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# LIE THEORY FOR HOPF OPERADS

MURIEL LIVERNET AND FRÉDÉRIC PATRAS

ABSTRACT. The present article takes advantage of the properties of algebras in the category of  $\mathbb{S}$ -modules (twisted algebras) to investigate further the fine algebraic structure of Hopf operads. We prove that any Hopf operad  $\mathcal{P}$  carries naturally the structure of twisted Hopf  $\mathcal{P}$ -algebra. Many properties of classical Hopf algebraic structures are then shown to be encapsulated in the twisted Hopf algebraic structure of the corresponding Hopf operad. In particular, various classical theorems of Lie theory relating Lie polynomials to words (i.e. elements of the tensor algebra) are lifted to arbitrary Hopf operads.

## INTRODUCTION

Let  $\mathcal{P}$  be an arbitrary algebraic operad, that is, a monoid in the category of  $\mathbb{S}$ -modules or, equivalently, the analytic functor associated to a given (suitable) class of algebras. One can extend the usual definition of algebras over  $\mathcal{P}$  in the category of vector spaces and define algebras over  $\mathcal{P}$  in the category of  $\mathbb{S}$ -modules. These algebras are classically referred to as *twisted  $\mathcal{P}$ -algebras*. The meaning and usefulness of twisted algebraic structures was pointed out already in 1978 by Barratt, whose work was motivated by the study of homotopy invariants [Bar78]. He introduced the notion of *twisted Lie algebras* (Lie algebras in the category of  $\mathbb{S}$ -modules). His constructions were later extended to various other algebraic structures by Joyal [Joy86].

The general definition of an algebra over an operad  $\mathcal{P}$  in the category of  $\mathbb{S}$ -modules is more recent. It appears for example in the work of Fresse [Fre04] under the categorical name of “left  $\mathcal{P}$ -modules”. We refer to his article, also for further bibliographical informations on operads.

The purpose of the present article is to take advantage of the properties of twisted algebraic structures to investigate further the internal algebraic structures of operads. We are mainly interested in Hopf operads, that is, the particular class of operads  $\mathcal{P}$  for which one can provide the tensor product of two  $\mathcal{P}$ -algebras with the structure of a  $\mathcal{P}$ -algebra.

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We first show that a Hopf operad  $\mathcal{P}$  has naturally a twisted Hopf  $\mathcal{P}$ -algebra structure. We then show that set of primitive elements for this twisted Hopf  $\mathcal{P}$ -algebra structure is a sub-operad of  $\mathcal{P}$ . For example, when  $\mathcal{P} = \mathcal{A}s$ , the associative operad, there is a twisted Hopf algebra structure on the direct sum  $S_*$  of the symmetric group algebras. As could be expected, its operad of primitive elements is the Lie operad, which makes more precise the results obtained in [PR04] on the fine twisted Hopf algebra structure of  $S_*$ .

We then turn to reciprocity laws, namely to the relations between the internal algebraic structure of an operad and the structure of the algebras over this operad. More precisely, we are interested in the link between the twisted Hopf  $\mathcal{P}$ -algebra structure of the Hopf operad  $\mathcal{P}$ , its primitive suboperad and the structural properties of Hopf  $\mathcal{P}$ -algebras. We show that many classical properties in the theory of free associative algebras and free Lie algebras go over to arbitrary Hopf operads and their primitive suboperads.

Our general results are illustrated on the most classical examples of Hopf operads, namely the associative and Poisson operads.

## 1. TWISTED ALGEBRAS OVER AN OPERAD

For this section we refer to May [May72], Barratt [Bar78], Joyal [Joy86], Fresse [Fre04] and Patras-Reutenauer [PR04].

### 1.1. $\mathbb{S}$ -modules and some related categories of modules.

**1.1.1.  $\mathbb{S}$ -modules.** A  $\mathbb{S}$ -module  $M = \{M(n)\}_{n \geq 0}$  is a collection of right  $S_n$ -modules over a ground field  $\mathbf{k}$ . Notice that we will write permutations in  $S_n$  as sequences:  $\sigma = (\sigma(1), \dots, \sigma(n))$ .

Following Joyal, a  $\mathbb{S}$ -module is equivalent to a *vector species* that is a contravariant functor from the category of finite sets  $\mathbf{Fin}$  (and set isomorphisms) to the category  $\mathbf{Vect}$  of vector spaces over a field  $\mathbf{k}$ . This equivalence goes as follows: from a  $\mathbb{S}$ -module  $M$  one defines the vector species

$$S \mapsto M(S) := \bigoplus_{i_*: S \rightarrow \{1, \dots, r\}} M(r) / \equiv$$

where  $r$  is the number of elements of  $S$  and  $i_*$  is a bijection. The equivalence relation is given by  $(m \cdot \sigma, i_*) \equiv (m, \sigma \circ i_*)$ .

Conversely, the skeleton of a vector species  $M$  is a  $\mathbb{S}$ -module. The action of  $S_n$  on  $M(n) := M(\{1, \dots, n\})$  is given by  $m \cdot \sigma = M(\sigma)(m)$ .

**1.1.2.  $\mathbb{I}$ -modules.** As it is well-known, and recalled in the next section,  $\mathbb{S}$ -modules or, equivalently, vector species, allow to study algebraic structures from the systematical point of view given by the operadic framework. As we will be interested with algebraic structures provided with additional properties, such as the existence of a unit, let us introduce a generalization of  $\mathbb{S}$ -modules suited for our purposes.

Let  $\mathbf{Inj}$  be the category of finite sets and injections. An *injective species* is a contravariant functor from  $\mathbf{Inj}$  to vector spaces. Correspondingly, a  $\mathbb{I}$ -module  $J = \{J(n)\}_{n \geq 0}$  is a collection of right  $S_n$ -modules together with *degeneracy maps*

$$\partial_i : J(n) \longmapsto J(n-1), i = 1, \dots, n$$

such that:

- For any  $1 \leq i \leq n$ ,  $1 \leq j \leq n-1$ , with  $j \geq i$ , we have the equality between maps from  $J(n)$  to  $J(n-2)$ :

$$\partial_j \circ \partial_i = \partial_i \circ \partial_{j+1}. \quad (1.1)$$

- For any  $1 \leq i \leq n$  and any  $m \in J(n)$ ,  $\sigma \in S_n$ , we have:

$$\partial_i(m \cdot \sigma) = \partial_{\sigma(i)}(m) \cdot \partial_i(\sigma), \quad (1.2)$$

with  $\partial_i(\sigma) = \text{st}(\sigma(1), \dots, \sigma(i-1), \sigma(i+1), \dots, \sigma(n))$  where  $\text{st}$  states for the standardization of a sequence of distinct integers.

Recall that, in general, the standardization of a sequence of length  $p$  of distinct nonnegative integers is the process by which the elements of the sequence are replaced by the integers  $1, \dots, p$  in such a way that the relative order of the elements in the sequence is preserved. The process is better understood by means of an example: if  $\sigma = (3, 2, 6, 1, 8, 7, 5, 4) \in S_8$ , we have

$$\text{st}(\sigma(1), \sigma(2), \sigma(4), \dots, \sigma(8)) = \text{st}(3, 2, 1, 8, 7, 5, 4) = (3, 2, 1, 7, 6, 5, 4) \in S_7.$$

The relation between the two notions of  $\mathbb{I}$ -modules and injective species is given by the same formula as the equivalence between the two notions of  $\mathbb{S}$ -modules and vector species. The map  $\partial_i$  from  $J(n)$  to  $J(n-1)$  corresponds to the *elementary injection*  $\text{inj}_i$  from  $[n-1]$  to  $[n]$  defined by:

$$\text{inj}_i(k) := \begin{cases} k & \text{if } k \leq i-1 \\ k+1 & \text{if } k \geq i. \end{cases}$$

The equivalence follows from the observation that any injection from  $[k]$  to  $[n]$ ,  $k < n$ , factorizes uniquely as a composition of permutations and elementary injections:

$$\text{inj}_{i_{n-k}} \circ \dots \circ \text{inj}_{i_1} \circ \sigma$$

with  $\sigma \in S_k$  and  $i_1 < \dots < i_{n-k}$ .

Notice that similar notions have been studied by Cohen, May and Taylor in [CMT78] and Berger [Ber96].

**1.1.3. The symmetric groups as an  $\mathbb{I}$ -module.** Consider now the  $\mathbb{S}$ -module given by  $\mathbf{k}[S_n]$  for all  $n \geq 0$  and the right action given by right multiplication. Then  $\partial_i(\sigma) = \text{st}(\sigma(1), \dots, \sigma(i-1), \sigma(i+1), \dots, \sigma(n))$  satisfies relations (1.1) and (1.2), thus this  $\mathbb{S}$ -module can be provided with the structure of an  $\mathbb{I}$ -module.

The property also follows from the observation that  $\mathbb{S}$  can be viewed, from the categorical point of view, as the skeleton of the full subcategory  $\mathbf{Fin}$  of  $\mathbf{Inj}$ , whereas  $\mathbb{I}$  identifies, categorically, with the skeleton of  $\mathbf{Inj}$ . In this point of view,  $\mathbb{S}$ -modules and  $\mathbb{I}$ -modules identify with contravariant functors from  $\mathbb{S}$  and  $\mathbb{I}$ .

#### 1.1.4. Operations on $\mathbb{S}$ -modules and $\mathbb{I}$ -modules.

i) The categories of vector species and injective species are linear symmetric monoidal categories with the following *tensor product*:

$$(M \hat{\otimes} N)(U) = \bigoplus_{I \sqcup J = U} M(I) \otimes N(J)$$

where  $I \sqcup J$  runs over the partitions of  $U$ .

Explicitly, a map  $\phi$  from  $V$  to  $U$  in  $\mathbf{Inj}$  induces a map from  $(M \hat{\otimes} N)(U)$  to  $(M \hat{\otimes} N)(V)$ . Its restriction to  $M(I) \otimes N(J)$ , where  $I \sqcup J = U$ , is the tensor product of the maps from  $M(I)$  to  $M(I')$  and from  $N(J)$  to  $N(J')$  that are induced by the restriction of  $\phi$  to a map from  $I' := \phi^{-1}(I)$  to  $I$  (resp. from  $J' := \phi^{-1}(J)$  to  $J$ ).

Translated to  $\mathbb{S}$ -modules and  $\mathbb{I}$ -modules, it gives

$$(M \hat{\otimes} N)(n) = \bigoplus_{I \sqcup J = [n]} M(I) \otimes M(J).$$

This definition makes use of the equivalence between  $\mathbb{S}$ -modules and vector species and between  $\mathbb{I}$ -modules and injective species. The equivalent definition using  $\mathbb{S}$ -modules and  $\mathbb{I}$ -modules only is the following

$$\begin{aligned} (M \hat{\otimes} N)(n) &= \bigoplus_{p+q=n} (M(p) \otimes N(q)) \otimes_{S_p \times S_q} \mathbf{k}[S_n] \\ &= \bigoplus_{p+q=n} (M(p) \otimes M(q)) \otimes \mathbf{k}[\text{Sh}_{p,q}] \end{aligned}$$

where  $\text{Sh}_{p,q}$  is the set of  $(p, q)$ -shuffles that is permutations of  $S_n$  written  $(\tau_1, \dots, \tau_p, \rho_1, \dots, \rho_q)^{-1}$  with  $\tau_1 < \dots < \tau_p$  and  $\rho_1 < \dots < \rho_q$ . The second equality is a consequence of the unique decomposition of any permutation  $\sigma \in S_n$  as  $\sigma = (\sigma_1 \times \sigma_2) \cdot \alpha$  where  $\alpha$  is a  $(p, q)$ -shuffle. The unit for this tensor product is the  $\mathbb{S}$ -module (resp.  $\mathbb{I}$ -module)  $\mathbf{1}$  given by

$$\mathbf{1}(n) = \begin{cases} \mathbf{k}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

When  $M$  and  $N$  are  $\mathbb{I}$ -modules, the structure of  $\mathbb{I}$ -module on  $M \hat{\otimes} N$  is given as follows: for any  $m \in M(p)$ , any  $n \in N(q)$  with  $p + q = n$ ,

$$\partial_i(m \otimes n) = \begin{cases} \partial_i(m) \otimes n & \text{if } 1 \leq i \leq p, \\ m \otimes \partial_{i-p}(n) & \text{if } p+1 \leq i \leq p+q. \end{cases} \quad (1.3)$$

Notice that this relation determines entirely the  $\mathbb{L}$ -module structure on  $M \hat{\otimes} N$  thanks to equation (1.2), since  $M(p) \otimes N(q)$  generates  $(M \hat{\otimes} N)(p+q)$  as an  $S_{p+q}$ -module.

The symmetry isomorphism  $\tau_{M,N} : M \hat{\otimes} N \rightarrow N \hat{\otimes} M$  is given by

$$\tau_{M,N} : M(I) \otimes N(J) \rightarrow N(J) \otimes M(I)$$

in the species framework, that is, in the  $\mathbb{S}$  and  $\mathbb{L}$ -modules framework

$$\tau_{M,N}(m \otimes n) = (n \otimes m) \cdot \zeta_{p,q} \quad (1.4)$$

where  $m \in M(p), n \in N(q)$  and  $\zeta_{p,q} = (q+1, \dots, q+p, 1, \dots, q)$ . The symmetry isomorphism is a morphism of  $\mathbb{L}$ -modules that is

$$\partial_i \tau_{M,N} = \tau_{M,N} \partial_i. \quad (1.5)$$

Indeed, if  $1 \leq i \leq p$  one has

$$\begin{aligned} \partial_i \tau_{M,N}(m \otimes n) &= \partial_i((n \otimes m) \cdot \zeta_{p,q}) = \partial_{\zeta_{p,q}(i)}(n \otimes m) \cdot \partial_i(\zeta_{p,q}) \\ &= \partial_{q+i}(n \otimes m) \cdot \zeta_{p-1,q} = (n \otimes \partial_i(m)) \cdot \zeta_{p-1,q} = \tau_{M,N}(\partial_i(m) \otimes n) \\ &= \tau_{M,N} \partial_i(m \otimes n). \end{aligned}$$

The same computation holds if  $p+1 \leq i \leq p+q$ .

For any  $\sigma \in S_n$ , the symmetry isomorphism induces an isomorphism  $\tau_\sigma$  of  $\mathbb{S}$ -modules and  $\mathbb{L}$ -modules from  $M_1 \hat{\otimes} \dots \hat{\otimes} M_n$  to  $M_{\sigma^{-1}(1)} \hat{\otimes} \dots \hat{\otimes} M_{\sigma^{-1}(n)}$  given by

$$\tau_\sigma(m_1 \otimes \dots \otimes m_n) = (m_{\sigma^{-1}(1)} \otimes \dots \otimes m_{\sigma^{-1}(n)}) \cdot \sigma(l_1, \dots, l_n) \quad (1.6)$$

where  $m_i \in M_i(l_i)$  and  $\sigma(l_1, \dots, l_k)$  is the permutation of  $S_{l_1+\dots+l_k}$  obtained by replacing  $\sigma(i)$  by the block  $\text{Id}_{l_i}$ . More precisely

$$\sigma(l_1, \dots, l_k) = (B_1, \dots, B_k)$$

where  $B_i$  is the block  $l_{\sigma^{-1}(1)} + \dots + l_{\sigma^{-1}(\sigma(i)-1)} + [l_i]$ . For instance

$$(2, 3, 1)(a, b, c) = (c+1, \dots, c+a, c+a+1, \dots, c+a+b, 1, \dots, c).$$

ii) The notation  $M \otimes N$  is devoted to the  $\mathbb{S}$ -module or  $\mathbb{L}$ -module given by the collection  $M(n) \otimes N(n)$  together with the diagonal action of maps. It is the vector or injective species  $(M \otimes N)(U) = M(U) \otimes N(U)$ .

iii) The category of  $\mathbb{S}$ -modules is endowed with another monoidal structure (which is not symmetric): the *plethysm*  $\circ$  defined by

$$(M \circ N)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} (N^{\hat{\otimes} k})(n),$$

where  $S_k$  acts on the left of  $(N^{\hat{\otimes} k})$  by formula (1.6).

An alternative definition using vector species is given by

$$(M \circ N)(U) := \bigoplus_{k \geq 1} M(k) \otimes_{S_k} \left( \bigoplus_{I_1 \sqcup \dots \sqcup I_k = U} N(I_1) \otimes \dots \otimes N(I_k) \right).$$

when  $U \neq \emptyset$  and where the action of  $S_k$  is given by

$$\sigma \cdot (N(I_1) \otimes \dots \otimes N(I_k)) = N(I_{\sigma^{-1}(1)}) \otimes \dots \otimes N(I_{\sigma^{-1}(k)}).$$

The unit for the plethysm is the  $\mathbb{S}$ -module  $I$  given by

$$I(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Because of relation (1.5), the left action of  $S_k$  on  $N^{\hat{\otimes} k}$  commutes with the action of the morphisms in **Inj**. Therefore, one can also define the plethystic product  $M \circ N$  of a  $\mathbb{S}$ -module  $M$  with a  $\mathbb{I}$ -module  $N$  by the same formula

$$(M \circ N)(U) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} N^{\hat{\otimes} k}(U),$$

together with a  $\mathbb{I}$ -module structure defined for any  $m \in M(k), \bar{n} \in N^{\hat{\otimes} k}(l)$ , by:

$$\partial_i(m \otimes \bar{n}) := m \otimes \partial_i(\bar{n}).$$

In particular, since any  $\mathbb{I}$ -module is also a  $\mathbb{S}$ -module, the plethystic product of two injective species is an injective species. However some properties that hold for vector species break down for injective species. For example, the  $\mathbb{S}$ -module  $I$  can be provided with a unique  $\mathbb{I}$ -module structure and is still a left unit for the plethystic product of  $\mathbb{I}$ -modules. However, it is not a right unit: the  $\mathbb{I}$ -module structure of  $M \circ I$  is not the  $\mathbb{I}$ -module structure of  $M$ , except in the very particular case when the degeneracy maps from  $M(n)$  to  $M(n-1)$  are all zero.

**1.1.5. Natural transformations relating the operations on  $\mathbb{S}$  and  $\mathbb{I}$ -modules.** One has two natural transformations in  $A, B, C$  and  $D$ :

$$\begin{aligned} T_1 : (A \otimes B) \circ (C \otimes D) &\rightarrow (A \circ C) \otimes (B \circ D) \\ T_2 : (A \otimes B) \circ (C \hat{\otimes} D) &\rightarrow (A \circ C) \hat{\otimes} (B \circ D) \end{aligned}$$

obtained by interverting terms in the corresponding direct sums. The symmetry isomorphism  $\tau$  has to be taken into account in order to define  $T_2$ . Since  $\tau$  commutes with degeneracies,  $T_2$  is a morphism of  $\mathbb{I}$ -modules if  $C$  and  $D$  are  $\mathbb{I}$ -modules.

In terms of  $\mathbb{S}$ -modules, one has the following description: let  $a \in A(k)$  and  $b \in B(k), c_i \in C(l_i), d_i \in D(l_i), e_i \in C(r_i), f_i \in D(s_i)$ :

$$\begin{aligned} T_1((a \otimes b) \otimes (c_1, d_1, c_2, d_2, \dots, c_k, d_k)) &= (a \otimes c_1, \dots, c_k) \otimes (b \otimes d_1, \dots, d_k) \\ T_2((a \otimes b) \otimes (e_1, f_1, e_2, f_2, \dots, e_k, f_k)) &= \\ &= (a \otimes e_1, \dots, e_k) \otimes (b \otimes f_1, \dots, f_k) \cdot \sigma(r_1, s_1, r_2, s_2, \dots, r_k, s_k) \end{aligned}$$

where  $\sigma = (1, k+1, 2, k+2, 3, \dots, 2k-1, 2k)$ .

The relations are even more natural when written in terms of vector species. Let us consider, for example, the relation defining  $T_2$ . We have, for  $e_i \in C(S_i)$  and  $f_i \in D(T_i)$  with  $\coprod_{i=1}^k (S_i \amalg T_i) = U$ :

$$\begin{aligned} & T_2 : (A(k) \otimes B(k)) \otimes_{S_k} ((C(S_1) \otimes D(T_1)) \otimes \dots \otimes (C(S_k) \otimes D(T_k))) \\ \longrightarrow & (A(k) \otimes_{S_k} (C(S_1) \otimes \dots \otimes C(S_k))) \otimes (B(k) \otimes_{S_k} (D(T_1) \otimes \dots \otimes D(T_k))) \\ & T_2(a \otimes b \otimes (e_1, f_1, e_2, f_2, \dots, e_k, f_k)) = (a \otimes e_1, \dots, e_k) \otimes (b \otimes f_1, \dots, f_k). \end{aligned}$$

As a consequence of the definitions, the following relations hold

$$(A \hat{\otimes} B) \circ C = (A \circ C) \hat{\otimes} (B \circ C), \quad (1.7)$$

$$T_1(T_1 \circ \text{Id}) = T_1(\text{Id} \circ T_1), \quad (1.8)$$

$$T_2(T_1 \circ \text{Id}) = T_2(\text{Id} \circ T_2). \quad (1.9)$$

Notice that the last identity expresses in two different ways the natural transformation

$$(A \otimes B) \circ (C \otimes D) \circ (E \hat{\otimes} F) \rightarrow (A \circ C \circ E) \hat{\otimes} (B \circ D \circ F).$$

## 1.2. Operads.

**1.2.1. Definition.** An operad is a monoid in the category of  $\mathbb{S}$ -modules with respect to the plethysm. Hence an operad is a  $\mathbb{S}$ -module  $\mathcal{P}$  together with a product  $\mu_{\mathcal{P}} : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$  and a unit  $u_{\mathcal{P}} : I \rightarrow \mathcal{P}$  satisfying

$$\begin{aligned} \mu_{\mathcal{P}}(\mathcal{P} \circ \mu_{\mathcal{P}}) &= \mu_{\mathcal{P}}(\mu_{\mathcal{P}} \circ \mathcal{P}) \\ \mu_{\mathcal{P}}(\mathcal{P} \circ u_{\mathcal{P}}) &= \mu_{\mathcal{P}}(u_{\mathcal{P}} \circ \mathcal{P}) = \mathcal{P}. \end{aligned}$$

In other terms an operad  $\mathcal{P}$  is a collection of  $S_n$ -modules  $(\mathcal{P}(n))_{n \geq 0}$  together with an element  $1_1 \in \mathcal{P}(1)$  and compositions

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(l_1) \otimes \dots \otimes \mathcal{P}(l_k) \rightarrow \mathcal{P}(l_1 + \dots + l_k)$$

satisfying associativity, unitary conditions and equivariance conditions reflecting the action of  $S_k \times S_n$  on  $\mathcal{P}^{\hat{\otimes} k}(n)$  that is:

$$\begin{aligned} \gamma(p \cdot \sigma, p_1, \dots, p_k) &= \gamma(p, p_{\sigma^{-1}(1)}, \dots, p_{\sigma^{-1}(k)}) \cdot \sigma(l_1, \dots, l_k), \\ \gamma(p, p_1 \cdot \tau_1, \dots, p_k \cdot \tau_k) &= \gamma(p, p_1, \dots, p_k) \cdot (\tau_1 \oplus \dots \oplus \tau_k). \end{aligned}$$

Most of the time  $\gamma(p, p_1, \dots, p_k)$  is written  $p(p_1, \dots, p_k)$ . When  $p_j = 1_1$  for all  $j$  except  $i$  the latter composition is written  $p \circ_i p_i$ .

With this notation the associativity, unitarity and equivariance write, for  $p \in \mathcal{P}(n), q \in \mathcal{P}(m), r \in \mathcal{P}(l), \sigma \in S_n, \tau \in S_m$ :

$$(p \circ_i q) \circ_{j+i-1} r = p \circ_i (q \circ_j r), \quad (1.10)$$

$$(p \circ_i q) \circ_{j+m-1} r = (p \circ_j r) \circ_i q, \quad i < j, \quad (1.11)$$

$$p \circ_i 1_1 = p = 1_1 \circ_1 p, \quad (1.12)$$

$$(p \cdot \sigma) \circ_i (q \cdot \tau) = (p \circ_{\sigma(i)} q) \cdot (\sigma \circ_i \tau), \quad (1.13)$$



where  $\sigma \circ_i \tau$  is the permutation of  $S_{n+m-1}$  obtained by replacing  $\tau$  for  $\sigma(i)$ . For instance

$$(3, 4, 2, 5, 1) \circ_2 (a, b, c) = (3, a + 3, b + 3, c + 3, 2, 7, 1).$$

**1.2.2. Pre-unital and connected operads.** A *pre-unital* operad is an injective species  $\mathcal{I}$  together with an operad structure such that the product  $\mu_{\mathcal{I}} : \mathcal{I} \circ \mathcal{I} \rightarrow \mathcal{I}$  is a morphism of  $\mathbb{I}$ -modules.

An operad  $\mathcal{P}$  is said to be *connected* if  $\mathcal{P}(0) = \mathbf{k}$  (unital in the terminology of May [May72]). With  $\mathcal{P}$  connected, let  $1_0$  be the image of  $1 \in \mathbf{k} = \mathcal{P}(0)$ . The operad  $\mathcal{P}$  is endowed with degeneracy maps  $\partial_i$  for  $1 \leq i \leq n$  defined by

$$\begin{aligned} \partial_i : \mathcal{P}(n) &\rightarrow \mathcal{P}(n-1) \\ \mu &\mapsto \mu \circ_i 1_0. \end{aligned}$$

The composition of degeneracies induces restriction maps: for any subset  $S \subset [n]$  of size  $k$  one defines

$$|_S : \mathcal{P}(n) \rightarrow \mathcal{P}(k)$$

by

$$\mu|_S = \begin{cases} \mu, & \text{if } S = [n], \\ \partial_{t_1} \partial_{t_2} \dots \partial_{t_{n-k}}(\mu), & \text{otherwise,} \end{cases}$$

where  $\{t_1 < t_2 < \dots < t_{n-k}\} = [n] \setminus S$ . Hence

$$\mu|_S = \mu(x_1, \dots, x_n) \text{ where } x_i = \begin{cases} 1_1, & \text{if } i \in S, \\ 1_0, & \text{otherwise.} \end{cases}$$

**1.2.3. The associative operad.** The operad  $\mathcal{A}s$  is defined by  $\mathcal{A}s(n) = \mathbf{k}[S_n]$  for all  $n \geq 0$ . The composition  $\sigma \circ_i \tau$  is the one given in definition 1.2.1. This operad is connected. The degeneracy maps were defined in paragraph 1.1.3:

$$\partial_i(\sigma) = \sigma \circ_i 1_0 = \text{st}(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n).$$

For  $S = \{s_1 < \dots < s_k\}$  the permutation  $\sigma|_S$  is the standardisation of  $(\sigma(s_1), \dots, \sigma(s_k))$ . For instance

$$(3, 2, 6, 1, 8, 7, 5, 4)|_{\{1,4,6,7\}} = \text{st}(3, 1, 7, 5) = (2, 1, 4, 3).$$

Furthermore, for any  $\sigma \in S_n$

$$\sigma|_{\emptyset} = 1_0.$$

**1.2.4. Proposition.** *Any connected operad is a pre-unital operad.*

*Proof.* By definition  $\partial_i(\mu) = \mu \circ_i 1_0$ . The formula (1.11) implies equation (1.1) and the formula (1.13) implies equation (1.2). This proves that  $\mathcal{P}$  is a  $\mathbb{I}$ -module.

Let  $\mu \in \mathcal{P}(n)$ ,  $\nu_k \in \mathcal{P}(l_k)$  and  $i \in [R_{j-1} + 1, R_{j-1} + l_j]$  with  $R_{j-1} = l_1 + \dots + l_{j-1}$ . Iterating formula (1.10) gives

$$\begin{aligned} \partial_i(\mu_{\mathcal{P}}(\mu \otimes \nu_1 \otimes \dots \otimes \nu_n)) &= \mu(\nu_1, \dots, \nu_{j-1}, \partial_{i-R_{j-1}}(\nu_j), \dots, \nu_n) \\ &= \mu_{\mathcal{P}}(\partial_i(\mu \otimes \nu_1 \otimes \dots \otimes \nu_n)). \end{aligned}$$

Hence  $\mu_{\mathcal{P}}$  is a morphism of  $\mathbb{I}$ -modules.  $\square$

Given  $n$  sets  $S_i \subset [l_i]$  for  $1 \leq i \leq n$  the set  $S_1 \star \dots \star S_n$  is the subset of  $[l_1 + \dots + l_n]$  of elements  $R_{i-1} + \alpha$  for  $\alpha \in S_i$ . Iterating the last equation gives the relation

$$\mu(\nu_1, \dots, \nu_n)|_{S_1 \star \dots \star S_n} = \mu(\nu_1|_{S_1}, \dots, \nu_n|_{S_n}),$$

which yields the following lemma

**1.2.5. Lemma.** *Let  $\mathcal{P}$  be a connected operad. Let  $\mu \in \mathcal{P}(n), \nu_i \in \mathcal{P}(l_i)$  and  $S_i \subset [l_i]$ . For any set  $J = \{j_1 < \dots < j_l\} \subset [n]$  such that  $S_i$  is empty for all  $i \notin J$ , one has*

$$\mu(\nu_1, \dots, \nu_n)|_{S_1 \star \dots \star S_n} = \left( \prod_{i \notin J} \nu_i|_{\emptyset} \right) \mu|_J(\nu_{j_1}|_{S_{j_1}}, \dots, \nu_{j_l}|_{S_{j_l}}).$$

**1.3. Twisted algebras over an operad.** Let  $V$  be a vector space. It can be considered as a  $\mathbb{S}$ -module concentrated in degree 0. In particular, the notation  $\mathcal{P} \circ V$  makes sense, and an algebra over an operad is a vector space together with a product map  $\mathcal{P} \circ V \rightarrow V$  satisfying the usual monadic relations (see Ginzburg and Kapranov [GK94] or replace  $M$  by  $V$  in the diagrams below). The notion of twisted algebras over an operad generalizes this definition to  $\mathbb{S}$ -modules. Notice that, since the category of vector spaces identifies with the full subcategory of the category of  $\mathbb{S}$ -modules which objects are the  $\mathbb{S}$ -modules concentrated in degree 0, any result on twisted algebraic structures holds automatically for classical algebraic structures. We will refer to this property as “Restriction to **Vect**”.

**1.3.1. Definition.** Let  $\mathcal{P}$  be an operad. A  $\mathbb{S}$ -module  $M$  is a *twisted  $\mathcal{P}$ -algebra* if  $M$  is endowed with a product  $\mu_M : \mathcal{P} \circ M \rightarrow M$  such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} \circ M & \xrightarrow{\mathcal{P} \circ \mu_M} & \mathcal{P} \circ M \\ \mu_{\mathcal{P} \circ M} \downarrow & & \downarrow \mu_M \\ \mathcal{P} \circ M & \xrightarrow{\mu_M} & M \end{array} \qquad \begin{array}{ccc} I \circ M & \xrightarrow{=} & M \\ u_{\mathcal{P} \circ M} \downarrow & \nearrow \mu_M & \\ \mathcal{P} \circ M & & \end{array}$$

**1.3.2. Remark.** Twisted  $\mathcal{P}$ -algebras are called left  $\mathcal{P}$ -modules in the categorical, monadic, terminology (see e.g. [Fre04]), since they are also algebras over the monad

$$\begin{array}{ccc} \mathbb{P} : \mathbb{S} - mod & \rightarrow & \mathbb{S} - mod \\ & & M \mapsto \mathcal{P} \circ M \end{array}$$

Here, we prefer to stick to the more appealing terminology of twisted  $\mathcal{P}$ -algebras. In case  $\mathcal{P}$  is the operad defining associative or Lie algebras, the definition coincides with the notion of twisted associative or twisted Lie algebras in the sense of Barratt in [Bar78], see the example below and Joyal’s article [Joy86].

If  $M = M(0)$  is concentrated in degree 0 then  $M$  is a  $\mathcal{P}$ -algebra in the usual sense, according to the Restriction to **Vect** principle. For instance if  $\mathcal{P} = \mathcal{A}s$ ,  $\mathcal{A}s \circ M = T(M)$  is the free associative algebra over  $M$ . If  $M$  is a  $\mathbb{S}$ -module such that the action of  $S_n$  is the trivial action or the signature action, then  $M$  is a graded  $\mathcal{P}$ -algebra in the usual sense (when the action is trivial no signs are considered).

**1.3.3. Example: Twisted Lie algebras.** Let  $\mathcal{L}ie$  be the Lie operad. As an operad it is generated by  $\mu \in \mathcal{L}ie(2)$  satisfying the following relations:

$$\begin{aligned}\mu \cdot (2, 1) &= -\mu \\ \mu \circ_2 \mu \cdot ((1, 2, 3) + (2, 3, 1) + (3, 1, 2)) &= 0\end{aligned}$$

A *twisted Lie algebra* is a twisted algebra over the operad  $\mathcal{L}ie$ , that is a  $\mathbb{S}$ -module  $M$  endowed with a multiplication, the bracket, defined by  $[a, b] = \mu(a, b)$  satisfying the following relations: let  $a \in M(p)$ ,  $b \in M(q)$ ,  $c \in M(r)$ ,

$$\begin{aligned}[b, a] \cdot \zeta_{p,q} &= -[a, b] \\ [a, [b, c]] + [c, [a, b]] \cdot \zeta_{p+q,r} + [b, [c, a]] \cdot \zeta_{p,q+r} &= 0,\end{aligned}$$

where  $\zeta_{p,q}$  was defined in relation (1.4). This is exactly definition 4 in [Bar78]. These relations are a direct consequence of the computation of  $(\mu \cdot (2, 1))(a, b)$  and  $((\mu \circ_2 \mu) \cdot (1, 2, 3) + (2, 3, 1) + (3, 1, 2))(a, b, c)$  using the symmetry isomorphism  $\tau$ . For instance

$$((\mu \circ_2 \mu) \cdot (2, 3, 1))(a, b, c) = [c, [a, b]] \cdot \zeta_{p+q,r} = [c, [a, b]] \cdot \zeta_{p+q,r}.$$

**1.3.4. Unital  $\mathcal{P}$ -algebras.** Assume  $\mathcal{P}$  is a connected operad, then the  $\mathbb{S}$ -module  $\mathbf{1}$  is a twisted  $\mathcal{P}$ -algebra for the product

$$\mu(1_0, \dots, 1_0) = \mu|_{\emptyset} 1_0.$$

By definition a *unital twisted  $\mathcal{P}$ -algebra* is a twisted  $\mathcal{P}$ -algebra  $M$  together with a morphism of twisted  $\mathcal{P}$ -algebras called the unit morphism

$$\eta_M : \mathbf{1} \rightarrow M$$

satisfying

$$\mu(a_1, \dots, a_n) = \mu|_{\{i_1, \dots, i_r\}}(a_{i_1}, \dots, a_{i_r}) \quad (1.14)$$

for all  $\mu \in \mathcal{P}(n)$ ,  $a_i \in M$  as soon as  $a_j = \eta_M(1_0)$  for all  $j \notin \{i_1, \dots, i_r\}$ . A *morphism of unital twisted  $\mathcal{P}$ -algebras* is a morphism of twisted  $\mathcal{P}$ -algebras commuting with the unit morphism.

**1.3.5. Free twisted  $\mathcal{P}$ -algebras.** Since  $I$  is the unit for the plethysm, the  $\mathbb{S}$ -module  $\mathcal{P} = \mathcal{P} \circ I$  is the free twisted  $\mathcal{P}$ -algebra generated by  $I$ . More generally,  $\mathcal{P} \circ M$  is the free twisted  $\mathcal{P}$ -algebra generated by a  $\mathbb{S}$ -module  $M$ . If  $\mathcal{P}$  is connected then  $\mathcal{P} \circ M$  is a unital twisted  $\mathcal{P}$ -algebra since

$$\eta_{\mathcal{P} \circ M}(0) : \mathbf{k} \rightarrow (\mathcal{P} \circ M)(0) = \mathcal{P}(0) \oplus_{k \geq 1} \mathcal{P}(k) \otimes_{S_k} M(0)^{\otimes k}.$$

is given by the isomorphism  $\mathcal{P}(0) = \mathbf{k}$ .

## 2. TWISTED HOPF ALGEBRAS OVER A HOPF OPERAD

**2.1. Hopf operads.** Let **Coalg** be the category of coassociative counital coalgebras, that is vector spaces  $V$  endowed with a coassociative coproduct  $\delta : V \rightarrow V \otimes V$  and a linear map, the counit,  $\epsilon : V \rightarrow \mathbf{k}$  satisfying

$$(\epsilon \otimes V)\delta = (V \otimes \epsilon)\delta = V.$$

**2.1.1. Definition.** A *Hopf operad*  $\mathcal{P}$  is an operad in the category of coalgebras:  $\mu_{\mathcal{P}}$  and  $u_{\mathcal{P}}$  are morphisms of coalgebras. More precisely, for each  $n$  there exists a coassociative coproduct

$$\delta(n) : \mathcal{P}(n) \rightarrow \mathcal{P}(n) \otimes \mathcal{P}(n) = (\mathcal{P} \otimes \mathcal{P})(n)$$

which is  $S_n$ -equivariant and such that the following two diagrams commute,

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} & \xrightarrow{\mu_{\mathcal{P}}} & \mathcal{P} \\ \delta \circ \delta \downarrow & & \delta \downarrow \\ (\mathcal{P} \otimes \mathcal{P}) \circ (\mathcal{P} \otimes \mathcal{P}) & \xrightarrow{T_1} (\mathcal{P} \circ \mathcal{P}) \otimes (\mathcal{P} \circ \mathcal{P}) \xrightarrow{\mu_{\mathcal{P}} \otimes \mu_{\mathcal{P}}} & \mathcal{P} \otimes \mathcal{P} \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{u_{\mathcal{P}}} & \mathcal{P} \\ \delta_I \downarrow & & \delta \downarrow \\ I \otimes I & \xrightarrow{u_{\mathcal{P}} \otimes u_{\mathcal{P}}} & \mathcal{P} \otimes \mathcal{P} \end{array}$$

and for each  $n$  there exists  $\epsilon(n) : \mathcal{P}(n) \rightarrow \mathbf{k}$  (abbreviated to  $\epsilon$  when no confusion can arise) such that

$$\begin{aligned} \epsilon(\mu(\nu_1, \dots, \nu_n)) &= \epsilon(\mu)\epsilon(\nu_1) \dots \epsilon(\nu_n) \\ \epsilon(1_1) &= 1_{\mathbf{k}}. \end{aligned}$$

A *connected Hopf operad* is a Hopf structure on a connected operad such that the counit  $\epsilon$  is given by

$$\epsilon(\mu) = \mu|_{\emptyset} 1_{\mathbf{k}}.$$

**2.1.2. Example.** The operad  $\text{Com}$  is defined by  $\text{Com}(n) = \mathbf{k}$  with the trivial action of the symmetric group. The composition map is given by  $e_n \circ_i e_m = e_{n+m}$  where  $e_n$  states for  $1_{\mathbf{k}} \in \text{Com}(n)$ . It is a connected Hopf operad for the coproduct  $\delta(e_n) = e_n \otimes e_n$  and the counit  $\epsilon(e_n) = 1_{\mathbf{k}}$ .

The operad  $\mathcal{A}s$  is a connected Hopf operad for the coproduct  $\delta(\sigma) = \sigma \otimes \sigma$  and  $\epsilon(\sigma) = 1_{\mathbf{k}}$  for all  $\sigma \in S_n$ .

**2.1.3. Theorem.** Let  $\mathcal{P}$  be a Hopf operad and  $V, W$  be twisted  $\mathcal{P}$ -algebras. The  $S$ -module  $V \hat{\otimes} W$  is a twisted  $\mathcal{P}$ -algebra for the following product:  $\mu_{V \hat{\otimes} W}$  is the composite of

$$\mathcal{P} \circ (V \hat{\otimes} W) \xrightarrow{\delta \circ (V \hat{\otimes} W)} (\mathcal{P} \otimes \mathcal{P}) \circ (V \hat{\otimes} W) \xrightarrow{T_2} (\mathcal{P} \circ V) \hat{\otimes} (\mathcal{P} \circ W) \xrightarrow{\mu_V \hat{\otimes} \mu_W} V \hat{\otimes} W.$$

If the coalgebra structure on  $\mathcal{P}$  is cocommutative, then the symmetry isomorphism  $\tau$  is a morphism of  $\mathcal{P}$ -algebras.

*Proof.* Let  $\delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$  be the Hopf coproduct of  $\mathcal{P}$  and  $M := V \hat{\otimes} W$ . One has to prove that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{P} \circ M & \xrightarrow{\mathcal{P} \circ \mu_M} & \mathcal{P} \circ M \\ \mu_{\mathcal{P} \circ M} \downarrow & & \downarrow \mu_M \\ \mathcal{P} \circ M & \xrightarrow{\mu_M} & M \end{array} \qquad \begin{array}{ccc} I \circ M & \xrightarrow{=} & M \\ \eta_{\mathcal{P} \circ M} \downarrow & \nearrow \mu_M & \\ \mathcal{P} \circ M & & \end{array}$$

For the first diagram, one has

$$\begin{aligned} \mu_M(\mu_{\mathcal{P}} \circ M) &= (\mu_V \hat{\otimes} \mu_W) T_2(\delta \circ (V \hat{\otimes} W))(\mu_{\mathcal{P}} \circ (V \hat{\otimes} W)) \\ &= (\mu_V \hat{\otimes} \mu_W) T_2((\delta \mu_{\mathcal{P}}) \circ (V \hat{\otimes} W)). \end{aligned}$$

The following relations hold

$$\begin{aligned} (\delta \mu_{\mathcal{P}}) \circ (V \hat{\otimes} W) &= (\mu_{\mathcal{P}} \otimes \mu_{\mathcal{P}}) T_1(\delta \circ \delta) \circ (V \hat{\otimes} W) \\ T_2((\mu_{\mathcal{P}} \otimes \mu_{\mathcal{P}}) \circ V \hat{\otimes} W) &= ((\mu_{\mathcal{P}} \circ V) \hat{\otimes} (\mu_{\mathcal{P}} \circ W)) T_2 \\ (\mu_V \hat{\otimes} \mu_W)((\mu_{\mathcal{P}} \circ V) \hat{\otimes} (\mu_{\mathcal{P}} \circ W)) &= (\mu_V \hat{\otimes} \mu_W)((\mathcal{P} \circ \mu_V) \hat{\otimes} (\mathcal{P} \circ \mu_W)) \end{aligned}$$

The first equality comes from definition 2.1.1. The second one comes from the naturality of  $T_2$ . The last one comes from the definition of twisted  $\mathcal{P}$ -algebras applied on  $V$  and  $W$  (see 1.3.1). Combining these equalities, one gets

$$\begin{aligned} \mu_M(\mu_{\mathcal{P}} \circ M) &= \\ &= (\mu_V \hat{\otimes} \mu_W)((\mathcal{P} \circ \mu_V) \hat{\otimes} (\mathcal{P} \circ \mu_W)) T_2(T_1 \circ M)((\delta \circ \delta) \circ M) \end{aligned}$$

The relation (1.9) implies

$$T_2(T_1 \circ M)((\delta \circ \delta) \circ M) = T_2((\mathcal{P} \otimes \mathcal{P}) \circ T_2)((\delta \circ \delta) \circ M)$$

The naturality of  $T_2$  implies

$$((\mathcal{P} \circ \mu_V) \hat{\otimes} (\mathcal{P} \circ \mu_W)) T_2 = T_2((\mathcal{P} \otimes \mathcal{P}) \circ (\mu_V \hat{\otimes} \mu_W))$$

As a consequence one has

$$\begin{aligned} \mu_M(\mu_{\mathcal{P}} \circ M) &= (\mu_V \hat{\otimes} \mu_W) T_2((\mathcal{P} \otimes \mathcal{P}) \circ (\mu_V \hat{\otimes} \mu_W))((\mathcal{P} \otimes \mathcal{P}) \circ T_2)((\delta \circ \delta) \circ M) \\ &= (\mu_V \hat{\otimes} \mu_W) T_2(\delta \circ (\mu_V \hat{\otimes} \mu_W))(\mathcal{P} \circ T_2)((\mathcal{P} \circ \delta) \circ M) \\ &= (\mu_V \hat{\otimes} \mu_W) T_2(\delta \circ M)(\mathcal{P} \circ \mu_M) = \mu_M(\mathcal{P} \circ \mu_M). \end{aligned}$$

It is easy to check that  $\mu_M$  makes the second diagram commute.

In case  $\delta$  is cocommutative, using diagrams one sees easily that  $\tau : V \hat{\otimes} W \rightarrow W \hat{\otimes} V$  is a morphism of  $\mathcal{P}$ -algebras.  $\square$

One could deduce the theorem from Moerdijk [Moe02, Proposition 1.4], since a part of the proof above implies that the monad

$$\begin{aligned} \mathbb{P} : \mathbb{S}\text{-mod} &\rightarrow \mathbb{S}\text{-mod} \\ M &\mapsto \mathcal{P} \circ M \end{aligned}$$

is a *Hopf monad*. However the proof above is still valid if we require the operad to be an operad in the category of coassociative coalgebras (non necessarily counital). The counit is necessary for the category of twisted  $\mathcal{P}$ -algebras to be a tensor category: the unit for  $\hat{\otimes}$  is given by the  $\mathcal{P}$ -algebra  $\mathbf{1}$  endowed with the product

$$\mu_{\mathbf{1}} : (\mathcal{P} \circ \mathbf{1})(\emptyset) = \bigoplus_{l \geq 0} \mathcal{P}(l)/S_l \rightarrow \mathbf{1}(\emptyset) = \mathbf{k}$$

given by  $\mu_{\mathbf{1}}(\nu) = \epsilon(n)(\nu)$  for  $\nu \in \mathcal{P}(n)$ .

When the operad is a Hopf connected operad the  $\mathcal{P}$ -algebra structure on  $\mathbf{1}$  coincides with the one given in 1.3.4. This implies the following

**2.1.4. Theorem.** *Let  $\mathcal{P}$  be a connected Hopf operad. Then the category of unital twisted  $\mathcal{P}$ -algebras is a tensor category, symmetric if  $\mathcal{P}$  is cocommutative.*

## 2.2. Twisted Hopf algebras over a connected Hopf operad.

**2.2.1. Definition.** Let  $\mathcal{P}$  be a connected Hopf operad. A unital twisted  $\mathcal{P}$ -algebra  $M$  is a *twisted Hopf  $\mathcal{P}$ -algebra* if  $M$  is endowed with a coassociative counital coproduct

$$\Delta : M \rightarrow M \hat{\otimes} M, \quad \epsilon_M : M \rightarrow \mathbf{1}$$

where  $\Delta$  and  $\epsilon_M$  are morphisms of unital  $\mathcal{P}$ -algebras.

**2.2.2. Remark.** In case  $\mathcal{P} = \mathcal{A}s$  one gets twisted bialgebras as defined in [Sto93, PR04].

**2.2.3. Theorem.** *If  $\mathcal{P}$  is a connected Hopf operad then  $\mathcal{P}$  is a twisted Hopf  $\mathcal{P}$ -algebra for the coproduct*

$$\Delta(\mu) = \sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \cdot \sigma(S, T)^{-1} \quad (2.1)$$

where the Hopf structure on  $\mathcal{P}$  is given by  $\delta(\mu) = \sum_{(1),(2)} \mu_{(1)} \otimes \mu_{(2)}$  and for  $S = \{s_1 < \dots < s_k\}$  and  $T = \{t_1 < \dots < t_{n-k}\}$  the permutation  $\sigma(S, T)$  is  $(s_1, \dots, s_k, t_1, \dots, t_{n-k})$ . The coproduct  $\Delta$  is cocommutative if  $\mathcal{P}$  is a cocommutative Hopf operad.

*Proof.* Consider the map of  $\mathbb{S}$ -modules induced by the embeddings  $I \oplus I \subset \mathcal{P}(1) \oplus \mathcal{P}(1) = \mathcal{P}(1) \otimes \mathcal{P}(0) \oplus \mathcal{P}(0) \otimes \mathcal{P}(1) \subset \mathcal{P} \hat{\otimes} \mathcal{P}$ :

$$\begin{aligned} \phi : I &\rightarrow \mathcal{P} \hat{\otimes} \mathcal{P} \\ 1_1 &\mapsto (1_1 \otimes 1_0) + (1_0 \otimes 1_1). \end{aligned}$$

Since  $\mathcal{P}$  is the free twisted  $\mathcal{P}$ -algebra on  $I$  and since  $\mathcal{P}\hat{\otimes}\mathcal{P}$  is endowed with a twisted  $\mathcal{P}$ -algebra structure thanks to theorem 2.1.3, there is a unique morphism of twisted  $\mathcal{P}$ -algebras

$$\Phi : \mathcal{P} \circ I = \mathcal{P} \rightarrow \mathcal{P}\hat{\otimes}\mathcal{P}$$

extending  $\phi$ . Indeed

$$\Phi = \mu_{\mathcal{P}\hat{\otimes}\mathcal{P}}(\mathcal{P} \circ \phi) = (\mu_{\mathcal{P}}\hat{\otimes}\mu_{\mathcal{P}})T_2(\delta \circ \phi)$$

As a consequence

$$\begin{aligned} \Phi(\mu) &= (\mu_{\mathcal{P}}\hat{\otimes}\mu_{\mathcal{P}})T_2\left(\sum_{(1),(2)} \mu_{(1)} \otimes \mu_{(2)} \otimes (1_0 \otimes 1_1 + 1_1 \otimes 1_0, \dots, 1_0 \otimes 1_1 + 1_1 \otimes 1_0)\right) \\ &= (\mu_{\mathcal{P}}\hat{\otimes}\mu_{\mathcal{P}})\left(\sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} \mu_{(1)} \otimes (x_1 \otimes \dots \otimes x_n) \otimes \mu_{(2)} \otimes (y_1 \otimes \dots \otimes y_n) \cdot \sigma(S, T)^{-1}\right) \\ &= \sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \cdot \sigma(S, T)^{-1} \end{aligned}$$

where

$$\begin{cases} x_i = 1_1 \text{ and } y_i = 1_0, & \text{if } i \in S, \\ x_i = 1_0 \text{ and } y_i = 1_1, & \text{if } i \in T, \end{cases}$$

and  $\sigma_{S,T}^{-1}$  is the shuffle coming from  $T_2$ . The unit  $\eta_{\mathcal{P}}$  and counit  $\epsilon_{\mathcal{P}}$  are given by the isomorphism between  $\mathbf{k}$  and  $\mathcal{P}(0)$ . Indeed

$$(\epsilon_{\mathcal{P}} \otimes \text{Id})\Delta(\mu) = \sum_{(1),(2)} (\mu_{(1)}|_{\emptyset})\mu_{(2)} = \mu$$

because  $\mu_{(1)}|_{\emptyset} = \epsilon(\mu_{(1)})$  and  $\epsilon$  is the counit for  $\delta$ .

The coassociativity of  $\Delta$  follows from the coassociativity of  $\phi$  and the unicity of the extension of  $\phi$  as a morphism of  $\mathcal{P}$ -algebras. The cocommutativity is clear in case  $\mathcal{P}$  is a cocommutative Hopf operad.  $\square$

**2.2.4. Example.** In case  $\mathcal{P} = \mathcal{A}$ s one has

$$\Delta(\sigma) = \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T \cdot \sigma(S, T)^{-1}.$$

This is the twisted Hopf algebra structure on the direct sum of the symmetric group algebras described by Patras and Reutenauer in [PR04].

**2.2.5. Corollary.** *Let  $\mathcal{P}$  be a connected Hopf operad. Any free twisted  $\mathcal{P}$ -algebra has a canonical twisted Hopf  $\mathcal{P}$ -algebra structure, cocommutative whenever  $\mathcal{P}$  is.*

*Proof.* The map  $\mathcal{P} \circ M \rightarrow (\mathcal{P} \circ M)\hat{\otimes}(\mathcal{P} \circ M)$  is given by the composite

$$\mathcal{P} \circ M \rightarrow (\mathcal{P}\hat{\otimes}\mathcal{P}) \circ M = (\mathcal{P} \circ M)\hat{\otimes}(\mathcal{P} \circ M).$$

thanks to relation (1.7). The unit  $\eta_{\mathcal{P} \circ M}$  and counit  $\epsilon_{\mathcal{P} \circ M}$  are induced by the one on  $\mathcal{P}$ : since  $\mathbf{1} \circ M = \mathbf{1}$ , one has  $\eta_{\mathcal{P} \circ M} := \eta_{\mathcal{P}} \circ M : \mathbf{1} = \mathbf{1} \circ M \rightarrow \mathcal{P} \circ M$  and  $\epsilon_{\mathcal{P} \circ M} := \epsilon_{\mathcal{P}} \circ M : \mathcal{P} \circ M \rightarrow \mathbf{1} \circ M = \mathbf{1}$ .  $\square$

### 2.3. Primitive elements.

**2.3.1. Definition.** Let  $\mathcal{P}$  be a connected Hopf operad and  $M$  be a twisted Hopf  $\mathcal{P}$ -algebra. Then  $M$  has a unit  $\eta_M : \mathbf{1} \rightarrow M$  and a counit  $\epsilon_M : M \rightarrow \mathbf{1}$  such that  $\epsilon_M \eta_M = \mathbf{1}$ . Let  $1_M$  be the image of  $1_0$  by  $\eta_M$ . Let  $\overline{M} := \text{Ker } \epsilon_M$ . One has

$$M = \mathbf{1} \oplus \overline{M}$$

and for all  $x \in \overline{M}$  one has

$$\Delta_M(x) = 1_M \otimes x + x \otimes 1_M + \sum_{i,j} x_i \otimes x_j$$

with  $x_i, x_j \in \overline{M}$ . The space  $\text{Prim}(M)$  of *primitive elements* of  $M$  is

$$\text{Prim}(M) = \{x \in \overline{M} \mid \Delta(x) = 1_M \otimes x + x \otimes 1_M\}.$$

Since  $\Delta_M$  is a morphism of  $\mathbb{S}$ -modules,  $\text{Prim}(M)$  is a sub  $\mathbb{S}$ -module of  $M$ . In the sequel,  $\overline{\Delta}$  denotes the projection of  $\Delta$  onto  $\overline{M} \hat{\otimes} \overline{M}$ . The space of primitive elements is then  $\text{Prim}(M) = \ker(\overline{\Delta} : \overline{M} \rightarrow \overline{M} \hat{\otimes} \overline{M})$ . Note that if  $V$  is a  $\mathbb{S}$ -module then  $\mathcal{P} \circ V$  is a twisted Hopf  $\mathcal{P}$ -algebra and

$$\overline{\mathcal{P} \circ V} = \overline{\mathcal{P}} \circ V \tag{2.2}$$

Since  $\overline{\Delta}$  is coassociative, one can define consistently  $\overline{\Delta}^{[n]}$  as a map from  $\overline{M}$  to  $\overline{M}^{\hat{\otimes} n}$  (e.g.  $\overline{\Delta}^{[3]} := \overline{\Delta} \hat{\otimes} \overline{M} = \overline{M} \hat{\otimes} \overline{\Delta}$ ). The twisted Hopf  $\mathcal{P}$ -algebra  $M$  is said to be *connected* if for any  $x \in \overline{M}$  there exists  $n$  such that

$$\overline{\Delta}^{[n]}(x) = 0.$$

For instance,  $\mathcal{P}$  is a connected twisted Hopf  $\mathcal{P}$ -algebra.

**2.3.2. Theorem.** *Let  $\mathcal{P}$  be a connected Hopf operad. The space of primitive elements of the twisted Hopf  $\mathcal{P}$ -algebra  $\mathcal{P}$  is a sub-operad of  $\mathcal{P}$ .*

*Proof.* Notice that  $\mu \in \mathcal{P}(n)$  is primitive if and only if

$$\sum_{S \sqcup T = [n], S, T \neq \emptyset} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \cdot \sigma(S, T)^{-1} = 0,$$

that is if and only if  $\Delta_{S,T}(\mu) := \mu_{(1)}|_S \otimes \mu_{(2)}|_T = 0$ .

As pointed out in the definition 2.3.1, the space  $Q = \text{Prim}(\mathcal{P})$  is a sub- $\mathbb{S}$ -module of  $\mathcal{P}$ ; moreover  $1_1 \in Q(1)$ .

Assume  $p \in Q(n)$ ,  $q_i \in Q(l_i)$ ,  $l_i > 0$  for all  $1 \leq i \leq n$  and  $S \sqcup T = [l_1 + \dots + l_n]$ ,  $S, T \neq \emptyset$ . Let us write  $S = S_1 \star \dots \star S_n$  and  $T = T_1 \star \dots \star T_n$  with  $S_i \sqcup T_i = [l_i]$ . Since  $\Delta : \mathcal{P} \rightarrow \mathcal{P} \hat{\otimes} \mathcal{P}$  is a morphism of  $\mathcal{P}$ -twisted algebras one has



$$\Delta_{S,T}(p(q_1, \dots, q_n)) = \sum_{(a),(b),(a_1, \dots, a_n), (b_1, \dots, b_n)} p_{(a)}(q_{1(a_1)}|_{S_1}, \dots, q_{n(a_n)}|_{S_n}) \otimes p_{(b)}(q_{1(b_1)}|_{T_1}, \dots, q_{n(b_n)}|_{T_n}).$$

Since the  $q_i$  are primitive elements, the displayed quantity is equal to 0 if there exists  $i \leq n$  with  $S_i \neq \emptyset$  and  $T_i \neq \emptyset$ . So, let us assume that, for any  $i$ ,  $S_i = \emptyset$  or  $T_i = \emptyset$ , and let us set:  $\{i_1, \dots, i_k\} = \{i, S_i \neq \emptyset\}$  and  $\{j_1, \dots, j_{n-k}\} = [n] - \{i_1, \dots, i_k\} = \{j, T_j \neq \emptyset\}$ . We get by lemma 1.2.5

$$\Delta_{S,T}(p(q_1, \dots, q_n)) = \sum_{(1),(2)} p_{(1)}|_{\{i_1, \dots, i_k\}}(q_{i_1}, \dots, q_{i_k}) \otimes p_{(2)}|_{\{j_1, \dots, j_{n-k}\}}(q_{j_1}, \dots, q_{j_{n-k}})$$

which is equal to zero since  $p$  is primitive, and  $S, T \neq \emptyset$  implies that  $k, n-k \neq 0$ .  $\square$

For example, if  $\mathcal{P} = \mathcal{A}s$ , it follows from the theorem and from [PR04, Prop.17] that the operad of primitive elements of  $\mathcal{A}s$ , viewed as a twisted Hopf algebra, is the Lie operad. This is not a surprising result in view of the classical Lie theory and structure theorems for Hopf algebras such as the Cartier-Milnor-Moore theorem [MM65, Pat94], however it shows that many properties of classical Hopf algebraic structures are encapsulated in the twisted Hopf algebra structure of the corresponding Hopf operad.

The next sections are devoted to the systematical study of these properties.

Before turning out to this systematical study, let us consider the example of the magmatic operads considered by Holtkamp in [Hol05].

**2.4. Magmatic operads.** In [Hol05], Holtkamp considers free operads  $\text{Mag}_N$  and  $\text{Mag}_\omega$  which are generated by an operation in  $k$  variables  $\vee^k$  for each  $2 \leq k \leq N$  or for each  $2 \leq k$ . These operads can be turned out to connected operads by setting  $\vee^k \circ_i 1_0 = \vee^{k-1}$  and more generally  $\vee^k|_S = \vee^{|S|}$ , where  $\vee^1 = 1_1$  and  $\vee^0 = 1_0$ . Moreover these operads are connected Hopf operad with  $\delta(\vee^k) = \vee^k \otimes \vee^k$  and  $\epsilon(\vee^k) = 1_{\mathbf{k}}$ . As a consequence, we recover from the previous theorem some results of his paper. For instance,  $\text{Prim}(\text{Mag}_N)$  is a suboperad of  $\text{Mag}_N$  and the same property holds for  $\text{Mag}_\omega$ .

### 3. RECIPROCITY LAWS

**3.1. Hopf algebras over an operad.** In the present section, we investigate the relations between the structure of a connected Hopf operad and the structure of Hopf algebras (twisted and classical) over this operad.

In this section, and the following one,  $\mathcal{P}$  is a connected Hopf operad, and  $Q$  is the operad of primitive elements of  $\mathcal{P}$ .

**3.1.1. Theorem.** *Let  $H$  be a twisted Hopf  $\mathcal{P}$ -algebra. Then, the  $\mathbb{S}$ -module  $\text{Prim}(H)$  is a  $Q$ -algebra.*

*Proof.* Since any element in  $\text{Prim}(H)$  satisfies  $\Delta_H(h) = 1_H \otimes h + h \otimes 1_H$  the same proof as in theorem 2.3.2 holds using relation (1.14) instead of lemma 1.2.5.  $\square$

As a direct consequence, due to the Restriction to **Vect** principle, the set of primitive elements of any Hopf algebra over  $\mathcal{P}$  is a  $\text{Prim}(\mathcal{P})$ -algebra.

**3.1.2. Generalized Lie monomials.** Recall that, in the classical theory of free Lie algebras, a *Lie polynomial* in a free associative algebra (or tensor algebra)  $T(V)$  over a vector space  $V$  is an arbitrary element in the free Lie algebra over  $V$ , where the latter is viewed as a sub Lie algebra of  $T(V)$ .

Due to a slightly misleading convention, a Lie monomial is a non commutative monomial of Lie polynomials. Lie monomials and Lie polynomials are one of the basic tools in the study of free Lie algebras, and, actually, most of the properties of free Lie algebras can be deduced from the behaviour of Lie monomials. We refer to the book by Reutenauer [Reu93], where this point of view is developed in a systematic way. More generally, Lie monomials can be defined in an arbitrary enveloping algebra  $U(L)$ , as non commutative monomials on  $L$  (where, as usual,  $U(L)$  is viewed as a quotient of  $T(L)$ ).

From the combinatorial point of view, the fundamental property of Lie monomials is their behaviour with respect to the coproduct in the tensor algebra and, more generally, in an arbitrary graded connected cocommutative Hopf algebra: besides [Reu93], we refer to [Pat94], and the computations therein, for more informations on the subject.

The purpose of the next paragraphs is to extend this property to arbitrary Hopf operads.

**3.1.3. Definition.** Let  $H$  be a twisted Hopf  $\mathcal{P}$ -algebra and  $\text{Prim}(H)$  the  $Q$ -algebra of primitive elements in  $H$ . By analogy with the case of Lie monomials, we call  *$Q$ -monomials* the elements of the free  $\mathcal{P}$ -algebra over  $\text{Prim}(H)$ .

**3.1.4. Theorem.** *We have the identity for  $Q$ -monomials:*

$$\forall \mu \in \mathcal{P}(n), \forall h_1, \dots, h_n \in \text{Prim}(H)$$

$$\Delta(\mu(h_1, \dots, h_n)) = \sum_{S, T} \Delta_{S, T}(\mu)(h_{i_1}, \dots, h_{i_k} \otimes h_{j_1}, \dots, h_{j_{n-k}}) \quad (3.1)$$

where  $S = \{i_1, \dots, i_k\}$  and  $T = \{j_1, \dots, j_{n-k}\}$  run over the partitions of  $[n]$ , and where we write  $\Delta_{S, T}(\mu)$  for the component of  $\Delta(\mu)$  in  $\mathcal{P}(S) \otimes \mathcal{P}(T) \subset (\mathcal{P} \hat{\otimes} \mathcal{P})(n)$ .

Once again, the theorem follows by adapting the proof of 2.3.2.

Due to the Reduction to **Vect** principle, the theorem includes for example, as a particular case, the computation of coproducts of Lie monomials in an

enveloping algebra. The interested reader may check that one recovers, in that case, the unshuffling coproduct formula familiar in Lie theory [Reu93].

**3.2. Free algebras.** In the present section, we assume that  $\mathbf{k}$  is a field of characteristic zero.

**3.2.1. Theorem.** *The  $Q$ -algebra of primitive elements of the free twisted Hopf  $\mathcal{P}$ -algebra  $\mathcal{P} \circ V$  over a  $\mathbb{S}$ -module  $V$  is canonically isomorphic to the free twisted  $Q$ -algebra  $Q \circ V$ .*

In particular, due to the Reduction to **Vect** principle, it follows that the primitive elements of the free  $\mathcal{P}$ -algebra over a vector space, viewed as a twisted Hopf  $\mathcal{P}$ -algebra, identify with the elements of the free  $Q$ -algebra over  $V$ . The result generalizes to arbitrary Hopf operads the fundamental property of associative algebras: the primitive part of a tensor algebra (i.e. of a free associative algebra, naturally provided with a cocommutative Hopf algebra structure) is a free Lie algebra [Reu93].

*Proof.* By definition of  $Q$ , the following sequence of  $\mathbb{S}$ -modules is left exact:

$$Q \xrightarrow{i} \overline{\mathcal{P}} \xrightarrow{\overline{\Delta}} \overline{\mathcal{P}} \hat{\otimes} \overline{\mathcal{P}}$$

(see sections 2.3).

In particular, for any  $n$ , the sequence

$$Q(n) \xrightarrow{i} \overline{\mathcal{P}}(n) \xrightarrow{\overline{\Delta}} \overline{\mathcal{P}} \hat{\otimes} \overline{\mathcal{P}}(n)$$

is a left exact sequence of right  $S_n$ -modules. Recall besides that, for any finite group and any field  $\mathbf{k}$  of characteristic 0, every  $\mathbf{k}[G]$ -module is projective. In particular, for any left  $S_n$ -module, the tensor product  $- \otimes_{S_n} M$  is an exact functor (see e.g. [Bro94, Sect. I.8]) and we have finally, for any  $\mathbb{S}$ -module  $V$ , a left exact sequence:

$$Q \circ V \rightarrow \overline{\mathcal{P}} \circ V \xrightarrow{\overline{\Delta} \circ V} (\overline{\mathcal{P}} \hat{\otimes} \overline{\mathcal{P}}) \circ V$$

From (2.2) one has  $\overline{\mathcal{P}} \circ V = \overline{\mathcal{P} \circ V}$  and from (1.7) one has  $(\overline{\mathcal{P}} \hat{\otimes} \overline{\mathcal{P}}) \circ V = \overline{\mathcal{P} \circ V} \hat{\otimes} \overline{\mathcal{P} \circ V}$ . As a consequence,  $Q \circ V = \text{Prim}(\mathcal{P} \circ V)$ .  $\square$

#### 4. A CARTIER-MILNOR-MOORE THEOREM

**4.1. Multiplicative Hopf operads.** Let  $\mathcal{P}$  be a connected Hopf operad. Then,  $\mathcal{P}$  has naturally the structure of a twisted Hopf  $\mathcal{P}$ -algebra. That is, there is a coproduct map from  $\mathcal{P}$  to  $\mathcal{P} \hat{\otimes} \mathcal{P}$  which is a morphism of twisted  $\mathcal{P}$ -algebras.

Assume that  $\phi : U \rightarrow \mathcal{P}$  is a morphism of connected Hopf operad. In view of theorem 2.1.3, this requirement amounts to the following condition. The morphism  $\phi$ , as any morphism of operads from  $U$  to  $\mathcal{P}$ , provides an arbitrary  $\mathcal{P}$ -algebra with the structure of a  $U$ -algebra. In particular, the tensor product  $A \hat{\otimes} B$  of two  $\mathcal{P}$ -algebras, which is a  $\mathcal{P}$ -algebra (since  $\mathcal{P}$  is a Hopf operad) carries naturally the structure of a  $U$ -algebra.

On the other hand,  $\phi$  induces on  $A$  and  $B$  a structure of  $U$ -algebra and, since  $U$  is a connected Hopf operad, the tensor product  $A \hat{\otimes} B$  carries the structure of a  $U$ -algebra. The hypothesis that  $\phi$  is a morphism of connected Hopf operads ensures that the two structures of  $U$ -algebras on  $A \hat{\otimes} B$  are identical.

It follows in particular that the coproduct map from  $\mathcal{P}$  to  $\mathcal{P} \hat{\otimes} \mathcal{P}$  is also, by restriction, a morphism of  $U$ -algebras, and  $\mathcal{P}$  inherits from this construction the structure of a twisted Hopf  $U$ -algebra. More generally, we have:

**4.1.1. Proposition.** *Let  $\phi : U \rightarrow \mathcal{P}$  be a morphism of connected Hopf operad. Then,  $\mathcal{P}$  and, more generally, any twisted Hopf  $\mathcal{P}$ -algebra, is naturally provided with the structure of a twisted Hopf  $U$ -algebra.*

**4.1.2. Definiton.** A multiplicative Hopf operad  $\mathcal{P}$  is a connected Hopf operad together with a morphism of connected Hopf operads  $\phi : \mathcal{A}s \rightarrow \mathcal{P}$ .

Notice that the map  $\mathcal{A}s \rightarrow \mathcal{P}$  induces a structure of twisted Hopf algebra on  $\mathcal{P}$  in view of theorem 2.2.3 and proposition 4.1.1, since  $\mathcal{P}$  is a twisted connected Hopf  $\mathcal{P}$ -algebra.

**4.2. Theorem.** *Any multiplicative co-commutative Hopf operad is –as a twisted Hopf algebra– the twisted enveloping algebra of its primitive elements.*

*Proof.* Let  $\mathcal{P}$  be a multiplicative co-commutative Hopf operad. Recall first that the notion of enveloping algebra holds in the category of  $\mathbb{S}$ -modules – that is, a twisted (associative) algebra is naturally associated to any twisted Lie algebra (with the usual universal properties of enveloping algebras, see e.g. [Joy86]). Recall also that the Cartier-Milnor-Moore theorem holds for twisted connected Hopf algebras in any characteristic. More precisely, if  $H$  is a connected cocommutative twisted Hopf algebra,  $\text{Prim}(H)$  carries naturally the structure of a twisted Lie algebra (this follows e.g. from our theorem 3.1.1). The embedding  $\text{Prim}(H) \hookrightarrow H$  of the primitive elements of  $H$  into  $H$  induces an isomorphism of twisted Hopf algebras between the twisted enveloping algebra of  $\text{Prim}(H)$  and  $H$ . This was proven by Stover [Sto93]; alternatively, as has been pointed out by Fresse [Fre98, Appendix A], the combinatorial proof of the classical Cartier-Milnor-Moore given in [Pat94] holds in any graded linear symmetric monoidal category, and therefore applies to twisted Hopf algebras, which are Hopf algebras in the category of  $\mathbb{S}$ -modules. Our theorem follows.  $\square$

## 5. THE POISSON OPERAD

In this section we assume that  $\mathbf{k}$  is of characteristic 0.

Recall that a Poisson algebra  $A$  is a commutative algebra with a unit 1 provided with a Lie bracket  $[\cdot, \cdot]$  which is a biderivation. That is, we have, besides the antisymmetry and Jacobi identities for  $[\cdot, \cdot]$ , the Poisson distributivity formula:

$$[f, gh] = [f, g]h + g[f, h] \tag{5.1}$$

In particular, we have  $[f, 1] = 0$ .

The simplest way to describe the Poisson operad  $\mathcal{Pois}$  is through the corresponding functor:

$$\mathcal{Pois}(V) = \mathcal{Com} \circ \mathcal{Lie}(V) \quad (5.2)$$

where  $\mathcal{Com}$  is the operad of commutative algebras with a unit, and  $\mathcal{Lie}$  the Lie operad. In concrete terms, an element of the free Poisson algebra over  $V$  is a commutative polynomial in the Lie polynomials (the elements of  $\mathcal{Lie}(V)$ ), and the bracket of two such commutative polynomials is computed using (iteratively) the Poisson distributivity formula and the Lie bracket in  $\mathcal{Lie}(V)$ .

Due to the Poincaré-Birkhoff-Witt theorem, which states that  $\mathcal{As}(V)$  is isomorphic to  $\mathcal{Com} \circ \mathcal{Lie}(V)$  as analytic functors,  $\mathcal{Pois}(V)$  and  $\mathcal{As}(V)$  are isomorphic as analytic functors. Therefore they are also isomorphic as  $\mathbb{S}$ -modules, as a consequence of the correspondance between polynomial functors and symmetric group representations. We refer to [Mac95, Appendix A] for further details on analytic functors, polynomial functors and symmetric group representations. In particular,  $\mathcal{Pois}(n)$  is isomorphic, as a right  $S_n$ -module to the regular representation of  $S_n$ .

Recall however that the Poincaré-Birkhoff-Witt theorem also holds in the category of  $\mathbb{S}$ -modules: the enveloping algebra of a connected twisted Lie algebra  $L$  is isomorphic, as a  $\mathbb{S}$ -module, to the free twisted commutative algebra over  $L$  [Joy86, theorem 2].

We are going to show that the (classical) Poincaré-Birkhoff-Witt isomorphism between  $\mathcal{Pois}(V)$  and  $\mathcal{As}(V)$  can be lifted to the Hopf operadic setting, and understood directly by means of the Joyal's Poincaré-Birkhoff-Witt theorem for the twisted enveloping algebras of twisted Lie algebras.

**5.1. Primitive elements of the Poisson operad.** Recall that the Poisson operad is a connected cocommutative Hopf operad. Let  $[, ]$  and  $\mu$  be the two generators in  $\mathcal{Pois}(2)$  representing the Lie structure and commutative structure. Then one has the following

$$\begin{cases} \mu|_{\emptyset} = 1_0, \\ \mu|_S = 1_1, \text{ for } |S| = 1 \end{cases} \quad \text{and} \quad [, ]|_T = 0, \text{ for } |T| < 2.$$

and

$$\delta(\mu) = \mu \otimes \mu, \quad \delta([, ]) = [, ] \otimes \mu + \mu \otimes [, ].$$

The last two equations mean in terms of Poisson algebras, that if  $A$  and  $B$  are two Poisson algebras, the tensor product  $A \otimes B$  is provided with a Poisson structure as follows. As a commutative algebra,  $A \otimes B$  is provided with the usual commutative product,  $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ . The bracket on  $A \otimes B$  is defined by:  $[a_1 \otimes b_1, a_2 \otimes b_2] = a_1 a_2 \otimes [b_1, b_2] + [a_1, a_2] \otimes b_1 b_2$ .

It follows in particular from the definition of  $\delta$  that the natural inclusion gives a morphism of connected Hopf operad  $\mathcal{Com} \rightarrow \mathcal{Pois}$ , so that the proposition 4.1.1 applies: any twisted Hopf  $\mathcal{Pois}$ -algebra is naturally provided with the structure of a commutative twisted Hopf algebra.

**5.2. Lemma.** *The Lie operad is a suboperad of the operad of primitive elements in  $\mathcal{Pois}$ .*

*Proof.* Since the Lie operad is a sub-operad of  $\mathcal{Pois}$  and is generated by the Lie bracket  $[\cdot, \cdot]$ , it is enough, to prove the lemma, to check that  $[\cdot, \cdot] \in \mathcal{Pois}(2)$  is a primitive element. Thanks to theorem 2.2.3 one has to compute

$$\Delta([\cdot, \cdot]) = \sum_{(1),(2)} \sum_{S \sqcup T = [2]} ([\cdot, \cdot]_{(1)})|_S \otimes ([\cdot, \cdot]_{(2)})|_T.$$

Since  $[\cdot, \cdot]|_S = 0$  for  $|S| < 2$  the latter equality writes

$$\Delta([\cdot, \cdot]) = 1_0 \otimes [\cdot, \cdot] + [\cdot, \cdot] \otimes 1_0,$$

and  $[\cdot, \cdot]$  is primitive.  $\square$

**5.3. Theorem.** *The Poisson operad is naturally provided with the structure of a commutative and cocommutative twisted Hopf algebra. The suboperad of primitive elements of  $\mathcal{Pois}$  is the Lie operad. Moreover, as a twisted Hopf algebra,  $\mathcal{Pois}$  is isomorphic to the free commutative twisted algebra over the  $\mathbb{S}$ -module  $\mathcal{L}ie$ .*

*Proof.* As pointed out before, the inclusion  $\mathcal{Com} \subset \mathcal{Pois}$  induces a morphism of connected Hopf operads  $\mathcal{Com} \rightarrow \mathcal{Pois}$ , thus, by composition with  $\mathcal{As} \rightarrow \mathcal{Com}$ , a morphism of connected Hopf operads  $\mathcal{As} \rightarrow \mathcal{Pois}$ . Then we can apply the theorem 4.2 to  $\mathcal{Pois}$ . Since  $\mathcal{Pois}$  is also twisted commutative, the twisted Lie algebra structure on  $\text{Prim}(\mathcal{Pois})$  is trivial (see [Sto93]), and  $\mathcal{Pois}$  is isomorphic, as a Hopf algebra, to the free twisted commutative algebra over the  $\mathbb{S}$ -module of its primitive elements (recall that a free twisted  $\mathcal{P}$ -algebra over a connected Hopf operad  $\mathcal{P}$  is naturally provided with a twisted Hopf  $\mathcal{P}$ -algebra structure, so that a free commutative twisted algebra is naturally provided with a twisted commutative Hopf algebra structure).

It remains to prove that  $\mathcal{L}ie$  is the set of primitive elements in  $\mathcal{Pois}$ . We conclude by a dimensionality argument based on the remark that, if the  $\mathbb{S}$ -module  $A$  is a sub  $\mathbb{S}$ -module of  $B$ , and if  $\dim_{\mathbf{k}} A(n) < \infty$  for all  $n$ , then, if the free twisted commutative algebras over  $A$  and  $B$  have the same dimension over  $\mathbf{k}$  in each degree, then  $A = B$ .

Recall from [PR04, Prop.17] that  $\mathcal{As}$  is, as a twisted Hopf algebra, the enveloping algebra of  $\mathcal{L}ie$ . Due to Joyal's Poincaré-Birkhoff-Witt theorem, it follows that the dimension of the component of degree  $n$  of the free commutative twisted algebra over  $\mathcal{L}ie$  is equal to the dimension of  $\mathcal{As}(n)$ , that is, to  $n!$ . Besides,  $\mathcal{Pois}(n)$  is isomorphic to the regular representation of  $S_n$  as a  $S_n$ -module and, in particular, has dimension  $n!$  as a vector space. Since  $\mathcal{L}ie$  is contained in  $\text{Prim}(\mathcal{Pois})$ , and since, according to our previous

arguments, the dimensions of the graded components of the free twisted commutative algebras over  $\mathcal{L}ie$  and over  $\text{Prim}(\mathcal{P}ois)$  are equal, the theorem follows:  $\mathcal{L}ie = \text{Prim}(\mathcal{P}ois)$ .  $\square$

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