\circ_1(\epsilon)}{\epsilon}-\tfrac{\circ_1(\epsilon)}{\epsilon}v(A^c)+\tfrac{\circ_2(\epsilon)}{\epsilon}v(A^c)\right] \\ &= \epsilon\left[v(A^c)u'(a)\left(v(A)+v(A^c)-1\right)+\beta(\epsilon)\right] \text{ with } lim_{\epsilon\to 0}\beta(\epsilon)=0. \end{aligned}$$

But, since $v(A)+v(A^c)>1,\ u'(a)>0$ and $v(A^c)>0,\ \exists\epsilon_1>0$ with

$$I\left(u\left(X_\epsilon\right)\right)-I(u(a)1_\Omega)>0\ if\ \epsilon\in(0,\epsilon_1].$$

We obtain in the same way that

$$I\left(u\left(Y_\epsilon\right)\right)-I(u(a)1_\Omega)>0\ if\ \epsilon\in(0,\epsilon_2]\ \ for\ some\ \epsilon_2>0.$$

Then, taking $\epsilon=min\left(\epsilon_1,\epsilon_2\right)$, we obtain $a1_\Omega\prec X_\epsilon$ and $a1_\Omega\prec Y_\epsilon$ which contradicts perfect hedging.

$\star$ Let us prove that (2) implies (4) (cf. Chateauneuf and Tallon [6] theorem 3 (ii) $\Rightarrow$ (i)):

Let $X,\ Y\in\mathcal{B}_\infty$ be comonotone and such that $X\sim Y$ and let $\alpha\in(0,1)$.

We want to prove that $\alpha X+(1-\alpha)Y\succsim Y$ i.e. $I(u(\alpha X+(1-\alpha)Y))\geq I(u(Y))$.

Thus,

$$\begin{aligned} I(u(Y)) &= \alpha I(u(Y))+(1-\alpha)I(u(Y)) \\ &= \alpha I(u(X))+(1-\alpha)I(u(Y))\ since\ X\sim Y \\ &= I(\alpha u(X)+(1-\alpha)u(Y))\ since\ u\ is\ non-decreasing\ and\ so\ preserves\ comonotony \\ &\leq I(u(\alpha X+(1-\alpha)Y))\ since\ u\ is\ concave. \end{aligned}$$

$\star$ Let us prove that (4) implies (2):

The proof given on $\mathbb{R}_{++}$ by Chateauneuf and Tallon for (i) $\Rightarrow$ (ii) in theorem 3 of [6] remains valid on $\mathbb{R}$, hence (4) implies (2). □

Proof of Theorem 3.9:

$\star$ Let us prove that (1) implies (5):

Let $X,\ Y \in \mathcal{B}_\infty$ with $X \succsim_{SM} Y$, so that there exists $Z \in \mathcal{B}_\infty$ comonotone with X such that $I(Z) = I(-Z)$ and $Y = X + Z$.

We want to prove that $X \succsim Y$ i.e. $I(u(X)) \geq I(u(Y))$.

1. Suppose that for all $t \in \Omega,\ Z(t) \geq 0$.

   For $t \in \Omega$ such that $Z(t) > 0$, since u is concave, we have

   $$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq u'(X(t)) \leq M := sup_{x \in X(\Omega)} u'(x).$$

   M is finite since u' is continuous and X bounded. Also $M \geq 0$ since $u' \geq 0$.

   So, $u(X(t) + Z(t)) \leq u(X(t)) + MZ(t)\ \ \forall t \in \Omega$ such that $Z(t) > 0$.

   This inequality is also obviously true for t such that $Z(t) = 0$.

2. Suppose that for all $t \in \Omega,\ Z(t) \leq 0$.

   For $t \in \Omega$ such that $Z(t) < 0$, since u is concave, we have

   $$\frac{u(X(t)) - u(X(t) + Z(t))}{-Z(t)} \geq u'(X(t)) \geq m := inf_{x \in X(\Omega)} u'(x).$$

   For the same reason as in 1., $0 \leq m \leq +\infty$.

   So, $u(X(t) + Z(t)) \leq u(X(t)) + mZ(t)\ \ \forall t \in \Omega$ such that $Z(t) < 0$.

   This inequality is also obviously true for t such that $Z(t) = 0$.

3. Suppose that there exist s and t in Ω such that $Z(s) < 0$ and $Z(t) > 0$.

   Since Z is comonotone with X, $X(t) - X(s) \geq 0$.

   Furthermore, since u is concave,

   $$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq \frac{u(X(s)) - u(X(s) + Z(s))}{-Z(s)} \qquad (\alpha)$$

   Let $M' = Sup_F \left\{ \frac{u(X(t)+Z(t)) - u(X(t))}{Z(t)} \right\}$ where $F = \{t \in \Omega \mid Z(t) > 0\}$.

   Clearly $M' \geq 0$ since u is non-decreasing.

   We have, thanks to (α), for all $s,\ t \in \Omega$ such that $Z(s) < 0$ and $Z(t) > 0$,

   $$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq M' \leq \frac{u(X(s)) - u(X(s) + Z(s))}{-Z(s)}$$

   and then

   $$u(X(t) + Z(t)) \leq u(X(t)) + M'Z(t) \quad \forall t \mid Z(t) > 0$$

   $$u(X(s) + Z(s)) \leq u(X(s)) + M'Z(s) \quad \forall s \mid Z(s) < 0$$

   We also have the same inequality if $Z(s) = 0$ or $Z(t) = 0$.

   Therefore, in all cases, there exists an $M \geq 0$ such that

   $$u(X + Z) \leq u(X) + MZ.$$

   So, since $Y = X + Z$ and $u(X)$ is comonotone with MZ,

   $$I(u(Y)) = I(u(X + Z)) \leq I(u(X) + MZ) = I(u(X)) + MI(Z).$$

   Since $v(A) + v\left(A^c\right) \leq 1$ for all A in $\mathcal{A},\ I(Z) + I(-Z) \leq 0$ (see theorem 3.4) and since $I(Z) = I(-Z),\ I(Z) \leq 0$.

Thus,

$$I(u(Y)) \leq I(u(X)) + MI(Z) \leq I(u(X)) \; i.e. \; X \succsim Y.$$

$\star$ Let us prove that (5) implies (1):

Let $x \in \mathbb{R}$ such that $u'(x) > 0,\ 0 < \epsilon$ and $A \in \mathcal{A}$.

If $v(A) = 1$ and $v(A^c) = 0$, there is nothing to prove.

Otherwise $1 + v(A^c) - v(A) > 0$, which we will assume from now on.

Let

$$Y_\epsilon = (x - \epsilon)1_A + \left(x + \frac{v(A^c) - v(A) - 1}{v(A) - v(A^c) - 1}\epsilon\right) 1_{A^c}$$

and

$$Z_\epsilon = -\epsilon 1_A + \frac{v(A^c) - v(A) - 1}{v(A) - v(A^c) - 1}\epsilon 1_{A^c}.$$

One can easily check that $I\left(Z_\epsilon\right) = I\left(-Z_\epsilon\right)$ and therefore $x1_\Omega \succsim_{SM} Y_\epsilon$.

$$I(u\left(Y_\epsilon\right) = u(x - \epsilon) + \left(u\left(x + \frac{v(A^c) - v(A) - 1}{v(A) - v(A^c) - 1}\epsilon\right) - u(x - \epsilon)\right) v(A^c)$$

Since the DM is SMUA and $x1_\Omega \succsim_{SM} Y_\epsilon$, $I\left(u\left(x1_\Omega\right)\right) \geq I(u\left(Y_\epsilon\right)$ so that,

$$\frac{u(x) - u(x - \epsilon)}{\epsilon} \geq \frac{v(A^c)}{\epsilon}\left(u\left(x + \frac{v(A^c) - v(A) - 1}{v(A) - v(A^c) - 1}\epsilon\right) - u(x - \epsilon)\right).$$

Passing to the limit when ϵ goes to 0, we obtain

$$u'(x) \geq \frac{2u'(x)v(A^c)}{1 + v(A^c) - v(A)}$$

and since $u'(x) > 0$, we conclude that

$$\frac{1 - v(A) - v(A^c)}{1 + v(A^c) - v(A)} \geq 0.$$

And therefore,

$$v(A) + v(A^c) \leq 1 \; since \; 1 + v(A^c) - v(A) > 0.$$

□

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