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A simple theory of the optimal number of immigrants

Chi-Chur Chao[*] , Bharat R. Hazari[+] and Jean-Pierre Laffargue[#]

<u>Abstract:</u> This paper develops a simple model to explain two stylised facts about immigration. First, some countries have a low ratio of migrants in their population, while other wealthy countries have a high number of migrants. In fact such migrants are of the same order of magnitude as their domestic workforce. Secondly, migrants are often segregated in jobs. The domestic residents do not wish to be employed in these jobs due to their unattractive working conditions and payments.

The model assumes that domestic residents are all identical in terms of their skills and wealth and furthermore that native and foreign workers have the same skills. However, foreign migrants cannot be excluded from the use of public services, the quality of which decreases due to congestion created or enhanced by migrants. On the basis of our model we show that the stylised facts are consistent with an optimal immigration policy, defined by domestic residents who have neither altruistic feelings nor ethnic prejudice toward foreign migrants.

Keywords: Congestion, Immigration, Public good, Segregation
JEL Classification: F22, H41, O41

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Introduction

Immigration is a very important issue in many countries, both developed and developing. Legislation to grant amnesty to migrants was submitted and failed before the US Congress. Fences have also been erected in the USA to stop the flow of migrants from Mexico. Even a country like India with an enormously large population and high unemployment receives a large number of illegal migrants from neighbouring countries. Given the importance of migration in many countries, it is important to ask: what are the effects of foreign migration on the economy of the host country, and in particular the effects on the welfare of the domestic residents. Several economists dealing with this question have focused on the income distribution effects of migration. For instance, do wages decrease, or do the cost and the sustainability of the income transfer policy of the government worsen? One reason for discussing this aspect of the migration issue is that the anti-immigration feelings are especially strong in specific social groups[1]. *Find more recent illustrative facts.*

The motivation of this paper comes from the following stylised facts. First, some countries have a moderate ratio of migrants in the population. OECD countries are a typical example of this moderate percentage, about 13% of the US population is foreign born. Second, other countries have a very large body of migrants. The *International Migration Outlook* of the OECD (2006) notes that most foreign migrants in OECD countries are allowed to settle in OECD countries for non-economic reasons: asylum seekers, family reunification, the regularisation of illegal migrants and so on. People who migrate for economic reasons, in general are hired in specific jobs, often but not always with low or no skills, which domestic workers do not want to take because of their unattractive working conditions

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[1] Benhabib(1996) examines how immigration policies would be determined under majority voting when native agents differ in their wealth holding and vote to maximise their income. Razin, Sadka and Swagel (2002) complement this approach and investigate the link between migration and tax transfer policy when domestic residents differ in ability and hold different skills. These papers are based on one-sector models. Other papers use a Heckscher-Ohlin framework, for instance Wellisch and Walz (1998). Scheve and Slaughter (1999) analyse opinion surveys and show that people with less skill (who directly compete with foreign migrants) are more hostile to immigration. Facchini and Mayda (2006) confirm this result, and complete it by showing that people with high income (who pay the taxes, which finance the welfare transfers to migrants) are also more hostile to immigration.

and payments. The number of migrants who are given these working permits is low, especially in Western Europe. However, a few countries admit a much higher number of migrants. In its report for 2005, the International Organisation for Migration writes: "It is estimated that by 2000 international migrants constituted 38 per cent of the population in Bahrain, 49 per cent in Kuwait, 26 per cent in Oman, 70 per cent in Qatar, 24 per cent in Saudi Arabia and 68 per cent in the United Arab Emirates." Other evaluations given in the report are higher, and the share of migrants in labour force is still higher: it is equal to 55.8 per cent in Saudi Arabia and represents 70 per cent in all Gulf Cooperation Council (GCC) countries.

The model we suggest has its roots in the literature on migration theory starting with Berry and Soligo (1969). They proved the important result that the income of domestic residents is an increasing and unbounded function of the number of migrants. However, the marginal effect of immigration on this income is zero when in the initial equilibrium the number of migrants is zero. Borjas (1999) confirmed that this result remains approximately valid when the number of migrants is of the order of magnitude observed in OECD countries. He concludes that the presence of migrants, who represented about 10 per cent of the labour force of the U.S. economy, increased at most by 0.4 per cent the income of American citizens [2]. This result is the second reason for the lack of interest of economists for the effect of immigration on the income of the natives.

This paper will suggest an explanation of the stylised facts and provides reasons as to why, in spite of the theoretical result that the income of domestic residents is an increasing and unbounded function of the number of migrants, migrants are unwelcome in OECD countries, the exception being those who are segregated in unattractive and poorly paid jobs. It will also explain why in other wealthy countries the number of legal migrants is very high. The analysis and the explanation given in this paper eliminate all considerations related to income distribution: domestic residents are identical and there is no place for income redistribution policy. We will also assume that there is no difference in skills

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[2] However, the distribution effect can be very high; the income of capital may increase by until 12 per cent and the income of domestic workers may decrease by 6 per cent.

between domestic and foreign workers. Thus, our paper differs from most previous approaches. We will use a simple neo-classical model and compute the optimal number of migrants. Our domestic residents are selfish and bear no altruistic feeling for migrant workers. They also are not prejudiced against them. Thus, our analysis excludes motivations based on a taste for discrimination or racism[3].

The optimal number of migrants in the model is not infinite because domestic residents consume a public service, which is a complement of a private good in consumption. Migrants cannot be excluded from the use of the public service. Then, for a given level of this service, the benefits obtained by its users decrease with the number of migrants because its quality decreases due to congestion. The supply of the public service can be increased at a cost. However, as the output of the public service also depends on fixed inputs such as land and space, its marginal cost is increasing. In this scenario although migrants make a positive contribution to national income, they will reduce the welfare of the domestic residents when too many of these migrants are present in the economy.

Public service has a very wide meaning in this paper. It includes health and education. As the costs of providing them are approximately constant, migrants do not create congestion if they pay a fair amount of taxes. However, the supply of other services can be raised only at increasing cost, because their input includes land, air, water and other natural resources with inelastic supply. OECD countries, with no lack of space such as Australia and Canada, are also those with the highest proportion of foreign born in total population. In this paper, the only externalities created by migrants work through the congestion effect. We will not consider other externalities, positive or negative, such as the taste for for cultural diversity, or the desire to keep the society homogenous, of domestic residents[4].

We begin by assuming that the domestic residents and migrants are perfect substitutes. When the number of migrants in the country is zero, we know that a small inflow of migrants has no effect on

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[3] There exists an empirical literature, using opinion surveys, which tries to disentangle economic reasons from racist and xenophobic sentiments in the reluctance of public opinion against a softening of immigration controls. See for instance Dustman and Preston (2004) and Malchow-Moller and *alii* (2006).
[4] The approach used in this paper is connected to but different from the important literature evaluating the impact of immigration on the costs of the welfare systems. Nannestad (2007) gives an excellent review of this literature.

the income of domestic residents. However, the quality of the public service declines, and the welfare of domestic residents decreases. As the number of immigrants approaches infinity, the quality of the public service converges to zero, and the welfare of domestic residents converges to its minimum value.

The first result of the paper is that if the preference of domestic residents for the consumption of the public service relative to the consumption of the private good is low enough, the welfare of domestic residents will start by decreasing with the number of migrants, then increases, reaches a maximum and finally decreases. A corollary of this result is that when the preference for the public service decreases, the optimum number of migrants will start from zero and remain at this value. Then it will jump to a positive value, and will increase indefinitely. However, the jump is very large, which means that the optimum number of migrants is either zero or very large. Moreover, we obtain a positive optimal number of migrants only for very low preferences for the public service, and so for an unreasonable small share of the population working in the government sector. Thus, under the assumption that migrants and domestic workers are perfect substitute, the model cannot explain the immigration policies of OECD and even GCC countries.

If the Government can segregate migrants in specific jobs, where jobs are imperfect substitute for each other (but, which requires the same skill), then results turn out to be different. A typical example of this is the refusal of medical practitioners to practice in rural areas in Australia. Immigrant doctors are happy to go to rural areas. Their visas are issued on the understanding that they practice for a few years in the rural areas. Then, migrants displace the domestic residents who held these jobs, and considerably expand the number of these jobs. Thus, the wages they pay become much lower. Under these circumstances, the optimal number of migrants will take reasonably small positive values, provided the size of the sectors of the labour market where they are allowed to work is small. The higher proportion of immigrants in the population of GCC countries can be explained by the access of these immigrants to a wider range of jobs (so by less segregation) than in industrialised countries.

The paper assumes that for the government, domestic residents and immigrants are different and the last group is used only for extracting rent. Migrants do not invest in their host country. So, either they will consume their whole income or they will remit part of it to their country of origin. However, many migrants become permanent residents and later citizens. They may have planned that when they decided to immigrate, or they could have made the decision of permanently settling in the country several years after their arrival. The assumptions of this paper do not apply to these cases. The definition of an optimal policy of immigration in this situation is also very different from the definition we give in this paper. It is an open and important question as to what stage of his "integration" a migrant would be considered as a fellow domestic resident by the nationals of the host country.

The model used in this paper is presented in section 1, under the assumption of perfect substitutability between the two kinds of labour. The effects of immigration on the welfare of domestic residents are investigated in section 2 at the theoretical level and in section 3 by running a series of simulations. Section 4 extends the previous results to the case of the imperfect substitutability of labour. The last section concludes.

1. The model

The model assumes an economy, which produces a private consumption good and a public good. The production technology of each good uses labour and a specific factor, which is supplied in a fixed quantity and normalised to 1. Domestic and foreign workers are perfect substitutes. The economy is closed to the rest of the world, except for international immigration and the remittance that immigrants may send home. The current output of private and public goods are respectively given by

$$Y = AL_Y^b \text{, with } 0 < b < 1 \text{ and } A > 0 \tag{1}$$

$$G = BL_G^g \text{, with } 0 < g < 1 \text{ and } B > 0 \tag{2}$$

where L_Y and L_G are the labour inputs used by each sector. There is no technical progress.

The quantity of domestic labour is fixed and normalised to 1. M denotes the number of migrants working in the country. The equilibrium of the labour market gives

$$L_Y + L_G = 1 + M \qquad (3)$$

The private good is produced by the private sector, which sets the wage rate w to the marginal productivity of labour. We have

$$w = \boldsymbol{b} A / L_Y^{1-\boldsymbol{b}} \qquad (4)$$

The public good is produced by the government, which owns (or can use freely) the specific factor and pays his workers the same wage w as the private sector. National workers and migrants are paid the same wage. Setting lower wages for migrants would require the introduction of mechanisms that justify this difference and would bring little insight to the problems we are investigating.

The public good is made available freely to all by the government and migrants cannot be excluded from its consumption. This non exclusion has humanitarian and other reasons. The production cost of the public good is equal to the wages paid by the government, wL_G. This cost is financed by a lump sum tax, which is equally shared between domestic residents and migrants.

The specific factor used by the private sector is entirely owned by domestic residents. Their net income is equal to the rent they earn on this factor plus their wages, minus their taxes, $(Y - wL_Y) + w - wL_G/(1 + M)$. This income is entirely spent on their consumption of private good C. We have

$$C = Y - \frac{M}{1 + M} w L_Y \qquad (5)$$

The net income of migrants is equal to their wages minus their taxes $wM - \dfrac{M}{1+M}wL_G = \dfrac{M}{1+M}wL_Y$.

It is spent on the purchase of private good, which they consume or transfer to their families at home[5].

We assume that the public good is subject to congestion, and that its quantity available to an individual declines as the number of users increases. The simplest way to model the congestion effect is to assume that the public good creates a public service S, equally enjoyed by each individual, which is equal to the total quantity of the public good divided by a function of the number of users: $G/(1+M)^j$, with $j > 0$. So public services are rival and non-excludable.

We can deduce from the previous equations the following expression of the supply set of the economy

$$\frac{(1+M)^{(1-b)/b}}{[1+(1-b)M]^{1/b}}(C/A)^{1/b} + (1+M)^{(j-g)/g}(S/B)^{1/g} = 1 \qquad (6)$$

This expression shows how the trade off between the consumption of private good and of public service by the domestic residents, C and S, depends on the number of migrants M. Domestic residents can obtain a zero amount of public service and a consumption of private good $C = A[1+(1-b)M]/(1+M)^{1-b}$. This consumption increases with the number of migrants. On the other hand domestic residents can obtain a zero amount of private consumption good and a consumption of public service $S = B(1+M)^{-(j-g)}$. This consumption decreases with the number of migrants if the congestion effect is high enough: $j > g$. Finally, when we move along the supply set, from $C = 0$ to $C = A[1+(1-b)M]/(1+M)^{1-b}$, the slope of the tangent to this set decreases from 0 to minus infinity. Figure 1 represents the supply set of the economy for two values of the number of migrants. This figure was computed for the calibrated version of the model presented in section 3 (and

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[5] Migrants have lower income per head than domestic residents because they do not get a share of the rent earned by the specific factor used by the private sector. However, they pay the same taxes as domestic residents. We made this assumption to simplify the analysis. In a previous version of this paper we explored the case when migrants pay no taxes. The results we obtained were qualitatively the same, although the optimal number of migrants was lower under this assumption.

under the assumption $j > g$). We can see how an increase in the number of immigrants expands the supply set in the direction of the consumption of private good and contracts it in the direction of the consumption of public service.

The utility of domestic residents, u, is a CES function of their consumption of private good and of public service.

$$u = \left[C^{s/(1+s)} + (S/b)^{s/(1+s)} \right]^{(1+s)/s} \text{, with } b > 0 \text{ if } -1 < s < 0$$

$$u = \left[C^{s/(1+s)} + (bS)^{s/(1+s)} \right]^{(1+s)/s} \text{, with } b \geq 0 \text{ if } s > 0$$

$$u = CS^{b} \text{, if } s = 0 \text{ and } u = Min(C, S/b) \qquad \text{if } s = -1 \tag{7}$$

Along an indifference curve, the substitution rate $, -dS/dC$, decreases with the value of parameter b. Thus, this parameter can be interpreted as a measure of the relative taste of the public service by domestic residents.

The government sets the number of migrants and the production of public good, by maximising the utility of domestic residents (7) under the resource constraint (6). Thus, we implicitly assume that the number of foreign workers who want to enter the country is always greater than the number the government wishes to allow in the country. Even when the optimal number of migrants is very high, the decrease in wages will be insufficient to discourage workers from foreign countries. Moreover, we give the government strong empowerment and the ability to control the legal situation of any worker without cost.

2. The effects of immigration on the welfare of domestic residents

The computation of the effects of immigration is especially easy and revealing when the private consumption good and the public service are complementary in the preferences of domestic residents

that is when $s = -1$. Then, the utility function of domestic residents becomes $u = Min(C, S/b)$ and the Government sets the production of public good G such that $u = C = S/b$. We also assume in this section that the elasticity of output to employment is the same in both sectors: $g = b$. Then, we deduce from the equation of the supply set of the economy the expression of the welfare of domestic residents

$u = f(M)^{-b}$, with

$$f(M) = (1 + M)^{(1-b)/b} [1 + (1 - b)M]^{-1/b} (1/A)^{1/b} + (1 + M)^{(j-b)/b} (b/B)^{1/b} \qquad (8)$$

Migrants have two opposite effects on the welfare of domestic residents. We will have a better understanding of the first effect by considering the extreme case where the taste for the public service b is zero. Then the government sets the production of public good and taxes to zero. The utility of domestic residents is equal to their consumption of private good, itself equal to their income $u = C = A[1 + (1 - b)M]/(1 + M)^{1-b}$. It is an increasing function in the number of migrants and tends to infinity with this number. More precisely we have: $\partial C/\partial M = -w'(M)M$. Thus a small increase in the number of migrants decreases the wage of all migrants and increases the income and the consumption of domestic residents by the same amount. The effect of a small increase in the number of migrants on the utility of domestic residents is zero when there are no migrants present in the country; it is maximum when the number of migrants already in the country is $1/(1 - b)$ (the utility function is convex in the number of migrants when $M < 1/(1 - b)$ and concave otherwise).

When the taste for the public service b is positive, a second effect appears which is that migrants create congestion, which reduces the welfare of domestic residents. The following proposition describes how under these two effects the optimal number of migrants is determined. It shows how the second effect leads to a finite optimal number of migrants, which becomes zero when the taste for the public service is high enough

Proposition 1. If the congestion effect is weak that is if $j \leq b$ then the optimal number of migrants is infinity. If $j > b$, there exist two thresholds for b: $0 < b_1 < b_2$, such that when the number of migrants increases from zero to infinity, then: a) if the taste for the public service b is larger than b_2, the utility of domestic residents decreases; b) if the taste for the public service b is between b_1 and b_2, the utility of domestic residents first decreases, then increases and then decreases. More importantly, the optimal number of migrants, which is the number of migrants that maximises the utility of domestic residents, is still zero; c) if the taste for the public service b is smaller than b_1, the utility of domestic residents still first decreases, then increases and then decreases. However, the optimal number of migrants is positive and equal to the largest of the two roots of the equation in M

$$(1 - b)(B / bA)^{1/b} M = (j / b - 1)(1 + M)^{(j-1)/b}[1 + (1 - b)M]^{1/b+1} \qquad (9)$$

This number is a decreasing function of b and increases indefinitely when b tends to 0.

Proof. See the appendix.

When the congestion effect is weak, the increase in congestion created by a higher number of migrants can be cancelled by affecting a part of these migrants to the production of more public good. The other migrants work in the private sector and increase the net income and the consumption of private good of domestic residents. Thus, the supply set expands in both directions when the number of migrants increases.

When the congestion effect is stronger we are in the configuration of Figure 1 that we commented on earlier. We saw that, when there are no migrants in the country, a small inflow of immigrants has no effect on the welfare of domestic residents, according to the first effect of immigration. However, the second effect, which is the increase of the congestion of the public good, leads to a decrease of this welfare. This explains why an inflow of migrants always reduces the welfare of domestic residents when the number of migrants already present in the country is low. When there are more migrants in

the country, the first effect turns positive, which compensates the higher congestion of the public service if the taste for this service is low enough. This could explain why, as Boeri and Brücker (2005) notice, immigration policy has become less restrictive from 1994 to 2004 in Spain and in Greece (these countries had very few migrants in 1994, then the number increased substantially).

If the taste for the public service is low enough, $b < b_1$, then a decrease in this taste leads to a higher optimal number of migrants. The change in the production of public good will result from two opposite effects. On one hand, the labour force in the economy is higher, which should imply a higher production of both goods. On the other hand the composition of this production changes in favour of more private consumption good and less public good. We cannot establish which effect is the strongest. The amount of public service delivered to domestic residents, will suffer of a third effect, which is that a higher number of immigrants increases the congestion of the public good.

Proposition 1 implies that when the taste for the public service decreases from infinity to zero, then the optimal number of migrants will start from zero and remain at this value, then it will jump to a positive value and increase indefinitely. The following corollary gives a lower bound for the size of this jump.

Corollary. When the utility function of domestic residents has a local maximum associated to a positive number of migrants, the minimum value of this number of migrants, is the positive root of the equation of variable M

$$j = \frac{b(1+2M)}{M[1+(1-b)M]} \ (10)$$

This root is a decreasing function of the strength of the congestion effect, j, and tends to zero when j increases indefinitely.

Proof. We have $f_2'(M) = f_2(M)/M = (1-b)(B/b_1A)^{1/(1-b)}$. So, M is the solution of the equation

$$\frac{j-1}{b}\frac{1}{1+M} + \left(\frac{1}{b}+1\right)\frac{1-b}{1+(1-b)M} = \frac{1}{M}.$$ The right side of equation (10) decreases from infinity to

zero when M increases from zero to infinity. The expression of the associated value of the preference

for public services, b_2, can be deduced from equation (9). ¦

Of course the smallest positive optimal number of immigrants, associated to a taste for the public

service equal to b_1, is largest than this bound.

The results of this section are based on the assumption that the two kinds of consumption are perfect

complement in the preferences of domestic residents. If we assume instead a Cobb Douglas utility

function that is a unitary elasticity of substitution between both kinds of consumption, we get the

following expressions of their optimal values, for a given number of migrants

$C = A[1/(1+b)]^b[1+(1-b)M](1+M)^{-1+b}$ and $S = B[b/(1+b)]^b(1+M)^{-j+b}$. So, the utility of

domestic residents is

$$u = \frac{AB^b}{(1+b)^b}\left(\frac{b}{1+b}\right)^{bb}\frac{1+(1-b)M}{(1+M)^{1-b+b(j-b)}} \quad (11)$$

If $b < b/(j-b)$, u first decreases, then increases and tends to infinity when M increases from zero

to infinity then. In the opposite case, u decreases and tends to zero. Thus, the optimal number of

migrants is either infinity (if the preference for the public service is low enough) or zero.

If, instead of the two assumptions made in the beginning of this section we assume that the elasticity

of substitution between the two kinds of consumption is larger than one ($s > 0$), then, for a given

level of the number of migrants, a feasible choice by the government is to produce a zero output of

public good $S = G = 0$. Then the welfare of domestic residents and their consumption of private good

is $u = C = A[1 + (1 - b)M]/(1 + M)^{1-b}$. Both increase indefinitely with the number of migrants, which establishes that its optimal value is infinity.

We will continue the analysis by simulation methods.

3. Simulations

We set the share of labour in private production $b = 2/3$ and we assume that $g = b$. We also set $A = 11.604$, $B = 7.31$. These two values are consistent with the productions $Y = 10$ and $G = 2.5$, obtained with the respective quantities of labour $L_1 = 0.8$ and $L_2 = 0.2$. We set the congestion parameter $j = 1$. For these values of parameters we can draw Figure 1, which represents the supply set of the economy, given by equation (6), when there are no migrants and when the number of migrants is equal to 30% of the population of domestic residents. This figure was commented in section 1.

The results of last section suggest that the most favourable case to obtain a reasonable positive value for the optimal number of migrants is when both kinds of consumption are perfect complement in the preferences of domestic residents. We will make this assumption in the rest of the section. Simulations not presented here, run for elasticities of substitution higher than -1, confirm this intuition. However, even in this favourable case, we will see that when the optimal number of migrants is positive and finite, its value is unreasonable high. Moreover, to obtain nonzero optimal numbers of migrants, we must assume a preference for public services extremely low, which implies a extremely small size of the government sector in the absence of immigration.

We set the value of the congestion parameter $j = 3$. Equation (10) shows that this value implies that the positive optimal levels of immigration will be more than 0.33. Smaller values for this parameter

would increase the value of this lower bound, and to obtain a smaller lower bound we would have to set this parameter to a still higher value.

Figure 2 represents the utility of domestic residents in function of the number of immigrants, for three values of the preferences for the public good, b. If $b = 0.030$, then the utility of domestic residents is a decreasing function of the number of migrants. If $b = 0.028$, this utility first decreases, then increases and finally decreases with the number of migrants. However, their optimal number is still zero (a local maximum in utility is reached for a number of migrants equal to 0.53 times the population of domestic residents ; for a value of b included between 0.028 and 0.030 this local maximum would reach the lower bound 0.33). If $b = 0.027$, then the utility function has the same shape as before, but the optimal number of migrants is positive (equal to 0.60 times the population of domestic residents). Other simulations show that if the preference for the public good is above 0.0274 , then the optimal number of migrants is zero. If this preference is less than 0.0273 the optimal number of migrants is more than 58% of the population of domestic residents. A preference for the public service as low as 0.0273 implies that, without migrants, less than 1% of the population would work in the production of public good.

If we set the congestion effect at the more reasonable value $j = 1$, equation (10) shows that the optimal number of migrants will be either zero or more than $b /(1 - b) = 2$ times the population of domestic residents. The associated value of the preference parameter for the public service associated to this lower bound is $b = 0.326$. However, the smallest positive optimal number of migrants is 4.57 times the population of domestic residents, which is associated to $b = 0.283$. This value of parameter b is consistent with reasonable values for the allocation of the labour force between the two sectors when there are no migrants (77% in the private sector and 23% in the public good sector). But the optimal number of migrants is unreasonably high.

We ran a series of simulations for a smaller elasticity of the production of public good to its labour input, $g = 1/3$. We adjusted the value of parameter B to 4.275, so that without migrants the economy was still able to produce the respective amounts $Y = 10$ and $G = 2.5$ of goods, with the allocation of labour 0.8 and 0.2 between the two sectors. We keep the last value of the congestion effect $j = 1$. Then, we get less dramatic results than in last paragraph. The smaller positive value for the number of migrants is equal to 1.14 times the population of domestic residents, and the associated value of the preference for the public good is $b = 0.111$. However, with this new parameterization, the allocation of labour between the two sectors becomes 97.4 and 2.6%, which gives an unreasonably low value to the size of the government sector.

These results suggest that the optimal number of migrants in industrialised countries is zero or is extremely large. The observed numbers of migrants in GCC countries are of an order of magnitude more consistent with the results of the simulations of the model. However, we should assume for these countries a strong congestion effect, a strong complementarity between both kinds of consumption in the preferences of domestic residents and a low value for the preference for the public service. The validity of these conditions is debatable.

We cannot accept the validity of the conclusion that all migrants who enter OECD countries cause a decline in the welfare of their domestic residents. In the next section we will assume that foreign and national workers are imperfect substitute, and we will reach less extreme conclusions.

4. Imperfect substitutability between national and foreign labour

4.1. Extension of the model

In this section we will assume that there are two kinds of jobs in the economy, denoted 1 and 2. These two jobs differ by their nature, but they ask for the same skill. A domestic resident can take any of the

two jobs, but the government can freely allocate immigrants between these jobs. We start by considering the private sector, which produces the consumption good. We assume that this sector employs N_Y domestic residents and M_Y immigrants. We have N_{Y1} domestic residents and M_{Y1} immigrants working in jobs of the first kind, while N_{Y2} domestic residents and M_{Y2} immigrants have taken the second kind of jobs, with $N_{Y1}, N_{Y2}, M_{Y1}, M_{Y2} \geq 0$, $N_{Y1} + N_{Y2} = N_Y$ and $M_{Y1} + M_{Y2} = M_Y$. The production function of the private sector becomes

$$Y = A\left(N_{Y1} + M_{Y1}\right)^{b} \left(N_{Y2} + M_{Y2}\right)^{d} \tag{12}$$

with the following conditions on the parameters: $0 < d < b < 1$, $d + b < 1$. We assume that there are relatively few of the second kinds of jobs, which means that the value of d / b is small. To simplify the exposition we will only consider the case where the number of migrants satisfies the condition

$$M_Y \leq \left(b / d\right)N_Y \tag{13}$$

The two jobs pay the respective wages

$$w_1 = bA\left(N_{Y1} + M_{Y1}\right)^{b-1}\left(N_{Y2} + M_{Y2}\right)^{d} \tag{14}$$

$$w_2 = dA\left(N_{Y1} + M_{Y1}\right)^{b} \left(N_{Y2} + M_{Y2}\right)^{d-1} \tag{15}$$

We can easily see that the wage rate of domestic residents (and of migrants) working on jobs 1 increases with the number of immigrants working on jobs 2.

The government allocates migrants between the two jobs so as to maximise the gross income that domestic residents obtain from the private sector: $R = Y - w_1 M_{Y1} - w_2 M_{Y2}$. The economy is in one of the three following regimes

- $w_1 = w_2$, $(b/d)(N_{Y2} + M_{Y2})/(N_{Y1} + M_{Y1}) = 1$ and $N_{Y1}, N_{Y2} \geq 0$[6].

- $w_1 > w_2$, $(b/d)(N_{Y2} + M_{Y2})/(N_{Y1} + M_{Y1}) > 1$, $N_{Y1} = N_Y$ and $N_{Y2} = 0$.

- $w_1 < w_2$, $(b/d)(N_{Y2} + M_{Y2})/(N_{Y1} + M_{Y1}) < 1$, $N_{Y1} = 0$ and $N_{Y2} = N_Y$.

In the first regime, the two kinds of jobs pay the same wages, and domestic residents are indifferent between them. In the second regime, the second kind of jobs pays lower wages than the first kind, no domestic residents take the former jobs, but migrants may be authoritatively allocated to them by the government. In the third regime the first kind of jobs pays lower wages and no domestic residents take them. We can establish the following proposition.

Proposition 2. If the number of migrants working in the private sector M_Y belongs to the interval

$$(d/b)N_Y < M_Y \leq y N_Y \text{ with } y = 2\frac{d/b + \sqrt{(d/b)(1 + d/b)(1 - b)}}{1 - d - b}, \text{ then it will be optimal for the}$$

government to segregate all the immigrants in the second kind of jobs. No domestic residents will take these jobs, which pay a lower wage than the jobs of the first kind. If $M_Y > y N_Y$, still no domestic residents will work on the second kind of jobs, which pay lower wages, but some migrants might be allocated to the first kind of jobs. If $M_Y \leq (d/b)N_Y$, the economy is in the first regime and the allocation of migrants and domestic residents between the two kinds of jobs, which pay the same wages, is a matter of indifference.

Proof. See the appendix.

Ottaviano and Peri (2006) note that, for given education and experience attainment the choice of occupation of foreign-born residents is quite different from that of natives. They conclude that overall

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[6] We can easily how that under this regime the production function (12) can be rewritten as $Y = A\dfrac{b^b d^d}{(d + b)^{d+b}}(N_Y + M_Y)^{b+d}$. Formally, we are in the same situation as in the model of section 1.

immigration over the 1980-2000 period significantly increased the average wages of U.S. born workers (by around 2%).

Now, we will complete the model. The production function for the public good is

$$G = B(N_{G1} + M_{G1})^b (N_{G2} + M_{G2})^d \qquad (16)$$

N_{G1} and M_{G1} respectively denote the number of domestic residents and immigrants working in jobs of the first kind, while N_{G2} and M_{G2} represent the number of domestic residents and immigrants having taken the second kind of jobs, with $N_{G1}, N_{G2}, M_{G1}, M_{G2} \geq 0$, $N_{G1} + N_{G2} = N_G$ and $M_{G1} + M_{G2} = M_G$. In order to better focus on the problem of the segregation of migrants we limit the analysis of the section to the case where the elasticities of production to its two inputs of labour are the same in both sectors. We assume that the Government pays the same wages as the private sector for both kinds of jobs, w_1 and w_2. It allocates its manpower between these jobs such as to minimise its production cost that is its wage bill. We have

$$\frac{w_1}{w_2} = \frac{\partial G / \partial (N_{G1} + M_{G1})}{\partial G / \partial (N_{G2} + M_{G2})} \qquad (17)$$

The production cost of the public good is covered by a lump sum tax. We make the simplifying assumption that the taxes paid by migrants cover the wages cost of migrants working for the government. The specific factor used by the private sector is entirely owned by domestic residents. Their net income is equal to the rent they earn on this factor plus their wages, minus their taxes (which are equal to the wages cost of domestic workers for the government). This income is entirely spent on their consumption of private good C. We have

$$Y - w_1 M_{Y1} - w_2 M_{Y2} = C \qquad (18)$$

Finally, the equilibrium of the two job markets gives the two last equations

$N_Y + N_G = 1$ (19)

$M_Y + M_G = M$ (20)

We will solve the model under the assumption that

$(d/b)N_Y < M_Y \leq yN_Y$ (21)

Under this condition, the second kind of jobs pays lower wages and migrants are segregated in these jobs and forbidden to take the first kind of jobs. Of course, domestic residents do not accept to work on the second kind of jobs. We deduce from equations (14), (15) and (17) the ratios between the two kinds of labour inputs in the two sectors

$M_Y / N_Y = M_G / N_G = M$ (22)

Thus, condition (21) can expressed in the terms of the total number of migrants who have entered the economy

$d/b < M \leq y$ (23)

After a succession of eliminations we can express the outputs of the private consumption good and of the public good in the terms of the employment of domestic residents by the private sector

$C = (1-d)AN_Y^{b+d}M^d$ (22)

$G = B(1-N_Y)^{b+d}M^d$ (23)

with $0 \leq N_Y \leq 1$

If we eliminate N_Y between these two equations, and if we remember that $S = G/(1+M)^j$, we get the following expression of the supply set of the economy

$M^{-d(b+d)}\left[(C/A(1-d))^{1/(b+d)} + (1+M)^{j/(b+d)}(S/B)^{1/(b+d)}\right] = 1$ (24)

This expression, as equation (6) in section 1, shows how the trade off between the consumption of private good and of public service by domestic residents, C and S, depends on the number of migrants M. Domestic residents can obtain a zero amount of public service and a consumption of private good $C = (1-d)AM^d$. This consumption increases with the number of migrants. On the other hand domestic residents can obtain a zero amount of private consumption good and a consumption of public service $S = BM^d(1+M)^{-j}$. This consumption decreases with the number of migrants if we introduce the assumption that the congestion effect is high enough: $j > b + d$. Finally, when we move along this set from $C = 0$ to $C = (1-d)AM^d$, the slope of the tangent to this set decreases from 0 to minus infinity.

4.2. Extension of the results

The computation of the effects of immigration is still especially easy and revealing when the private consumption good and the public service are complementary in the preferences of domestic residents that is when $s = -1$. Then, the Government sets the production of public good G such that $u = C = S/b$. We deduce from the equation of the supply set of the economy the expression of the welfare of domestic residents

$u = g(M)^{-(b+d)}$, with

$$g(M) = M^{-d/(b+d)} \left[(1/A(1-d))^{1/(b+d)} + (1+M)^{j/(b+d)}(b/B)^{1/(b+d)} \right] \qquad (25)$$

Migrants have two opposite effects on the welfare of domestic residents. In the extreme case where the taste for the public service b is zero, the government sets the production of public good and taxes to zero. The utility of domestic residents is equal to their consumption of private good, itself equal to their income $u = C = A(1-d)M^d$. It is an increasing function in the number of migrants and tends to infinity with this number. The reason is that a higher number of migrants drive the wages and the rent of domestic residents up. When the taste for the public service b is positive, a second effect appears

which is that migrants create congestion, which reduces the welfare of domestic residents. Proposition 1 has to be revised in the following way.

Proposition 3. If the taste for the public service is included between the two values b_3 and b_4, given

by $\dfrac{A(1-d)}{B}b_3 = \dfrac{(1+d/b)^{b+d-j}}{[(j-d)/b-1]^{d+b}}$ *and* $\dfrac{A(1-d)}{B}b_4 = \dfrac{(1+y)^{b+d-j}}{[(j/d-1)y-1]^{d+b}}$, *then the utility of*

domestic residents reaches a local maximum for a number of migrants included between

d/b *and* y.

Proof. We have $g(M) \to +\infty$ *when* $M \to 0$ *and* $g(M) \to +\infty$ *when* $M \to +\infty$ *if* $b > 0$ *and*

$g(M) \to 0$ *when* $M \to \infty$ *if* $b = 0$. *The derivative of function* g *has the same sign as*

$g_1(M) - g_2(M)$, *with* $g_1(M) = j M (1+M)^{j/(b+d)-1}$ *and*

$g_2(M) = d\left[\dfrac{B}{bA(1-d)}\right]^{1/(b+d)} + d(1+M)^{j/(b+d)}$. *When* M *incrases from* 0 *to infinity,* $g_1(M)$

increases from 0 to infinity and $g_2(M)$ *increases from* $d\left[\dfrac{B}{bA(1-d)}\right]^{1/(b+d)} + d$ *to infinity. So,*

when $M = 0$, *we have* $g_1(M) < g_2(M)$. *But when* $M \to \infty$ *then*

$g_1(M) \sim j M^{j/(b+d)} > g_2(M) \sim d M^{j/(b+d)}$. *So, the derivative of function* g *has at least one*

positive root. The derivatives of functions $g_1(M)$ *and* $g_2(M)$ *are*

$g_1{}'(M) = \dfrac{g_1(M)}{M(1+M)}\left[1 + \dfrac{j}{b+d}M\right]$ *and* $g_2{}'(M) = \dfrac{dj}{b+d}(1+M)^{j/(+d)-1}$. *At their interscetin*

point $g_1(M) = g_2(M)$, *we have* $g_2{}'(M) = \dfrac{j}{b+d}\dfrac{1}{1+M}\left[g_1(M) - d\left[\dfrac{B}{bA(1-d)}\right]^{1/(b+d)}\right]$. *Then, we*

have $g_1'(M) - g_2'(M) = \dfrac{g_1(M)}{M(1+M)} + \dfrac{j\,d}{b+d}\dfrac{1}{1+M}\left[\dfrac{B}{bA(1-d)}\right]^{1/(b+d)} > 0$. *So, the intersection*

point is unique.

Now, we will look for the range of variation of parameter b consistent with condition (23). If function g reaches is minimum for $M = d/b$, then $b = b_3$. If the minimum of the function is reached for $M = y$, then $b = b_4$.

4.3. Simulations

We set the share of labour in private production $b + d = 2/3$ and $d = 0.025$. We also set $A = 12.909604$, $B = 8.132$. These two values are consistent with the productions $Y = 10$ and $G = 2.5$ obtained with the quantities of labour 0.8 and 0.2 in the two sectors, and the same wages for both jobs. This allocation of production between the two goods is consistent with a preference for the public service $b = 0.25$. We set the congestion parameter $j = 1$.

Figure 3 represents the utility of domestic residents in function of the number of immigrants M, for $0 \le M \le y$. We know that for a number of migrants $M \le d/b = 0.039$, both jobs pay the same wages and we are in a situation similar to the one investigated in previous sections. If $d/b = 0.039 < M \le y = 0.332$, we are in a situation where all migrants are segregated in jobs 2, which pay lower wages. We can see that the optimal number of immigrants is 12.7%, which is quite a reasonable value.

We obtain a local maximum of utility in the case of total segregation of the immigrants on jobs 2, for $b_3 = 0.112 \le b \le b_4 = 0.987$. The two extreme values of this range are respectively associated to a number of migrants equal to y and to d/b. We computed that the range of values for the preferences for the public services, which are consistent with a global maximum of utility is

$b_3 = 0.112 \leq b \leq 0.409$. If $b = 0.409$, the optimal number of migrants is equal to 6.2% of the population of domestic residents. However, this high value of b implies that without migrants, the economy would allocate 34.3% of its manpower to the production of public good. If $b > 0.409$, the optimal number of immigrants is zero.

We[2] also simulated the model for a positive elasticity of substitution between the two goods in the preferences of domestic residents (that is for $-1 < s < 0$. We set the value of parameter b such that, without migrants, the optimal allocation of production between the two goods was still $Y = 10$ and $G = 2.5$ (of course the value of b has to be changed with the value of s). We found that when s increases from -1 to 0, the optimal number of migrants only increases from 12.7% to 14.1% of the population of domestic residents.

Thus, we see that this new version of the model can explain optimal numbers of migrants of the same order of magnitude as what we can see in reality.

Conclusion

This paper offers a simple explanation of important features of the immigration policies of host countries, which are observed in many countries these days. Many countries admit few foreign migrants for economic reasons and segregate them in jobs, which domestic workers do not wish to undertake. Some other countries admit a much larger number of immigrants and segregate them to a wider range of jobs. The paper assumes that domestic residents are all identical. Thus, it eliminates considerations based on the specific interest of a class of natives and on economic and political conflicts arising in a heterogeneous society. Moreover, it assumes that although domestic residents have no altruistic feelings toward immigrants, they do not bear xenophobic or racist sentiments against them.

There exist episodes in the past when such an explanation would have been insufficient. For instance, if a large number of foreign migrants were allowed to settle in Western Europe in the fifties and the sixties, this was probably to prevent a rise in industrial wages and to allow firms to make enough profits to modernise and expand their equipment. This policy strengthened the depressing effects on wages induced by domestic migration from the countryside to urban areas. Both movements stopped with the wage explosions, which took place in Europe at the end of the sixties. The current debate on immigration in the U.S. widely reflects a contradiction between the economic interests of different social classes. However, we are convinced that much can be explained on the policy of immigration controls by focusing the investigation on the defence of national interest.

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Appendix

Proof of Proposition 1. We have $f(0) = (1/A)^{1/b} + (b/B)^{1/b} > 0$, $f(M) \to \infty$ when $M \to \infty$ if

$b > 0$ and $f(M) \to 0$ when $M \to \infty$ if $b = 0$. The derivative of function f is

$$f'(M) = (b/B)^{1/b}(1+M)^{(1-b)/b-1}[1+(1-b)M]^{-1/b-1}[-f_1(M) + f_2(M)],$$

with $f_1(M) = (1-b)(B/bA)^{1/b}$ and $f_2(M) = (j/b-1)(1+M)^{(j-1)/b}[1+(1-b)M]^{1/b+1}$.

$f_1(M)$ is a linear function of M with a positive slope $(1-b)(B/bA)^{1/b}$, which decreases with

parameter b. If $j \leq b$, $f_2(M)$ is negative or zero. Then, the optimal number of migrants is infinity.

If $j > b$, $f_2(M)$ is an increasing function of M with $f_2(0) = j/b - 1 > 0$,

$f_2'(0) = f_2(0)(j - b^2)/b > 0$ and $f_2(M), f_2'(M) \to +\infty$ when $M \to +\infty$. $f_2(M)$ is convex for

M high enough, may be concave for $M = 0$, but has at most one inflection point.

Thus, if parameter b is low enough, $f'(M)$ has two roots, M_1 and M_2. When the number of

migrants M increases from 0 to infinity, $f(M)$ increases from $f(0)$ to $f(M_1)$, then decreases

from $f(M_1)$ to $f(M_2)$, then increases from $f(M_2)$ to infinity. When the value of b increases, M_1

increases and M_2 decreases. Finally for $b = b_2$ they converge to a common value $M_{\min}$. Then, when

the number of migrants M increases from 0 to infinity, $f(M)$ increases from $f(0)$ to infinity, with

$f'(M_{\min}) = f''(M_{\min}) = 0$. For $b > b_2$, we have $f'(M) > 0$, which means that when the number of

migrants M increases from 0 to infinity, $f(M)$ increases from $f(0)$ to infinity.

We still have to compare the local minimum $f(M_2)$ when $b < b_2$, to $f(0)$. We use equation (9) to

eliminate b from the expression of $f(M)$. We get

$$f(M_2) = A^{-1/b}(1+M_2)^{(1-b)b}[1+(1-b)M_2]^{-1/b-1}\left[1+(1-b)M_2\frac{j}{j-b}\right].$$ $f'(M_2)$ has the same

sign as $b/j - (1-2b/j)M_2 - (1-b)M_2^2$. This expression is equal to zero for $M_2 = M_{\min}$, as can

be deduced from equation (10). It is negative for $M_2 > M_{\min}$. So, when M_2 increases from $M_{\min}$ to

infinity, $f(M_2)$ decreases from $f(M_{min})$ to zero. We noticed before that $f(M_{min}) > f(0)$. b_1 is the value of parameter b such that $f(M_2) = f(0)$./

Proof of Proposition 2. a) In the third regime we have the condition $(b/d)(N_Y + M_{Y2})/M_{Y1} < 1$, which is equivalent to $M_{Y1} > [b/(b+d)](N_Y + M_Y)$. As $M_Y \geq M_{Y1}$ we obtain for the existence of this regime the condition $M_Y > (b/d)N_Y$. This condition is not satisfied.

b) In the second regime the wages paid by the two jobs are $w_1 = bA(N_Y + M_{Y1})^{b-1}M_{Y2}{}^d$ and $w_2 = dA(N_Y + M_{Y1})^b M_{Y2}{}^{d-1}$. We also have the condition $(b/d)M_{Y2}/(N_Y + M_{Y1}) > 1$, which is equivalent to $M_{Y2} > [d/(b+d)](N_Y + M_Y)$. As $M_Y \geq M_{Y2}$ we obtain the condition $M_Y > (d/b)N_Y$.

The production function of the private sector is $Y = A(N_Y + M_{Y1})^b M_{Y2}{}^d$. The income of domestic residents created by the activity of this sector is

$$R = A(N_Y + M_Y - M_{Y2})^{b-1}M_{Y2}^d[(1-d)N_Y + (1-d-b)(M_Y - M_{Y2})].$$

The government sets the number of migrants working on jobs 2, M_{Y2}, under the constraint $0 \leq M_{Y2} \leq M_Y$, so as to maximise function R. The derivative of this function is

$$\frac{1}{R}\frac{dR}{dM_{Y2}} = \frac{1-b}{N_Y + M_Y - M_{Y2}} + \frac{d}{M_{Y2}} - \frac{1}{[(1-d)/(1-d-b)]N_Y + M_Y - M_{Y2}}.$$ It has the same sign

as $\dfrac{d+b}{d}M_{Y2}^2 - 2\left[N_Y + M_Y + \dfrac{b/2}{d}M_Y\right]M_{Y2} + (N_Y + M_Y)\left[\dfrac{1-d}{1-d-b}N_Y + M_Y\right]$.

The minimum of this function of M_{Y2} is reached for $M_{Y2} = \dfrac{d}{d+b}\left[N_Y + M_Y + \dfrac{b/2}{d}M_Y\right]$. Then,

the value of the function is $-\dfrac{d}{d+b}\left(N_Y + \dfrac{d+b/2}{d}M_Y\right)^2 + (N_Y + M_Y)\left(\dfrac{1-d}{1-d-b}N_Y + M_Y\right)$. This

function of M_Y is non negative, that is $-\dfrac{b(1-d-b)}{4d}M_Y^2 + N_Y M_Y + N_Y^2 \geq 0$, if

$$M_Y \leq y N_Y = 2N_Y \frac{d/b + \sqrt{d/b}\sqrt{(1+d/b)(1-b)}}{1-d-b} \text{. Then } M_{Y2} = M_Y \text{ .}$$

If $M_Y > y N_Y$, then M_{Y2} is equal to the smallest root of dR/dM_{Y2}, $\hat{M}_{Y2}$, if $R(\hat{M}_{Y2}) > R(M_Y)$.

Otherwise, we have $M_{Y2} = M_Y$. We have

$$\hat{M}_{Y2} = \frac{1}{b+d}\left[d\left(N_Y + \frac{d+b/2}{d}M_Y \right) - \sqrt{\frac{bd}{1-b-d}\left[\frac{b(1-b-d)}{4d}M_Y^2 - N_Y M_Y - N_Y^2 \right]} \right]. \text{ We easily}$$

deduce from the condition $M_Y > y N_Y$ and from the expression of y that $\hat{M}_{Y2} < M_Y$.

c) In regime 1 we have $M_{Y2} = (N_Y + M_Y)d/(d+b) - N_{Y2}$. So, we can give M_{Y2} any arbitrary

value in the interval $Max\left(0, \dfrac{dM_Y - bN_Y}{b+d} \right) < M_{Y2} < M_Y + Min\left(0, \dfrac{dN_Y - bM_Y}{b+d} \right)$, and the previous

equation gives the value of N_{Y2}.

Then, if $M_Y > (d/b)N_Y$, we can set M_{Y2} to its upper bound $[d/(b+d)](N_Y + M_Y)$, which

is the lower bound of regime 2. So, we can see that this regime dominates regime 1. If

$M_Y < (d/b)N_Y$, then we can give M_{Y2} any arbitrary value included between 0 and M_Y.
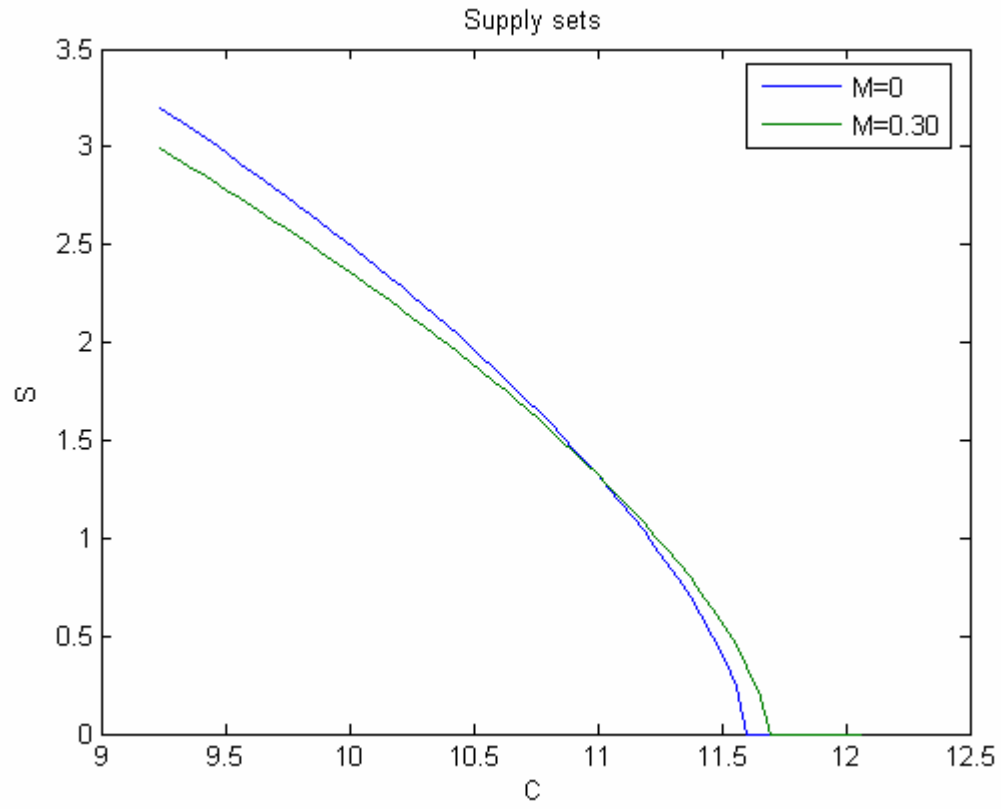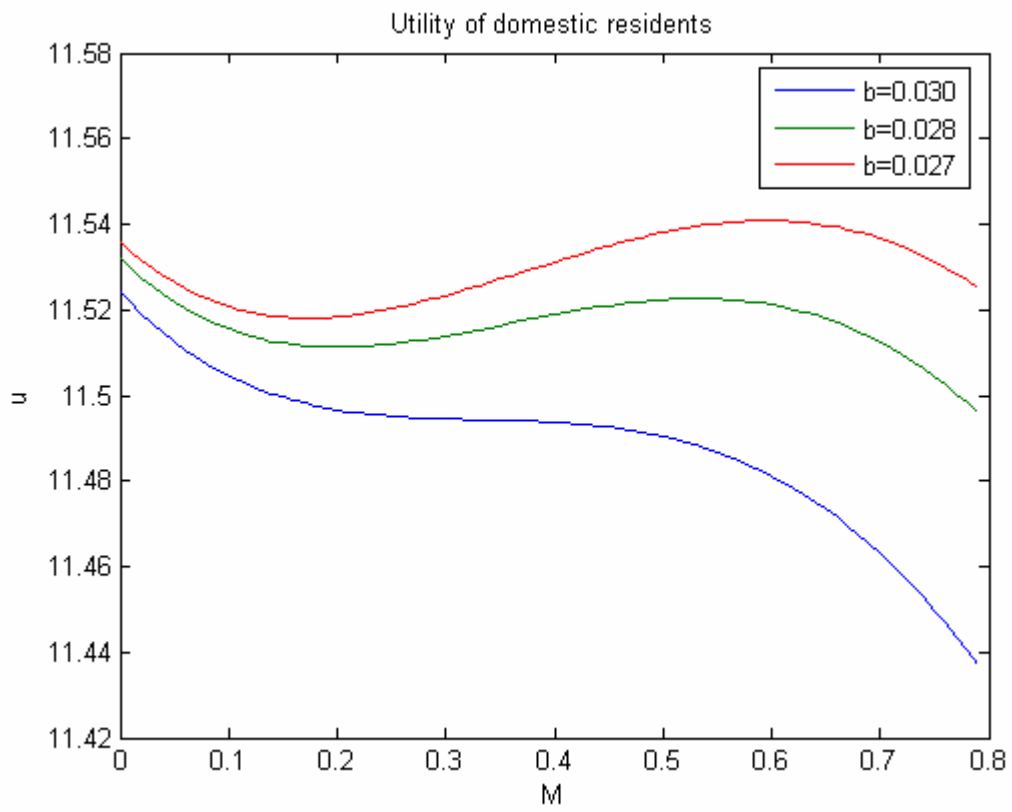
Figure 1

Figure 2

Figure 3