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Regular economies with ambiguity aversion

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Abstract

We consider a family of exchange economies where consumers have multiprior preferences representing their ambiguity aversion. Under a linear independence assumption, we prove that regular economies are generic. Regular economies exhibit enjoyable properties: odd finite number of equilibrium prices, local constancy of this number and local differentiable selections of the equilibrium prices.

Thus, even if ambiguity aversion is represented by non-differentiable multiprior preferences, economies retain generically the properties of the differentiable approach.

Keywords: demand function, general equilibrium, ambiguity aversion, multiprior preferences, regular economies, Lipschitz behavior.

JEL Classification: C6, D4, D5.

1 Introduction

Classically, the global analysis of the general economic equilibrium is based on well known differential techniques. Basically, one requires the differentiability of the demand functions. We refer the reader to Debreu [7], Mas Colell [12] and Balasko [1] for much details.

This differentiability is often well derived from well known assumptions on the utility functions. Indeed, the utility functions are supposed to be C^2

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to obtain C^1 demand functions. This does not allow the presence of kinks on indifference curves that arise in uncertainty context.

In the maxmin expected utility model due to Gilboa and Schmeidler [11], the agents face ambiguity modeled by the multiplicity of the priors of the agents. Each agent considers the minimum expected utility over his set of priors. This "minimum" generates kinks on the indifference curves when more than one probability realize the minimum, this leads to the non-differentiability of the demand functions. These kinks cannot be removed since they are genuinely linked to uncertainty not to modelling issues. The main objective of this paper is to get the genericity of regular economies despite that the demand functions are non-differentiable.

In this paper, we consider an exchange economy with a finite number ℓ of commodities and a finite number m of consumers. The preferences of a consumer i are represented by a utility function u_i from $\mathbb{R}^\ell_{++}$ to $\mathbb{R}$. The function u_i is the minimum of a finite number n_i of functions that satisfy the usual differentiability requirements and a linear independence assumption on the gradient vectors. This framework encompasses the case of multiprior preferences defined by a Bernoulli function and a finite set of linearly independent probability vectors.

We first study the properties of the demand functions. This systematic study constitutes in itself a new result concerning consumers with multiprior preferences. We first prove that the demand functions are locally Lipschitz continuous. We then prove that the demand functions are continuously differentiable on an open set of full Lebesgue measure.

The first part of the proof relies on a result of Cornet and Vial [5] concerning the Lipschitz behavior of the solution of a mathematical programming problem. The second part of the proof is based on the following result: *The demand function is differentiable at some point if and only if it is continuously differentiable on a neighborhood of this point.*

In the second part of the paper, we follow Balasko's program. We define and parametrize the equilibrium manifold. We show that it is indeed a smooth manifold at almost every point. As in the classical case, we can propose a global parametrization from which we deduce that the equilibrium manifold is lipeomorphic[1] to an open connected subset of an Euclidean space denoted by $\mathcal{U}$ using similar approach than Bonnisseau and Rivera-Cayupi [4].

We can define an extended natural projection using the parametrization. This mapping is continuously differentiable almost everywhere and locally

[1]Two sets are *lipeomorphic* if it exists a one-to-one, onto and locally Lipschitz continuous mapping from the first set to the second one with a locally Lipschitz continuous inverse.

Lipschitz continuous.

Contrary to the classical case, we have to take into account the kinks to define regular economies. A singular economy is either the image of a point where the differential mapping does not exist or the image of a point where the differential mapping is not onto. A regular economy is, by definition, an economy that is not singular. By Sard's theorem since the set $\mathcal{U}$ and the space of the economies are two manifolds of same dimension, the set of singular economies is a set of Lebesgue measure zero[2]. By the Implicit Function Theorem, each regular economy has a finite number of equilibria and, around a regular economy, there exist continuously differentiable selections of the equilibrium prices.

Computing the degree of the extended projection by an homotopy argument, we obtain that every regular economy has a finite odd number of equilibrium prices.

We now mention earlier contributions. Rader [13] showed that, when the consumers have demand functions a.e. differentiable satisfying property (N): "*The image of a null set is a null set.*", almost every economy has a finite number of equilibrium prices. In our paper, we prove that Rader's properties are satisfied by multiprior preferences but we get more with the local continuously differentiable selections.

Shannon and Rigotti [14] study market implications of the presence of ambiguity modelled by variational preferences. Variational preferences encompass multiprior preferences. They show that almost all economies are determinate which means that there exist a finite number of equilibrium prices and local continuous selections[3]. Note that regularity and determinacy are two distinct concepts, the first one implying the second one. In particular, the number of equilibria may not be constant around a determinate economy. We need the linear independence of *Assumption 2* to get regularity instead of determinacy.

In [6], Dana study agents that are Choquet expected-utility maximizers. She is interested in equilibrium welfare properties and indeterminacy of the equilibrium. She provides a sufficient condition on equilibrium implying that there exists a continuum of equilibrium prices. But, she does not address the issue of genericity.

In [4], Bonnisseau and Rivera-Cayupi study a non-smooth model although the failure of differentiability was not in the utility function. They obtain demand functions with properties similar to ours.

In Section 2, we present the model, give the assumptions on the prefer-

[2] Actually, we also use that the image of a null set by a Lipschitz mapping is a null set.
[3] They also obtain a Lipschitz behavior in a particular case.

ences of the consumers. We also show how our model allows to deal with multiprior preferences. In Section 3, we study extensively the demand function of a consumer with multiprior preferences. The fourth section is devoted to the global analysis of the equilibrium manifold and to the genericity analysis. Some concluding remarks are given in Section 5 and finally, some technical proofs are given in Appendix.

2 Model and assumptions

We study a family of economies parametrized by strictly positive endowments with m consumers and ℓ commodities. We denote respectively by M and L the set of consumers and the set of commodities. Let $M \equiv \{1, \ldots, m\}$ and $L \equiv \{1, \ldots, \ell\}$.

We assume that the preferences of consumer $i \in M$ are represented by a utility function u_i from $\mathbb{R}^\ell_{++}$ to $\mathbb{R}$ which is the minimum of n_i functions:

$$u_i = \min\{u_i^1, u_i^2, \ldots, u_i^{n_i}\}$$

with $n_i \in \mathbb{N}^*$. Let us be more precise on the assumptions concerning the utility functions.

Assumption 1 *For all $i \in M$, for all $k \in \{1, .., n_i\}$,*

1. *u_i^k is C^2 on $\mathbb{R}^\ell_{++}$,*
2. *$D^2 u_i^k(x)$ is negative definite on $\nabla u_i^k(x)^\perp$ for all $x \in \mathbb{R}^\ell_{++}$,*
3. *u_i^k satisfy $\nabla u_i^k(x) \gg 0$ for all $x \in \mathbb{R}^\ell_{++}$.*

For all $x \in \mathbb{R}^\ell_{++}$, $M_i(x)$ denotes the set of the indices of the functions realizing the minimum, i.e. $M_i(x) := \{k \in \{1, \ldots, n_i\} : u_i^k(x) = u_i(x)\}$.

Assumption 2 *For all $i \in M$, for all $x \in \mathbb{R}^\ell_{++}$, the vectors $(\nabla u_i^k(x))_{k \in M_i(x)}$ are linearly independent.*

Assumption 3 *For all $i \in M$, for all $k \in \{1, \ldots, n_i\}$, for all $x \in \mathbb{R}^\ell_{++}$, the closure in $\mathbb{R}^\ell$ of the set $\{x' \in \mathbb{R}^\ell_{++} | u_i^k(x') \geq u_i^k(x)\}$ is contained in $\mathbb{R}^\ell_{++}$.*

Remark 1 Assumption 1 tells us that the preferences are continuous, monotone and strictly convex. Moreover, each commodity is desirable.

The demand of consumer i with respect to the price and to his initial endowment is the solution of the following problem:

$$\begin{cases} \max u_i(x) \\ \text{subject to } x \gg 0 \text{ and } p \cdot x \leq p \cdot e_i \end{cases} \tag{2.1}$$

In this paper, since the preferences of the consumers are fixed, we define an economy as an element $\mathbf{e} := (e_i)_{i=1}^m$ of $(\mathbb{R}^\ell_{++})^m$. Therefore, the space of the economies is $\Omega := (\mathbb{R}^\ell_{++})^m$.

Multiprior preferences

The above framework encompasses the case of multiprior preferences used to represent the ambiguity aversion of an agent facing uncertainty. Let us present briefly the model. There are two dates $t = 0$ and $t = 1$. There is uncertainty at date 0 about which state will occur at date 1. At date 1, there are S states of nature. We denote by $\Delta(S)$ the set of probabilities on $\{1, \ldots, S\}$.

We model the ambiguity aversion by a multiplicity of probabilities. To each agent $i \in M$, we associate a closed convex set $\mathcal{P}^i \subset \Delta(S)$. We suppose that the set $\mathcal{P}^i$ has n_i extremal points $(\pi_i^k)_{1 \leq k \leq n_i}$. We also suppose that the set $\mathcal{P}^i$ is contained in $\mathbb{R}^S_{++}$ to get the strict monotony of preferences. This can in particular correspond to the convex case of the C.E.U. (Choquet Expected Utility) model of Schmeidler since the core of a convex capacity has at most $S!$ extremal points (Shapley [15]).

We assume that there is no consumption at date 0 and that the agent choose a contingent consumption vector $(x_s)_{1 \leq s \leq S}$. The utility of the agent i is given by:

$$u_i(x) = \min_{\pi_i \in \mathcal{P}^i} E_{\pi_i}[b_i(x)] = \min_{1 \leq k \leq n_i} E_{\pi_i^k}[b_i(x)] \tag{2.2}$$

where $b_i :]0, +\infty[\longrightarrow \mathbb{R}$ is the Bernoulli function of agent i. We define, for $k \in \{1, \ldots, n_i\}$, the function u_i^k by:

$$u_i^k(x) = E_{\pi_i^k}[b_i(x)] \text{ for } x \in \mathbb{R}^S_{++}.$$

We need to add an assumption on the probability vectors $(\pi_i^k)_{1 \leq k \leq n_i}$ to ensure that the functions $(u_i^k)_{1 \leq k \leq n_i}$ satisfy Assumption 2.

Assumption 4 *For every $i \in M$, the probability vectors $(\pi_i^k)_{1 \leq k \leq n_i}$ are linearly independent.*

Note that his assumption holds true when $\mathcal{P}^i$ is an ε-contamination of a probability $\bar{\pi}$. Indeed, the extremal points of $\mathcal{P}^i$ are $\bar{\pi} + \varepsilon\pi^s$ for $s = 1, \ldots, S$, where π^s is the probability such that $\pi^s(s) = 1$. We now can present the result of the section:

Proposition 1 *Let $i \in M$. Suppose that the Bernoulli function b_i is of class C^2 and satisfies: $b_i' > 0$, $b_i'' < 0$ and $\lim_{y \to 0^+} b_i(y) = -\infty$. Under Assumption 4, the family of functions $(u_i^k)_{1 \le k \le n_i}$ satisfies Assumptions 1, 2 and 3.*

The proof of Proposition 1 is given in Appendix.

Consequently, all results below apply to a general equilibrium model with multiprior preferences satisfying Assumption 4. In particular, we get the generic regularity of economies with multiprior preferences.

3 Properties of the individual demand

In this section, we study the individual demand of consumer $i \in M$ defined as the solution of the program:

$$\begin{cases} \max u_i(x) \\ \text{subject to } x \gg 0 \text{ and } p \cdot x \le w \end{cases} \tag{3.1}$$

where $w \in]0, +\infty[$ and $p \in \mathbb{R}^\ell_{++}$. Let us present the main result of the section:

Proposition 2 *Under Assumptions 1, 2 and 3, $f_i(p, w)$ is a singleton for all $i \in M$, $p \in \mathbb{R}^\ell_{++}$ and $w > 0$. The function f_i is locally Lipschitz continuous on $\mathbb{R}^\ell_{++} \times]0, +\infty[$. Furthermore, there exists an open subset Ω_i^0 of $\mathbb{R}^\ell_{++} \times]0, +\infty[$ of full Lebesgue measure on which f_i is C^1.*

Proof To simplify the notation, we skip the index i during the proof and denote the function u_i^k by u^k. The set $f(p, w)$ is nonempty by Weierstrass Theorem and Assumption 3. $f(p, w)$ is a singleton since the function u is strictly quasi-concave.

The function f is continuous on $\mathbb{R}^\ell_{++} \times]0, +\infty[$ by Berge's Theorem. We use a result of Cornet and Vial [5] to obtain that the function f is locally Lipschitz continuous on $\mathbb{R}^\ell_{++} \times]0, +\infty[$. By Rademacher's Theorem [9], the function f is differentiable almost everywhere. Details of the proofs are given in Appendix.

Now, we show that the set on which the function f is differentiable is an open set and that the function f is continuously differentiable on this set.

We first remark that, for all $(p, w) \in \mathbb{R}^{\ell}_{++} \times]0, +\infty[$, $f(p, w)$ is the solution of the following problem:

$$\begin{cases} \min -t \\ \text{ subject to} \\ t - u^k(x) \leq 0, k = 1, \dots, n \\ p \cdot x - w \leq 0 \end{cases} \tag{3.2}$$

The first order conditions associated to this problem are the following: there exists $\lambda \in \mathbb{R}^{n+1}_{+}$ such that

$$\begin{cases} t - u^k(x) \leq 0, \lambda_k(t - u^k(x)) = 0, k = 1, \dots, n \\ p \cdot x - w \leq 0, \lambda_{n+1}(p \cdot x - w) = 0 \\ \sum_{k=1}^{n} \lambda_k = 1 \\ \lambda_{n+1}\, p = \sum_{k=1}^{n} \lambda_k \nabla u^k(x) \end{cases} \tag{3.3}$$

Note that this maximization problem is not necessarily convex since the constraint $t - u^k(x)$ may not be quasi-convex. Nevertheless, the first order conditions are necessary since the Mangasarian-Fromovitz qualification condition is satisfied and sufficient as shown in Appendix in the proof of Proposition 2.

We will show that the set Ω^0 defined by:

$$\Omega^0 := \left\{ (p, w) \in \mathbb{R}^{\ell}_{++} \times]0; +\infty[\;\middle|\; \forall k \in M(f(p, w)), \lambda_k(p, w) > 0 \right\}$$

is an open set on which the function f is continuously differentiable and that the function f is not differentiable at any point outside Ω^0.

The result is a consequence of the two following lemmas. □

For the remaining of the section, $\bar{\lambda}_k := \lambda_k(\bar{p}, \bar{w})$ for $k \in \{1, \dots, n+1\}$.

Lemma 1 *If the multipliers $(\bar{\lambda}_k)_{k \in M(f(\bar{p}, \bar{w}))}$ are all positive, then the function f and the multipliers are continuously differentiable on an open neighborhood of $(\bar{p}, \bar{w})$.*

The proof of this lemma as a consequence of the Implicit Function Theorem is quite standard borrowing ideas from Fiacco McCormick [10]. The continuity of the multipliers implies that the set Ω^0 is an open set. The proof of Lemma 1 is given in Appendix.

Now we turn ourselves to the second lemma.

Lemma 2 *If some multiplier $\bar{\lambda}_k$ with $k \in M(f(\bar{p}, \bar{w}))$ is equal to zero then the function f is not differentiable at $(\bar{p}, \bar{w})$.*

Note that these two lemmas imply that Ω^0 is an open set and that the function f is differentiable at $(\bar{p}, \bar{w})$ if and only if $(\bar{p}, \bar{w})$ belongs to Ω^0 and is continuously differentiable on Ω^0. Furthermore Ω^0 has full Lebesgue measure in $\mathbb{R}^\ell_{++} \times]0, +\infty[$ since f is locally Lipschitz continuous.

Proof Let $(\bar{p}, \bar{w}) \in \mathbb{R}^\ell_{++} \times]0, +\infty[$ such that: $M(f(\bar{p}, \bar{w})) = K \cup K'$ with K and K' subsets of $\{1, \ldots, n\}$, $\bar{\lambda}_k > 0$ for $k \in K$, $\bar{\lambda}_k = 0$ for $k \in K'$ and $K' \neq \emptyset$. Note that we have: $K \neq \emptyset$ since $\bar{p} \neq 0$ and $\sum_{k \in K} \bar{\lambda}_k = 1$ since $\sum_{K \cup K'} \bar{\lambda}_k = 1$ and $\bar{\lambda}_k = 0$ for every $k \in K'$.

Let $\bar{x} := f(\bar{p}, \bar{w})$ and $\bar{v} = (\bar{v}^k)_{k \in K} := (u^k(\bar{x}))_{k \in K}$. We define the function $\bar{f}$ on $\mathbb{R}^\ell_{++}$ by: $\bar{f}(p) := f(p, p \cdot \bar{x})$. Let us first recall some results.

A generalized Hicksian demand and the related expenditure function:

For $p \in \mathbb{R}^\ell_{++}$ and $(v^k)_{k \in K} \in \mathbb{R}^K$, let $\Delta_K(p, (v^k)_{k \in K})$ be the solution[4] of the problem:

$$\left\{ \begin{array}{l} \min p \cdot x \\ \text{ subject to } u^k(x) \geq v^k \ \forall k \in K \\ x \gg 0 \end{array} \right. \tag{3.4}$$

The related expenditure function is defined by:

$$e_K(p, v) := p \cdot \Delta_K(p, v).$$

The map Δ_K has been extensively studied in [3]. This map is continuously differentiable with respect to (p, v) around the point $(\bar{p}, (u^k(\bar{x}))_{k \in K})$ whenever the gradients $(\nabla u^k(\bar{x}))_{k \in K}$ are linearly independent which holds true thanks to Assumption 2.

Note also that $\Delta_K(\bar{p}, \bar{v}) = \bar{x}$. Indeed the necessary and sufficient first order conditions are satisfied by $\bar{x}$.

Like the classical expenditure function, the function e_K is concave with respect to p so a.e. twice differentiable and satisfies $D^2_p e_K = D_p \Delta_K$ whenever this expression makes sense[5].

The next claim is a generalization of the well known result about the negative definiteness of the Slutsky matrix.

Claim 1 *The matrix $D_p \Delta_K(\bar{p}, \bar{v})$ has rank $\ell - \sharp K$[6] and its kernel is the linear space $\mathcal{L}\left(\nabla u^k(\bar{x}) \ , \ k \in K\right)$ spanned by the family $\left(\nabla u^k(\bar{x})\right)_{k \in K}$.*

[4] This solution exists and is unique because the functions u^k for $k \in K$ are strictly quasi-concave when there exists $x \gg 0$ such that $u^k(x) \geq v^k$ for all $k \in K$.

[5] See [3].

[6] $\sharp K$ denotes the cardinal of the set K.

Proof We first recall that $D_p\Delta(\bar{p},\bar{v})$ is the Hessian matrix of the expenditure function $e_K(.,\bar{v})$. Since the map $e_K(.,\bar{v})$ is concave, $D_p\Delta(\bar{p},\bar{v})$ defines a symmetric negative semi-definite bilinear form.

For $p \in \mathbb{R}^{\ell}_{++}$ sufficiently near from $\bar{p}$, $\Delta_K(p,\bar{v})$ is characterized by the first order conditions:

- $u^k(\Delta_K(p,\bar{v})) = \bar{v}^k, \ \forall k \in K,$
- $p = \sum_{k\in K} \mu_k(p)\nabla u^k(\Delta_K(p,\bar{v}))$ with $\mu_k(p) > 0 \ \forall k \in K.$

We differentiate the first condition with respect to p and obtain at $\bar{p}$ for all $q \in \mathbb{R}^{\ell}$:

$$\nabla u^k(\Delta_K(\bar{p},\bar{v})) \cdot D_p\Delta_K(\bar{p},\bar{v})(q) = \nabla u^k(\bar{x}) \cdot D_p\Delta_K(\bar{p},\bar{v})(q) = 0 \ \forall k \in K.$$

These equalities tell us that the image of $D_p\Delta_K(\bar{p},\bar{v})$ is contained in the linear subspace $\cap_{k\in K}\nabla u^k(\bar{x})^{\perp}$ of dimension $\ell - \sharp K$[7]. Furthermore, since $D_p\Delta_K(\bar{p},\bar{v})$ is negative semi-definite, $\nabla u^k(\bar{x})$ belongs to the kernel of $D_p\Delta_K(\bar{p},\bar{v})$ for all $k \in K$. Thus, the dimension of the image of $D_p\Delta_K(\bar{p},\bar{v})$ is at most $\ell - \sharp K$. We differentiate the second condition with respect to p. We have for $q \in \mathbb{R}^{\ell}$:

$$q = \sum_{k\in K} \mu_k(p)D^2u^k(\Delta_K(p,\bar{v}))D_p\Delta_K(p,\bar{v})(q) + \sum_{k\in K}(\nabla\mu_k(p)\cdot q)\nabla u^k(\Delta_K(p,\bar{v})).$$

For all $q \in \cap_{k\in K}\nabla u^k(\bar{x})^{\perp}$, we have:

$$q = \left[\sum_{k\in K} \mu_k(\bar{p})D^2u^k(\bar{x})\right] D_p\Delta_K(\bar{p},\bar{v})(q).$$

So, we have for $q \in \cap_{k\in K}\nabla u^k(\bar{x})^{\perp}$:

$$D_p\Delta_K(\bar{p},\bar{v})(q) = 0 \Longrightarrow q = 0.$$

So the kernel of the restriction on $\cap_{k\in K}\nabla u^k(\bar{x})^{\perp}$ of $D_p\Delta_K(\bar{p},\bar{v})$ is reduced to zero. So, the rank of $D_p\Delta_K(\bar{p},\bar{v})$ is at least $\ell - \sharp K$. Hence, the rank of $D_p\Delta_K(\bar{p},\bar{v})$ is equal to $\ell - \sharp K$ and the kernel of $D_p\Delta_K(\bar{p},\bar{v})$ is equal to $\mathcal{L}\left(\nabla u^k(\bar{x})\ ,\ k \in K\right)$.$\square$

An auxiliary demand function:

[7] This is a consequence of Assumption 2. Indeed, $K \subset M(\bar{x})$ implies that the family $(\nabla u^k(\bar{x}))_{k\in K}$ is linearly independent.

We introduce the demand function f_K associated to the utility functions $(u^k)_{k\in K}$. $f_K(p,w)$ is the solution of the optimization problem:

$$\begin{cases} \min -t \\ \text{ subject to} \\ t - u^k(x) \leq 0 \ k \in K \\ p \cdot x - w \leq 0 \\ x \gg 0 \end{cases} \tag{3.5}$$

Note that we have: $f_K(\bar{p},\bar{w}) = \bar{x} = f(\bar{p},\bar{w})$ since the necessary and sufficient first order conditions are satisfied by $\bar{x}$. Indeed, we have: $\bar{\lambda}_k = 0$ for all $k \in K'$. The function f_K is continuously differentiable around $(\bar{p},\bar{w})$ since all the multipliers are positive. See Lemma 1.

Remark 2 In a neighborhood of $(\bar{p},\bar{w})$, the binding constraints are the same since the utility functions are continuous and the multipliers are all positive and continuous with respect to (p,w). Thus, for all $(k,k') \in K^2$, $u^k(f_K(p,w)) = u^{k'}(f_K(p,w))$.

We define on $\mathbb{R}^\ell_{++}$ the function $\bar{f}_K$ by: $\bar{f}_K(p) = f_K(p, p\cdot\bar{x})$. We also define the function $\bar{v}$ from $\mathbb{R}^\ell_{++}$ to $\mathbb{R}^K$ by $\bar{v}(p) = (\bar{v}^k(p) = u^k[\bar{f}_K(p)])_{k\in K}$. Note that, for $(k,k') \in K^2$, $\bar{v}^k(p) = \bar{v}^{k'}(p)$ in a neighborhood of $\bar{p}$, and from the first order necessary and sufficient conditions, $\bar{f}_K(p) = \Delta_K(p,\bar{v}(p))$.

We first prove the following claim:

Claim 2 *For all $k \in K$, $\nabla\bar{v}^k(\bar{p}) = 0$.*

Proof Indeed, $p\cdot\bar{f}_K(p) = p\cdot\bar{x}$ by Walras law.

Differentiating with respect to p, we obtain for all $q \in \mathbb{R}^\ell$:

$$q \cdot \bar{f}_K(p) + p \cdot D\bar{f}_K(p)(q) = q \cdot \bar{x}\ .$$

Since $\bar{x} = \bar{f}_K(\bar{p})$, this implies: $\bar{p}\cdot D\bar{f}_K(\bar{p})(q) = 0$. Let $k \in K$. By the chain rule: $\nabla\bar{v}^k(p)\cdot q = \nabla u^k(\bar{f}_K(p))\cdot D\bar{f}_K(p)(q)$.

From the first order conditions of the utility maximization problem, we have:

$$\sum_{k\in K} \bar{\lambda}_k \nabla u^k(\bar{x}) = \bar{\lambda}_{n+1}\bar{p}\ .$$

Recalling that $\bar{f}(\bar{p}) = \bar{x}$, we have:

$$\begin{aligned} \textstyle\sum_{k\in K} \bar{\lambda}_k \nabla\bar{v}^k(\bar{p})\cdot q &= \left(\textstyle\sum_{k\in K}\bar{\lambda}_k\nabla u^k(\bar{x})\right)\cdot D\bar{f}_K(\bar{p})(q) \\ &= \bar{\lambda}_{n+1}\bar{p}\cdot D\bar{f}_K(\bar{p})(q) = 0 \end{aligned}$$

Since the equality is true for all $q \in \mathbb{R}^\ell$, we conclude that $\sum_{k\in K}\bar{\lambda}_k\nabla\bar{v}^k(\bar{p}) = 0$. Recalling that $\sum_{k\in K}\bar{\lambda}_k = 1$, since the functions $(\bar{v}^k)_{k\in K}$ coincide on a neighborhood of $\bar{p}$, one concludes $\nabla\bar{v}^k(\bar{p}) = 0$ for all $k \in K$. □

The following result states the equality of the differentials with respect to p of the compensated demand and the demand at the price $\bar{p}$.

Claim 3

$$D\bar{f}_K(\bar{p}) = D_p\Delta_K(\bar{p}, \bar{v}).$$

Proof Since $\bar{f}_K(p) = \Delta_K(p, \bar{v}(p))$ in an open neighborhood of $\bar{p}$, it suffices to use the chain rule for differential mappings and the above claim to conclude. Indeed, let $q \in \mathbb{R}^\ell$,

$$D\bar{f}_K(\bar{p})(q) = D_p\Delta_K(\bar{p}, \bar{v})(q) + \left(\sum_{k\in K} \nabla\bar{v}^k(\bar{p}) \cdot q\right) D_v\Delta_K(\bar{p}, \bar{v}).$$

□

Recall that we have: $M(f(\bar{p}, \bar{w})) = K \cup K'$ with K and K' subsets of $\{1, \ldots, n\}$ such that $\bar{\lambda}_k > 0$ for $k \in K$ and $\bar{\lambda}_k = 0$ for $k \in K'$.

We will now consider a particular price path to get the desired result. For all $t \in I :=]-a, a[$[8], let :

$$p(t) := \sum_{k\in K} \bar{\lambda}_k \nabla u^k(\bar{x}) + t\left(\sum_{k\in K'} \alpha_k \nabla u^k(\bar{x})\right). \tag{3.6}$$

for well chosen coefficients $(\alpha_k)_{k\in K'}$.

To choose the coefficients $(\alpha_k)_{k\in K'}$, we use the following linear algebra result:

Proposition 3 *Let E be a vector space and Φ be a symmetric positive definite bilinear form on E. Let $(\xi_i)_{i=1}^k$ be a linearly independent family of vectors of E. There exists $\alpha \in R_{++}^k$ such that for all $i = 1, \ldots k$,*

$$\Phi(\xi_i, \sum_{j=1}^k \alpha_j \xi_j) > 0.$$

This result is proved in Appendix.

We apply this algebraic result to the vector space E spanned by the family $\left(\nabla u^k(\bar{x})\right)_{k\in K'}$. The matrix $D_p\Delta_K(\bar{p}, \bar{v})$ is negative semi-definite on $\mathbb{R}^\ell$. Moreover, we have:

[8]The number a is a positive number sufficiently small to ensure that $p(t)$ belongs to $\mathbb{R}_{++}^\ell$ for all $t \in I$.

$\mathrm{Ker} D_p\Delta_K(\bar{p},\bar{v}) = \mathcal{L}\left(\nabla u^k(\bar{x})\ ,\ k \in K\right)$. The family $(\nabla u^k(\bar{x}))_{k\in K\cup K'}$ is linearly independent. Hence $D_p\Delta_K(\bar{p},\bar{v})$ is negative definite on E since $\mathrm{Ker} D_p\Delta_K(\bar{p},\bar{v}) \cap E = \{0\}$.

Thanks to Proposition 3, there exists $(\alpha_k)_{k\in K'}$ in $\mathbb{R}^{K'}_{++}$ such that, for all $k' \in K'$,

$$\nabla u^{k'}(\bar{x}) \cdot D_p\Delta_K(\bar{p},\bar{v}) \left(\sum_{k\in K'} \alpha_k \nabla u^k(\bar{x})\right) < 0. \tag{3.7}$$

For $k \in K$, we introduce the function defined on I by: $\hat{v}^k(t) := v^k(p(t))$. For $k \in K'$, we define the functions φ_k and $\tilde{\varphi}_k$ on I by: $\varphi_k(t) := u^k(\bar{f}(p(t)))$ and $\tilde{\varphi}_k(t) := u^k(\bar{f}_K(p(t)))$[9].

Claim 4 *If $t > 0$ is small enough, $\bar{f}(p(t)) = f(p(t), p(t)\cdot\bar{x}) = \bar{x}$ so $\varphi'_k(t) = 0$ for all $k \in K'$.*

Proof For $t > 0$ small enough, thanks to the formula 3.6, one remarks that $\bar{x}$ satisfies the necessary and sufficient first order conditions associated to the problem:

$$\begin{cases} \max u(x) \\ \ \text{s.t. } x \gg 0 \\ p(t)\cdot x \le p(t)\cdot \bar{x} \end{cases}$$

with the associated multipliers

$$\frac{\bar{\lambda}_k}{\sum_{\kappa\in K}\bar{\lambda}_\kappa + t(\sum_{\kappa\in K'}\alpha_\kappa)} \text{ for } k \in K, \frac{t\alpha_k}{\sum_{\kappa\in K}\bar{\lambda}_\kappa + t(\sum_{\kappa\in K'}\alpha_\kappa)} \text{ for } k \in K',$$

0 for $k \notin M(\bar{x}) \cup \{n+1\}$ and $\dfrac{1}{\sum_{\kappa\in K}\bar{\lambda}_\kappa + t(\sum_{\kappa\in K'}\alpha_\kappa)}$ for the budget constraint[10]. Therefore we can conclude that $\bar{x}$ is a solution of the problem, that is: $f(p(t), p(t)\cdot\bar{x}) = \bar{f}(p(t)) = \bar{x}$. □

From Claim 3 and the choice of the family $(\alpha_k)_{k\in K'}$, we obtain the following lemma:

Claim 5 *For $k \in K'$, we have:*

$$\tilde{\varphi}_k{}'(0) < 0.$$

[9] To avoid any confusion, we denote with "φ"s the functions for $k \in K'$ and with "v"s the functions for $k \in K$.

[10] See (5.1).

Proof By the chain rule, Claim 3 and 3.7, we have for $k \in K'$:

$$\begin{array}{ll} \tilde{\varphi_k}'(0) & = \nabla u^k(\bar{x}) \cdot D\bar{f}_K(\bar{p}) \left(\sum_{k \in K'} \alpha_{k'} \nabla u^k(\bar{x})\right) \\ & = \nabla u^k(\bar{x}) \cdot D_p\Delta_K(\bar{p}) \left(\sum_{k \in K'} \alpha_{k'} \nabla u^k(\bar{x})\right) \\ & < 0. \end{array}$$

□

To complete the proof of Lemma 2, let us summarize the above results:

- $(\hat{v}^k)'(0) = 0$ for $k \in K$,
- $\tilde{\varphi}_k'(0) < 0$ for $k \in K'$,
- $\hat{v}^k(0) = \tilde{\varphi}_k(0)$ for $k \in K'$.

The last equality comes from the following equalities: for all $(k, k') \in K \times K'$,

$$p(0) = \bar{p}, f_K(\bar{p}, \bar{w}) = f_K(\bar{p}, \bar{p} \cdot \bar{x}) = f(\bar{p}, \bar{w}) = \bar{x}, u^k(\bar{x}) = u^{k'}(\bar{x}).$$

Now, we get: $\tilde{\varphi}_{k'}(t) > \hat{v}^k(t)$ for all $(k, k') \in K \times K'$ and $t < 0$ sufficiently near from zero. From the necessary and sufficient first order conditions, since the constraints corresponding to K are the only binding constraints, we can conclude: $\bar{f}_K(p(t)) = \bar{f}(p(t))$ for $t < 0$ sufficiently near from zero. Therefore, we have: $\varphi_k(t) = \tilde{\varphi}_k(t)$ for all $k \in K'$ and $t < 0$ sufficiently near from zero. Finally, for every $k \in K'$, from Claim 5, the left derivative of φ_k at 0 is negative and the right derivative of φ_k at 0 is equal to zero by Claim 4. So the functions φ_k are not differentiable, hence the function f is not differentiable. To conclude, when some multiplier $\bar{\lambda}_k$ (for $k \in M(f(\bar{p}, \bar{w}))$) is equal to zero, the function f is not differentiable at $(\bar{p}, \bar{w})$.□

4 The equilibrium manifold

In this section, we study the equilibrium price vectors from a global point of view following Balasko [1]. The monotony of the utility functions implies that the equilibrium prices are always strictly positive. Moreover they are defined up to a normalization. We use the simplex normalization, i.e., we take prices in $\mathbb{S}$ with $\mathbb{S} := \left\{p \in \mathbb{R}^\ell_{++} \ , \ \sum_{h=1}^{\ell} p_h = 1\right\}$. We now define the equilibrium manifold and the natural projection.

Definition 1 (i) $p \in \mathbb{S}$ is an *equilibrium price* of the economy $\mathbf{e} = (e_i)_{i\in M}$ if:

$$\sum_{i=1}^{m} f_i(p, p.e_i) = \sum_{i=1}^{m} e_i.$$

(ii) The *equilibrium manifold* E_{eq} is the set of the pairs $(p, \mathbf{e}) \in \mathbb{S} \times \Omega$ with p equilibrium price for the economy $\mathbf{e}$. An element $(p, \mathbf{e})$ of the set E_{eq} is called an *equilibrium point*.

In our framework, the equilibrium manifold[11] is not necessarily smooth and we cannot directly apply the classical arguments of differential topology. Nevertheless, we parametrize the equilibrium manifold and exploit the results of the previous section.

We denote by $\mathbf{1}$ the vector whose coordinates are all equal to 1 and by $\mathbf{1}^\perp$ the vector space orthogonal to the vector $\mathbf{1}$. We define the mapping θ^2 from $\mathbb{S} \times \mathbb{R}^m \times (\mathbf{1}^\perp)^{m-1}$ to $(\mathbb{R}^\ell)^m$ as follows: for $\xi = (p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1})$,

$$\begin{cases} \theta_i^2(\xi) := \eta_i + (w_i - p \cdot \eta_i)\mathbf{1} \ \forall i = 1, \dots, m-1 \\ \theta_m^2(\xi) := f_m(p, w_m) + \sum_{i=1}^{m-1}(f_i(p, w_i) - \theta_i^2(\xi)) \end{cases} \tag{4.1}$$

Let us define the sets: $\mathcal{X} := \mathbb{S} \times \mathbb{R}_{++}^m \times (\mathbf{1}^\perp)^{m-1}$ and the set $\mathcal{U}$ by

$$\mathcal{U} := \left\{ \xi \in \mathcal{X} \mid \theta^2(\xi) \in (\mathbb{R}_{++}^\ell)^m \right\}.$$

Now we define the subset $\mathcal{V}$ of $\mathcal{U}$, as follows: an element $(p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1})$ of $\mathcal{U}$ belongs to $\mathcal{V}$ if and only if (p, w_i) belongs to Ω_i^0 for all $i = 1, \dots, m$, Ω_i^0 is given by Proposition 2.

Proposition 4 *The set $\mathcal{U}$ is an open connected subset of $\mathbb{S} \times \mathbb{R}^m \times (\mathbf{1}^\perp)^{m-1}$.*

Proof The set $\mathcal{U}$ is clearly open in $\mathbb{S} \times \mathbb{R}^m \times (\mathbf{1}^\perp)^{m-1}$ as θ^2 is continuous. For the remaining of the proof, we will use extensively the following result. For all $\epsilon \in \mathbb{R}^\ell$ and all $p \in \mathbb{S}$, we have:

$$\epsilon = \mathrm{proj}_{\mathbf{1}^\perp}\epsilon + (p \cdot \epsilon - p \cdot \mathrm{proj}_{\mathbf{1}^\perp}\epsilon)\mathbf{1}. \tag{4.2}$$

We show that the set $\mathcal{U}$ is arcconnected. Let $\xi^k := (p^k, (w_i^k)_{i=1}^m, (\eta_i^k)_{i=1}^{m-1})$, $k = 1, 2$, two elements of $\mathcal{U}$. Our goal is to connect ξ^1 to ξ^2. We introduce two intermediate points: $\chi^k := (p^k, (w_i^k)_{i=1}^m, (\mathrm{proj}_{\mathbf{1}^\perp} f_i(p^k, w_i^k))_{i=1}^{m-1})$, $k = 1, 2$.

We show that we can construct a continuous path between ξ^k and χ^k, $k = 1, 2$ and another one between χ^1 and χ^2, theses paths taking values in

[11] We prove that it is indeed a smooth manifold at almost every point.

$\mathcal{U}$ which gives us the result. We first remark that Formula 4.2 implies that $\theta^2(\chi^k) = (f_i(p^k, w_i^k))_{i=1}^m$ for $k = 1, 2$.

Paths between ξ^k and χ^k

For $k = 1, 2$, for all $t \in [0, 1]$, let $\zeta^{kt} := (1-t)\xi^k + t\chi^k$. This defines a continuous path between ξ^k and χ^k. ζ^{kt} belongs to $\mathcal{U}$ for all t since θ^2 is linear with respect to the variables $(\eta_i)_{i=1}^{m-1}$ and $(\mathbb{R}^\ell_{++})^m$ is convex.

Path between χ^1 and χ^2

For all $t \in [0, 1]$, we define $p^t := (1-t)p^1 + tp^2$ and $w_i^t := (1-t)w_i^1 + tw_i^2$ for all $i = 1, \ldots, m$. The vector $\chi^{t+1} := (p^t, (w_i^t)_{i=1}^m, (\mathrm{proj}_{\mathbf{1}^\perp} f_i(p^t, w_i^t))_{i=1}^{m-1})$ defines a continuous path between χ^1 and χ^2 thanks to the continuity of the demand functions and belongs to $\mathcal{U}$ since $\theta^2(\chi^{t+1}) = (f_i(p^t, w_i^t))_{i=1}^m \in (\mathbb{R}^\ell_{++})^m$.

We conclude that the set $\mathcal{U}$ is arcconnected.$\square$

Now, we introduce the map θ from $\mathcal{U}$ to E_{eq} to parametrize the equilibrium manifold. For $\xi = (p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1}) \in \mathcal{U}$,

$$\theta(\xi) = \left(p, \theta^2(\xi)\right). \tag{4.3}$$

Note that the range of the map θ is contained in E_{eq}. Let $e_i = \theta_i^2(\xi)$. From the formula defining θ^2, one has $p \cdot e_i = w_i$, from the definition of $\mathcal{U}$, $e_i \gg 0$ for all i in M and from the formula for θ_m^2, $\sum_{i=1}^m f_i(p, p \cdot e_i) = \sum_{i=1}^m e_i$.

The other map that we consider is the following:

$$\begin{aligned} \phi : E_{eq} &\longrightarrow \mathcal{U} \\ (p, (e_i)_{i=1}^m) &\longmapsto (p, (p.e_i)_{i=1}^m, (\mathrm{proj}_{\mathbf{1}^\perp} e_i)_{i=1}^{m-1}). \end{aligned} \tag{4.4}$$

Note that the range of the map ϕ is contained in $\mathcal{U}$. Indeed, from Formula 4.2, $\theta^2(p, (p \cdot e_i)_{i=1}^m, (\mathrm{proj}_{\mathbf{1}^\perp} e_i)_{i=1}^{m-1}) = (e_i)_{i=1}^m$ belongs to $(\mathbb{R}^\ell_{++})^m$.

We now state the properties of θ and ϕ, which imply that E_{eq} is a manifold parametrized by θ.

Proposition 5

1. *The maps θ and ϕ are one-to-one, onto and $\theta^{-1} = \phi$.*

2. *The maps θ and ϕ are locally Lipschitz and continuous.*

3. *The set $\mathcal{U} \setminus \mathcal{V}$ is closed in $\mathcal{U}$ and has Lebesgue measure zero.*

4. *θ is continuously differentiable on $\mathcal{V}$.*

5. *E_{eq} is lipeomorphic to $\mathcal{U}$.*

Proof The proof is based on the properties of the demand functions.

1. As already noticed above, from Formula 4.2, for all $(p, (e_i)_{i=1}^m) \in E_{eq}$, $\theta \circ \phi(p, (e_i)_{i=1}^m) = (p, (e_i)_{i=1}^m)$.

Conversely, for all $\xi = (p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1}) \in \mathcal{U}$, the definition of θ^2 and Walras law imply that $p \cdot \theta_i^2(\xi) = w_i$ for $i = 1, \ldots, m$, and $\text{proj}_{\mathbf{1}^\perp} \theta_i^2(\xi) = \eta_i$ for $i = 1, \ldots, m-1$. So, $\phi \circ \theta(\xi) = \xi$.

Hence, $\phi = \theta^{-1}$ and ϕ and θ are one-to-one and onto.

2. θ and ϕ are locally Lipschitz and continuous since the demand functions f_i are so.

3. Since the set $(\mathbb{S} \times \mathbb{R}) \setminus \Omega_i^0 \cap (\mathbb{S} \times \mathbb{R})$ is a closed set of Lebesgue measure zero for each $i \in M$, the set $\mathcal{U} \setminus \mathcal{V}$ is closed in $\mathcal{U}$ and has Lebesgue measure zero from Fubini's Theorem.

4. The map θ is C^1 on $\mathcal{V}$ from the definition of $\mathcal{V}$ and the properties of the demand function f_i on Ω_i^0.

5. This is a consequence of 1. and 2.$\square$

Let us write $\mathbf{e} := (e_i)_{i=1}^m$. Following Balasko [1], let us introduce the natural projection as well as the extended natural projection.

Definition 2 (The natural projection) The *natural projection* π is the map from E_{eq} to Ω defined by:

$$\pi : \begin{array}{c} E_{eq} \longrightarrow \Omega \\ (p, \mathbf{e}) \longmapsto \mathbf{e} \end{array} \tag{4.5}$$

The map $\Pi := \pi \circ \theta$ is called the *extended natural projection*.
$\Pi : \mathcal{U} \longrightarrow \Omega$ is defined by: $\Pi(p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1}) := \theta^2(p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1})$.

Proposition 6 *The mapping Π is proper, locally Lipschitz continuous. Moreover the mapping Π is continuously differentiable on $\mathcal{V}$. $\Pi(\mathcal{U} \setminus \mathcal{V})$ is closed and has Lebesgue measure zero in Ω.*

Proof The mapping Π is locally Lipschitz continuous and continuously differentiable on the set $\mathcal{V}$ by the properties of θ. The properness of Π is a particular case of the properness of F, the proof of which is given below as part of the proof of Theorem 3.

The set $\Pi(\mathcal{U} \setminus \mathcal{V})$ is closed in Ω because the set $\mathcal{U} \setminus \mathcal{V}$ is closed in $\mathcal{U}$ and the map Π is proper on $\mathcal{U}$. The set $\Pi(\mathcal{U} \setminus \mathcal{V})$ has Lebesgue measure zero because the map Π is locally Lipschitz continuous[12] and the set $\mathcal{U} \setminus \mathcal{V}$ is a null set. $\square$

[12] The image of a null set by a locally Lipschitz map is a null set. See Federer [9].

A regular economy is a regular value of the natural projection. There are different concepts of regularity for non-smooth mappings. In this paper, a regular point is a point where the mapping is differentiable and the differential mapping is onto. A value is regular if all pre-images are regular points. A value is singular, by definition, if it is not regular.

Definition 3 The economy $\mathbf{e} := (e_i)_{i \in M}$ is called *regular* if $\mathbf{e} \notin \Pi(\mathcal{U} \setminus \mathcal{V})$ and if the differential of Π at all the pre-images of $\mathbf{e}$ is onto. An economy which is not regular is called *singular*. E^r denotes the set of regular economies and E^s the set of singular economies.

The following result is the extension of one of the cornerstones of the differentiable approach of general equilibrium theory.

Theorem 1 *The set of regular economies E^r is an open dense subset of Ω of full Lebesgue measure.*

Proof *E^r is open and has full Lebesgue measure.* We have already seen that $\Pi(\mathcal{U} \setminus \mathcal{V})$ is a closed null set. The set of the critical points of $\Pi_{|\mathcal{V}}$ is closed in $\mathcal{V}$, hence this set has the form $\mathcal{V} \cap C$ where C is a set closed in $\mathcal{U}$. Remark that we have the equalities: $E^s = \Pi(C \cap \mathcal{V}) \cup \Pi(\mathcal{U} \setminus \mathcal{V}) = \Pi(C) \cup \Pi(\mathcal{U} \setminus \mathcal{V})$. We deduce that the set E^s is a closed set since the map Π is proper. This set has Lebesgue measure zero by Sard's Theorem and the previous theorem. Hence, E^r is an open set of full Lebesgue measure.

E^r is dense. Indeed, the set E^s is a set of Lebesgue measure zero, so its complement E^r is dense. $\square$

The following result summarizes the properties of regular economies. It is a direct consequence of the Implicit Function Theorem.

Theorem 2

1. *For $\mathbf{e} \in E^r$, there exists a finite number n of equilibrium prices.*

2. *For $\mathbf{e} \in E^r$, there exists an open neighborhood $U \subset E^r$ of $\mathbf{e}$ such that the inverse image of U is the union of a finite number of pairwise disjoint subsets $(V_k)_{k=1}^n$ of $\mathcal{V}$ and such that the restriction of Π to V_k is a diffeomorphism for all $k = 1, \dots, n$.*

3. *For $\mathbf{e} \in E^r$, there exist an open neighborhood $U \subset E^r$ of $\mathbf{e}$ and a finite number n of continuously differentiable maps $s_k : U \longrightarrow \mathbb{S}$ such that the union $\cup_{k=1}^n s_k(\mathbf{e}')$ is the set of equilibrium price vectors associated with the economy $\mathbf{e}' \in U$.*

Remark 3 Note that, around a regular economy $\mathbf{e}$, the number of equilibrium prices is constant by the previous theorem.

We now turn ourselves to the computation of the degree of Π. We first remark that the mapping Π is not continuously differentiable. So we cannot use the classical definition of the degree. Therefore we consider the degree for continuous mappings[13]. Since the set $\mathcal{U}$ is unbounded, the definition of the degree needs some properness assumption[14].

Theorem 3

1. *The map* Π *is of degree one and onto.*
2. *For all* $\mathbf{e} \in \Omega$*, there exists an equilibrium.*
3. *For all* $\mathbf{e} \in E^r$*, there exists a finite odd number of equilibrium prices.*

Proof We consider the natural projection Π^1 associated with consumers having as utility functions the functions u_i^1.[15]

The map[16] F from $\mathbb{S} \times \mathbb{R}^m_{++} \times (\mathbf{1}^\perp)^{m-1} \times [0,1]$ to $\mathbb{R}^{\ell m}$ is defined for all $(\xi, t) = ((p, (w_i)_{i=1}^m, (\eta_i)_{i=1}^{m-1}), t) \in \mathbb{S} \times \mathbb{R}^m_{++} \times (\mathbf{1}^\perp)^{m-1} \times [0,1]$ by:

$$F(\xi, t) := t\Pi(\xi) + (1-t)\Pi^1(\xi).$$

We first show that the inverse image of every compact subset of Ω is a compact subset of $\mathbb{S} \times \mathbb{R}^m_{++} \times (\mathbf{1}^\perp)^{m-1} \times [0,1]$. Let us consider a compact subset of $(\mathbb{R}^\ell_{++})^m$ denoted by $\mathcal{K}$ and a sequence $\left(\xi^\nu := \left(p^\nu, (w_i^\nu)_{i=1}^m, (\eta_i^\nu)_{i=1}^{m-1}\right), t^\nu\right)_{\nu \geq 0}$ of $F^{-1}(\mathcal{K})$. We denote by $(\mathbf{e}^\nu)_{\nu \geq 0}$ the sequence of $\mathcal{K}$ defined by $\mathbf{e}^\nu := F\left(\xi^\nu, t^\nu\right)$ for $\nu \geq 0$. The sequence $\left(\xi^\nu, t^\nu\right)_{\nu \geq 0}$ remains in a compact set of $\bar{\mathbb{S}} \times \mathbb{R}^m \times (\mathbf{1}^\perp)^{m-1} \times [0,1]$. Indeed, the first and the last components lie in a compact set by definition. Moreover, the set $\mathbb{A} := \{(p \cdot e_i)_{i=1}^m | (e_i)_{i=1}^m \in \mathcal{K}\ ,\ p \in \bar{\mathbb{S}}\}$ is a compact set. Since $p \in \bar{\mathbb{S}}$ and since the set $\mathcal{K}$ is contained in Ω, every element $a \in \mathbb{A}$ is positive i.e. $a \in \mathbb{R}^m_{++}$. By Walras law and the definition of F, $w_i^\nu = p^\nu \cdot e_i^\nu$ for all $i = 1, \ldots, m$ and all $\nu \geq 0$. Hence, the sequence $((w_i^\nu)_{i=1}^m)_{\nu \geq 0}$ lies in the compact set $\mathbb{A}$.

[13] A good reference is Deimling [8] for example.

[14] See Deimling [8] p.27.

[15] As it can be easily understood, the definitions of Π^1, f_i^1 and θ^1 are analogous to those of Π, f_i and θ. The main difference is that those maps are smooth.

[16] By a slight abuse of notation, we denote by Π and Π^1 the extensions of those maps to $\mathcal{X}$. Both are defined with the same formulas as Π (respectively Π^1). Nevertheless, note that these extensions are not proper, that is why we only consider here the inverse images of compact subsets of Ω. At the end, we obtain the existence of an equilibrium only for positive endowments.

The compactness of the set $\mathcal{K}$ and the continuity of the projection map imply that the sequence $(\eta_i^\nu = \mathrm{proj}_{\mathbf{1}^\perp} e_i^\nu)_{\nu \geq 0}$ lies in a compact subset of $\mathbf{1}^\perp$. To conclude, up to a subsequence, the sequence $(\xi^\nu, t^\nu)_{\nu \geq 0}$ converges to a vector $(\bar{\xi} := (\bar{p}, (\bar{w}_i)_{i=1}^m, (\bar{\eta}_i)_{i=1}^{m-1}), \bar{t}) \in \bar{\mathbb{S}} \times \mathbb{R}^m \times (\mathbf{1}^\perp)^{m-1} \times [0,1]$. Remark that $(\bar{w}_i)_{i=1}^m$ belongs to $\mathbb{A}$ so the real number $\bar{w}_i$ is positive for all $i = 1, \dots, m$.

Now, we have to prove that this vector belongs to $F^{-1}(\mathcal{K})$. Thanks to the continuity of F and to the closedness of $\mathcal{K}$, it suffices to prove that the price $\bar{p}$ belongs to $\mathbb{S}$. Otherwise it would have a component equal to zero. By definition of F, Π and Π^1, the m-th component of F, $F_m(\xi^\nu, t^\nu)$ is equal to $(1 - t^\nu)[f_m^1(p^\nu, w_m^\nu) + \sum_{i=1}^{m-1}(f_i^1(p^\nu, w_i^\nu) - e_i^\nu)] + t^\nu[f_m(p^\nu, w_m^\nu) + \sum_{i=1}^{m-1}(f_i(p^\nu, w_i^\nu) - e_i^\nu)]$ for $\nu \geq 0$.

We first remark that, for all $i = 1, \dots, m$, $(e_i^\nu)_{\nu \geq 0}$ is bounded since $(\mathbf{e}^\nu)_{\nu \geq 0}$ belongs to the compact set $\mathcal{K}$. We also remark that $f_i(p^\nu, w_i^\nu)$ and $f_i^1(p^\nu, w_i^\nu)$ are positive. If $(p^\nu)_{\nu \geq 0}$ converged to $\bar{p}$ in $\partial\mathbb{S}$ and $(w_i^\nu)_{i \in M, \nu \geq 0}$ converged to some element $(\bar{w}_i)_{i \in M} \in \mathbb{R}_{++}^m$ then $\|f_i^1(p^\nu, w_i^\nu)\|$ and $\|f_i(p^\nu, w_i^\nu)\|$ would go to $+\infty$ as ν goes to infinity for all $i \in M$ by monotony of the functions $(u_i)_{i \in M}$ and $(u_i^1)_{i \in M}$. So $\|F_m(\xi^\nu, t^\nu)\|$ would go to $+\infty$, which contradicts that $F(\xi^\nu, t^\nu)$ belongs to the compact set $\mathcal{K}$ for all $\nu \geq 0$.

Since Ω is connected, the degree does not depend on the choice of the element where it is computed. Let $\bar{\mathbf{e}} \in \Omega$ defined , for $i \in M$, by $\bar{e}_i := f_i^1(p, w_i)$ for some $(p, (w_i)_{i=1}^m) \in \mathbb{S} \times \mathbb{R}_{++}^m$. Let $\bar{B} := B_c(\bar{\mathbf{e}}, r)$ be a closed ball[17] of center $\bar{\mathbf{e}}$ contained in Ω[18] and $B := B_o(\bar{\mathbf{e}}, r)$ the open ball of same center and same radius. We know that the set $F^{-1}(\bar{B})$ is a compact set. The set $F^{-1}(B)$ contains $F^{-1}(\{\bar{\mathbf{e}}\})$ and is an open set, by the continuity of F, contained in $F^{-1}(\bar{B})$. Hence $F^{-1}(\{\bar{\mathbf{e}}\})$ is contained in the interior of $F^{-1}(\bar{B})$.

We now define the set $\vartheta := \mathrm{proj}_{\mathbb{S} \times \mathbb{R}_{++}^m \times (\mathbf{1}^\perp)^{m-1}} F^{-1}(B)$. This set is an open set[19]. The mapping F is obviously continuous on $\bar{\vartheta} \times [0,1]$. Let $\hat{F}$ be the restriction of F to $\bar{\vartheta} \times [0,1]$. From Balasko[1], the degree modulo 2 at $\bar{\mathbf{e}}$ of F_0 is equal to 1[20]. Since $F_0^{-1}(\{\bar{\mathbf{e}}\})$ is contained in ϑ, the degree of $\hat{F}_0$ is also equal to the degree of F_0 by Property (d2) of the degree (See Deimling [8].). The degree of $\hat{F}_1$ at $\bar{\mathbf{e}}$ is also equal to 1 since $\hat{F}$ is a continuous homotopy. Since $F_1^{-1}(\{\bar{\mathbf{e}}\})$ is contained in ϑ, the degree of Π at $\bar{\mathbf{e}}$ is equal to the degree of $\hat{F}_1$. In conclusion, the degree of Π is equal to 1.

Hence, for all $\mathbf{e} \in \Omega$, $\Pi^{-1}(\{\mathbf{e}\}) \neq \emptyset$, which means that there exists an equilibrium for every economy $\mathbf{e} \in \Omega$. The third point is a consequence of

[17] $B_o(a, r)$ (respectively $B_c(a, r)$) denotes the open (resp. closed) ball of center a and of radius r.

[18] The radius r has to be sufficiently small.

[19] The image of an open set by a projection map is open.

[20] See [1] pp 103-106.

the fact that Π is locally C^1 around all the pre-images of a regular economy using that the degree of Π is equal to 1. $\square$

5 Concluding remarks

This paper provides a contribution in the analysis of the individual behavior and in the global analysis of the equilibrium with multiprior preferences. The first main result is that, under an assumption of linear independence of the priors, the demand of a consumer with multiprior preferences is locally Lipschitz and continuously differentiable on an open set of full Lebesgue measure. Using Rader [13] and this result, we derive that almost every economy has a finite number of equilibrium prices.

The second main result concerns the genericity of regular economies. We have recovered the usual results of the global analysis of economic equilibrium. Contrary to most of the previous contributions in non-smooth cases, we have obtained not only that almost every economy has a finite number of equilibrium prices but that there exists an open set of full Lebesgue measure on which the result holds true. Moreover, we have proven that the equilibrium price selections are continuously differentiable. Furthermore, regular economies enjoy good properties like the local constancy of the number of equilibrium prices.

The only restriction is the requirement of linear independence of the gradients. First note that this requirement is easy to check. Then, remark that this requirement is always satisfied when the agents have at most two "extremal" priors. It remains open to study cases where not all but just some of the "extremal" priors are linearly independent.

Appendix: Proofs

Proof of Proposition 1

The assumptions on the function b_i imply straightforwardly that the functions $(u_i^k)_{1 \leq k \leq n_i}$ satisfy Assumption 1.

Let us prove that Assumption 2 is satisfied. Let $\bar{x} \in \mathbb{R}^S_{++}$.

If $\sum_{k \in M(\bar{x})} \gamma_k \nabla u_i^k(\bar{x}) = 0$ for some $(\gamma_k)_{k \in M(\bar{x})}$, then, for all $s \in \{1, \ldots, S\}$,

$$\sum_{k \in M(\bar{x})} \gamma_k \pi_i^k(s) b_i'(\bar{x}_s) = 0.$$

Dividing by $b_i'(\bar{x}_s)$ for $s \in \{1, \dots, S\}$, we get for $s \in \{1, \dots, S\}$:

$$\sum_{k \in M(\bar{x})} \gamma_k \pi_i^k(s) = 0$$

By Assumption 4, we conclude that $\gamma_k = 0$ for $k \in M(\bar{x})$. Hence, the gradient vectors $(\nabla u_i^k(\bar{x}))_{k \in M(\bar{x})}$ are linearly independent. So, the functions $(u_i^k)_{1 \le k \le n_i}$ satisfy Assumption 2.

It is straightforward that: $\lim_{y \to 0^+} b_i(y) = -\infty$ implies that Assumption 3 is satisfied for the function u_i^k for $k \in \{1, \dots, n_i\}$.$\square$

Proof of Proposition 2

Step 1: The set $f(p, w)$ is a singleton and the function f is continuous.

Let us first prove that $f(p, w)$ is a singleton. Indeed, $f(p, w)$ is the solution of the following optimization problem:

$$\begin{cases} \max u(x) \\ \text{ subject to } x \gg 0 \text{ and } p \cdot x \le w \end{cases}$$

The condition defines the budget set $B(p, w)$ with respect to the price p and the wealth w. Since this budget set is not a compact set, let us introduce the set:

$$\tilde{B}(p, w) := \{x \in \mathbb{R}^\ell | x \gg 0, p \cdot x \le w, u(x) \ge u(\underline{x}(p, w))\}$$

where $\underline{x}(p, w) := \left(\dfrac{w}{2p_1}, \dots, \dfrac{w}{2p_\ell} \right)$. As usual, we remark that maximizing u on the set $B(p, w)$ is equivalent to maximizing u on the set $\tilde{B}(p, w)$. From Assumption 1 and Assumption 3, the set $\tilde{B}(p, w)$ is a compact set. Then, by Weierstrass Theorem, the set $f(p, w)$ is not empty. By strict quasi-concavity of the function u[21], the set $f(p, w)$ is a singleton.

We now prove that the function $(p, w) \longmapsto f(p, w)$ is continuous. We use Berge's Theorem and remark that the set $\tilde{B}(p, w)$ is nonempty for every $p \gg 0$ and every $w > 0$. We have to show that the correspondence $\tilde{B}$ is both upper semi-continuous and lower semi-continuous.

First, let us show that the correspondence $\tilde{B}$ is upper semi-continuous. Let $(\bar{p}, \bar{w}) \in \mathbb{R}_{++}^\ell \times]0, +\infty[$. For every (p, w) in a neighborhood of $(\bar{p}, \bar{w})$,

[21] Indeed a differentiably strictly quasi-concave function is strictly quasi-concave (See Balasko [2].). And the minimum of strictly quasi-concave functions is strictly quasi-concave.

the set $\tilde{B}(p,w)$ remains in a fixed compact set K. Hence the upper semi-continuity of $\tilde{B}$ is equivalent to the closedness of its graph, which is a consequence of the continuity of the utility function u, the mapping $\underline{x}$ and the budget constraint.

We now have to show that the correspondence $\tilde{B}$ is lower semi-continuous. Let us first introduce the correspondence $\hat{B}$ defined on $\mathbb{R}^{\ell}_{++}\times]0,+\infty[$ by: $\hat{B}(p,w) := \{x \in \mathbb{R}^{\ell} | x \gg 0, p\cdot x < w, u(x) > u(\underline{x}(p,w))\}$. The correspondence $\hat{B}$ has an open graph by the continuity of the functions u and $\underline{x}$ and the budget constraint. So $\hat{B}$ is lower semi-continuous. From the monotony of u, since $p \cdot \underline{x}(p,w) < w$, $\hat{B}(p,w)$ is nonempty for every $(p,w) \in \mathbb{R}^{\ell}_{++}\times]0,+\infty[$. So, since the function u is strictly quasi-concave and continuous, the closure of $\hat{B}(p,w)$ is $\tilde{B}(p,w)$. Thus, the correspondence $\tilde{B}$ is lower semi-continuous since the closure of a lower semi-continuous correspondence is lower semi-continuous.

Berge's Theorem implies that the function f is continuous on the set $\mathbb{R}^{\ell}_{++}\times]0,+\infty[$.

Step 2: The function f is locally Lipschitz on $\mathbb{R}^{\ell}_{++}\times]0;+\infty[$.

Let us show that the function f is locally Lipschitz. The proof relies on Cornet and Vial's result [5]. To apply it, we rewrite the problem:

$$\begin{cases} \max u(x) \\ \text{ subject to } x \gg 0 \text{ and } p \cdot x \leq w \end{cases}$$

since the function u is not differentiable. For all $(p,w) \in \mathbb{R}^{\ell}_{++}\times]0,+\infty[$, $f(p,w)$ is the solution of the following problem:

$$\begin{cases} \min -t \\ \text{ subject to} \\ t - u^k(x) \leq 0 \;\; k = 1,\ldots,n \\ p \cdot x - w \leq 0 \end{cases} \tag{5.1}$$

The function that we now minimize is C^2 and we have modified the constraints.

Let us introduce the functions $\tau : \mathbb{R} \times \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell}_{++}\times]0,+\infty[\longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell}_{++}\times]0,+\infty[\longrightarrow \mathbb{R}^{n+1}$ defined by:

- $\tau(t,x,p,w) := -t$,
- $g_k(t,x,p,w) := t - u^k(x)$ for $k \in \{1,\ldots,n\}$,
- $g_{n+1}(t,x,p,w) := p \cdot x - w$.

(t,x) are the variables and (p,w) the parameters. The gradients and the Hessian matrix with respect to (t,x) are given for all $(t,x,p,w) \in \mathbb{R} \times \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \times]0,+\infty[$ by:

- $\nabla \tau(t,x,p,w) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,
- $\nabla g_k(t,x,p,w) = \begin{pmatrix} 1 \\ -\nabla u^k(x) \end{pmatrix}$ for $k \in \{1,\dots,n\}$,
- $\nabla g_{n+1}(t,x,p,w) = \begin{pmatrix} 0 \\ p \end{pmatrix}$,
- $D^2\tau(t,x,p,w) \equiv 0$,
- $D^2 g_k(t,x,p,w) = \begin{bmatrix} 0 & 0 \\ 0 & -D^2u^k(x) \end{bmatrix}$ for $k \in \{1,\dots,n\}$,
- $D^2 g_{n+1}(t,x,p,w) \equiv 0$.

The function τ is linear, the constraint functions $(g_k)_{k=1}^{n+1}$ are twice differentiable and satisfy: $\nabla g_k(t,x,p,w) \neq 0$ for all $k = 1,\dots,n+1$ and $(t,x,p,w) \in \mathbb{R} \times \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \times]0,+\infty[$.

We remark that the Mangasarian-Fromovitz condition for the qualification of the constraints is satisfied everywhere. Indeed, for $\alpha > 0$ large enough, $(-\alpha, -\mathbf{1}) \cdot \nabla g_k(t,x,p,w) < 0$ for all $k = 1,\dots,n+1$. So the first order conditions are necessary. So if (t,x) is a solution of 5.1, there exists $\lambda = (\lambda_k)_{1 \leq k \leq n+1} \in \mathbb{R}^{n+1}_+$ such that :

$$\left\{ \begin{array}{l} \nabla \tau(t,x,p,w) + \displaystyle\sum_{k=1}^{n+1} \lambda_k \nabla g_k(t,x,p,w) = 0 \\ \lambda_k g_k(t,x,p,w) = 0 \ \forall k \in \{1,\dots,n+1\} \\ g_k(t,x,p,w) \leq 0 \ \forall k \in \{1,\dots,n+1\} \\ (t,x,p,w) \in \mathbb{R} \times \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \times]0,+\infty[\end{array} \right. \tag{5.2}$$

Note that we have: $\{k \in \{1,\dots,n\} | g_k(t,x,p,w) = 0\} = M(x)$. Thus, the multipliers λ_k are equal to zero for $k \notin M(x)$. From the above formula for the gradients, these conditions can be rewritten as follows:

$$\left\{ \begin{array}{l} \sum_{k=1}^n \lambda_k = 1 \\ \lambda_k(t - u^k(x)) = 0, k = 1,\dots,n \\ \lambda_{n+1}(p \cdot x - w) = 0 \\ t - u^k(x) \leq 0, k = 1,\dots,n \\ p \cdot x - w \leq 0 \\ \lambda_{n+1} p = \sum_{k \in M(x)} \lambda_k \nabla u^k(x) \end{array} \right. \tag{5.3}$$

By the first equation and the fact that the vectors $\nabla u_i^k(x)$ and p are elements of $\mathbb{R}^\ell_{++}$, we deduce that λ_{n+1} is necessarily positive, which means that the budget constraint is binding.

We now check that these conditions are sufficient. Let (t, x, λ) satisfying these conditions. If (t, x) is not a solution of 5.1, there exists (t', x') such that $t' > t$, $t' \leq u^k(x')$ for $k = 1, \ldots, n$ and $p \cdot x' \leq w$. Since u^k is strictly quasi-concave with a non-vanishing gradient, for all $k \in M(x)$, one has $t = u^k(x) < t' \leq u^k(x')$, so $\nabla u^k(x) \cdot (x' - x) > 0$. Hence,

$$\lambda_{n+1} p \cdot (x' - x) = \left(\sum_{k \in M(x)} \lambda_k \nabla u^k(x) \right) \cdot (x' - x) > 0$$

which implies that $p \cdot x' > p \cdot x = w$ in contradiction with $p \cdot x' \leq w$. So, the first order conditions are sufficient.

To show that the function f is locally Lipschitz, we check that Assumptions (A.0), (C.1) and (C.2) of Corollary 2.3. of [5] are satisfied.

Assumptions (A.0) are satisfied. We take $U = \mathbb{R} \times \mathbb{R}^\ell_{++}$ and $P = \mathbb{R}^\ell_{++} \times]0, +\infty[$. The set U is open and obviously a metric space. So Assumption (A.0) (i) is satisfied. Assumptions (A.0) (ii), (iii), (iv) and (v) are satisfied because the functions are C^2 on the set $U \times P$. Assumption (A.0) (vi) is satisfied with $Q = C = -\mathbb{R}^{n+1}_+$.

Assumption (C.1) is satisfied. We show that, at a solution (t, x) of 5.1, the vectors $(\nabla g_k(t, x, p, w))_{k \in M(x)}$ and $\nabla g_{n+1}(t, x, p, w)$ are linearly independent. Let a vector $\left((\gamma_k)_{k \in M(x)}, \gamma_{n+1}\right) \in \mathbb{R}^{\sharp M(x)+1}$ such that:

$$\sum_{k \in M(x)} \gamma_k \nabla g_k(t, x, p, w) + \gamma_{n+1} \nabla g_{n+1}(t, x, p, w) = 0$$

We obtain:

$$\begin{cases} \sum_{k \in M(x)} \gamma_k = 0 \\ \gamma_{n+1} p = \displaystyle\sum_{k \in M(x)} \gamma_k \nabla u^k(x) \end{cases}$$

If $\gamma_{n+1} = 0$, then $\gamma_k = 0$ for every $k \in M(x)$ because the vectors $(\nabla u^k(x))_{k \in M(x)}$ are linearly independent by Assumption 2.

If $\gamma_{n+1} \neq 0$, we get by the first order conditions:

$$\sum_{k \in M(x)} \left(\frac{\lambda_k}{\lambda_{n+1}} - \frac{\gamma_k}{\gamma_{n+1}} \right) \nabla u^k(x) = 0.$$

So we have: $\dfrac{\lambda_k}{\lambda_{n+1}} = \dfrac{\gamma_k}{\gamma_{n+1}}$ for every $k \in M(x)$ by Assumption 2. But we get a contradiction since:

$$0 = \sum_{k \in M(x)} \frac{\gamma_k}{\gamma_{n+1}} = \sum_{k \in M(x)} \frac{\lambda_k}{\lambda_{n+1}} = \frac{1}{\lambda_{n+1}} > 0.$$

Assumption (C.2) is satisfied. Let (t,x) be a solution of 5.1 with an associated multiplier $\lambda := (\lambda_k)_{k=1}^{n+1}$. Let us introduce the set $K(x) := \{k \in M(x) | \lambda_k > 0\}$. We have to check that, for all $h \in \mathbb{R}^{\ell+1}$, $h \neq 0$ such that: $\nabla \tau(t,x,p,w) \cdot h = 0$ and $\nabla g_k(t,x,p,w) \cdot h = 0$ for $k \in K(x) \cup \{n+1\}$, we have:

$$\left[D^2 \tau(t,x,p,w) + \sum_{k \in K(x) \cup \{n+1\}} \lambda_k D^2 g_k(t,x,p,w) \right] h \cdot h > 0.$$

Since we have : $D^2 \tau \equiv 0$ and $D^2 g_{n+1} \equiv 0$. It remains to show that[22]:

$$\sum_{k \in K(x)} \lambda_k D^2 g_k(t,x,p,w) h \cdot h > 0.$$

This reduces to:

$$-\sum_{k \in K(x)} \lambda_k D^2 u^k(x) \acute{h} \cdot \acute{h} > 0$$

which is true because of Assumption 1 and because $\nabla u^k(x) \cdot \acute{h} = 0$ for $k \in K(x)$. Indeed, since we have: $\nabla \tau(t,x,p,w) \cdot h = 0$ and $\nabla g_k(t,x,p,w) \cdot h = 0$ for every $k \in K(x) \cup \{n+1\}$, one obtains $h_0 = 0$ with the first equality and it remains $\nabla u^k(x) \cdot \acute{h} = 0$ for $k \in K(x)$.

From [5], the function f is locally Lipschitz on $\mathbb{R}^{\ell}_{++} \times]0, +\infty[$ and by Rademacher's Theorem, the function f is almost everywhere differentiable.$\square$

Proof of Lemma 1. Recall that we have: $\forall k \in M(f(\bar{p}, \bar{w}))$, $\lambda_k(\bar{p}, \bar{w}) > 0$. In this proof, we use the following notations: $\bar{x} := f(\bar{p}, \bar{w})$ and $\bar{\lambda}_k := \lambda_k(\bar{p}, \bar{w})$ for $k \in \{1, \ldots, n\}$. To simplify the notation, without loss of generality, we suppose that we have: $M(f(\bar{p}, \bar{w})) = \{1, \ldots, r\}$. Then, locally around $(\bar{p}, \bar{w})$, by the continuity of the demand function and of the functions $(u^k)_{1 \le k \le n}$, $(u(f(p,w), f(p,w))$ is the solution of the following problem:

$$\begin{cases} \min -t \\ \text{ subject to} \\ t - u^k(x) \le 0, \ k \in \{1, \ldots, r\} \\ p \cdot x - w \le 0 \end{cases}$$

[22] We write $h = (h_0, h_1, \ldots, h_\ell)$ and $\acute{h} = (h_1, \ldots, h_\ell)$.

As already shown, since the first order optimality conditions are necessary and sufficient, the element $(u(f(p,w)), f(p,w))$ and the associated multipliers $(\lambda_1(p,w),\ldots,\lambda_r(p,w),\lambda_{n+1}(p,w))$ are solution of the equation $G(t,x,\lambda,p,w)=0$ where G is defined by:

$$G(t,x,\lambda,p,w)=\begin{cases}(\sum_{k=1}^r \lambda_k)-1\\ \sum_{k=1}^r \lambda_k \nabla u^k(x)-\lambda_{n+1}p\\ t-u^k(x), k=1,\ldots,r\\ p\cdot x-w\end{cases} \tag{5.4}$$

To show that the function f and the multipliers are continuously differentiable on a neighborhood of $(\bar{p},\bar{w})$, from the Implicit Function Theorem, it suffices to show that the partial Jacobian matrix of G with respect to (t,x,λ) has full column rank[23].

$$A:=\begin{array}{c} \begin{array}{cccccc} \mathbf{t} & \mathbf{x} & \lambda_{\mathbf{1}} & \ldots & \lambda_{\mathbf{r}} & \lambda_{\mathbf{n+1}} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & \ldots & 1 & 0\\ 0 & \sum_{k=1}^r \bar{\lambda}_k D^2u^k(\bar{x}) & \nabla u^1(\bar{x}) & \ldots & \nabla u^r(\bar{x}) & -\bar{p}\\ 1 & -\nabla u^1(\bar{x})^T & 0 & \ldots & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & -\nabla u^r(\bar{x})^T & 0 & \ldots & \ldots & 0\\ 0 & \bar{p}^T & 0 & \ldots & \ldots & 0 \end{pmatrix} \end{array}$$

It is sufficient to prove that $A(\Delta t,\Delta x,\Delta\lambda)=0$ implies: $\Delta t=0$, $\Delta x=0$ and $\Delta\lambda=0$. Δt is a real number, Δx is a column vector of dimension ℓ and $\Delta\lambda$ is a column vector of dimension $r+1$.

We obtain the system:

$$\sum_{k=1}^r \Delta\lambda_k=0 \tag{5.5}$$

$$\sum_{k=1}^r [\bar{\lambda}_k D^2u^k(\bar{x})\Delta x+\Delta\lambda_k\nabla u^k(\bar{x})]-\Delta\lambda_{n+1}\bar{p}=0 \tag{5.6}$$

$$\Delta t-\Delta x\cdot\nabla u^k(\bar{x})=0\ \forall k\in\{1,\ldots,r\} \tag{5.7}$$

$$\Delta x\cdot\bar{p}=0 \tag{5.8}$$

Using Equations 5.7 and 5.8, we get for all $k\in\{1,\ldots,r\}$:

$$\bar{\lambda}_k\Delta t-\bar{\lambda}_k\Delta x\cdot\nabla u^k(\bar{x})\ \ =0$$

[23]The vectors are, by convention, column vectors and the transpose of a vector x is denoted by x^T.

Summing, we obtain:

$$\left(\sum_{k=1}^{r}\bar{\lambda}_k\right)\Delta t - \Delta x \cdot \left(\sum_{k=1}^{r}\bar{\lambda}_k \nabla u^k(\bar{x})\right) = 0$$

We recall the equality: $\bar{\lambda}_{n+1}\bar{p} = \sum_{k=1}^{r}\bar{\lambda}_k \nabla u^k(\bar{x})$.

So, we have:

$$\left(\sum_{k=1}^{r}\bar{\lambda}_k\right)\Delta t - \Delta x \cdot \bar{\lambda}_{n+1}\bar{p} = 0$$

From 5.8 and the fact that $\sum_{=1}^{r}\bar{\lambda}_k = 1$, we deduce: $\Delta t = 0$. Thanks to 5.7,

$$\Delta x \cdot \nabla u^k(\bar{x}) = 0 \ \forall k \in \{1, \dots, r\}$$

which implies $\bar{p} \cdot \Delta x = 0$ since $\bar{\lambda}_{n+1}\bar{p} = \sum_{k=1}^{r}\bar{\lambda}_k \nabla u^k(\bar{x})$ and $\bar{\lambda}_{n+1} > 0$.

So if $\Delta x \neq 0$,

$$\Delta x \cdot D^2 u^k(\bar{x})\Delta x < 0 \ \forall k \in \{1, \dots, r\}.$$

Doing an inner product of 5.6 by Δx, we get:

$$\sum_{k=1}^{r}\bar{\lambda}_k \Delta x \cdot D^2 u^k(\bar{x})\Delta x + \sum_{k=1}^{r}(\Delta\lambda_k \nabla u^k(\bar{x}) \cdot \Delta x) - \Delta\lambda_{n+1}\bar{p} \cdot \Delta x = 0$$

which becomes:

$$\sum_{k=1}^{r}\bar{\lambda}_k \Delta x \cdot D^2 u^k(\bar{x})\Delta x = 0,$$

which is in contradiction with $\Delta x \cdot D^2 u^k(\bar{x})\Delta x < 0$ for all $k \in \{1, \dots, r\}$ recalling that the multipliers $\bar{\lambda}_k$ are all positive. Hence, we get $\Delta x = 0$.

Since $\bar{\lambda}_{n+1}\bar{p} = \sum_{k=1}^{r}\bar{\lambda}_k \nabla u^k(\bar{x})$, 5.6 becomes:

$$\sum_{k=1}^{r}\left(\Delta\lambda_k - \frac{\Delta\lambda_{n+1}}{\bar{\lambda}_{n+1}}\bar{\lambda}_k\right)\nabla u^k(\bar{x}) = 0.$$

By Assumption 2, for every $k \in \{1, \dots, r\}$, $\Delta\lambda_k = \dfrac{\Delta\lambda_{n+1}}{\bar{\lambda}_{n+1}}\bar{\lambda}_k$. From 5.5, we have:

$$0 = \sum_{k=1}^{r}\Delta\lambda_k = \sum_{k=1}^{r}\frac{\Delta\lambda_{n+1}}{\bar{\lambda}_{n+1}}\bar{\lambda}_k = \frac{\Delta\lambda_{n+1}}{\bar{\lambda}_{n+1}}.$$

So $\Delta\lambda_{n+1} = 0$ and finally, $\Delta\lambda_k = 0$ for $k = 1, \dots, r$.$\square$

Proof of Proposition 3

Let S be the simplex of R^k and C the subset of R^k defined by:

$$C = \{(\Phi(\xi_i, \sum_{j=1}^{k} \alpha_j \xi_j))_{i=1}^{k} \mid \alpha \in S\}.$$

Since Φ is bilinear, its restriction to the space generated by the family $(\xi_i)_{i=1}^k$ is continuous, so C is a nonempty compact polyhedral subset of R^k.

Let us first show that the conclusion of the proposition holds true if $C \cap R^k_{++}$ is nonempty. Indeed, if there exists $c \in C \cap R^k_{++}$, then there exists $\alpha \in S$ such that for all i, $c_i = \Phi(\xi_i, \sum_{j=1}^k \alpha_j \xi_j) > 0$. Since Φ is continuous on the space generated by the family $(\xi_i)_{i=1}^k$, there exists $t > 0$ small enough so that for all i, $\Phi(\xi_i, \sum_{j=1}^k (\alpha_j + t)\xi_j) > 0$, hence $(\alpha_j + t)_{j=1}^k \in R^k_{++}$ and the conclusion of the proposition holds true.

We now prove by contraposition that $C \cap R^k_{++}$ is nonempty. If it is not true, we apply a separation theorem between C and R^k_{++}, so there exists an element $\lambda \in R^k \setminus \{0\}$ such that for $(c, d) \in C \times R^k_{++}$, $\lambda \cdot c \leq \lambda \cdot d$. Using usual arguments, one deduces that $\lambda \in R^k_+$ and $\lambda \cdot c \leq 0$ for all $c \in C$. Let $\bar{\alpha} = (1/\sum_{j=1}^k \lambda_j)\lambda$. Then $\bar{\alpha} \in S$. Let $\bar{c} = (\Phi(\xi_i, \sum_{j=1}^k \bar{\alpha}_j \xi_j))_{i=1}^k$, then $\bar{c} \in C$. We remark that:

$$\begin{aligned} \lambda \cdot \bar{c} &= \sum_{i=1}^k \lambda_i \Phi(\xi_i, \sum_{j=1}^k \bar{\alpha}_j \xi_j) = \Phi(\sum_{i=1}^k \lambda_i \xi_i, \sum_{j=1}^k \bar{\alpha}_j \xi_j) \\ &= (1/\sum_{j=1}^k \lambda_j)\Phi(\sum_{i=1}^k \lambda_i \xi_i, \sum_{j=1}^k \lambda_j \xi_j). \end{aligned}$$

So $\Phi(\sum_{i=1}^k \lambda_i \xi_i, \sum_{j=1}^k \lambda_j \xi_j) \leq 0$ and since Φ is positive definite, this implies that $\sum_{j=1}^k \lambda_j \xi_j = 0$. Hence, since the family $(\xi_i)_{i=1}^k$ is linearly independent, one concludes that $\lambda_j = 0$ for $j = 1, \ldots, k$, which contradicts $\lambda \neq 0$.□

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