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**Abstract**

We describe an algorithm for implicitizing rational hypersurfaces with at most a finite number of base points, based on a technique described in Busé and Jouanolou [2003], where implicit equations are obtained as determinants of certain graded parts of an approximation complex. We detail and improve this method by providing an in-depth study of the cohomology of such a complex. In both particular cases of interest of curve and surface implicitization we also present algorithms which involve only linear algebra routines.

## 1. Introduction

The implicitization problem asks for the computation of an implicit equation of a rational hypersurface given by a parameterization map $\phi : \mathbb{P}^{n-1} \to \mathbb{P}^n$, with $n \geq 2$. This problem has recently received particular attention, especially in the cases $n = 2$ and $n = 3$, because it is a key-point in computer aided geometric design and modeling. There are basically three kinds of methods to compute such an implicit equation for rational curves and surfaces. The first kind of methods use Gröbner basis computations. Even if they work, they are known to be quite slow in practice and hence are rarely used in geometric modeling (see e.g. Hoffmann [1989]). The second kind of methods are based on resultant matrices. Such methods have the advantage of yielding square matrices whose determinant is an implicit equation. This more compact formulation of an implicit equation is very

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useful, and well behaved algorithms are known to work. However such methods are known only if there are no base points (see [Jouanolou, 1996, §5.3.17]), or the base points are isolated and local complete intersections with some additional technical hypothesis (see Busé [2001]). The third methods are based on the syzygies of the parameterization. This technique was introduced in Sederberg and Chen [1995] as the method of "moving surfaces". For curve implicitization this method, also called "moving lines", is very efficient and general. This generality is no longer true for surface implicitization, but the method remains very efficient. In Cox et al. [2000] and D'Andrea [2001] its validity is proved in the absence of base points. An extension in the presence of base points is presented in Busé et al. [2003], assuming five hypotheses on the geometry of the base points, one being that the base points are isolated and locally a complete intersection.

A similar approach involving syzygies was recently explored in Busé and Jouanolou [2003] using more systematic tools from algebraic geometry and commutative algebra, among them the approximation complexes (see Vasconcelos [1994]). A new method was given to solve the hypersurface (and hence curve and surface) implicitization problem in the case where the base points are isolated and locally a complete intersection, without any additional hypothesis.

In this paper we detail and improve the results described in Busé and Jouanolou [2003] for implicitizing rational hypersurfaces in the case where the ideal of base points is finite and a projective local almost complete intersection. We also give some other properties based on the exactness of approximation complexes. Section 2 and section 3 provide a complete and detailed description of our results in the particular cases of interest of curve and surface implicitization. Algorithms are completely described and we present some examples. Section 4 deals with the general case of hypersurface implicitization, and is much more technical than both previous sections. In particular all the proofs are given in this section.

Hereafter $\mathbb{K}$ denotes any field.

## 2. Implicitization of rational parametric curves

In this section we present a method to implicitize any parameterized plane curve. Let $f_0, f_1, f_2$ be three homogeneous polynomials in $\mathbb{K}[s,t]$ of the same degree $d \geq 1$, and consider the rational map

$$\begin{array}{rccl} \phi: & \mathbb{P}^1_{\mathbb{K}} & \rightarrow & \mathbb{P}^2_{\mathbb{K}} \\ & (s:t) & \mapsto & (f_0(s,t):f_1(s,t):f_2(s,t)). \end{array}$$

We denote by $x, y, z$ the homogeneous coordinates of $\mathbb{P}^2$. The algebraic closure of the image of $\phi$ is a curve in $\mathbb{P}^2$ if and only if $\phi$ is generically finite onto its image, that is $\delta := \gcd(f_0, f_1, f_2)$ is not of degree $d$. We assume this minimal hypothesis and denote by $\mathcal{C}$ the scheme-theoretic closed image of $\phi$. It is known that $\mathcal{C}$ is an irreducible and reduced curve of degree $(d - \deg(\delta))/\beta$, where $\beta$ denotes the degree of $\phi$ onto its image (that is the number of points in a general fiber). Thus any implicit equation of $\mathcal{C}$ is of the form $P(x, y, z) = 0$ where $P$ is an irreducible homogeneous polynomial of degree $(d - \delta)/\beta$ (actually there is a unique implicit equation of $\mathcal{C}$ up to multiplication by a nonzero constant). By abuse of language, we will call $P(x, y, z)$ an implicit equation of $\mathcal{C}$.

In what follows we describe an algorithm for computing explicitly an implicit equation $P$ of $\mathcal{C}$ (in fact we will compute $P^\beta$) without any other hypothesis on the polynomials $f_0, f_1, f_2$. In the case $\deg(\delta) = 0$ we recover the well-known method of moving lines (see Sederberg and Chen [1995], Cox et al. [1998]).

### 2.1. The method

Let us denote by $A$ the polynomial ring $\mathbb{K}[s, t]$ with its natural grading obtained by setting $\deg(s) = \deg(t) = 1$. From the polynomials $f_0, f_1, f_2$ of the parameterization $\phi$, we can build the following well-known graded Koszul complex (the notation $[-]$ stands for the degree shift in $A$)

$$0 \to A[-3d] \xrightarrow{d_3} A[-2d]^3 \xrightarrow{d_2} A[-d]^3 \xrightarrow{d_1} A, \tag{1}$$

where the differentials are given by

$$d_3 = \begin{pmatrix} f_2 \\ -f_1 \\ f_0 \end{pmatrix}, d_2 = \begin{pmatrix} f_1 & f_2 & 0 \\ -f_0 & 0 & f_2 \\ 0 & -f_0 & -f_1 \end{pmatrix}, d_1 = \begin{pmatrix} f_0 & f_1 & f_2 \end{pmatrix}.$$

In what follows we will not consider exactly this complex, but the complex obtained by tensoring it by $A[x, y, z]$ over $A$. This complex, which we denote by $(K_\bullet(f_0, f_1, f_2), u_\bullet)$, is of the form

$$0 \to A[x, y, z][-3d] \xrightarrow{u_3} A[x, y, z][-2d]^3 \xrightarrow{u_2} A[x, y, z][-d]^3 \xrightarrow{u_1} A[x, y, z],$$

where the matrices of the differentials $d_i$ and $u_i$ are the same, for all $i = 1, 2, 3$. Note that the ring $A[x, y, z]$ is naturally bi-graded, having one grading coming from $A = \mathbb{K}[s, t]$, and another one coming from $\mathbb{K}[x, y, z]$ with $\deg(x) = \deg(y) = \deg(z) = 1$; we hereafter adopt the notation $(-)$ for the degree shift in $\mathbb{K}[x, y, z]$.

We form another bi-graded Koszul complex on $A[x, y, z]$, the one associated to the sequence $(x, y, z)$. We denote it by $(K_\bullet(x, y, z), v_\bullet)$, it is of the form

$$0 \to A[x, y, z](-3) \xrightarrow{v_3} A[x, y, z](-2)^3 \xrightarrow{v_2} A[x, y, z](-1)^3 \xrightarrow{v_1} A[x, y, z],$$

and the matrices of its differentials are obtained from the matrices of the differentials of (1) by replacing $f_0$ by $x$, $f_1$ by $y$ and $f_2$ by $z$. Observe that since $(x, y, z)$ is a regular sequence in $A[x, y, z]$, the previous complex $K_\bullet(x, y, z)$ is acyclic, that is to say all its homology groups $H_i(K_\bullet(x, y, z))$ vanish for $i > 0$.

We can now construct a new bi-graded complex of $A[x, y, z]$-modules, denoted $\mathcal{Z}_\bullet$, from both Koszul complexes $(K_\bullet(f_0, f_1, f_2), u_\bullet)$ and $(K_\bullet(x, y, z), v_\bullet)$ (observe that these complexes differ only by their differentials). Define $Z_i := \ker(d_i)$ for all

$i = 0, \dots, 3$ (with $d_0 : A \to 0$), and set $\mathcal{Z}_i := Z_i[id] \otimes_A A[x,y,z]$ for $i = 0, \dots, 3$, which are bi-graded $A[x,y,z]$-modules. The map $v_1$ induces the bi-graded map, that we denote also by $v_1$,

$$\begin{array}{rcl} \mathcal{Z}_1(-1) & \xrightarrow{v_1} & \mathcal{Z}_0 = A[x,y,z] \\ (g_1, g_2, g_3) & \mapsto & g_1 x + g_2 y + g_3 z. \end{array}$$

Using the differential $v_2$ of $K_\bullet(x,y,z)$ we can map $\mathcal{Z}_2$ to $A[x,y,z](-d)^3$, but since $u_1 \circ v_2 + v_1 \circ u_2 = 0$ (which follows from a straightforward computation), we have $v_2(\mathcal{Z}_2) \subset \mathcal{Z}_1$. And in the same way we can map $\mathcal{Z}_3$ to $\mathcal{Z}_2$ with the differential $v_3$, since $u_2 \circ v_3 + v_2 \circ u_3 = 0$. Thus we obtain the following bi-graded complex (it is a complex since $(K_\bullet(x,y,z), v_\bullet)$ is):

$$(\mathcal{Z}_\bullet, v_\bullet) \ : 0 \to \mathcal{Z}_3(-3) \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[x,y,z].$$

This complex is known as the *approximation complex of cycles* associated to the polynomials $f_0, f_1, f_2$ in $\mathbb{K}[s,t]$. It was originally introduced in Simis and Vasconcelos [1981] for studying Rees algebras through symmetric algebras (see also Vasconcelos [1994]).

*Remark:* In the language of moving lines, the image of a triple $(g_1, g_2, g_3) \in \mathcal{Z}_{1[\nu](0)}$ by $v_1$ is nothing but a moving line of degree $\nu$ following the curve parameterized by the polynomials $f_0, f_1, f_2$ (see e.g. Cox [2001]).

Since we have supposed that at least one of $f_0, f_1, f_2$ is nonzero, we have $Z_3 = 0$, and consequently $\mathcal{Z}_3 = 0$, and the following theorem (recall $\delta := \gcd(f_0, f_1, f_2)$):

Theorem 2.1: *The determinant of the graded complex of free $\mathbb{K}[x,y,z]$-modules*

$$0 \to \mathcal{Z}_{2[d-1]}(-2) \xrightarrow{v_2} \mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]} = A[x,y,z]_{[d-1]},$$

*is $P(x,y,z)^\beta$, where $P$ is an implicit equation of the curve $\mathcal{C}$.*

*Moreover, if $\deg(\delta) = 0$ then $\mathcal{Z}_{2[d-1]} = 0$; thus the $d \times d$ determinant of the map $\mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]}$ equals $P(x,y,z)^\beta$.*

*Proof:* See section 5 in Busé and Jouanolou [2003] for a proof. We only mention how the last statement follows from the description of the approximation complex $\mathcal{Z}_\bullet$. If $\deg(\delta) = 0$ then $\mathrm{depth}_{(s,t)}(f_0, f_1, f_2) = 2$, which means that $f_0, f_1, f_2$ have no base points in $\mathbb{P}^1$. This implies that not only the third homology group of the Koszul complex (1) vanishes, but also the second. It follows that $Z_2 \simeq A[-3d]$, and hence $\mathcal{Z}_2 \simeq A[x,y,z][-d]$. □

The second assertion of this theorem gives exactly the matrix constructed by the method of moving lines. The first assertion shows that this method can be extended even if we do not assume $\deg(\delta) = 0$, $P^\beta$ being obtained as the quotient of two determinants of respective size $d$ and $\delta$. In Busé and Jouanolou [2003] it is in fact proved that for any integer $\nu \geq d-1$ the determinant of the complex $(\mathcal{Z}_\bullet)_{[\nu]}$ equals $P^\beta$. In case $\deg(\delta)=0$, this and the graded isomorphism $\mathcal{Z}_2 \simeq A[x,y,z][-d]$ explain clearly, in our point of view, why the method of moving lines works so well *only* with moving lines of degree $d-1$.

### 2.2. The algorithm

Here we convert theorem 2.1 into an algorithm, where each step reduces to well-known and efficient linear algebra routines.

ALGORITHM (implicitization of a rational parametric curve):
**Input:** Three homogeneous polynomials $f_0(s,t), f_1(s,t), f_2(s,t)$ of the same degree $d \geq 1$ such that $\delta = \gcd(f_0, f_1, f_2)$ is not of degree $d$.
**Output:** Either:

- a square matrix $\Delta_1$ such that $\det(\Delta_1)$ equals $P^\beta$, in case $\deg(\delta)=0$,
- two square matrices $\Delta_1$ and $\Delta_2$, respectively of size $d$ and $\deg(\delta)$, such that $\frac{\det(\Delta_1)}{\det(\Delta_2)}$ equals $P^\beta$.

1. *Compute the matrix* $\mathtt{F}_1$ *of the first map of* (1): $A^3_{d-1} \xrightarrow{d_1} A_{2d-1}$. *Each entry is either 0 or a coefficient of* $f_0, f_1$ *or* $f_2$*, and* $\mathtt{F}_1$ *is of size* $2d \times 3d$.
2. *Compute a kernel matrix* $\mathtt{K}_1$ *of the transpose of* $\mathtt{F}_1$. *It has* $3d$ *columns and* $\operatorname{rank}(\mathcal{Z}_{1[d-1]})$ *rows.*
3. *Construct the matrix* $\mathtt{Z}_1$ *by* $\mathtt{Z}_1(i,j) = x\mathtt{K}_1(j,i) + y\mathtt{K}_1(j,i+d) + z\mathtt{K}_1(j,i+2d)$, *with* $i=1,\dots,d$ *and* $j=1,\dots,\operatorname{rank}(\mathcal{Z}_{1[d-1]})$. *It is the matrix of the map* $\mathcal{Z}_{1[d-1]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[d-1]}$.
4. ***If*** $\mathtt{Z}_1$ *is square* ***then*** *set* $\Delta_1 := \mathtt{Z}_1$ ***else***
   (a) *Compute a list* $L_1$ *of* $d$ *integers indexing* $d$ *independent columns in* $\mathtt{Z}_1$. *Let* $\Delta_1$ *be the* $d \times d$ *submatrix of* $\mathtt{Z}_1$ *obtained by removing columns not in* $L_1$.
   (b) *Compute the matrix* $\mathtt{F}_2$ *of the second map of* (1): $A^3_{d-1} \xrightarrow{d_2} A^3_{2d-1}$, *and a kernel matrix* $\mathtt{K}_2$ *of its transpose. The matrix* $\mathtt{K}_2$ *has* $3d$ *columns and* $\operatorname{rank}(\mathcal{Z}_{2[d-1]})$ *rows.*
   (c) *Construct the matrix* $\mathtt{Z}'_2$ *by, for all* $j=1,\dots,\operatorname{rank}(\mathcal{Z}_{2[d-1]})$,
$$\begin{aligned} i=1,\dots,d &: \mathtt{Z}'_2(i,j) = y\mathtt{K}_2(j,i) + z\mathtt{K}_2(j,i+d), \\ i=d+1,\dots,2d &: \mathtt{Z}'_2(i,j) = -x\mathtt{K}_2(j,i-d) + z\mathtt{K}_2(j,i+d), \\ i=2d+1,\dots,3d &: \mathtt{Z}'_2(i,j) = -x\mathtt{K}_2(j,i-d) - y\mathtt{K}_2(j,i). \end{aligned}$$
   (d) *Construct the* $\operatorname{rank}(\mathcal{Z}_{1[d-1]}) \times \operatorname{rank}(\mathcal{Z}_{2[d-1]})$ *matrix* $\mathtt{Z}_2$ *whose* $j^{\text{th}}$ *column* $\mathtt{Z}_2(\bullet,j)$ *is the solution of the linear system* ${}^t\mathtt{Z}_2(\bullet,j).\mathtt{K}_1 = \mathtt{Z}'_2(\bullet,j)$. *It is the matrix of the map* $\mathcal{Z}_{2[d-1]}(-1) \xrightarrow{v_2} \mathcal{Z}_{1[d-1]}$.

(e) *Define $\Delta_2$ to be the square submatrix of $\mathtt{Z}_2$ obtained by removing the rows indexed by $L_1$.*

***endif***

*Remark:* In order to make this algorithm easy to understand, we kept the symbolic variables $t$ and $z$, but they are of course unnecessary and should be specialized to 1.

### 2.3. An example

As we have already said, the previous algorithm is exactly the well-known method of moving lines in case $\deg(\delta) = 0$. The only thing new is its extension to the case where $\deg(\delta) > 0$. We illustrate the method with the following (very simple) example.

Let $f_0(s,t) = s^2$, $f_1(s,t) = st$ and $f_2(s,t) = t^2$. Applying our algorithm we found that the matrix $\mathtt{Z}_1$ (step 3) is square:

$$\mathtt{Z}_1 = \begin{pmatrix} -y & -z \\ x & y \end{pmatrix}.$$

We deduce that an implicit equation is given by $xz - y^2$. Now let us multiply artificially each $f_i$ by $s$. We obtain new entries for our algorithm which are: $f_0(s,t) = s^3$, $f_1(s,t) = s^2t$ and $f_2(s,t) = st^2$. In this case the matrix $\mathtt{Z}_1$ is not square, it is the $3 \times 4$ matrix:

$$\mathtt{Z}_1 = \begin{pmatrix} -y & -z & -z & 0 \\ x & 0 & y & -z \\ 0 & x & 0 & y \end{pmatrix}.$$

The matrix $\Delta_1$ can be chosen to be the submatrix of $\mathtt{Z}_1$ given by $\begin{pmatrix} -y & -z & -z \\ x & 0 & y \\ 0 & x & 0 \end{pmatrix}$, with determinant $-x^2z + xy^2$. Continuing the algorithm we obtain the matrices $\mathtt{Z}_2 = \begin{pmatrix} -z \\ y \\ 0 \\ -x \end{pmatrix}$, and $\Delta_2 = \begin{pmatrix} -x \end{pmatrix}$. It follows that an implicit equation is $-xz + y^2$.

## 3. Implicitization of rational parametric surfaces

We arrive at the much more intricate problem of surface implicitization. Let $f_0, f_1, f_2, f_3$ be four homogeneous polynomials in $\mathbb{K}[s,t,u]$ of the same degree $d \geq 1$, and consider the rational map

$$\begin{array}{cccc} \phi : & \mathbb{P}^2_{\mathbb{K}} & \to & \mathbb{P}^3_{\mathbb{K}} \\ & (s:t:u) & \mapsto & (f_0(s,t,u) : f_1(s,t,u) : f_2(s,t,u) : f_3(s,t,u)). \end{array}$$

We denote by $x, y, z, w$ the homogeneous coordinates of $\mathbb{P}^2$. The closure of the image of $\phi$ is a surface if and only if the map $\phi$ is generically finite onto its image, which we assume hereafter. Thus let $\mathcal{S}$ denote the surface in $\mathbb{P}^3$ obtained as the closed image of $\phi$. Our problem is to compute an implicit equation of $\mathcal{S}$, which is more difficult than curve implicitization because of the presence of isolated base points arising from the parameterization. If it exists, the one dimensionnal component of the base locus can be easily removed by substituting each $f_i$, $i = 0, \ldots, 3$, by $f_i/\delta$, where $\delta := \gcd(f_0, f_1, f_2, f_3)$. Consequently, we assume that the ideal $I = (f_0, f_1, f_2, f_3)$ in $\mathbb{K}[s, t, u]$ is at least of codimension 2, that is defines $n$ isolated points $p_1, \ldots, p_n$ in $\mathbb{P}^2$, and we have (see section 2 and theorem 2.5 in Busé and Jouanolou [2003] for a proof and more details):

Theorem 3.1: *Let $e_{p_i}$ denote the algebraic multiplicity of the point $p_i$ for all $i = 1, \ldots, n$, and $\beta$ denote the degree of $\phi$ onto its image. Then*

$$d^2 - \sum_{i=1}^{n} e_{p_i} = \begin{cases} \beta \deg(\mathcal{S}) & \text{if } \phi \text{ is generically finite,} \\ 0 & \text{if } \phi \text{ is not generically finite.} \end{cases}$$

This theorem gives us the degree of the closed image of $\phi$ if it is a surface (which we have denoted by $\mathcal{S}$), and also that this image is not a surface if and only if $d^2 = \sum_{i=1}^{n} e_{p_i}$. We will call an *implicit equation* of $\mathcal{S}$ an equation of its associated divisor, which is an irreducible and homogeneous polynomial $P(x, y, z, w)$ of total degree $(d^2 - \sum_{i=1}^{n} e_{p_i})/\beta$.

In the following subsections we mainly present a method based on approximation complexes for computing an implicit equation of $\mathcal{S}$ (in fact we will compute $P^\beta$) in case the ideal $I$ defines a zero-dimensionnal scheme in $\mathbb{P}^2$ locally defined by at most three equations (in other words, this scheme is locally an almost complete intersection), and we also give some other related results. Then we provide an explicit description of the algorithm and illustrate it with some examples.

### 3.1. The method

We denote by $A$ the polynomial ring $\mathbb{K}[s, t, u]$ which is naturally graded by $\deg(s) = \deg(t) = \deg(u) = 1$. Let us consider the Koszul complex of $f_0, f_1, f_2, f_3$ in $A$:

$$0 \to A[-4d] \xrightarrow{d_4} A[-3d]^4 \xrightarrow{d_3} A[-2d]^6 \xrightarrow{d_2} A[-d]^4 \xrightarrow{d_1} A, \tag{2}$$

where the differentials are given by

$$d_4 = \begin{pmatrix} -f_3 \\ f_2 \\ -f_1 \\ f_0 \end{pmatrix}, \quad d_3 = \begin{pmatrix} f_2 & f_3 & 0 & 0 \\ -f_1 & 0 & f_3 & 0 \\ f_0 & 0 & 0 & f_3 \\ 0 & -f_1 & -f_2 & 0 \\ 0 & f_0 & 0 & -f_2 \\ 0 & 0 & f_0 & f_1 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -f_1 & -f_2 & 0 & -f_3 & 0 & 0 \\ f_0 & 0 & -f_2 & 0 & -f_3 & 0 \\ 0 & f_0 & f_1 & 0 & 0 & -f_3 \\ 0 & 0 & 0 & f_0 & f_1 & f_2 \end{pmatrix}, \quad d_1 = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \end{pmatrix}.$$

As for curve implicitization, we denote with $(K_\bullet(f_0, f_1, f_2, f_3), u_\bullet)$ this Koszul complex tensored by $A[\underline{x}] := A[x, y, z, w]$ over $A$, which is of the form:

$$0 \to A[\underline{x}][-4d] \xrightarrow{u_4} A[\underline{x}][-3d]^4 \xrightarrow{u_3} A[\underline{x}][-2d]^6 \xrightarrow{u_2} A[\underline{x}][-d]^4 \xrightarrow{u_1} A[\underline{x}],$$

where the matrices of the differentials $d_i$ and $u_i$ are the same, $i = 1, 2, 3, 4$ (here again we set $\deg(x) = \deg(y) = \deg(z) = \deg(w) = 1$). We also consider the bi-graded Koszul complex on $A[\underline{x}]$ associated to the sequence $(x, y, z, w)$, and denote it $(K_\bullet(x, y, z, w), v_\bullet)$:

$$0 \to A[\underline{x}](-4) \xrightarrow{v_4} A[\underline{x}](-3)^4 \xrightarrow{v_3} A[\underline{x}](-2)^6 \xrightarrow{v_2} A[\underline{x}](-1)^4 \xrightarrow{v_1} A[\underline{x}].$$

The matrices of its differentials are obtained from the matrices of the differentials of (2) by replacing $f_0$ by $x$, $f_1$ by $y$, $f_2$ by $z$ and $f_3$ by $w$. Note that since $(x, y, z, w)$ is a regular sequence in $A[\underline{x}]$, the complex $K_\bullet(x, y, z, w)$ is acyclic. From both Koszul complexes $(K_\bullet(f_0, f_1, f_2, f_3), u_\bullet)$ and $(K_\bullet(x, y, z, w), v_\bullet)$ we can build, as we did for curves, the approximation complex $\mathcal{Z}_\bullet$. We define $Z_i := \ker(d_i)$ and $\mathcal{Z}_i := Z_i[id] \otimes_A A[\underline{x}]$ for all $i = 0, 1, 2, 3, 4$ (where $d_0 : A \to 0$), they are naturally bi-graded $A[\underline{x}]$-modules. Since for all $i = 1, 2, 3$ we have $u_i \circ v_{i+1} + v_i \circ u_{i+1} = 0$, we obtain the bi-graded complex:

$$(\mathcal{Z}_\bullet, v_\bullet) : 0 \to \mathcal{Z}_4(-4) \xrightarrow{v_4} \mathcal{Z}_3(-3) \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[\underline{x}],$$

where, since we have supposed $d \geq 1$, $\mathcal{Z}_4 = 0$.

*Remark:* In the language of moving surfaces (see Sederberg and Chen [1995], Cox [2001]), an element $(g_1, g_2, g_3, g_4) \in \mathcal{Z}_{1[\nu](0)}$ is nothing but a moving plane of degree $\nu$ following the surface $\mathcal{S}$.

To state the main result of this section, we need some notation. If $p$ is an isolated base point defined by the ideal $I$, we denote by $d_p$ its geometric multiplicity (also called its degree); note that we have already denoted by $e_p$ its algebraic multiplicity. Recall that if $M$ is a $\mathbb{Z}$-graded $R$-module, where $R$ is a $\mathbb{Z}$-graded ring, its initial degree is defined as $\mathrm{indeg}(M) = \inf\{\nu \in \mathbb{Z} : M_\nu \neq 0\}$.

THEOREM 3.2: *Suppose that the ideal $I = (f_0, f_1, f_2, f_3) \subset A$ is of codimension at least 2. Let $\mathcal{P} := \mathrm{Proj}(A/I)$ and denote by $I_\mathcal{P}$ the saturated ideal of $I$ w.r.t. the maximal ideal $\mathfrak{m} = (s, t, u)$ of $A$. Then,*

- *The complex $\mathcal{Z}_\bullet$ is acyclic if and only if $\mathcal{P}$ is locally defined by (at most) 3 equations.*

- *Assume that $\mathcal{P}$ is locally defined by 3 equations, then for any integer*

$$\nu \geq \nu_0 := 2(d-1) - \mathrm{indeg}(I_{\mathcal{P}})$$

*the determinant of the graded complex of free $\mathbb{K}[\underline{x}]$-modules*

$$0 \to \mathcal{Z}_{3[\nu]}(-3) \xrightarrow{v_3} \mathcal{Z}_{2[\nu]}(-2) \xrightarrow{v_2} \mathcal{Z}_{1[\nu]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[\nu]} = A_{[\nu]}[\underline{x}]$$

*is a homogeneous element of $\mathbb{K}[\underline{x}]$ of degree $d^2 - \sum_{p\in\mathcal{P}} d_p$, and is a multiple of $P^\beta$ independent of $\nu$, where $P$ is an implicit equation of $\mathcal{S}$. It is exactly $P^\beta$ if and only if $I$ is locally a complete intersection. Moreover, for all $\nu \in \mathbb{Z}$, $\mathcal{Z}_{3[\nu]}$ is always a free $\mathbb{K}[\underline{x}]$-module of rank $\max(\binom{\nu-d+2}{2}, 0)$; in particular $\mathcal{Z}_{3[\nu]} = 0$ if and only if $\nu \leq d-1$.*

*Proof:* See theorem 4.1 for the complete proof. Note that an argument similar to the one given in the proof of theorem 2.1 shows easily that $\mathcal{Z}_3 \simeq A[\underline{x}][-d]$, and hence the last statement of the second assertion of the theorem. □

By standard properties of determinants of complexes (see e.g. appendix A in Gelfand et al. [1994]) we deduce the

Corollary 3.1: *Suppose that $I = (f_0, f_1, f_2, f_3) \subset A$ is of codimension at least 2, and $\mathcal{P} = \mathrm{Proj}(A/I)$ is locally defined by 3 equations. Then for all $\nu \geq \nu_0 := 2(d-1) - \mathrm{indeg}(I_{\mathcal{P}})$ any non-zero minor of (maximal) size $(\nu+2)(\nu+1)/2$ of the matrix of the surjective map*

$$\begin{array}{rcl} \mathcal{Z}_{1[\nu]}(-1) & \xrightarrow{v_1} & A_{[\nu]}[\underline{x}] \\ (g_1, g_2, g_3, g_4) & \mapsto & xg_1 + yg_2 + zg_3 + wg_4 \end{array}$$

*is a non-zero multiple of $P^\beta$. Moreover, if $\mathcal{P}$ is locally a complete intersection, then the gcd of all these minors equals $P^\beta$.*

Before going through an algorithmic version of this theorem we make few remarks on the integer $\nu_0$. First note that $\nu_0$ depends geometrically on the ideal $I$, since $\mathrm{indeg}(I_{\mathcal{P}})$ is the smallest degree of a hypersurface in $\mathbb{P}^2$ containing the closed subscheme defined by $I$.

Notice that this initial degree is very fastly determined by dedicated computer algebra softwares (for instance Macaulay, Singular or Cocoa) and the complexity of the computation is also proved to be small, as it follows from Chardin [2004] Corollary 3.3 that $\mathrm{reg}(I) \leq 3(d-1)+1$ and $\mathrm{reg}(I_{\mathcal{P}}) \leq 2(d-1)+1$.

Let us also give few observations on how the initial degree of $I_{\mathcal{P}}$ behaves. If $I$ has no base points, that is $I_{\mathcal{P}} = A$, then $\mathrm{indeg}(I_{\mathcal{P}}) = 0$. If there exists base points, then this initial degree is always greater than or equal to 1 since $I_{\mathcal{P}}$ is generated in degree at least 1, and is always bounded by $d$ since the $f_i$'s are in $I_{\mathcal{P}}$. Also if $I$ is saturated, then its initial degree is exactly $d$. We deduce that

- $\nu_0 = 2d-2$ if $I$ has no base point,

- $d-2 \leq \nu_0 \leq 2d-3$ if $I$ has base points,
- $\nu_0 = d-2$ if $I$ is saturated (note that in this case we know that $\mathcal{Z}_{3[\nu_0]} = 0$, and hence $\det(\mathcal{Z}_{\bullet[\nu_0]})$ is always either a single determinant or a quotient of two determinants).

This shows, in particular, that the presence of base points simplifies the complexity of the computation of the implicit surface. Finally recall that the explicit bound (and not the theoretical one) given in Busé and Jouanolou [2003] is only $2d-2$, no matter if there exists base points or not.

### 3.2. The algorithm

We now develop the algorithm suggested by theorem 3.2, and then discuss some computational aspects.

ALGORITHM (implicitization of a rational parametric surface with local almost complete intersection isolated base points, possibly empty):
**Input:** Four homogeneous polynomials $f_0, f_1, f_2, f_3$ in $A$ of the same degree $d \geq 1$ such that the ideal $(f_0, f_1, f_2, f_3) \subset A$ is locally generated by 3 elements outside $V(s,t,u)$ and at least of codimension 2. An integer $\nu$ with default value $\nu := 2d-2$.
**Output:** Either:

- a square matrix $\Delta_1$ such that $\det(\Delta_1)$ is the determinant of $(\mathcal{Z}_\bullet)_{[\nu]}$,
- two square matrices $\Delta_1$ and $\Delta_2$ such that $\frac{\det(\Delta_1)}{\det(\Delta_2)}$ is the determinant of $(\mathcal{Z}_\bullet)_{[\nu]}$,
- three square matrices $\Delta_1, \Delta_2$ and $\Delta_3$ such that $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$ is the determinant of $(\mathcal{Z}_\bullet)_{[\nu]}$.

1. *Compute the matrix* $\mathtt{F_1}$ *of the first map of* (2): $A_\nu^4 \xrightarrow{d_1} A_{\nu+d}$, *and a kernel* $\mathtt{K_1}$ *of its transpose which has* $\operatorname{rank}(\mathcal{Z}_{1[\nu]})$ *rows.*
2. *Set* $m := \frac{(\nu+2)(\nu+1)}{2}$. *Construct the matrix* $\mathtt{Z_1}$ *defined by:*
$$\mathtt{Z_1}(i,j) = x\mathtt{K_1}(j,i) + y\mathtt{K_1}(j,i+m) + z\mathtt{K_1}(j,i+2m) + w\mathtt{K_1}(j,i+3m),$$
*with* $i = 1,\dots,m$ *and* $j = 1,\dots,\operatorname{rank}(\mathcal{Z}_{1[\nu]})$, *which is the matrix of the map* $\mathcal{Z}_{1[\nu]}(-1) \xrightarrow{v_1} \mathcal{Z}_{0[\nu]}$.
3. ***If*** $\mathtt{Z_1}$ *is square* ***then*** *set* $\Delta_1 := \mathtt{Z_1}$ ***else***
   (a) *Compute a list* $L_1$ *of integers indexing independent columns in* $\mathtt{Z_1}$. $L_1$ *consists in* $m$ *integers. Let* $\Delta_1$ *be the* $m \times m$ *submatrix of* $\mathtt{Z_1}$ *obtained by removing columns not in* $L_1$.
   (b) *Compute the matrix* $\mathtt{F_2}$ *of the second map of* (2): $A_\nu^6 \xrightarrow{d_2} A_{\nu+d}^3$, *and a kernel* $\mathtt{K_2}$ *of its transpose which has* $\operatorname{rank}(\mathcal{Z}_{2[\nu]})$ *rows.*

(c) *Construct the matrix* $\mathtt{Z}_2'$ *defined by, for all* $j = 1, \ldots, \mathrm{rank}(\mathcal{Z}_{2[\nu]})$,

$$\begin{array}{l} \mathtt{Z}_2'(i,j) = -y\mathtt{K}_2(j,i) - z\mathtt{K}_2(j,i+m) - w\mathtt{K}_2(j,i+3m), i = 1, \ldots, m, \\ \mathtt{Z}_2'(i,j) = x\mathtt{K}_2(j,i-m) - z\mathtt{K}_2(j,i+m) - w\mathtt{K}_2(j,i+3m), i = m+1, \ldots, 2m, \\ \mathtt{Z}_2'(i,j) = x\mathtt{K}_2(j,i-m) + y\mathtt{K}_2(j,i) - w\mathtt{K}_2(j,i+3m), i = 2m+1, \ldots, 3m, \\ \mathtt{Z}_2'(i,j) = x\mathtt{K}_2(j,i) + y\mathtt{K}_2(j,i+m) + z\mathtt{K}_2(j,i+2m), i = 3m+1, \ldots, 4m. \end{array}$$

(d) *Construct the* $\mathrm{rank}(\mathcal{Z}_{1[\nu]}) \times \mathrm{rank}(\mathcal{Z}_{2[\nu]})$ *matrix* $\mathtt{Z}_2$ *whose* $j^{\mathrm{th}}$ *column* $\mathtt{Z}_2(\bullet, j)$ *is the solution of the linear system* ${}^t\mathtt{Z}_2(\bullet, j).\mathtt{K}_1 = \mathtt{Z}_2'(\bullet, j)$*, which is the matrix of the map* $\mathcal{Z}_{2[\nu]}(-1) \xrightarrow{v_2} \mathcal{Z}_{1[\nu]}$.

(e) *Define* $\Delta_2'$ *to be the submatrix of* $\mathtt{Z}_2$ *obtained by removing the rows indexed by* $L_1$.
***If*** $\Delta_2'$ *is square* ***then*** *set* $\Delta_2 := \Delta_2'$ ***else***

i. *Compute a list* $L_2$ *of integers indexing independent columns in* $\Delta_2'$*. Define* $\Delta_2$ *to be the square submatrix of* $\Delta_2'$ *obtained by removing columns not in* $L_2$.

ii. *Construct the matrix* $\mathtt{F}_3$ *of the third map of* (2)*:* $A_\nu^4 \xrightarrow{d_3} A_{\nu+d}^6$*, and the kernel* $\mathtt{K}_3$ *of its transpose which has* $\mathrm{rank}(\mathcal{Z}_{3[\nu]})$ *rows.*

iii. *Construct the matrix* $\mathtt{Z}_3'$ *defined by, for all* $j = 1, \ldots, \mathrm{rank}(\mathcal{Z}_{3[\nu]})$,

$$\begin{array}{l} \mathtt{Z}_3'(i,j) = z\mathtt{K}_3(j,i) + w\mathtt{K}_3(j,i+m), i = 1, \ldots, m, \\ \mathtt{Z}_3'(i,j) = -y\mathtt{K}_3(j,i-m) + w\mathtt{K}_3(j,i+m), i = m+1, \ldots, 2m, \\ \mathtt{Z}_3'(i,j) = x\mathtt{K}_3(j,i-2m) + w\mathtt{K}_3(j,i+m), i = 2m+1, \ldots, 3m, \\ \mathtt{Z}_3'(i,j) = -y\mathtt{K}_3(j,i-2m) - z\mathtt{K}_3(j,i-m), i = 3m+1, \ldots, 4m, \\ \mathtt{Z}_3'(i,j) = x\mathtt{K}_3(j,i-3m) - z\mathtt{K}_3(j,i-m), i = 4m+1, \ldots, 5m, \\ \mathtt{Z}_3'(i,j) = x\mathtt{K}_3(j,i-3m) + y\mathtt{K}_3(j,i-2m), i = 5m+1, \ldots, 6m. \end{array}$$

iv. *Construct the* $\mathrm{rank}(\mathcal{Z}_{2[\nu]}) \times \mathrm{rank}(\mathcal{Z}_{3[\nu]})$ *matrix* $\mathtt{Z}_3$ *whose* $j^{\mathrm{th}}$ *column* $\mathtt{Z}_3(\bullet, j)$ *is the solution of the linear system* ${}^t\mathtt{Z}_3(\bullet, j).\mathtt{K}_2 = \mathtt{Z}_3'(\bullet, j)$*, which is the matrix of the map* $\mathcal{Z}_{3[\nu]}(-1) \xrightarrow{v_3} \mathcal{Z}_{2[\nu]}$*. Define* $\Delta_3$ *to be the square submatrix of* $\mathtt{Z}_3$ *obtained by removing the rows indexed by* $L_2$*.* $\Delta_3$ *is of size* $\frac{(\nu-d+2)(\nu-d+1)}{2}$.

***endif***

***endif.***

In the language of moving surfaces, the matrix $\mathtt{Z}_1$ obtained at step 2 gather all the moving *planes* of degree $\nu$ following the surface $\mathcal{S}$. Assume hereafter that $\mathcal{P}$ is locally a complete intersection. From a computational point of view, corollary 3.1 implies that this matrix can be taken as a *representation* of the surface $\mathcal{S}$, replacing an expanded implicit equation (even if it is generally non-square). For instance, to test if a given point $p = (x_0 : y_0 : z_0 : w_0) \in \mathbb{P}^3$ is in the surface $\mathcal{S}$, we just have to substitute $x, y, z, w$ respectively by $x_0, y_0, z_0, w_0$ in $\mathtt{Z}_1$ and check its rank; $p$ is on $\mathcal{S}$ if and only if the rank of $\mathtt{Z}_1$ does not drop. Of course some numerical aspects have to be taken into account here, but the use of numerical linear algebra seems to be very promising in this direction. Note also that such matrices, whose computation is very fast, are much more compact representations of implicit equations compared to expanded polynomials which can have a lot of monomials.

Even if the use of $\mathtt{Z}_1$ seems to be, in the opinion of the authors, the best way to work quickly with implicit equations, the previous algorithm also returns an explicit description of $P^\beta$, where $P$ is as usual an implicit equation of $\mathcal{S}$ (always if $\mathcal{P}$ is locally a complete intersection). With the convention $\det(\Delta_i) = 1$ if $\Delta_i$ does not exists, for $i = 2, 3$, $P^\beta$ can be computed as the quotient $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$. Since $\det(\Delta_1)$ is always a multiple of $P^\beta$, we have $\det(\Delta_1) = P^\beta Q$, where $Q$ is a homogeneous polynomial in $\mathbb{K}[\underline{x}]$. Notice that this extraneous factor $Q$ divides $\det(\Delta_2)$; in fact $\det(\Delta_2) = Q\det(\Delta_3)$.

### 3.3. Examples

We illustrate our algorithm with four particular examples. The algorithm has been implemented in the software MAGMA, which offers very powerful tools to deal with linear algebra, and experimentally seems to be very efficient.

#### *3.3.1. An example without base point*

Consider the following example:

$$\left\{\begin{array}{lcl} f_0 & = & s^2t, \\ f_1 & = & t^2u, \\ f_2 & = & su^2, \\ f_3 & = & s^3+t^3+u^3, \end{array}\right.$$

is a parameterization without base point of a surface of degree 9. Applying our algorithm we find, in degree $\nu = 2.3 - 2 = 4$ a matrix $\mathtt{Z}_1$ of size $15 \times 24$ which represents our surface. Continuing the algorithm we obtain a matrix $\Delta_1$ of size $15 \times 15$, a matrix $\Delta_2$ of size $9 \times 9$ and a matrix $\Delta_3$ of size $3 \times 3$. Computing the quotient $\frac{\det(\Delta_1)\det(\Delta_3)}{\det(\Delta_2)}$ we obtain an expanded implicit equation:

$$x^6z^3 + 3x^5y^2z^2 + 3x^4y^4z + 3x^4yz^4 + x^3y^6 + 6x^3y^3z^3 + 3x^2y^5z^2 + 3x^2y^2z^5 - x^2y^2z^2w^3 + 3xy^4z^4 + y^3z^6.$$

Trying our algorithm empirically with $\nu = 3$, we obtain a matrix $\Delta_1$ which is not square, it has only 9 columns (instead of the expected 10). The algorithm hence fails here, showing that in this case the degree bound $2d - 2$ is the lowest possible.

#### *3.3.2. An example where the ideal of base points is saturated*

This example is taken from Sederberg and Chen [1995] (see also Busé et al. [2003]); it is the parameterization of a cubic surface with 6 local complete intersection base points:

$$\begin{cases} f_0 &= s^2t + 2t^3 + s^2u + 4stu + 4t^2u + 3su^2 + 2tu^2 + 2u^3, \\ f_1 &= -s^3 - 2st^2 - 2s^2u - stu + su^2 - 2tu^2 + 2u^3, \\ f_2 &= -s^3 - 2s^2t - 3st^2 - 3s^2u - 3stu + 2t^2u - 2su^2 - 2tu^2, \\ f_3 &= s^3 + s^2t + t^3 + s^2u + t^2u - su^2 - tu^2 - u^3. \end{cases}$$

The ideal $I = (f_0, f_1, f_2, f_3)$ is saturated, so that we have $\nu_0 = d - 2 = 1$, and we hence can apply the algorithm with $\nu = 1$. The matrix $\mathtt{Z}_1$ is then square and the algorithm stops in step 2, giving

$$\begin{pmatrix} x & -z - w & y + w \\ y & x - 2y + z - 2w & 2y - z \\ z & -x - 2w & y + 2w \end{pmatrix}.$$

*3.3.3. An example where the method of moving quadrics fails*

We here consider the example 3.2 in Busé et al. [2003]. This example was introduced to show how the method of moving quadrics (introduced in Sederberg and Chen [1995]) generalized in this paper to the presence of base points may fail. Consider the parameterization

$$\begin{cases} f_0 &= su^2, \\ f_1 &= t^2(s + u), \\ f_2 &= st(s + u), \\ f_3 &= tu(s + u). \end{cases}$$

We can directly apply our algorithm in degree $\nu = 2.3 - 2 = 4$; we obtain a matrix $\mathtt{Z}_1$ of size $15 \times 30$, matrices $\Delta_1$ and $\Delta_2$ of size $15 \times 15$, and a matrix $\Delta_3$ of size $3 \times 3$. An expanded implicit equation is then $xyz + xyw - zw^2$.

If now we take into account that the ideal $I = (f_0, f_1, f_2, f_3)$ has base points, we can apply the algorithm with $\nu = 3$. Even better, the saturated ideal of $I$, denoted $I_{\mathcal{P}}$, is generated by $t(s + u)$ and $su^2$, showing that $\mathrm{indeg}(I_{\mathcal{P}}) = 2$ (since $t(s + u)$ is of degree 2), and hence that we can apply the algorithm with $\nu = 2$. In this case $\Delta_1$ is $6 \times 6$, $\Delta_2$ is $3 \times 3$ and the algorithm stops in step $(e)$.

*3.3.4. An example with a fat base point*

With this example we would like to illustrate theorem 3.2 when there is a base point minimally generated by three polynomials. Consider the parameterization given by

$$\begin{cases} f_0 &= s^3 - 6s^2t - 5st^2 - 4s^2u + 4stu - 3t^2u, \\ f_1 &= -s^3 - 2s^2t - st^2 - 5s^2u - 3stu - 6t^2u, \\ f_2 &= -4s^3 - 2s^2t + 4st^2 - 6t^3 + 6s^2u - 6stu - 2t^2u, \\ f_3 &= 2s^3 - 6s^2t + 3st^2 - 6t^3 - 3s^2u - 4stu + 2t^2u. \end{cases}$$

The ideal $I = (f_0, f_1, f_2, f_3)$ defines exactly one *fat* base point $p$ which is defined by $(s,t)^2$. Therefore any implicit equation of our parameterized surface is of degree $d^2 - e_p = 9 - 4 = 5$, and the degree of the determinant of the complex $(\mathcal{Z}_\bullet)_{[\nu_0]}$ is of degree $d^2 - d_p = 9 - 3 = 6$. Applying our algorithm with $\nu_0 = 2(d-1) - 2 = 2$ we find that the matrix $\mathtt{Z}_1$ is already square, of size $6 \times 6$. Expanding its determinant one obtains a product of a degree 5 irreducible polynomial (an implicit equation of our surface) and a linear (since $e_p - d_p = 1$) irreducible polynomial.

## 4. Implicitization of rational parametric hypersurfaces

We now turn to the general problem of hypersurface implicitization, always using approximation complexes. In this section we prove new results to solve this problem under suitable conditions, and obtain the proof of theorem 3.2 as a particular case.

We are going to consider hypersurfaces obtained as the closed image of maps from $\mathbb{P}^{n-1}$ to $\mathbb{P}^n$, where $n \geq 3$. Consequently we set $A := \mathbb{K}[X_1, \ldots, X_n]$, which will be the ring of the polynomials of the parameterizations we will consider. We also denote $\mathfrak{m} := (X_1, \ldots, X_n)$, and set $—^\vee := \mathrm{Hom}_A(—, A[-n])$ and $—^* := \mathrm{Homgr}_A(—, A/\mathfrak{m})$. Finally $\omega_-$ will denote the associated canonical module (see Bruns and Herzog [1993]). Our first result is a technical lemma on the cycles of certain Koszul complexes.

Lemma 4.1: *Let $f_0, \ldots, f_n$ be $n+1$ polynomials of positive degrees $d_0, \ldots, d_n$, $I$ the ideal generated by them, $\mathbf{K}_\bullet(f;A)$ the Koszul complex of the $f_i$'s on $A$ and denote by $Z_i$ and $H_i$ the $i$-th cycles and the $i$-th homology modules of $\mathbf{K}_\bullet(f;A)$, respectively. Denoting $\sigma := d_0 + \cdots + d_n$, if $\dim(A/I) \leq 1$ then we have*

- *$H_i \neq 0$ for $i = 0, 1$, $H_i = 0$ for $i > 2$ and $H_2$ is zero if and only if $\dim(A/I) = 0$. If $\dim(A/I) = 1$, $H_2 \simeq \omega_{A/I}[n-\sigma]$.*
- *If $n \geq 3$, then*

$$H^i_{\mathfrak{m}}(Z_p) \simeq \begin{cases} 0 & \text{for } i = 0,1 \\ H^0_{\mathfrak{m}}(H_{i-p})^*[n-\sigma] & \text{for } i = 2 \\ H^*_{i-p}[n-\sigma] & \text{for } 2 < i < n \\ Z^*_{n-p}[n-\sigma] & \text{for } i = n. \end{cases}$$

*Proof:* The first bullet point is classical, see e.g. Bruns and Herzog [1993] 1.6.16 and 1.2.4. For the second point we consider the truncated complexes:

$$\mathbf{K}^{>p}_\bullet : \qquad 0 \to K_{n+1} \to \cdots \to K_{p+1} \to Z_p \to 0.$$

If $p > 2$ this complex is exact, and gives rise to a spectral sequence $H^\bullet_{\mathfrak{m}}(\mathbf{K}^{>p}_\bullet) \Rightarrow 0$, which is at step 1:

$$\begin{array}{cccc}
0 & \cdots & 0 & H^0_{\mathfrak{m}}(Z_p) \\
\vdots & & \vdots & H^1_{\mathfrak{m}}(Z_p) \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & H^{n-1}_{\mathfrak{m}}(Z_p)
\end{array}$$

$$H^n_{\mathfrak{m}}(K_{n+1}) \xrightarrow{H^n_{\mathfrak{m}}(\partial_{n+1})} \cdots \xrightarrow{H^n_{\mathfrak{m}}(\partial_{p+2})} H^n_{\mathfrak{m}}(K_{p+1}) \xrightarrow{H^n_{\mathfrak{m}}(\partial_{p+1})} H^n_{\mathfrak{m}}(Z_p),$$

and modulo the identifications

$$H^n_{\mathfrak{m}}(K_\bullet) \simeq (K^\vee_\bullet)^* \simeq (K^\bullet[-n])^* \simeq (K_{n+1-\bullet}[\sigma - n])^*,$$

the last line becomes

$$K^*_0[n-\sigma] \xrightarrow{\partial^*_1} K^*_1[n-\sigma] \xrightarrow{\partial^*_2} \cdots \xrightarrow{\partial^*_{n-p}} K^*_{n-p}[n-\sigma] \to H^n_{\mathfrak{m}}(Z_p).$$

It follows

- $H^i_{\mathfrak{m}}(Z_p) = 0$ for $i < p$ and for $p+2 < i < n$,
- $d^{>p}_{n-p+1} : H^*_0[n-\sigma] \xrightarrow{\simeq} H^p_{\mathfrak{m}}(Z_p)$,
- $d^{>p}_{n-p} : H^*_1[n-\sigma] \xrightarrow{\simeq} H^{p+1}_{\mathfrak{m}}(Z_p)$,
- $d^{>p}_{n-p-1} : H^*_2[n-\sigma] \xrightarrow{\simeq} H^{p+2}_{\mathfrak{m}}(Z_p)$,

where the upper index indicates from which spectral sequence the isomorphism comes, and the lower index indicates at which step of the spectral sequence the map is obtained. We also obtain a short exact sequence

$$K^*_{n-p-1}[n-\sigma] \xrightarrow{\partial^*_{n-p}} K^*_{n-p}[n-\sigma] \to H^n_{\mathfrak{m}}(Z_p) \to 0$$

that gives the isomorphism $H^n_{\mathfrak{m}}(Z_p) \simeq Z^*_{n-p}[n-\sigma]$.

For $p = 2$, we have a spectral sequence $H^\bullet_{\mathfrak{m}}(\mathbf{K}^{>2}_\bullet) \Rightarrow H^1_{\mathfrak{m}}(H_2)$ which shows as in the previous cases that $H^n_{\mathfrak{m}}(Z_2) \simeq Z^*_{n-2}[n-\sigma]$, and that

- $H^i_{\mathfrak{m}}(Z_2) = 0$ for $4 < i < n$,
- $d^{>2}_{n-2} : H^*_1[n-\sigma] \xrightarrow{\simeq} H^3_{\mathfrak{m}}(Z_2)$,
- $d^{>2}_{n-3} : H^*_2[n-\sigma] \xrightarrow{\simeq} H^4_{\mathfrak{m}}(Z_2)$.

It also provides an exact sequence

$$0 \to H^1_{\mathfrak{m}}(Z_2) \xrightarrow{can} H^1_{\mathfrak{m}}(H_2) \xrightarrow{\tau} H^*_0[n-\sigma] \xrightarrow{d^{>2}_{n-1}} H^2_{\mathfrak{m}}(Z_2) \to 0, \qquad (3)$$

where $\tau$ is the transgression map of the spectral sequence. If $H_2 \neq 0$, we also look at the complex

$$\mathbf{K}_\bullet^{\leq 2}: \quad 0 \to Z_2 \to K_2 \to K_1 \to I \to 0,$$

whose only homology is $H_1$ and the corresponding spectral sequence $H^\bullet_\mathfrak{m}(\mathbf{K}_\bullet^{\leq 2}) \Rightarrow H^\bullet_\mathfrak{m}(H_1)$. Noticing that $H^i_\mathfrak{m}(H_1) = 0$ for $i > 1$, we get that $H^0_\mathfrak{m}(Z_2) = H^1_\mathfrak{m}(Z_2) = 0$ (this is also easily obtained by splitting $\mathbf{K}_\bullet^{\leq 2}$ into two short exact sequences, and taking cohomology). Together with (3) and local duality, which gives $H^1_\mathfrak{m}(H_2) \simeq \omega^*_{H_2} \simeq (A/I^{\mathrm{sat}})^*[n-\sigma]$ where $I^{\mathrm{sat}}$ denotes the saturation of the ideal $I$, we get the asserted isomorphism for $H^2_\mathfrak{m}(Z_2)$.

Note that we also get an isomorphism $H^0_\mathfrak{m}(H_0)^*[n-\sigma] \simeq H^2_\mathfrak{m}(Z_2) \simeq H^0_\mathfrak{m}(H_1)$, and an exact sequence:

$$0 \to H^1_\mathfrak{m}(H_1) \to H^3_\mathfrak{m}(Z_2) \to H^1_\mathfrak{m}(I) \to 0.$$

Finally for $p = 1$, the exact sequence

$$\mathbf{K}_\bullet^{\leq 1}: \quad 0 \to Z_1 \to K_1 \to I \to 0$$

shows that $H^{i-2}_\mathfrak{m}(H_0) \simeq H^{i-1}_\mathfrak{m}(I) \simeq H^i_\mathfrak{m}(Z_1)$ for $i < n$, so that $H^0_\mathfrak{m}(Z_1) = H^1_\mathfrak{m}(Z_1) = 0$, $H^2_\mathfrak{m}(Z_1) \simeq H^0_\mathfrak{m}(H_0) \simeq H^0_\mathfrak{m}(H_1)^*[n-\sigma]$, $H^3_\mathfrak{m}(Z_1) \simeq H^1_\mathfrak{m}(H_0) \simeq H_2^*[n-\sigma]$ and $H^i_\mathfrak{m}(Z_1) = 0$ for $3 < i < n$.

The spectral sequence $H^\bullet_\mathfrak{m}(\mathbf{K}_\bullet^{>1}) \Rightarrow H^\bullet_\mathfrak{m}(H_\bullet)$ (because only $H^1_\mathfrak{m}(H_2)$, $H^0_\mathfrak{m}(H_1)$ and $H^1_\mathfrak{m}(H_1)$ may not be zero) gives $H^n_\mathfrak{m}(Z_1) \simeq Z^*_{n-1}[n-\sigma]$, and this concludes the proof. □

*Remark:* Note that the last spectral sequence of the proof also identifies $H^2_\mathfrak{m}(Z_1)$ in another way by providing the exact sequence

$$0 \to H^1_\mathfrak{m}(H_1) \xrightarrow{\tau'} H_1^*[n-\sigma] \xrightarrow{d^{>1}_{n-1}} H^2_\mathfrak{m}(Z_1) \to 0$$

which shows that if $\dim(A/I) = 1$

$$H_1/H^0_\mathfrak{m}(H_1) \simeq \omega_{H_1}[n-\sigma].$$

Thus in this case $H_1/H^0_\mathfrak{m}(H_1)$ has a symmetric free resolution. Also, the spectral sequences derived from the complexes $\mathbf{K}_\bullet^{\leq p}$ provide exact sequences

$$0 \to H^0_\mathfrak{m}(H_1) \to H^p_\mathfrak{m}(Z_p) \to H^1_\mathfrak{m}(H_2) \to 0$$

for $p < n$, and

$$0 \to H^1_\mathfrak{m}(H_1) \to H^{p+1}_\mathfrak{m}(Z_p) \to H^0_\mathfrak{m}(H_0) \to 0$$

for $p < n-1$, as well as an isomorphism $H^{p+2}_\mathfrak{m}(Z_p) \simeq H^1_\mathfrak{m}(H_0)$ for $p < n-2$.

As we have already done in previous sections, we now consider approximation complexes. To do this, we introduce new variables $T_0,\dots,T_n$ that represent the homogeneous coordinates of the target $\mathbb{P}^n_{\mathbb{K}}$ of a given parameterization. Let $f_0,\dots,f_n$ be $n+1$ homogeneous polynomials in $A$ of the same degree $d \geq 1$. Denoting by $Z_i$ the $i^{\text{th}}$-cycles of the Koszul complex of the $f_i$'s on $A$, we set $\mathcal{Z}_i := Z_i[id] \otimes_A A[\underline{T}]$, where [—] stands for the degree shift in the $X_i$'s and (—) for the one in the $T_i$'s. The differentials $v_\bullet$ of the Koszul complex $K_\bullet(T_0,\dots,T_n;A[\underline{T}])$ induce maps between the $\mathcal{Z}_i$'s, and hence we can define the approximation complex (note that $\mathcal{Z}_{n+1}=0$)

$$(\mathcal{Z}_\bullet, v_\bullet) : 0 \to \mathcal{Z}_n(-n) \xrightarrow{v_n} \dots \xrightarrow{v_3} \mathcal{Z}_2(-2) \xrightarrow{v_2} \mathcal{Z}_1(-1) \xrightarrow{v_1} \mathcal{Z}_0 = A[\underline{T}].$$

This complex is naturally bi-graded, and $H_0(\mathcal{Z}_\bullet) \simeq \mathrm{Sym}_A(I)$. We now give some acyclicity criteria in case $I=(f_0,\dots,f_n)$ define isolated points in $\mathrm{Proj}(A)$. Recall that a sequence $x_1,\dots,x_m$ of elements in a ring $R$ is said to be a *proper sequence* if

$$x_{i+1}H_j(x_1,\dots,x_i;A)=0 \ \text{ for } i=0,\dots,m-1 \text{ and } j>0,$$

where the $H_j$'s denote the homology groups of the associated Koszul complex.

LEMMA 4.2: *Suppose that $I=(f_0,\dots,f_n)$ is of codimension at least $n-1$, $\mathbb{K}$ is infinite, and let $\mathcal{P} := \mathrm{Proj}(A/I)$. Then the following are equivalent:*

(1) *$\mathcal{Z}_\bullet$ is acyclic,*

(2) *$\mathcal{Z}_\bullet$ is acyclic outside $V(\mathfrak{m})$,*

(3) *$I$ is generated by a proper sequence,*

(4) *$\mathcal{P}$ is locally defined by a proper sequence,*

(5) *$\mathcal{P}$ is locally defined by $n$ equations.*

*Proof:* By Herzog et al. [1983], (1)⇔(3) and (2)⇔(4). Moreover (1)⇒(2) and (3)⇒(4). We will show that (4)⇒(5)⇒(3).

Assume (4). For each $p \in \mathcal{P}$ there exists a non empty open set $\Omega_p$ in $\mathbb{K}^{(n+1)^2}$ such that if $(a_{ij}) \in \Omega_p$ and $g_i := \sum_j a_{ij}f_j$, the $g_i$'s form a proper sequence locally at $p$. Taking $(a_{ij}) \in \Omega := \cap_{p\in\mathcal{P}}\Omega_p$, this gives $g_1,\dots,g_{n+1}$ so that the $g_i$'s form a proper sequence outside $V(\mathfrak{m})$. We may also assume, by shrinking $\Omega$ if necessary, that $g_1,\dots,g_{n-1}$ is a regular sequence, so that $g_1,\dots,g_n$ is clearly a proper sequence. Now $g_1,\dots,g_{n+1}$ is proper if and only if $g_{n+1}$ annihilates $H_1(g_1,\dots,g_n;A) \simeq \omega_{A/J}$ (locally outside $V(\mathfrak{m})$, *a priori*), where $J$ denotes the saturated ideal of $(g_1,\dots,g_n)$ w.r.t. $\mathfrak{m}$. But $\mathrm{Ann}_A(\omega_{A/J})=J$, so that $g_{n+1}\in J$, which proves (5).

Now assume (5). One considers (in the same spirit as above) $g_1,\dots,g_{n+1}$ so that $g_1,\dots,g_{n-1}$ is a regular sequence and $g_1,\dots,g_n$ defines $\mathcal{P}$. Then $g_{n+1}\in I_{\mathcal{P}}$, and therefore annihilates $\omega_{A/I_{\mathcal{P}}}$, so that the $g_i$'s forms a proper sequence. □

Notice that conditions (1), (2) and (5) of the lemma above are unaffected by extension of the base field. Therefore the equivalence of these three assertions remains if we drop the hypothesis that $\mathbb{K}$ is infinite.

We are now able to state our main result on the hypersurface implicitization problem.

THEOREM 4.1: *Let $n \geq 3$. Let $I$ be the ideal $(f_0, \ldots, f_n)$, $\mathcal{P} := \mathrm{Proj}(A/I)$ and $I_{\mathcal{P}}$ the saturation of $I$ w.r.t. $\mathfrak{m}$. Assume that $\dim \mathcal{P} \leq 0$, that is $\mathcal{P}$ is supported on a finite number of points in $\mathrm{Proj}(A)$. Then we have*

- *$\mathcal{Z}_\bullet$ is acyclic if and only if $\mathcal{P}$ is locally defined by $n$ equations.*
- *Let $\nu \geq \nu_0 := (n-1)(d-1) - \mathrm{indeg}(I_{\mathcal{P}})$. Then $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_{[\nu]} = 0$. Moreover $(\mathcal{Z}_\bullet)_{[\nu]} \otimes_{\mathbb{K}[\underline{T}]} \mathbb{K}(\underline{T})$ is acyclic if and only if $\mathcal{Z}_\bullet$ is acyclic.*
- *Let $\nu \geq \nu_0$ and assume that $\mathcal{P}$ is locally defined by $n$ equations. Then $D := \det((\mathcal{Z}_\bullet)_{[\nu]})$ is a non zero homogeneous element of $\mathbb{K}[\underline{T}]$, independent of $\nu$ (modulo $\mathbb{K}^\times$), of degree $d^{n-1} - \sum_{p \in \mathcal{P}} d_p$. Denoting by $\mathcal{H}$ the closed image of the rational map $\phi : \mathbb{P}^{n-1} \to \mathbb{P}^n$, $D = H^{\deg(\phi)} G$ where $H$ is an implicit equation of $\mathcal{H}$ and $G \in \mathbb{K}[\underline{T}]$ is a nonzero polynomial. Moreover $G \in \mathbb{K}^\times$ if and only if $\mathcal{P}$ is locally of linear type, and if and only if $\mathcal{P}$ is locally a complete intersection.*

Before giving the proof of the theorem we recall that, by definition, $\mathcal{P}$ is said to be locally of linear type if $\mathrm{Proj}(\mathrm{Sym}_A(I)) = \mathrm{Proj}(\mathrm{Rees}_A(I)) \subset \mathbb{P}^{n-1}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^n_{\mathbb{K}}$. By Busé and Jouanolou [2003] theorem 2.5, $\deg(H^{\deg(\phi)}) = d^{n-1} - \sum_{p \in \mathcal{P}} e_p$, and consequently we always have $\deg(G) = \sum_{p \in \mathcal{P}} (e_p - d_p)$. Also $e_p \geq d_p$ with equality if and only if $\mathcal{P}$ is locally a complete intersection at $p \in \mathcal{P}$.

*Proof:* The first bullet point follows from lemma 4.2. For the second point we consider the two spectral sequences associated to the double complex $H^\bullet_{\mathfrak{m}}(\mathcal{Z}_\bullet)$, both abutting to the hypercohomology of $\mathcal{Z}_\bullet$. One of them abuts at step two with:

$$
{}_2{}'E^p_q = {}_\infty{}'E^p_q = \left\{ \begin{array}{ll} H^p_{\mathfrak{m}}(H_q(\mathcal{Z}_\bullet)) & \text{for } p = 0, 1 \text{ and } q > 0 \\ H^p_{\mathfrak{m}}(\mathrm{Sym}_A(I)) & \text{for } q = 0 \\ 0 & \text{else.} \end{array} \right.
$$

The other one gives at step one:

$$
{}_1{}''E^p_q = H^p_{\mathfrak{m}}(Z_q)[qd] \otimes_A A[\underline{T}](-q).
$$

By lemma 4.1, $H^p_{\mathfrak{m}}(Z_q) = 0$ for $p < q$ and for $p < 2$. This shows that $H^p_{\mathfrak{m}}(H_q(\mathcal{Z}_\bullet)) = 0$ for $q > 0$ except possibly for $p = q = 1$, by comparing the two spectral sequences. Therefore, $H_q(\mathcal{Z}_\bullet) = 0$ for $q > 1$ and $H^0_{\mathfrak{m}}(H_1(\mathcal{Z}_\bullet)) = 0$.

If $p > 2$

$$
{}_1{}''E^p_p \simeq H^*_0[n - (n+1-p)d] \otimes_A A[\underline{T}](-p)
$$

so that $({}_1{''}E_p^p)_\nu = 0$ if $\nu > (n-2)d - n$. Also,

$$({}_1{''}E_2^2) \simeq (I_{\mathcal{P}}/I)^*[n-(n-1)d] \otimes_A A[\underline{T}](-2),$$

so that $({}_1{''}E_2^2)_\nu = 0$ for $\nu > (n-1)d - n - \mathrm{indeg}(I_{\mathcal{P}}/I)$.

Therefore, if $\nu \geq \nu_0$, $({}_1{''}E_q^p)_\nu = 0$ for $p \leq q$. Note also that we have the equalities

$$\min\{d, \mathrm{indeg}(I_{\mathcal{P}}/I)\} = \min\{d, \mathrm{indeg}(I_{\mathcal{P}})\} = \mathrm{indeg}(I_{\mathcal{P}}).$$

By comparing with the other spectral sequence, we have $H^1_{\mathfrak{m}}(H_1(\mathcal{Z}_\bullet))_\nu = 0$ and $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$.

As $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for $\nu \geq \nu_0$ we have

$$\mathrm{Ann}_{\mathbb{K}[\underline{T}]}(\mathrm{Sym}_A(I)_\nu) = \mathrm{Ann}_{\mathbb{K}[\underline{T}]}(\mathrm{Sym}_A(I)_{\nu_0})$$

for any $\nu \geq \nu_0$ (see for instance Busé and Jouanolou [2003], the proof of 5.1), so that this module is torsion if and only if $\mathcal{P}$ is locally defined by at most $n$ equations (because $\mathrm{Ann}_{\mathbb{K}[\underline{T}]}(\mathrm{Sym}_A(I)_\nu)$ is torsion for $\nu \gg 0$ if and only if $I$ is defined by $< n+1$ equations outside $V(\mathfrak{m})$; one may also use the study of the minimal primes of $\mathrm{Sym}_A(I)$ in Huneke and Rossi [1986]). Also in the case where $\mathcal{P}$ is locally defined by at most $n$ equations, the divisor associated to this module is independent of $\nu \geq \nu_0$. This finishes the proof of the second bullet point.

We come to the third bullet point. Note that we have just shown that $D$ is independent of $\nu$ (up to an element of $\mathbb{K}^\times$). To compute the degree of $D$, we may then take $\nu \gg 0$.

The matrices of the maps of $\mathcal{Z}_\bullet$ have entries which are linear forms in the $T_i$'s, so that the determinant of $(\mathcal{Z}_\bullet)_{[\nu]}$ is a form in the $T_i$'s of degree

$$\delta := \sum_{i=1}^{n} (-1)^{i+1} i \dim_{\mathbb{K}}(Z_{i[\nu+id]}).$$

We have canonical exact sequences, $i = 0, \ldots, n$,

$$0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$$

and

$$0 \to B_i \to Z_i \to H_i \to 0$$

which shows that $\delta$ can be expressed in terms of the Hilbert polynomials of the $K_i$'s and the $H_i$'s. The contribution of the $K_i$'s only depends on $n$ and $d$, and the one of the $H_i$'s only comes from $Z_1$ and $Z_2$ and is :

$$(H_0)_{\nu+d} - 2((H_1)_{\nu+2d} - (H_0)_{\nu+2d})$$

as $\deg H_1 = 2 \deg \mathcal{P}$ (one may use for example that $(H_0)_\nu - (H_1)_\nu + (H_2)_\nu = 0$

for $\nu \gg 0$ and that $H_2 \simeq \omega_{R/I_{\mathcal{P}}}$ up to a degree shift), this contribution is equal to $-\deg \mathcal{P}$ for $\nu \gg 0$. The contribution of the $K_i's$ is $d^{n-1}$ (for $\nu \gg 0$) in the case where the $H_i$'s are 0 for $\nu \gg 0$, and we get $\delta = d^{n-1} - \deg \mathcal{P}$.

Let $X := \mathrm{Proj}(\mathrm{Rees}_I(A)) \subseteq Y := \mathrm{Proj}(\mathrm{Sym}_A(I)) \subset \mathbb{P}^{n-1} \times \mathbb{P}^n$. The scheme $X$ is the closure of the graph of $\phi$ in $\mathbb{P}^{n-1} \times \mathbb{P}^n$. Therefore, the closed image of $\phi$ is $p_2(X)$, and $p_2(X) \subseteq p_2(Y)$. As $D \neq 0$, $p_2(Y) \neq \mathbb{P}^n$. If $\mathcal{P}$ is locally of linear type, then $X = Y$, so that $G \in \mathbb{K}^\times$ which implies that $d_p = e_p$ for any $p \in \mathcal{P}$ and therefore $\mathcal{P}$ is locally a complete intersection. If $\mathcal{P}$ is not locally a complete intersection at $p \in \mathcal{P}$, $d_p \neq e_p$ so that $\deg G > 0$ and $X \neq Y$. Note that these last facts also follow from the study of the minimal primes associated to $\mathrm{Sym}_A(I)$ in Huneke and Rossi [1986]. $\square$

This theorem easily yields an algorithm for implicitizing parameterized hypersurfaces under suitable assumptions, using only linear algebra routines. The case $n = 3$ has been made completely explicit in section 3.

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