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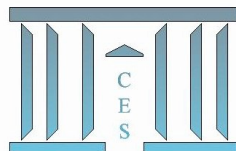
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**On the decomposition of Generalized Additive
Independence models**

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On the decomposition of Generalized Additive Independence models

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Abstract

The GAI (Generalized Additive Independence) model proposed by Fishburn is a generalization of the additive utility model, which need not satisfy mutual preferential independence. Its great generality makes however its application and study difficult. We consider a significant subclass of GAI models, namely the discrete 2-additive GAI models, and provide for this class a decomposition into nonnegative monotone terms. This decomposition allows a reduction from exponential to quadratic complexity in any optimization problem involving discrete 2-additive models, making them usable in practice.

Subject classification: Utility: multiattribute. Decision Analysis: multiple criteria. Games/group decisions: cooperative.

Area of review: Decision Analysis

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1 Introduction

The theory of multiattribute utility (MAUT) provides an adequate and widely studied framework for the representation of preferences in decision making with multiple objectives or criteria (let us mention here only the classic works of Keeney and Raiffa (1976), and Krantz et al. (1971) on conjoint measurement, among numerous other ones). The most representative models in MAUT are the additive utility model $U(x) = \sum_i u_i(x_i)$, and the multilinear model (see Dyer and Sarin (1979)), whose characteristic property is the (mutual) preferential independence, stipulating that the preference among two alternatives should not depend on the attributes where the two alternatives agree (see Abbas and Sun (2015) for a detailed study on MAUT models satisfying preferential independence).

However, it is well known that in real situations, preferential independence could be easily violated, because of the possible interaction between objective/criteria. Referring to the example of evaluation of students in Grabisch (1996) where students are evaluated on three subjects like mathematics, physics and language skills, the preference between two students may be inverted depending on their level in mathematics, assuming that the evaluation policy pays attention to scientific subjects. For instance, the following preference reversal is not unlikely (marks are given on a 0-100 scale, in the following order: mathematics, physics and language skills): $(40, 90, 60) \succ (40, 60, 90)$ and $(80, 90, 60) \prec (80, 60, 90)$, because if a student is weak in one of the scientific subject (e.g., 40 in mathematics), more attention is paid to the other scientific subject (here, physics), otherwise more attention is paid to language skills.

To escape preferential independence, Krantz et al. (1971) have proposed the so-called *decomposable model*, of the form $U(x) = F(u_1(x_1), \dots, u_n(x_n))$, where F is strictly monotone. This model, which is a generalization of the additive utility model, is characterized by a much weaker property than preferential independence, namely *weak independence* or *weak separability* (Wakker (1989)). This property amounts to requiring preferential independence only for one attribute versus the others, and is generally satisfied in practice. Taking F as the Choquet integral w.r.t. a capacity (Choquet (1953)) permits to have a versatile model, which has been well studied and applied in practice (see a survey in Grabisch and Labreuche (2010)). The drawback of these models is that in general they require commensurate utility functions, i.e., one should be able to compare $u_i(x_i)$ with $u_j(x_j)$ for every distinct i, j .

Another generalization of the additive utility model escaping preferential independence has been proposed by Fishburn (1967), under the name of *generalized additive independence (GAI)* model. It has the general form $U(x) = \sum_{S \in \mathcal{S}} u_S(x_S)$, where \mathcal{S} is any collection of subsets of attributes, and x_S is the vector of components of x belonging to S . This model is very general (it even need not satisfy weak independence, see below for an example) and does not need commensurate attributes.

Its great generality is also the Achille's heel of this model, making it difficult to use in practice, and so far it has not been so much considered in the MAUT community. Some developments, essentially focused on the identification of the parameters of the model, have been done in the field of artificial intelligence (see, e.g., Bacchus and Grove (1995); Boutilier et al. (2001); Bigot et al. (2012)).

The aim of this paper is to provide a first step in making GAI models usable in

practice, by proving a fundamental result on decomposition, in a subclass of GAI models which is significant for applications. Specifically, we are interested in GAI models where, first, the collection \mathcal{S} is made only of singletons and pairs, thus limiting the model to a sum of univariate or bivariate terms, and second, the attributes take discrete values. We call this particular class *2-additive discrete GAI* models.

Even restricting to this particular class, the practical identification of the model appears to be rapidly computationally intractable as the number of attributes and the cardinality of the attributes grow. Indeed, U being monotone nondecreasing, the number of monotonicity constraints on the parameters of the model grows exponentially fast in the number of attributes.

We show that by an adequate decomposition of the model, the complexity is reduced to be quadratic in the number of attributes. Indeed, for a given 2-additive GAI model, there exist several decompositions, i.e., several ways to express it as a sum of univariate and bivariate terms. We show that it is always possible to obtain a decomposition into nonnegative monotone nondecreasing terms, which considerably reduces the number of monotonicity constraints. The result is proved by using an equivalence between 2-additive discrete GAI models and 2-additive k -ary capacities, and amounts to finding the set of extreme points of the polytope of 2-additive k -ary capacities.

The paper is organized as follows. Section 2 introduces the necessary concepts and notation in multiattribute utility, capacities, k -ary capacities, and GAI models. Section 3 introduces p -additive GAI models, and shows the equivalence with p -additive k -ary capacities. Section 4 explains the complexity problem behind the identification of 2-additive discrete GAI models, and proves that a decomposition into nonnegative monotone nondecreasing terms is always possible, which constitutes the main result of the paper.

2 Background

2.1 Multi-Attribute Utility Theory

We consider n attributes X_1, \dots, X_n , letting $N = \{1, \dots, n\}$ be its index set. Alternatives are represented by a vector $x = (x_1, \dots, x_n)$ in $X = X_1 \times \dots \times X_n$. We denote by $(x_A, y_{-A}) \in X$ the compound alternative taking value x_i if $i \in A$ and value y_i otherwise.

One of the leading model in decision theory is Multi-Attribute Utility Theory (Keeney and Raiffa, 1976). The overall utility $U : X \rightarrow \mathbb{R}$ representing the preference relation \succsim of a decision maker (i.e. $x \succsim y$ iff $U(x) \geq U(y)$) is then supposed to satisfy *preferential independence*, whereby the comparison between two alternatives does not depend on the attributes having the same value. Accordingly, U can take the form of either an additive model $U(x) = \sum_{i \in N} k_i u_i(x_i)$, or a multilinear form $1 - k U(x) = \prod_{i \in N} (1 - k_i u_i(x_i))$, where u_i is a marginal utility function over attribute X_i .

As we explained in the introduction, preferential independence is a quite strong condition which is not always met in practice. A weaker condition is *weak independence* where for all $i \in N$, all $x_i, y_i \in X_i$ and all $z_{-i}, t_{-i} \in X_{-i}$

$$(x_i, z_{-i}) \succsim (y_i, z_{-i}) \iff (x_i, t_{-i}) \succsim (y_i, t_{-i})$$

(x_i is at least as good as y_i ceteris paribus). Under this condition, we can define a

preference relation \succsim_i on a single attribute X_i as follows: for all $x_i, y_i \in X_i$

$$x_i \succsim_i y_i \quad \text{iff} \quad (x_i, z_{-i}) \succsim (y_i, z_{-i}),$$

for some $z_{-i} \in X_{-i}$.

2.2 Generalized Additive Independence (GAI) model

The additive utility model $\sum_{i \in N} u_i(x_i)$ can be easily generalized by considering marginal utility functions over subsets of attributes, with potential overlap between the subsets (Fishburn, 1967; Bacchus and Grove, 1995):

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_S) \quad (x \in X), \quad (1)$$

where $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$. This model is called the *Generalized Additive Independence* (GAI) model. It is characterized by a condition stating that if two probability distributions P and Q over the alternatives X have the same marginals over every $S \in \mathcal{S}$, then the expected utility of P and Q are equal.

Unlike the additive utility model or the multilinear model, the GAI model does not necessarily satisfy weak independence. In the Artificial Intelligence community, researchers are interested in the representation of preferences that may violate weak independence. A well-known example of such a preference is the following: consider two attributes X_1, X_2 where X_1 pertains on the type of wine and X_2 to the type of main course in a restaurant. Then usually, one prefers ‘red wine’ to ‘white wine’ if the main course is ‘meat’, but ‘white wine’ is preferred to ‘red wine’ if the main course is ‘fish’ (the preference over attribute ‘wine’ is conditional on the value on attribute ‘main course’) (Boutillier et al., 2001).

In this work, we follow a more traditional view of Decision Theory and assume that weak independence holds, which is the case in most of the decision problems.

We make the following two assumptions:

- **Assumption 1:** Monotonicity:

$$\forall i \in N, x_i \succsim_i y_i \Rightarrow U(x) \geq U(y)$$

- **Assumption 2:** Boundaries: each X_i is bounded, in the sense that there exist $x_i^\top, x_i^\perp \in X_i$ which are the best and worst elements of X_i according to \succsim_i , and

$$U(x_i^\top, \dots, x_n^\top) = 1, \quad U(x_i^\perp, \dots, x_n^\perp) = 0.$$

2.3 Capacities and k-ary capacities

We consider a finite set $N = \{1, \dots, n\}$ (e.g., the index set of attributes as in Section 2.1). A *game* on N is a set function $v : 2^N \rightarrow \mathbb{R}$ vanishing on the empty set. A game v is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T$. Note that monotone games take nonnegative values, and if in addition $v(N) = 1$, the game is said to be *normalized*. In the sequel,

we will mainly deal with monotone normalized games, which are usually called *capacities* (Choquet, 1953)¹.

Making the identification of sets with their characteristic functions, i.e., $S \leftrightarrow 1_S$ for any $S \in 2^N$, with $1_S : N \rightarrow \{0, 1\}$, $1_S(i) = 1$ iff $i \in S$, games can be seen as functions on the set of binary functions. A natural generalization is then to consider functions taking values in $\{0, 1, \dots, k\}$, leading to the so-called multichoice or k -choice games (Hsiao and Raghavan, 1990) and k -ary capacities (Grabisch and Labreuche, 2003). Formally, a *k-choice game* is a mapping $v : \{0, 1, \dots, k\}^N \rightarrow \mathbb{R}$ satisfying $v(0, \dots, 0) = 0$. A *k-ary capacity* is a k -choice game being monotone and normalized, i.e., satisfying $v(y) \leq v(z)$ whenever $y \leq z$, and $v(k, \dots, k) = 1$.

Let $v : 2^N \rightarrow \mathbb{R}$ be a game. The *Möbius transform* of v (a.k.a. *Möbius inverse*) is the set function $m^v : 2^N \rightarrow \mathbb{R}$ which is the (unique) solution of the linear system

$$v(S) = \sum_{T \subseteq S} m^v(T) \quad (S \in 2^N)$$

(see Rota (1964)). It is given by

$$m^v(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T) \quad (S \in 2^N). \quad (2)$$

A capacity v is said to be (*at most*) p -*additive* for some $p \in \{1, \dots, n\}$ if its Möbius transform vanishes for subsets of more than p elements: $m^v(S) = 0$ for all $S \subseteq N$ such that $|S| > p$.

Similarly, given a k -ary game v , its Möbius transform is defined as the unique solution of the linear system

$$v(z) = \sum_{y \leq z} m^v(y) \quad (z \in \{0, 1, \dots, k\}^N). \quad (3)$$

It is shown in the appendix that its solution is given by

$$m^v(z) = \sum_{y \leq z : z_i - y_i \leq 1 \forall i \in N} (-1)^{\sum_{i \in N} (z_i - y_i)} v(y) \quad (z \in \{0, 1, \dots, k\}^N). \quad (4)$$

It follows that any k -ary game v can be written as:

$$v = \sum_{x \in L^N} m^v(x) u_x,$$

with u_x a k -ary capacity defined by

$$u_x(z) = \begin{cases} 1, & \text{if } z \geq x \\ 0, & \text{otherwise.} \end{cases}$$

By analogy with classical games, u_x is called the *unanimity game* centered on x . Note that this decomposition is unique as the unanimity games are linearly independent, and form a basis of the vector space of k -ary games.

¹Often capacities are defined as monotone games, not necessarily normalized.

3 Relation between GAI and k -ary capacities

3.1 Discrete GAI models are k -ary capacities

We consider discrete GAI models, i.e., where attributes can take only a finite number of values, and show that they are particular instances of k -ary capacities. We put

$$X_i = \{a_i^0, \dots, a_i^{m_i}\} \quad (i \in N),$$

with $a_i^0 \preceq_i \dots \preceq_i a_i^{m_i}$. Any alternative $x \in X$ is mapped to $\{0, \dots, m_1\} \times \dots \times \{0, \dots, m_n\}$ by the mapping φ which simply keeps track of the rank of the value of the attribute:

$$(a_1^{j_1}, \dots, a_n^{j_n}) \mapsto \varphi(a_1^{j_1}, \dots, a_n^{j_n}) = (j_1, \dots, j_n).$$

We consider now the smallest (discrete) hypercube $\{0, \dots, k\}^N$ containing $\{0, \dots, m_1\} \times \dots \times \{0, \dots, m_n\}$, with $k := \max_i m_i$. Given a GAI model U with discrete attributes as described above, we define the mapping $v : \{0, \dots, k\}^N \rightarrow \mathbb{R}$ by

$$U(x) =: v(\varphi(x)) \quad (x \in X)$$

and let $v(z) := v(m_1, \dots, m_n)$ when $z \in \{0, \dots, k\}^N \setminus \varphi(X)$. In words, v encodes the values of U for every alternative, and fills in the missing values in the hypercube by the maximum of U . By assumption 1 and 2 on U , it follows that v is a normalized k -ary capacity on N .

From now on, we put $L = \{0, 1, \dots, k\}$.

3.2 p -additive GAI models

Consider a GAI model U on X , where the attributes need not be discrete. As U is in general exponentially complex in the number of attributes, one is looking for simple particular cases. The simplest case would be to consider a classical additive model. The characteristic property of an additive model is that the variation of U in one attribute is unrelated to the value of the other fixed ones:

$$U(y_i, x_{-i}) - U(x_i, x_{-i}) = u_{\{i\}}(y_i) - u_{\{i\}}(x_i).$$

Calling the left member the (1st order) variation of U w.r.t. i from x_i to y_i at x , we define inductively the *variation of U w.r.t. $P \subseteq N$ from x_P to y_P at x* by

$$\Delta_{x_P}^{y_P} U(x) = \sum_{T \subseteq P} (-1)^{|P \setminus T|} U(y_T, x_{P \setminus T}, x_{-P})$$

For example, one has, abbreviating $\{i, j\}$ by ij :

$$\begin{aligned} \Delta_{x_i}^{y_i} U(x) &= U(y_i, x_{-i}) - U(x_i, x_{-i}) \\ \Delta_{x_{ij}}^{y_{ij}} U(x) &= U(y_{ij}, x_{-ij}) - U(x_i, y_j, x_{-ij}) - U(y_i, x_j, x_{-ij}) + U(x). \end{aligned}$$

Definition 1. A function U on X is said to be p -additive for some $p \in \{1, \dots, n\}$ if for every $P \subseteq N$ with $|P| \leq p$, for every $x \in X$, $x_P, y_P \in X_P$ and $x'_{-P} \in X_{-P}$,

$$\Delta_{x_P}^{y_P} U(x_P, x_{-P}) = \Delta_{x_P}^{y_P} U(x_P, x'_{-P}).$$

The above definition generalizes the notion of 2-additivity proposed in Labreuche and Grabisch (2013).

The next theorem relates p -additivity to the decomposition of U into terms involving at most p variables, and generalizes (Labreuche and Grabisch, 2013, Prop. 4).

Theorem 1. A function U on X is p -additive for some $p \in \{1, \dots, n\}$ if and only if there exist functions $u_A : X_A \rightarrow \mathbb{R}$, for every $A \subseteq N$ with $|A| \leq p$, such that U takes the form (1) with $\mathcal{S} = \{A \subseteq N, 0 < |A| \leq p\}$.

Proof. We suppose $p \neq n$ to discard the trivial case. The “if” part is easy to check. As for the “only if” part, fix $x \in X$ and define $v(A) = U(x_A, 0_{-A})$ for all $A \subseteq N$. By assumptions 1 and 2, v is a (non-normalized) capacity on N . Define its discrete derivative inductively as follows. For any $\emptyset \neq S \subset N$, $T \in 2^N$ and $i \notin S$,

$$\Delta_{S \cup i} v(T) = \Delta_i(\Delta_S v(T))$$

with $\Delta_i v(T) = v(T \cup i) - v(T)$. Then it is easy to see by (2) that $\Delta_S v(\emptyset) = m^v(S)$, and that for disjoint S and T

$$\Delta_S v(T) = \Delta_{0_S}^{x_S} U(x_T, 0_{-T}).$$

Take S such that $|S| = p$ and any $i \in N \setminus S$. Then for any $T \subseteq N \setminus (S \cup i)$,

$$\Delta_{S \cup i} v(T) = \Delta_i(\Delta_S v(T)) = \Delta_{0_S}^{x_S} U(x_{T \cup i}, 0_{-T \cup i}) - \Delta_{0_S}^{x_S} U(x_T, 0_{-T}) = 0$$

by assumption of p -additivity of U . Letting $T = \emptyset$, it follows that v is p -additive too (in the sense of capacities), hence we can write:

$$U(x) = v(N) = \sum_{S \subseteq N, 0 < |S| \leq p} m^v(S)$$

with $m^v(S) = \Delta_S v(\emptyset) = \Delta_{0_S}^{x_S} U(\mathbf{0})$. Since the latter term only depends on the variables x_S , the desired result follows. \square

3.3 p -additive k -ary capacities

By analogy with classical capacities, a k -ary capacity v is said to be (*at most*) p -additive if $m^v(z) = 0$ whenever $|\text{supp}(z)| > p$, where

$$\text{supp}(z) = \{i \in N \mid z_i > 0\}.$$

Lemma 1. Let $k \in \mathbb{N}$ and $p \in \{1, \dots, n\}$. A k -ary game v is p -additive if and only if it has the form

$$v(z) = \sum_{x \in L^N, 0 < |\text{supp}(x)| \leq p} v_x(x \wedge z) \quad (z \in L^N) \quad (5)$$

where $v_x : L^N \rightarrow \mathbb{R}$ with $v_x(\mathbf{0}) = 0$.

Proof. Suppose that v is p -additive. By the decomposition of v in the basis of unanimity games, it follows that

$$v = \sum_{x \in L^N, 0 < |\text{supp}(x)| \leq p} m^v(x) u_x,$$

hence we have the required form with $v_x = m^v(x)u_x$. Conversely, again by decomposition in the basis of unanimity games and since v_x is a game, (5) can be rewritten as:

$$\begin{aligned} \sum_{y \in L^N} m^v(y)u_y(z) &= \sum_{x \in L^N, 0 < |\text{supp}(x)| \leq p} \sum_{y \in L^N} m^{v_x}(y)u_y(x \wedge z) \\ &= \sum_{y \in L^N, 0 < |\text{supp}(y)| \leq p} \sum_{x \in L^N, 0 < |\text{supp}(x)| \leq p} m^{v_x}(y)u_y(x \wedge z) \\ &= \sum_{y \in L^N, 0 < |\text{supp}(y)| \leq p} \left(\sum_{x \geq y, 0 < |\text{supp}(x)| \leq p} m^{v_x}(y) \right) u_y(z). \end{aligned}$$

By uniqueness of the decomposition, it follows that v is p -additive. \square

Note that even if v is a capacity, the v_x are not necessarily capacities.

It follows from Theorem 1 and the above result that the set of p -additive discrete GAI models on X coincides with the set of (at most) p -additive k -ary capacities.

4 Monotone decomposition of a 2-additive GAI model

4.1 A complexity problem

In general, given a GAI model U on X under the form (1), its decomposition is not unique. For example, with two attributes,

$$U(x_1, x_2) = 2x_1 + x_2 - \max(x_1, x_2) \quad (x \in \mathbb{R}_+^2)$$

is equivalent to

$$U(x_1, x_2) = x_1 + \min(x_1, x_2) \quad (x \in \mathbb{R}_+^2)$$

Observe that in the second expression, all terms are nonnegative and monotone nondecreasing. Then the following question arises: *Given a GAI model, is it always possible to get a decomposition into nonnegative nondecreasing terms?* The main result of this paper will give a positive answer to this question, in the case of 2-additive GAI models. This case is of particular importance in practice, since it constitutes a good compromise between versatility and complexity. Experimental studies in multicriteria evaluation have shown that 2-additive capacities have almost the same approximation ability than general capacities (see, e.g., Grabisch et al. (2002)). A two-additive GAI model is considered in Bigot et al. (2012), and a very similar model is defined in Greco et al. (2014).

Before stating and proving the result, we explain why it is important to solve this problem, which is related to the complexity of the model.

We begin by computing the number of unknowns in a 2-additive GAI model equivalent to a k -ary capacity. By Theorem 1, such a model has the form (1) with \mathcal{S} being the set of singletons and pairs. Since $|L| = k + 1$, this yields

$$(k + 1) \binom{n}{1} + (k + 1)^2 \binom{n}{2} = \frac{n(k + 1)}{2} (2 + (k + 1)(n - 1))$$

unknowns. U being monotone nondecreasing, this induces a number of monotonicity constraints on the unknowns, of the type

$$U(a_1^{j_1}, \dots, a_{i-1}^{j_{i-1}}, a_i^{j_i+1}, a_{i+1}^{j_{i+1}}, \dots, a_n^{j_n}) \geq U(a_1^{j_1}, \dots, a_{i-1}^{j_{i-1}}, a_i^{j_i}, a_{i+1}^{j_{i+1}}, \dots, a_n^{j_n}) \quad (6)$$

for every $i \in N$, $j_1 \in \{0, \dots, m_1\}, \dots, j_{i-1} \in \{0, \dots, m_{i-1}\}, j_i \in \{0, \dots, m_i - 1\}, j_{i+1} \in \{0, \dots, m_{i+1}\}, \dots, j_n \in \{0, \dots, m_n\}$. The number of elementary conditions contained in (6) is equal to

$$\sum_{i \in N} \left(m_i \times \prod_{j \in N \setminus \{i\}} (m_j + 1) \right).$$

In the case where $m_i = k$ for every i , this number becomes

$$n \times k \times (k + 1)^{n-1}.$$

Although the number of variables was still quadratic in n and k , the number of constraints is exponential in n . It follows that any practical identification of a GAI model based on some optimization procedure², where the variables are the unknowns of the GAI model and the constraints are the monotonicity constraints (6) plus possibly some learning data, has to cope with an exponential number of constraints. The following tables, obtained with $k = 4$, shows that the underlying optimization problem becomes rapidly intractable.

n	4	6	8	10
# of variables	170	405	740	1175
# of constraints	2000	75 000	2 500 000	78 125 000

n	12	14	20
# of variables	1710	2345	4850
# of constraints	2 343 750 000	68 359 375 000	$1.526E + 15$

However, if a decomposition into nonnegative nondecreasing terms is possible, one has only to check monotonicity of each term. Then the number of monotonicity conditions drops to

$$\sum_{i \in N} m_i + \sum_{\{i,j\} \subseteq N} (m_i(m_j + 1) + m_j(m_i + 1)).$$

In the case where $m_i = k$ for every i , this number becomes

$$n \times k \times \left[(n - 1)(k + 1) + 1 \right],$$

which is quadratic in n . The following table ($k = 4$) shows that the optimization problem becomes tractable even for a large number of attributes.

n	4	6	8	10	12	14	20
# of constraints with monotone decomposition	256	624	1152	1840	2688	3696	7680

²The learning problem can be classically transformed into a linear program, where the training set is seen as linear constraints on the GAI variables (Bigot et al., 2012; Greco et al., 2014). It could also be possible to perform statistical learning, like in Fallah Tehrani et al. (2012), where the underlying optimization problem is a convex problem under linear constraints.

4.2 The main result

The following theorem states that a decomposition of a 2-additive GAI model into monotone nondecreasing terms is always possible.

Theorem 2. Let us consider a 2-additive discrete GAI model U satisfying assumptions 1 and 2. Then there exist nonnegative and nondecreasing functions $u_i : X_i \rightarrow [0, 1]$, $i \in N$, $u_{ij} : X_i \times X_j \rightarrow [0, 1]$, $\{i, j\} \subseteq N$, such that

$$U(x) = \sum_{i \in N} u_i(x_i) + \sum_{\{i, j\} \subseteq N} u_{ij}(x_i, x_j) \quad (x \in X)$$

The rest of this section is devoted to the proof of this theorem, which goes through a number of intermediary results. First, we remark that the problem is equivalent to the decomposition of a 2-additive k -ary capacity v into a sum of 2-additive k -ary capacities whose support has size at most 2, where the *support* of v is defined by

$$\text{supp}(v) = \bigcup_{x \in L^N : m^v(x) \neq 0} \text{supp}(x).$$

We consider $\mathcal{P}_{k,\cdot}$ the polytope of k -ary capacities, and $\mathcal{P}_{k,2}$ the polytope of 2-additive k -ary capacities. Our aim is to study the vertices of the latter, and we will show that these vertices are the adequate k -ary capacities to perform the decomposition.

A first easy fact is that the extreme points of $\mathcal{P}_{k,\cdot}$ are the 0-1-valued k -ary capacities.

Lemma 2. \hat{v} is an extreme point of $\mathcal{P}_{k,\cdot}$ iff \hat{v} is 0-1-valued.

Proof. Take \hat{v} in $\mathcal{P}_{k,\cdot}$ which is 0-1-valued, and consider $v, v' \in \mathcal{P}_{k,\cdot}$ such that $\frac{v+v'}{2} = \hat{v}$. Then, since \hat{v} is 0-1-valued,

$$v(x) + v'(x) = \begin{cases} 2, & \text{if } \hat{v}(x) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Since v, v' are normalized and monotone, the only possibility to get $v(x) + v'(x) = 2$ is to have $v(x) = v'(x) = 1$, and similarly, $v(x) + v'(x) = 0$ forces $v(x) = v'(x) = 0$. It follows that $v = v' = \hat{v}$, i.e., \hat{v} is an extreme point of $\mathcal{P}_{k,\cdot}$.

Conversely, consider a vertex \hat{v} which is not 0-1-valued, and let

$$\epsilon = \min(1 - \max_{x: \hat{v}(x) < 1} \hat{v}(x), \min_{x: \hat{v}(x) > 0} \hat{v}(x)).$$

Define

$$\begin{aligned} v'(x) &= \hat{v}(x) + \epsilon, \text{ for all } x \text{ s.t. } \hat{v}(x) \neq 0, 1 \\ v''(x) &= \hat{v}(x) - \epsilon, \text{ for all } x \text{ s.t. } \hat{v}(x) \neq 0, 1, \end{aligned}$$

and $v' = v'' = \hat{v}$ otherwise. Then $v', v'' \in \mathcal{P}_{k,\cdot}$ and $\hat{v} = \frac{v'+v''}{2}$, a contradiction. \square

Lemma 3. Let $k \in \mathbb{N}$ and $v \in \mathcal{P}_{k,2}$. Then v is 0-1-valued iff m^v is $\{-1, 0, 1\}$ valued.

Proof. \Leftarrow) By the assumption $\sum_{y \leq x} m^v(y) \in \mathbb{Z}$ for every $x \in \{0, 1, \dots, k\}^N$. Since $v \in \mathcal{P}_{k,2}$ it follows that v is 0-1-valued.

\Rightarrow) Assume v is 0-1-valued and use (4) to compute the Möbius transform. For $z = \ell_i$ with $\ell \in \{1, \dots, k\}$, we have $m^v(z) = v(\ell_i) - v((\ell - 1)_i)$, so that the desired result holds. Otherwise $z = \ell_i \ell'_j$ with $\ell, \ell' \in \{1, \dots, k\}$ and distinct $i, j \in N$. Then

$$m^v(z) = v(z) - v((\ell - 1)_i \ell'_j) - v(\ell_i (\ell' - 1)_j) + v((\ell - 1)_i (\ell' - 1)_j). \quad (7)$$

By the assumption and monotonicity of v , it follows that $m^v(z) \in \{-1, 0, 1\}$. \square

We recall that a $m \times n$ matrix is totally unimodular if the determinant of every square submatrix is equal to $-1, 0$ or 1 . A polyhedron is integer if all its extreme points have integer coordinates. Then a matrix A is totally unimodular iff the polyhedron $\{x \mid Ax \leq b\}$ is integer for every integer vector b . In particular it is known that the vertex-arc matrix M of a directed graph, i.e., whose entries are $M_{x,a} = 1$ if the arc a leaves vertex x , -1 if a enters x , and 0 otherwise, is totally unimodular (in other words, each column of M has exactly one $+1$ and one -1 , the rest being 0).

We are now in position to characterize the extreme points of $\mathcal{P}_{k,2}$.

Theorem 3. Let $k \in \mathbb{N}$. The set of extreme points of $\mathcal{P}_{k,2}$, the polytope of 2-additive k -ary capacities, is the set of 0-1-valued 2-additive k -ary capacities.

Proof. By Lemma 2, we need only to prove that any extreme point of $\mathcal{P}_{k,2}$ is 0-1-valued.

1. We prove that $A_{k,\cdot}$, the matrix defining the polytope of k -ary capacities, is totally unimodular. The argument follows the one given for classical capacities by Miranda et al. (Miranda et al., 2006, Th. 2). We prove that $A_{k,\cdot}^\top$ is totally unimodular, which is equivalent to the desired result. Since the monotonicity constraints are either of the form $v(1_i) \geq 0$ or $v(x) - v(x') \geq 0$ where x' is a lower neighbor of x (i.e. $x' = x - 1_i$ for some i), the matrix $A_{k,\cdot}^\top$ has the form (I, B) , where I is a submatrix of the $(k^n - 1)$ -dim identity matrix $I_{k^n - 1}$, and B is a matrix where each column has exactly one $+1$ and one -1 . Hence B is totally unimodular, and so is $(I_{k^n - 1}, B)$ as it is easy to check. Since $A_{k,\cdot}^\top$ is a submatrix of it, it is also totally unimodular.

2. It follows from Step 1 that the polytope $\mathcal{P}_{k,\cdot}(b)$ given by $A_{k,\cdot} v \leq b$ is integer for every integer vector b . Next, consider the $(k^n - 1) \times (k^n - 1)$ -matrix Z expressing the Zeta transform, i.e., $Zm^v = v$, as given by (3). This matrix has only 0 and 1 as entries, and its inverse Z^{-1} exists and its entries are $0, -1, +1$ only (see (4)). Consider the polytope $\mathcal{P}_{k,\cdot}^m(b)$ given by $A_{k,\cdot}^m m \leq b$ with $A_{k,\cdot}^m = A_{k,\cdot} Z$, the image by the linear transform Z of the polytope $\mathcal{P}_{k,\cdot}(b)$. It is easy to check that \hat{v} is an extreme point of $\mathcal{P}_{k,\cdot}(b)$ iff $Z^{-1}\hat{v}$ is an extreme point of $\mathcal{P}_{k,\cdot}^m(b)$. Evidently, the coordinates of $Z^{-1}\hat{v}$ are integer, therefore $\mathcal{P}_{k,\cdot}^m(b)$ is integer for every integer vector b . We conclude that $A_{k,\cdot}^m$ is totally unimodular.

3. Inasmuch as a submatrix of a totally unimodular matrix is itself totally unimodular, it follows from Step 2 that $A_{k,2}^m$, the matrix defining the set of 2-additive k -ary capacities in Möbius coordinates, is also totally unimodular. As a conclusion, the extreme points of $\mathcal{P}_{k,2}^m$ are integer-valued.

4. We show that the extreme points of $\mathcal{P}_{k,2}^m$ are $\{-1, 0, 1\}$ -valued. Then Lemma 3 permits to conclude. It suffices to show that $|m^v(z)| \geq 2$ cannot happen. If $z = \ell_i$ with $\ell \in \{1, \dots, k\}$, we find by (4) that $m^v(z) = v(\ell_i) - v((\ell - 1)_i)$, so that the claim holds since $v \in \mathcal{P}_{k,2}$. Otherwise, $z = \ell_i \ell'_j$ with $\ell, \ell' \in \{1, \dots, k\}$ and distinct i, j , and $m^v(z)$ is given by (7). Since v is monotone and normalized, the claim easily follows. \square

The last step is to prove that a 0-1-valued 2-additive k -ary capacity has a support of size at most 2.

Theorem 4. Consider a 2-additive k -ary capacity u on N which is 0-1-valued. Then the support of u is restricted to at most two attributes.

Proof. Preliminary Step. u being 2-additive, its expression is

$$u(x) = \sum_{\{i,j\} \subseteq N} u_{i,j}(x_i, x_j) \quad (x \in X). \quad (8)$$

If we set $u'_{i,j}(x_i, x_j) = u_{i,j}(x_i, x_j) - u_{i,j}(0, 0)$, we obtain $u(x) = \sum_{\{i,j\} \subseteq N} u'_{i,j}(x_i, x_j) + C$, where $C = -\sum_{\{i,j\} \subseteq N} u_{i,j}(0, 0)$. By assumption 2 and $u'_{i,j}(0, 0) = 0$, one gets $C = 0$. This proves that in decomposition (8), one can always assume that

$$\forall \{i, j\} \subseteq N \quad u_{i,j}(0, 0) = 0. \quad (9)$$

We wish to prove that u depends only on one term $u_{i,j}$ in (8). In order to avoid cases where such a term $u_{i,j}$ depends only on one variable (in which case u might also depend on another term $u_{k,l}$), we are interested in terms $u_{i,j}$ depending on its two variables x_i and x_j . We say that $u_{i,j}$ *depends on its two variables* if

$$\exists y_i \in X_i \exists y_j \in X_j \quad u_{i,j}(y_i, y_j) \neq u_{i,j}(y_i, 0) \quad (10)$$

$$\exists y'_i \in X_i \exists y'_j \in X_j \quad u_{i,j}(y'_i, y'_j) \neq u_{i,j}(0, y'_j) \quad (11)$$

Clearly, if (10) (resp. (11)) is not fulfilled, then $u_{i,j}$ does not depend on attribute x_j (resp. x_i).

The proof is organized as follows. We show in Step 1 that if there is no term $u_{i,j}$ that depends on its two variables, then u depends only on one variable. We then assume that at least one term $u_{i,j}$ depends on its two variables – denoted $u_{1,2}$ w.l.o.g. Step 2 shows that it is not possible to have a non-zero term $u_{i,j}$, with $\{i, j\} \subseteq N \setminus \{1, 2\}$. Step 3 proves that it is not possible to have a non-zero term $u_{i,j}$, with $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$. We conclude that $u_{1,2}$ is the only non-zero term in the decomposition. This proves that u depends only on two variables.

Step 1: case of the additive utility model. We first start with the case where there is no term $u_{i,j}$ that depends on its two variables.

Lemma 4. Assume that there is no term $u_{i,j}$ that depends on its two variables. Then the support of u is restricted to one attribute.

Proof. If there is no term $u_{i,j}$ that depends on its two variables, u takes the form of an additive utility:

$$u(x) = \sum_{i \in N} u_i(x_i)$$

where $u_i : X_i \rightarrow \mathbb{R}$ is not necessarily non-negative or monotone. By (9), we have $u_i(0) = 0$ for every $i \in N$.

Let $i \in N$, we write $u(x_i, 0_{-i}) = u_i(x_i)$. Hence u_i is 0-1-valued and monotone.

As u is not constant by Assumption 2, at least one term u_i is not constant. W.l.o.g. let us assume it is u_1 . Then there exists $x_1 \in X_1$ such that $u_1(x_1) = 1$.

Now for every $i \in N \setminus \{1\}$ and $x_i \in X_i$, $u(x_1, x_i, 0_{-1,i}) = 1 + u_i(x_i)$. As u_i is non-negative and u is 0-1-valued, we conclude that $u_i(x_i) = 0$. Hence u depends only on x_1 . \square

Step 2: Case where u has two non-zero terms with non-overlapping support, e.g., $u_{1,2}$ and $u_{3,4}$. We now focus on the situation where at least one term $u_{i,j}$ depends on its two variables. W.l.o.g., we assume it is $u_{1,2}$.

We consider the general case where there are at least 4 attributes. The restriction with only 3 attributes will be handled in Step 3. For every $j \in N \setminus \{1, 2\}$, we choose $k(j) \in N \setminus \{1, 2, j\}$ (where $k(j) \neq k(j')$ for $j \neq j'$). For every $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$, we set

$$\begin{aligned} u'_{i,j}(x_i, x_j) &= u_{i,j}(x_i, x_j) - u_{i,j}(x_i, 0) - u_{i,j}(0, x_j) \\ u'_{1,2}(x_1, x_2) &= u_{1,2}(x_1, x_2) + \sum_{j \in N \setminus \{1, 2\}} (u_{1,j}(x_1, 0) + u_{2,j}(x_2, 0)) \\ u'_{j,k(j)}(x_j, x_{k(j)}) &= u_{j,k(j)}(x_j, x_{k(j)}) + u_{1,j}(0, x_j) + u_{2,j}(0, x_j) \end{aligned}$$

Then $u(x) = \sum_{\{i,j\} \subseteq N} u'_{i,j}(x_i, x_j)$. Moreover $u'_{i,j}(x_i, 0) = 0$ and $u'_{i,j}(0, x_j) = 0$ for $i \in \{1, 2\}$, $j \in N \setminus \{1, 2\}$, $x_i \in X_i$ and $x_j \in X_j$. Hence in decomposition (8), we can assume that

$$\forall i \in \{1, 2\} \forall j \in N \setminus \{1, 2\} \forall x_i \in X_i \forall x_j \in X_j \quad u_{i,j}(x_i, 0) = 0 \text{ and } u_{i,j}(0, x_j) = 0. \quad (12)$$

Thanks to (9) and (12), we have

$$u(x_1, x_2, 0_{-1,2}) = u_{1,2}(x_1, x_2) \quad (13)$$

Hence

$$u_{1,2} \text{ is 0-1-valued and monotone.} \quad (14)$$

By (14), conditions (10) and (11) with $i = 1$, $j = 2$ give

$$\begin{aligned} u_{1,2}(y_1, y_2) &= 1, \quad u_{1,2}(y_1, 0) = 0 \\ u_{1,2}(y'_1, y'_2) &= 1, \quad u_{1,2}(0, y'_2) = 0 \end{aligned} \quad (15)$$

Assume by contradiction that there exists a non-zero $u_{i,j}$ for some $\{i, j\} \subseteq N \setminus \{1, 2\}$. W.l.o.g., we assume it is $u_{3,4}$. Then there exists $z_3 \in X_3$ and $z_4 \in X_4$ such that $u_{3,4}(z_3, z_4) \neq 0$. As for (12), we can transfer, for $i \in \{3, 4\}$ and $j \in N \setminus \{1, 2, 3, 4\}$, the term $u_{i,j}(x_i, 0)$ in $u_{3,4}$. Hence we can assume that

$$\forall i \in \{3, 4\} \forall j \in N \setminus \{1, 2, 3, 4\} \forall x_i \in X_i \quad u_{i,j}(x_i, 0) = 0. \quad (16)$$

Thanks to (9), (12) and (16), we have

$$u(x_3, x_4, 0_{-3,4}) = u_{3,4}(x_3, x_4) \quad (17)$$

Hence

$$u_{3,4} \text{ is 0-1-valued, monotone, and } u_{3,4}(z_3, z_4) = 1. \quad (18)$$

Lemma 5. If $u_{1,2}$ depends on its two variables, then $u_{3,4}$ is identically zero.

Proof. We set $v(x_1, x_2, x_3, x_4) = u(x_1, x_2, x_3, x_4, 0_{-1,2,3,4})$. We write

$$v(x_1, x_2, x_3, x_4) = \sum_{1 \leq i < j \leq 4} u_{i,j}(x_i, x_j).$$

Analysis with y and z :

- $v(y_1, y_2, z_3, z_4) = \underbrace{u_{1,2}(y_1, y_2)}_{=1} + \underbrace{u_{3,4}(z_3, z_4)}_{=1} + \sum_{i \in \{1,2\}, j \in \{3,4\}} u_{i,j}(y_i, z_j)$. We have $v(y_1, y_2, z_3, z_4) = 1$ as $v(y_1, y_2, z_3, z_4) \geq v(y_1, y_2, 0, 0) = u_{1,2}(y_1, y_2) = 1$. Hence

$$\sum_{i \in \{1,2\}, j \in \{3,4\}} u_{i,j}(y_i, z_j) = -1. \quad (19)$$

- $\underbrace{v(y_1, y_2, z_3, 0)}_{=1 \text{ by monotonicity}} = 1 + u_{3,4}(z_3, 0) + u_{1,3}(y_1, z_3) + u_{2,3}(y_2, z_3)$. Hence

$$u_{3,4}(z_3, 0) + u_{1,3}(y_1, z_3) + u_{2,3}(y_2, z_3) = 0. \quad (20)$$

- $\underbrace{v(y_1, y_2, 0, z_4)}_{=1 \text{ by monotonicity}} = 1 + u_{3,4}(0, z_4) + u_{1,4}(y_1, z_4) + u_{2,4}(y_2, z_4)$. Hence

$$u_{3,4}(0, z_4) + u_{1,4}(y_1, z_4) + u_{2,4}(y_2, z_4) = 0. \quad (21)$$

- $\underbrace{v(y_1, 0, z_3, z_4)}_{=1 \text{ by monotonicity by (15)}} = u_{1,2}(y_1, 0) + 1 + u_{1,3}(y_1, z_3) + u_{1,4}(y_1, z_4)$. Moreover, $u_{1,2}(y_1, 0) = 0$. Hence

$$u_{1,3}(y_1, z_3) + u_{1,4}(y_1, z_4) = 0. \quad (22)$$

- $\underbrace{v(0, y_2, z_3, z_4)}_{=1 \text{ by monotonicity}} = u_{1,2}(0, y_2) + 1 + u_{2,3}(y_2, z_3) + u_{2,4}(y_2, z_4)$. Hence

$$u_{1,2}(0, y_2) + u_{2,3}(y_2, z_3) + u_{2,4}(y_2, z_4) = 0. \quad (23)$$

- From (22), (23) and (19),

$$u_{1,2}(0, y_2) = 1. \quad (24)$$

- $v(0, y_2, z_3, 0) = \underbrace{u_{1,2}(0, y_2)}_{=1 \text{ by (24)}} + u_{3,4}(z_3, 0) + u_{2,3}(y_2, z_3)$. Moreover, $v(0, y_2, z_3, 0) \geq v(0, y_2, 0, 0) = u_{1,2}(0, y_2) = 1$. Hence

$$u_{3,4}(z_3, 0) + u_{2,3}(y_2, z_3) = 0 \text{ and } u_{2,3}(y_2, z_3) \in \{-1, 0\}. \quad (25)$$

- $v(0, y_2, 0, z_4) = 1 + u_{3,4}(0, z_4) + u_{2,4}(y_2, z_4)$. Moreover, $v(0, y_2, 0, z_4) \geq v(0, y_2, 0, 0) = u_{1,2}(0, y_2) = 1$. Hence

$$u_{3,4}(0, z_4) + u_{2,4}(y_2, z_4) = 0 \text{ and } u_{2,4}(y_2, z_4) \in \{-1, 0\}. \quad (26)$$

From (19) and (22), we get $u_{2,3}(y_2, z_3) + u_{2,4}(y_2, z_4) = -1$. As $u_{2,3}(y_2, z_3), u_{2,4}(y_2, z_4) \in \{-1, 0\}$ (by (25) and (26)), we have two cases:

- Case 1: $u_{2,3}(y_2, z_3) = -1$ and $u_{2,4}(y_2, z_4) = 0$. Then

$$\begin{aligned} u_{3,4}(z_3, 0) &= 1 && \text{by (25)} \\ u_{1,3}(y_1, z_3) &= 0 && \text{by (20)} \\ u_{1,4}(y_1, z_4) &= 0 && \text{by (22)} \\ u_{1,2}(y_1, 0) &= 0 && \text{by (15)} \\ u_{1,2}(0, y_2) &= 1 && \text{by (24)} \\ u_{3,4}(0, z_4) &= 0 && \text{by (26)} \end{aligned}$$

All values are determined.

- Case 2: $u_{2,3}(y_2, z_3) = 0$ and $u_{2,4}(y_2, z_4) = -1$. Then

$$\begin{aligned} u_{3,4}(0, z_4) &= 1 && \text{by (26)} \\ u_{3,4}(z_3, 0) &= 0 && \text{by (25)} \\ u_{1,3}(y_1, z_3) &= 0 && \text{by (20)} \\ u_{1,4}(y_1, z_4) &= 0 && \text{by (21)} \\ u_{1,2}(y_1, 0) &= 0 && \text{by (15)} \\ u_{1,2}(0, y_2) &= 1 && \text{by (24)} \end{aligned}$$

All values are determined.

Analysis with y' and z : The analyses with y and z , and with y' and z are similar. By (15), we just need to invert the two attributes 1 and 2. Hence a similar reasoning to the previous analysis can be done. We obtain thus the two cases 1' and 2' which are deduced from cases 1 and 2 just by switching attributes 1 and 2:

- Case 1':

$$\begin{aligned} u_{1,3}(y'_1, z_3) &= -1 \\ u_{1,4}(y'_1, z_4) &= 0 \\ u_{3,4}(z_3, 0) &= 1 \\ u_{2,3}(y'_2, z_3) &= 0 \\ u_{2,4}(y'_2, z_4) &= 0 \\ u_{1,2}(y'_1, 0) &= 1 \\ u_{1,2}(0, y'_2) &= 0 \\ u_{3,4}(0, z_4) &= 0 \end{aligned}$$

- Case 2':

$$\begin{aligned}
u_{1,3}(y'_1, z_3) &= 0 \\
u_{1,4}(y'_1, z_4) &= -1 \\
u_{3,4}(0, z_4) &= 1 \\
u_{3,4}(z_3, 0) &= 0 \\
u_{2,3}(y'_2, z_3) &= 0 \\
u_{2,4}(y'_2, z_4) &= 0 \\
u_{1,2}(y'_1, 0) &= 1 \\
u_{1,2}(0, y'_2) &= 0
\end{aligned}$$

Synthesis: Cases 1 and 2' are incompatible, and so are cases 2 and 1'. We have thus the alternative:

- Case 1 and 1'. Gathering the values of partial utilities, we get

$$\begin{aligned}
u_{1,2}(0, y_2) &= 1 & u_{1,4}(y'_1, z_4) &= 0 & u_{1,3}(y'_1, z_3) &= -1 \\
u_{2,3}(y_2, z_3) &= -1 & u_{2,4}(y_2, z_4) &= 0
\end{aligned}$$

As $u_{1,2}(y'_1, y_2) \geq u_{1,2}(0, y_2) = 1$, we have $u_{1,2}(y'_1, y_2) = 1$. Hence

$$\begin{aligned}
u(y'_1, y_2, z_3, z_4) &= \underbrace{u_{1,2}(y'_1, y_2)}_{=1} + \underbrace{u_{3,4}(z_3, z_4)}_{=1} + \underbrace{u_{1,3}(y'_1, z_3)}_{=-1} \\
&+ \underbrace{u_{1,4}(y'_1, z_4)}_{=0} + \underbrace{u_{2,3}(y_2, z_3)}_{=-1} + \underbrace{u_{2,4}(y_2, z_4)}_{=0} \\
&= 0
\end{aligned}$$

We obtain a contradiction as $u(y'_1, y_2, z_3, z_4) \geq u(0, 0, z_3, z_4) = 1$.

- Case 2 and 2'. Gathering the values of partial utilities, we get

$$\begin{aligned}
u_{1,2}(0, y_2) &= 1 & u_{1,4}(y'_1, z_4) &= -1 & u_{1,3}(y'_1, z_3) &= 0 \\
u_{2,3}(y_2, z_3) &= 0 & u_{2,4}(y_2, z_4) &= -1
\end{aligned}$$

As $u_{1,2}(y'_1, y_2) \geq u_{1,2}(0, y_2) = 1$, we have $u_{1,2}(y'_1, y_2) = 1$. Hence

$$\begin{aligned}
u(y'_1, y_2, z_3, z_4) &= \underbrace{u_{1,2}(y'_1, y_2)}_{=1} + \underbrace{u_{3,4}(z_3, z_4)}_{=1} + \underbrace{u_{1,3}(y'_1, z_3)}_{=0} \\
&+ \underbrace{u_{1,4}(y'_1, z_4)}_{=-1} + \underbrace{u_{2,3}(y_2, z_3)}_{=0} + \underbrace{u_{2,4}(y_2, z_4)}_{=-1} \\
&= 0
\end{aligned}$$

We obtain a contradiction as $u(y'_1, y_2, z_3, z_4) \geq u(0, 0, z_3, z_4) = 1$.

A contradiction is raised in all situations. Hence it is not possible to have $u_{3,4}$ non-zero, knowing that $u_{1,2}$ depends on its two variables. \square

Step 3: Case where u has two non-zero terms with overlapping support, e.g., $u_{1,2}$ and $u_{1,3}$. In the last case, term $u_{1,2}$ depends on its two variables, and there is no non-zero term $u_{i,j}$, with $i, j \neq 1, 2$, that depends on its two variables.

We proceed as in the beginning of Step 2, assuming that

$$\forall i \in \{1, 2\} \forall j \in N \setminus \{1, 2\} \forall x_i \in X_i \quad u_{i,j}(x_i, 0) = 0. \quad (27)$$

Then relations (13) through (15) also hold in this case.

Assume by contradiction that there exists a non-zero $u_{i,j}$ for some $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$. Wlog, we assume it is $u_{1,3}$. There exists thus $z_1 \in X_1$ and $z_3 \in X_3$ such that

$$u_{1,3}(z_1, z_3) \neq 0. \quad (28)$$

One can transfer term $u_{i,3}(0, x_3)$, for $i \neq 1, 3$, to $u_{1,3}$ (proceeding as in the beginning of Step 2). Hence we can assume that

$$\forall i \in N \setminus \{1, 3\} \forall x_3 \in X_3 \quad u_{i,3}(0, x_3) = 0. \quad (29)$$

Lemma 6. If $u_{1,2}$ depends on its two variables, then $u_{1,3}$ is identically zero.

Proof. We set $v(x_1, x_2, x_3) = u(x_1, x_2, x_3, 0_{-1,2,3})$. Then

$$v(x_1, x_2, x_3) = u_{1,2}(x_1, x_2) + u_{1,3}(x_1, x_3) + u_{2,3}(x_2, x_3).$$

Analysis with y and z : We write thanks to (13) and to the monotonicity of v

$$\begin{aligned} v(z_1, 0, z_3) &= u_{1,2}(z_1, 0) + u_{1,3}(z_1, z_3) \\ &\geq v(z_1, 0, 0) = u_{1,2}(z_1, 0) \end{aligned}$$

Hence $u_{1,3}(z_1, z_3) \geq 0$, which gives by (28)

$$u_{1,3}(z_1, z_3) = 1 \quad (30)$$

$$u_{1,2}(z_1, 0) = 0 \quad (31)$$

We have the following basic relations:

$$v(y_1, 0, z_3) = \underbrace{u_{1,2}(y_1, 0)}_{=0} + u_{1,3}(y_1, z_3) \quad (32)$$

$$v(y_1, y_2, z_3) = 1 + u_{1,3}(y_1, z_3) + u_{2,3}(y_2, z_3) \quad (33)$$

$$v(z_1, y_2, z_3) = u_{1,2}(z_1, y_2) + u_{1,3}(z_1, z_3) + u_{2,3}(y_2, z_3) \quad (34)$$

Analysis with compound alternatives: We distinguish between two cases:

- Assume first that $z_1 \geq y_1$. By (14) and (15), we have

$$u_{1,2}(z_1, y_2) = 1. \quad (35)$$

By monotonicity, $v(z_1, y_2, z_3) = 1$ (as $v(z_1, 0, z_3) = u_{1,2}(z_1, 0) + 1$ and thus $v(z_1, 0, z_3) = 1$). Hence (30) and (34) give

$$u_{2,3}(y_2, z_3) = -1. \quad (36)$$

By monotonicity, $v(y_1, y_2, z_3) = 1$ (as $v(y_1, y_2, 0) = u_{1,2}(y_1, y_2) = 1$). From (33) and previous relation, we have

$$u_{1,3}(y_1, z_3) = 1. \quad (37)$$

- Assume then that $z_1 < y_1$. We have $v(y_1, 0, z_3) = 1$ by monotonicity of v (as $v(z_1, 0, z_3) = 1$). Then (32) proves that (37) holds. This implies that (36) also holds, thanks to (33).

By monotonicity, $v(z_1, y_2, z_3) = 1$ (as $v(z_1, 0, z_3) = 1$). Hence (34) and (36) show that (35) is satisfied.

In the two cases, we have proved that relations (35), (36) and (37) are true.

We make the following reasoning.

- We write

$$\begin{aligned} v(0, y_2, z_3) &= u_{1,2}(0, y_2) + u_{1,3}(0, z_3) - 1 \\ &\geq v(0, y_2, 0) = u_{1,2}(0, y_2) \end{aligned}$$

Therefore $u_{1,3}(0, z_3) \geq 1$. We also see that $u_{1,3}(0, z_3) \in \{0, 1\}$ as $v(0, 0, z_3) = u_{1,3}(0, z_3)$. Hence

$$u_{1,3}(0, z_3) = 1 \tag{38}$$

$$v(0, 0, z_3) = 1 \tag{39}$$

- We write

$$\begin{aligned} v(0, y_2, z_3) &= u_{1,2}(0, y_2) + u_{1,3}(0, z_3) + u_{2,3}(y_2, z_3) = u_{1,2}(0, y_2) \\ &\geq v(0, 0, z_3) = 1 \end{aligned}$$

Hence

$$u_{1,2}(0, y_2) = 1. \tag{40}$$

- We have

$$\underbrace{v(0, y'_2, z_3)}_{=1 \text{ by monotonicity and (39)}} = \underbrace{u_{1,3}(0, z_3)}_{=1} + u_{2,3}(y'_2, z_3)$$

Hence

$$u_{2,3}(y'_2, z_3) = 0. \tag{41}$$

- We have

$$\underbrace{v(y'_1, y'_2, z_3)}_{=1 \text{ by monotonicity}} = 1 + u_{1,3}(y'_1, z_3) + \underbrace{u_{2,3}(y'_2, z_3)}_{=0 \text{ by (41)}}$$

Hence

$$u_{1,3}(y'_1, z_3) = 0. \tag{42}$$

- Finally

$$\begin{aligned} v(y'_1, y_2, z_3) &= \underbrace{u_{1,2}(y'_1, y_2)}_{=1 \text{ by (14) and (40)}} + \underbrace{u_{1,3}(y'_1, z_3)}_{=0 \text{ by (42)}} + \underbrace{u_{2,3}(y_2, z_3)}_{=-1 \text{ by (36)}} = 0 \end{aligned}$$

We obtain a contradiction as $v(y'_1, y_2, z_3) = 1$ (thanks to monotonicity of v , and to (39)).

A contradiction is raised in all situations. Hence it is not possible to have $u_{1,3}$ non-zero, knowing that $u_{1,2}$ depends on its two variables. \square

Finally, we have proved that if $u_{1,2}$ depends on its two variables, no other term can be non-zero. This proves that u depends only on two variables. \square

In summary, we have proved that the extreme points of $\mathcal{P}_{k,2}$ are the 2-additive 0-1-valued k -ary capacities, and that these capacities have a support of size at most 2. It follows that any $v \in \mathcal{P}_{k,2}$ can be written as a convex combination of 2-additive k -ary capacities with support of size at most 2, which proves Theorem 2.

4.3 Expression of the extreme points of the polytope of 2-additive k -ary capacities

We are now in position to determine all vertices of $\mathcal{P}_{k,2}$, for a fixed $k \in \mathbb{N}$. By Theorem 4, we know that any vertex has a support of at most two elements, hence w.l.o.g. we can restrict to elements 1 and 2. By Theorem 3, finding all vertices with support $\{1, 2\}$ amounts to finding all 0-1 k -ary capacities which are linear combinations of unanimity games u_x with $\text{supp}(x) \subseteq \{1, 2\}$. By analogy with classical simple games, a coalition $x \in L^N$ is *winning* for v if $v(x) = 1$. Minimal winning coalitions are those which are minimal w.r.t. the order \leq on L^N , and therefore they form an antichain in L^N . We show several properties of minimal winning coalitions.

Lemma 7. Let μ be a 0-1-valued k -ary capacity.

- (i) x is a minimal winning coalition if and only if $m^\mu(x) = 1$ and $m^\mu(y) = 0$ for all $y < x$.
- (ii) $\text{supp}(\mu) \subseteq \{1, 2\}$ if and only if its minimal winning coalitions have support included in $\{1, 2\}$.
- (iii) If $|\text{supp}(\mu)| = 2$, there are at most $k + 1$ distinct minimal winning coalitions.
- (iv) Suppose that $\text{supp}(\mu) \subseteq \{1, 2\}$. Denote by x^1, \dots, x^q the minimal winning coalitions of μ , arranged such that $x_1^1 < x_1^2 \cdots < x_1^q$. Then $m^\mu(x^\ell) = 1$ for all $\ell = 1, \dots, q$, $m^\mu(x^\ell \vee x^{\ell+1}) = -1$ for $\ell = 1, \dots, q - 1$, and $m^\mu(x) = 0$ otherwise.

Proof. (i) Suppose $m^\mu(x) = 1$ and $m^\mu(y) = 0$ for all $y < x$. Then clearly x is a minimal winning coalition. Conversely, suppose first that there exists $y < x$ such that $m^\mu(y) \neq 0$, and choose a minimal y with this property. Then $\mu(y) \neq 0$, a contradiction. Then, suppose there is no such $y < x$ but $m^\mu(x) \neq 1$. Then $\mu(x) = m^\mu(x) \neq 1$, again a contradiction.

- (ii) Suppose there exists a minimal winning coalition x such that $\text{supp}(x) \not\subseteq \{1, 2\}$. Then by (i), the support of μ is not included in $\{1, 2\}$.

Conversely, suppose that there exists $x \in L^N$ with $m^\mu(x) \neq 0$ and $\text{supp}(x) \not\subseteq \{1, 2\}$. Choose a minimal such x . By Lemma 3, $m^\mu(x) \in \{-1, 0, 1\}$. Observe that $m^\mu(x) = -1$ is impossible, because this would yield $\mu(x) = -1$. Then $m^\mu(x) = 1 = \mu(x)$, proving by (i) that x is a minimal winning coalition.

- (iii) Take x being a minimal winning coalition, and suppose w.l.o.g. that $\text{supp}(x) \subseteq \{1, 2\}$. Observe that any other minimal winning coalition y must satisfy $x_1 \neq y_1$, otherwise one of the two would not be minimal. Hence, there can be at most $k + 1$ distinct minimal winning coalitions.
- (iv) By uniqueness of the decomposition, it suffices to check that the computation of μ by $\mu(x) = \sum_{y \leq x} m^\mu(y)$ works. By construction, any $x \in L^N$ is greater or equal to a subset of consecutive minimal winning coalitions, say, $x^{i+1}, x^{i+2}, \dots, x^{i+j}$, so that there are $j - 1$ pairs $(x^{i+\ell}, x^{i+\ell+1})$, $\ell = 1, \dots, j - 1$. The result follows by the definition of m^μ . □

The various properties in the Lemma permit to say that the vertices of $\mathcal{P}_{k,2}$ with support included into $\{1, 2\}$ are in bijection with the antichains (which are of size at most $k + 1$) in the lattice $(k + 1)^2$. Moreover, their Möbius transform is known.

Lemma 8. Let $k \in \mathbb{N}$. Denote by $\kappa(\ell)$ the number of antichains of ℓ elements in the lattice $(k + 1)^2$, $\ell = 1, \dots, k + 1$. Then

$$\kappa(\ell) = \binom{k + 1}{\ell}^2.$$

Moreover, the total number of antichains on $(k + 1)^2$ is

$$\sum_{\ell=1}^{k+1} \kappa(\ell) = \binom{2k + 2}{k + 1} - 1.$$

Proof. Let $x \in (k + 1)^2$, with coordinates (x_1, x_2) . Considering that the 1st coordinate axis is on the left, we say that y is on the left of x if $y_1 > x_1$ and $y_2 < x_2$. Let us denote by $F_1(x_1, x_2)$ the number of points y to the left of x (i.e., $\{x, y\}$ is an antichain). We obtain

$$F_1(x_1, x_2) = \sum_{y_2=0}^{x_2-1} \sum_{y_1=x_1+1}^k 1 = x_2(k - x_1).$$

Note that $\kappa(1) = F_1(-1, k + 1)$ since any point in $(k + 1)^2$ is to the left of $(-1, k + 1)$.

Define $F_2(x_1, x_2)$ as the number of antichains $\{y, z\}$ to the left of x , with z to the left of y , i.e., $\{x, y, z\}$ forms an antichain. We obtain

$$F_2(x_1, x_2) = \sum_{y_2=1}^{x_2-1} \sum_{y_1=x_1+1}^{k-1} F_1(y_1, y_2).$$

(note that $y_2 = 0$ and $y_1 = k$ are impossible because z is on the left of y). Again remark that $\kappa(2) = F_2(-1, k + 1)$. More generally, the number of antichains of ℓ elements to the left of x is

$$F_\ell(x_1, x_2) = \sum_{y_2=\ell-1}^{x_2-1} \sum_{y_1=x_1+1}^{k-\ell+1} F_{\ell-1}(y_1, y_2) \quad (1 \leq \ell \leq k + 1),$$

and $\kappa(\ell) = F_\ell(-1, k+1)$. We show by induction that

$$F_\ell(x_1, x_2) = \binom{x_2}{\ell} \binom{k-x_1}{\ell}. \quad (43)$$

The result has already been verified for $\ell = 1$. We assume it is true up to some integer $1 \leq \ell \leq k$ and prove it for $\ell + 1$. We have

$$\begin{aligned} F_{\ell+1}(x_1, x_2) &= \sum_{y_2=\ell}^{x_2-1} \sum_{y_1=x_1+1}^{k-\ell} F_\ell(y_1, y_2) \\ &= \sum_{y_2=\ell}^{x_2-1} \sum_{y_1=x_1+1}^{k-\ell} \binom{y_2}{\ell} \binom{k-y_1}{\ell} \\ &= \sum_{y_2=\ell}^{x_2-1} \binom{y_2}{\ell} \sum_{y_1=x_1+1}^{k-\ell} \binom{k-y_1}{\ell} \\ &= \binom{x_2}{\ell+1} \binom{k-x_1}{\ell+1}, \end{aligned}$$

where we have used the fact that (see (Gradshteyn and Ryzhik, 2007, §0.151))

$$\sum_{k=0}^m \binom{n+k}{n} = \binom{n+m+1}{n+1}.$$

Hence (43) is proved. It remains to compute the total number of antichains. Using the fact that (see (Gradshteyn and Ryzhik, 2007, §0.157))

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n},$$

we find the desired result. \square

Observing that the antichain $\{\mathbf{0}\}$ does not correspond to a normalized capacity, we obtain directly from Lemma 8 and previous considerations the following result.

Theorem 5. Let $k \in \mathbb{N}$ and consider the polytope $\mathcal{P}_{k,2}$. The following holds.

- (i) For any $i \in N$, the number of vertices with support $\{i\}$ is k .
- (ii) For any distinct $i, j \in N$, the number of vertices with support included in $\{i, j\}$ is $\binom{2k+2}{k+1} - 2$.
- (iii) The total number of vertices of $\mathcal{P}_{k,2}$ is

$$\left[\binom{2k+2}{k+1} - 2 - 2k \right] \frac{n(n-1)}{2} + kn = \left[\binom{2k+2}{k+1} - 2 \right] \frac{n(n-1)}{2} - kn(n-2).$$

5 Conclusion

We have shown in this paper that it is always possible to write a 2-additive discrete GAI model as a sum of nonnegative and monotone nondecreasing terms, thus reducing the complexity of any optimization problem involving such models from exponential to quadratic complexity in the number of attributes. We believe that this result opens the way to the practical utilization of GAI models.

By the equivalence between 2-additive discrete GAI models and 2-additive k -ary capacities, as a by-product of our main result, we have obtained all extreme points of the polytope of 2-additive k -ary capacities, a result which is new, as far as we know, and which generalizes the results of Miranda et al. (2006) for classical 2-additive capacities.

Appendix: Möbius transform of a k -ary capacity

The result can be easily obtained by using standard results of the theory of Möbius functions (see, e.g., Aigner (1979)). Given a finite poset (partially ordered set) (P, \leq) , its *Möbius function* $\mu : P \times P \rightarrow \mathbb{R}$ is defined inductively by:

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq t < y} \mu(x, t), & \text{if } x < y \\ 0, & \text{otherwise} \end{cases} .$$

Then the solution of the system $f(x) = \sum_{y \leq x} g(y)$, $x \in P$, is given by

$$g(x) = \sum_{y \leq x} \mu(y, x) f(y) \quad (x \in P),$$

and g is called the Möbius transform (or inverse) of f . Note that in the case of capacities, (P, \leq) is taken as $(2^N, \subseteq)$.

Considering two posets (P, \leq) , (P', \leq') , and the product poset $(P \times P', \leq)$ where \leq is the product order, i.e., $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq' y'$, it is easy to show that the Möbius function on $P \times P'$ is the product of the Möbius functions on P and P' :

$$\mu((x, t), (y, z)) = \mu_P(x, y) \mu_{P'}(t, z) \quad (x, y \in P, t, z \in P').$$

Let us apply this result to k -ary capacities. It is easy to see that the Möbius function on the chain $\{0, 1, \dots, k\}$ is given by

$$\mu_{\{0, 1, \dots, k\}}(x, y) = \begin{cases} (-1)^{y-x}, & \text{if } 0 \leq y - x \leq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

It follows that the Möbius transform m^v of a k -ary capacity v is given by

$$m^v(x) = \sum_{y \leq x: x_i - y_i \leq 1 \forall i \in N} (-1)^{\sum_{i \in N} (x_i - y_i)} v(y).$$

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