Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature.

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Abstract

We show that complete n-manifolds whose part of Ricci curvature less than a positive number is small in L^p norm (for $p > n/2$) have bounded diameter and finite fundamental group. On the contrary, complete metrics with small $L^{n/2}$-norm of the same part of the Ricci curvature are dense in the set of metrics of any compact differentiable manifold.

KEYWORDS: Ricci curvature, comparison theorems, fundamental group

1 Introduction

A classical problem in Riemannian geometry is to find topological, geometrical or analytical necessary conditions for the existence on a manifold of a Riemannian metric satisfying a given set of curvature bounds. For instance, S. Myers showed that a complete n-manifold with $\mathrm{Ric} \geq k(n-1)$ (where $k>0$) is compact (the diameter is bounded by $\frac{\pi}{\sqrt{k}}$) and has finite π_1, whereas, on the contrary, J. Lohkamp showed in [11] that on every n-manifold with $n\geq 3$ there exists a metric with negative Ricci curvature. This paper is devoted to the study of the Riemannian manifolds satisfying only an L^p-pinching on the negative lower part of their Ricci curvature tensors. *Let* $\underline{\mathrm{Ric}}(x) = \inf_{X\in T_xM} \mathrm{Ric}_x(X,X)/g(X,X)$ *denote the lowest eigenvalue of the Ricci tensor at* $x \in M$*, and* $f_-(x) = \max(-f(x), 0)$*, for an arbitrary function* f*.*

Our first result is the following Bishop's type theorem,

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Theorem 1.1 *Let (M^n, g) be a complete manifold and $p>\frac{n}{2}$. If $\rho_p=\displaystyle\int_M \big(\underline{\mathrm{Ric}}-(n-1)\big)_-^p$ is finite then g is of finite volume and*

$$\operatorname{Vol} g \leq \operatorname{Vol}\mathbb{S}^n(1+\rho_p^{\frac{9}{10}})(1+C(p,n)\rho_p^{\frac{1}{10}}).$$

The classical version of the Bishop theorem assumes Ric $\geq n-1$ and so applies only for compact manifolds with finite π_1 which form a precompact family for the Gromov-Hausdorff distance on the length spaces. On the contrary, Theorem 1.1 applies for every compact Riemannian manifold and some non-compact ones (for instance hyperbolic manifolds with finite volume) which form a set of metric spaces that is Gromov-Hausdorff dense amongs the length spaces (see proposition 9.1). While the form of our majorant implies the classical Bishop theorem, it is certainly not optimal. However, the condition $p > n/2$ is optimal since we show that for any $V > 0$ and any $\epsilon > 0$, there exists a large (actually dense amongs the length spaces for the Gromov-Hausdorff distance) familly of Riemannian manifolds of volume V and with $\rho_{\frac{n}{2}} \leq \epsilon$ (see proposition 9.2).

Our second result is the following myers's type theorem.

Theorem 1.2 *Let (M^n, g) be a complete manifold and $p > n/2$. If $\frac{\rho_p}{\operatorname{Vol} M} \leq \frac{1}{C(p,n)}$, then M is compact with finite π_1 and*

$$\operatorname{Diam}(M, g) \leq \pi \times \Big(1 + C(p,n)\big(\frac{\rho_p}{\operatorname{Vol} M}\big)^{\frac{1}{10}}\Big).$$

A few comments are in order:
1) Such a diameter bound was obtained in [14] under stronger curvature assumptions but the finiteness of the π_1 was a conjecture (see also [18]). As noticed in [14], if L^∞ bounds on the curvature tranfer readily to the universal cover (even if it is non-compact), that is not the same for integral pinchings. That is the reason why there is, up to now, no property of the fundamental cover implied by purely integral pinching on the Ricci curvature, and it is the main point of this article to prove that if a manifold has $\rho_p/\operatorname{Vol} M$ small then its universal cover satisfies the same pinching.
2) For any $k > 0$, a renormalization argument readily shows that we can replace ρ_p by $\rho_p^k=\int_M \big(\underline{\mathrm{Ric}}-k(n-1)\big)_-^p$ in Theorems 1.2 and 1.1 provided we replace $C(p,n)$ by $C(p,n,k)$, and also $\operatorname{Vol}\mathbb{S}^n$ by $\frac{\operatorname{Vol}\mathbb{S}^n}{k^{\frac{n}{2}}}$ and π by $\frac{\pi}{\sqrt{k}}$. The n-Euclidean space makes obvious that it does not generalize to $k \leq 0$.
3) The cartesian product of a small $\mathbb{S}^1$ with a finite volume hyperbolic manifold show that the compactness and the π_1-finiteness cannot be obtained if we only assume that ρ_p is small (or that

$\frac{\rho_p}{\operatorname{Vol} M}$ is finite). We can also slightly modify the example A.2 of [8] to get a manifold with infinite topology, finite volume and finite ρ_p. By cartesian product with a small $\mathbb{S}^1$ we get a manifold with infinite topology, finite volume and ρ_p as small as we want.
4) In the case $p = 1$ and $n = 2$ the theorem is still valid (π_1-finiteness obviously follows from the Gauss-Bonnet theorem), but in case $p = n/2$ and $n \geq 3$ no generalization of the classical results valid under pointwise lower bound on the Ricci curvature can be expected, as shows the following theorem,

Theorem 1.3 *Let (M^n, g) be any compact Riemannian n-manifold ($n \geq 3$). There exists a sequence of complete Riemannian metrics (g_m) on M that converges to g in the Gromov-Hausdorff distance and such that*

$$\frac{\rho_{n/2}(g_m)}{\operatorname{Vol} g_m} \to 0$$

Since 1941 several generalizations of Myers's theorem appeared, under roughly three different kinds of hypothesis:
a) some integrals of the Ricci curvature along minimizing geodesics are controlled ([1], [5], [3], [10], [12]),
b) the Ricci curvature is almost bounded below by $n-1$ but not allowed to take values under a given negative number ([7], [19],[16], [18]),
c) the L^∞ lower Ricci curvature bound of case b) is replaced by bounds on other Riemannian invariants (for example the volume bounded below or the diameter bounded above or the sectional curvature bounded).

Since we do not assume an L^∞ lower bound on Ricci curvature, we cannot use the second variation formula for the length of geodesics, which is the classical tool in the proof of Myers theorems of type a) and b). Techniques, which need a priori bounds on some Sobolev constants, have been developed to get generalizations of the Myers theorem when the second variation formula fails (see [4], [14], [7], [16]). Until this present paper (see our proposition 8.1), only two bounds on Sobolev constants were known under an integral control of the Ricci curvature: one by S. Gallot requiring a bound on the diameter [8], one by D. Yang requiring a lower bound on the volume of the small balls [20]. Such extra hypotheses are natural (and necessary) for manifolds with almost nonnegative Ricci curvature, but are not pertinent in our context: for instance the lower bound on volume would bound the cardinality of π_1 whereas the set of n-manifolds with Ricci curvature bounded below by $n-1$ has finite but not bounded cardinalities of π_1.

To avoid these unnatural extra hypothesis and to be able to control the Ricci curvature of the universal cover, we first develope a technique based on measure concentration estimates (and which make no use of bounds on Sobolev constant) to prove the following local version of our diameter bound,

Lemma 1.4 *Let (M^n, g) be a manifold (not necessarily complete) which contains a subset T satisfying the following conditions:*

1. *T is is star-shaped at a point x (see definition 2).*
2. *$B(x, R_T) \supset T \supset B(x, R_0)$ for some $R_T \geq R_0 > \pi$.*
3. *$\epsilon = R_T^2 \Big[\frac{1}{\operatorname{Vol} T} \int_T \big(\underline{\operatorname{Ric}} - (n-1)\big)_-^p\Big]^{\frac{1}{p}} \leq B(p,n)\big(1 - \frac{\pi}{R_0}\big)^{100}$*

Then $\operatorname{Diam}(M^n, g) \leq \pi\big(1 + C(p,n)\epsilon^{\frac{1}{20}}\big)$ *(and $M \subset T$).*

Remark. — The connected sum of an n-sphere of diameter $2R_0 - \pi$ with a Euclidean n-space by a sufficiently small cylinder shows that in order to get the compactness of M, it is important that T contain a ball of radius $R_0 > \pi$ and also that the pinching required on $\frac{1}{\operatorname{Vol} T} \int_T \big(\underline{\operatorname{Ric}} - (n-1)\big)_-^p$ tend to 0 when R_0 tends to π.

To prove lemma 1.4, we show that $\operatorname{Vol} B\big(x, \pi\big) / \operatorname{Vol} B(x, R_0)$ goes to 1 when the L^p-norm of $\big(\underline{\operatorname{Ric}} - (n-1)\big)_-$ tends to 0 and that for any $B(y, r) \subset B(x, R_0)$ the quotient $\operatorname{Vol} B(y, r) / \operatorname{Vol} B(x, R_0)$ is uniformly bounded below by a positive increasing function of r. These two opposite behaviours of the concentration of the measure in $B(x, R_0)$ prevent the manifold from having points too far away from x.

To prove theorem 1.1, we construct a good decomposition of M into star-shaped subsets and show that either M has small volume or lemma 1.4 apply to at least one of these subsets. The bound on the volume is then infered by the volume estimates developed for the proof of lemma 1.4. To show the π_1-finiteness, we construct a star-shaped domain in the universal Riemannian cover of (M^n, g) which satisfies the assumptions of lemma 1.4.

Under our curvature assumptions, we also get generalizations of the Lichnerowicz and Bishop-Gromov theorems.

Proposition 1.5 *Let us denote by λ_1, λ_1^1 and $\tilde{\lambda}_1^1$ respectively the first nonzero eigenvalue of the Laplacian on functions, the first eigenvalue on* 1*-forms and on co-closed* 1*-forms of (M^n, g). Then:*

$$\lambda_1(M^n, g) = \lambda_1^1(M^n, g) \geq n \times \Big(1 - C(p,n)\big(\frac{\rho_p}{\operatorname{Vol} M}\big)^{\frac{1}{p}}\Big),$$
$$\tilde{\lambda}_1^1(M^n, g) \geq 2(n-1) \times \Big(1 - C(p,n)\big(\frac{\rho_p}{\operatorname{Vol} M}\big)^{\frac{1}{p}}\Big).$$

In the last section we show that this result becomes false with $p=\frac{n}{2}$ when $n \geq 3$. By adapting the proofs of lemmas 5.1 and 4.1 (see [2] for details), we further obtain:

Proposition 1.6 *If $\eta^{10} = \frac{\rho_p}{\operatorname{Vol} M} \leq \frac{1}{C(p,n)}$ then, for all $x \in M$ and all radii $0 \leq r \leq R$, we have:*

$$\Big(\frac{\operatorname{Vol}_{n-1} S(x,R)}{L_{1-\eta}(R)}\Big)^{\frac{1}{2p-1}} - \Big(\frac{\operatorname{Vol}_{n-1} S(x,r)}{L_{1-\eta}(r)}\Big)^{\frac{1}{2p-1}} \leq \eta^2 (R-r)^{\frac{2p-n}{2p-1}},$$
$$\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol} B(x,R)} \geq (1-\eta)\frac{A_1(r)}{A_1(R)},$$

where $L_k(t)$ (resp. $A_k(t)$) stands for the volume of a geodesic sphere (resp. ball) of radius t in $(\mathbb{S}^n, \frac{1}{k}g)$, hence also:

$$\operatorname{Vol}_{n-1} S(x,R) \leq \big(1+\eta^2\big) L_{1-\eta}(R)$$
$$\operatorname{Vol} B(x,R) \leq \big(1+\eta\big) A_1(R).$$

In contrast to the case $\operatorname{Ric} \geq (n-1)$, our assumptions do not yield an upper bound on the quotient $\frac{\operatorname{Vol}_{n-1} S(x,.)}{L_1}$ for all possible values of r because the diameter of our manifolds can be greater than π. This results are similar to the results obtained in [15] and [14] under stronger curvature assumptions.

Theorem 1.2 and proposition 1.6 imply that the set of n-manifolds satisfying $\frac{\rho_p}{\operatorname{Vol} M} \leq C(p,n)$, for a $p > n/2$, is pre-compact for the Gromov-Hausdorff distance. We show in the last section that this property is false in the case $n \geq 3$ and $p = n/2$, even for the pointed Gromov-Hausdorff distance.

This article is organised as follows. For our proof of theorem 1.2, we need to improve the estimates on volume established in [14] (see also [8], [20] and [15] for other similar estimates and technics). Section 2 is devoted to a brief survey on the properties of the volume of star-shaped domains we need subsequently. In section 3, we establish a comparison lemma (see lemma 3.1), improving the similar comparison lemma of [14], and which is fundamental for our proof of theorem 1.2: it provides a bound from above by a curvilinear integral of $\big(\underline{\operatorname{Ric}} - (n-1)\big)_-$ on the part less than $(n-1)\frac{\cos r}{\sin r}$ of the mean curvature of geodesic spheres of radius r. This lemma is used in sections 4 and 5 to get some bounds from above and below on the volume of geodesic balls. The proofs of the diameter and volume bounds of theorems 1.1 and 1.2 are given in section 6. Section 7 is devoted to the proof of the finiteness of $\pi_1(M)$, and section 8 to the proof of proposition 1.5. Finally, we discuss in section 9 the case $p = n/2$.

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2 Volume and mean curvature of spheres

Notation. Let $x \in M$. We denote by U_x the *injectivity domain* of the exponential map at x and we identify points of $U_x \backslash \{0_x\}$ with their polar coordinates $(r, v) \in \mathbb{R}_+^* \times \mathbb{S}_x^{n-1}$ (where $\mathbb{S}_x^{n-1}$ is the set of normal vectors at x). *We wright v_g for the Riemannian measure and set $\omega = \exp_x^* v_g = \theta(r, v)\, dr\, dv$, where dv and dr are the canonical measures of $\mathbb{S}_x^{n-1}$ and $\mathbb{R}_+^*$.* Henceforth, *we extend θ to $\big(\mathbb{R}_+^* \times \mathbb{S}^{n-1}\big) \backslash U_x$ by 0.*

For all (r, v) in $U_x \backslash \{0\}$, *we denote by $h(r, v)$ the mean curvature at $\exp_x(rv)$ (for the exterior normal $\frac{\partial}{\partial r}$) of the sphere centered at x and of radius r.* This function h is defined on U_x and satisfies the formula $\frac{\partial \theta}{\partial r}(t, v) = h(t, v)\theta(t, v)$ (*cf* [17], p. 329).

For all real k, we set $h_k = (n-1)\frac{s_k'(r)}{s_k(r)}$ for the corresponding function on the model space (S_k^n, g_k) (n-dimensional, simply connected, with sectional curvature k) where, as usual,

$$s_k(r) = \frac{\sinh(\sqrt{|k|}r)}{\sqrt{|k|}} \text{ when } k < 0, \qquad s_k(r) = r \text{ when } k = 0,$$

$$s_k(r) = \begin{cases} \frac{\sin(\sqrt{k}r)}{\sqrt{k}} & \text{if } r \leq \frac{\pi}{\sqrt{k}} \\ 0 \ \text{ if } r > \frac{\pi}{\sqrt{k}}, & \end{cases} \quad \text{when } k > 0.$$

On U_x (*resp.* on $U_x \cap B(0, \frac{\pi}{\sqrt{k}})$ if $k > 0$), *we set $\psi_k = (h_k - h)_-$.* Following [15], we will use:

Lemma 2.1 *Let u be an element of $\mathbb{S}_x^{n-1}$ and $I_u =]0, r(u)[$ the interval of values t such that $(t, u) \in U_x$. The function $t \mapsto \psi_k(t, u)$ is continuous, right and left differentiable everywhere in $I_u \cap]0, \frac{\pi}{\sqrt{k}}[$ and it satisfies:*

$$\begin{cases} 1) \ \lim_{t \to 0^+} \psi_k(t, u) = 0, \\ 2) \ \frac{\partial \psi_k}{\partial r} + \frac{\psi_k^2}{n-1} + \frac{2\psi_k h_k}{n-1} \leq \rho_k, \end{cases}$$

(where this differential inequality is satisfied by the right and left derivatives of ψ_k and where $\rho_k = \big(\underline{\mathrm{Ric}} - k(n-1)\big)_-$).

Proof. — Apply the well known Bochner formula $g(\nabla \triangle f, \nabla f) = \frac{1}{2}\triangle |\nabla f|^2 + |Ddf|^2 + \mathrm{Ric}(\nabla f, \nabla f)$ to the distance to x function d_x.

Since $|\nabla d_x| = 1$ and the Hessian $Dd(d_x)$ is zero on $\mathbb{R}\nabla d_x$ and equal to the second fondamental form of the geodesic sphere of center x on $\nabla d_x^{\perp}$, we infer that h satisfies the following Riccati inequation,

$$\frac{\partial h}{\partial r} + \frac{h^2}{n-1} + \mathrm{Ric}\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) \leq 0$$

This inequation becomes an equation on the model spaces (S^n_k, g_k), which easily gives inequality 2) of lemma 2.1. Since $h \sim (n-1)/r + o(1)$ (see [17] for details), we also easily get 1).

q.e.d.

Volume of star-shaped domains:

Definition. — Let $x\in M$ and $T \subset M$. We say that T is star-shaped at x if for all $y \in T$ there exists a minimizing geodesic from x to y contained in T. Equivalently, we may assume that $T = \exp_x\big(T_x\big)$, where T_x is an affine star-shaped subset of $\overline{U_x} \subset T_xM$.

Given T, a subset of M star-shaped at x, let $A_T(r)$ denote the volume of $B(x,r) \cap T$. In the same way, $L_T(r)$ stands for the $(n-1)$-dimensional volume of $(r\,\mathbb{S}^{n-1}_x)\cap U_x \cap T_x$ for the measure $\theta(r,.)\,dv$. Note that $L_T(r) = \int_{\mathbb{S}^{n-1}_x} \mathbb{1}_{T_x}\theta(r,v)dv$ and $A_T(r) = \int_0^r L_T(t)\,dt$. Finally, the functions corresponding to θ, A and L on the model manifold (S^n_k, g_k) will be denoted by θ_k, A_k and L_k respectively. The regularity properties of the functions L_T and A_T used subsequently are summarized in the following lemma:

Lemma 2.2 *Let T a star-shaped subset of (M,g).*
(i) L_T is a right continuous, left lower semi-continuous function,
(ii) A_T is a continuous, right differentiable function of derivative L_T.

(iii) Given $\alpha \in]0,1]$, the function

$$f(r) = \left(\frac{L_T(r)}{L_k(r)}\right)^{\alpha} - \frac{\alpha}{\mathrm{Vol}\,\mathbb{S}^{n-1}} \int_0^r \int_{\mathbb{S}^{n-1}_x} \Big(\frac{L_T(s)}{L_k(s)}\Big)^{\alpha-1} \mathbb{1}_{T_x}\psi_k \frac{\theta}{\theta_k}$$

is decreasing either on $\mathbb{R}^*_+$ (if $k \leq 0$) or on $]0, \frac{\pi}{\sqrt{k}}[$ (if $k > 0$).

Proof. — To prove (i), note that $\theta(r,v)\mathbb{1}_{T_x}$ is the product of $r^{n-1}\mathbb{1}_{T_x}(r,v)$ by the Jacobian of $\exp_x$, hence $r\mapsto\theta(r,v)\mathbb{1}_{T_x}$ is positive on an interval $]0, r(v)[$, vanishes on $[r(v), +\infty[$, and so is right continuous and left lower semi-continuous on $\mathbb{R}$. We infer also that $\mathbb{1}_{T_x}\theta$ is bounded on every compact of T_xM. This yields the boundedness of L_T on every compact subset of $[0, +\infty[$. We infer (i) from the Lebesgue dominated convergence theorem and the Fatou lemma. Property (ii) now follows (i) by the definition of A_T.

To complete the proof of lemma 2.2, we note that, by definition of L_T, and since $\mathrm{Vol}\, M \backslash \exp_x(U_x) = 0$, we may assume that $T_x \subset U_x$. For all integers $m \geq 1$ let $T_x^{(m)} = (1 - \frac{1}{m}) T_x \subset T_x$ be the image of T_x by the homothety of center 0 and factor $(1 - \frac{1}{m})$ in $T_{x_0}M$ and set $T^{(m)} = \exp_x(T_x^{(m)})$. By the monotone convergence theorem, we have $A_T = \lim_{m\to\infty} A_{T^{(m)}}$ and $L_T = \lim_{m\to\infty} L_{T^{(m)}}$. Hence, it only remains to show (iii) for $T^{(m)}$. We will use the following elementary lemma:

Lemma 2.3 *A function* $f : [a,b] \to \mathbb{R}$ *is decreasing if and only if it satisfies the two conditions*
(a) for all $x \in [a, b[$, $\limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \leq 0$,
(b) for all $x \in]a, b]$, $\liminf_{h\to 0^-} f(x+h) \geq f(x)$.

As for L_T and A_T, the function $r \mapsto \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x^{(m)}} \psi_k \frac{\theta}{\theta_k}(r,v)\, dv$ is right continuous, left lower semi-continuous on $I_k =]0, +\infty[$ if $k \leq 0$ (resp. on $I_k =]0, \frac{\pi}{\sqrt{k}}[$ if $k > 0$), and $r \mapsto \int_0^r \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x^{(m)}} \psi_k \frac{\theta}{\theta_k}$ is continuous, right differentiable on I_k; so the function f satisfies the inequality (b) of lemma 2.3. We now prove (a):

For all $r > 0$ let $\mathbb{S}^r_{T^{(m)}} = \{ v \in \mathbb{S}_x^{n-1} /\ rv \in T_x^{(m)} \}$. We denote by $\tilde{L}(r+t)$ the volume of $(r+t).\mathbb{S}^r_{T^{(m)}}$ for the measure $\theta(r+t, .)dv$. Since $T_x^{(m)}$ is star-shaped at x, we have $\tilde{L}(r+t) \geq L_{T^{(m)}}(r+t)$ (with equality if $t = 0$). Hence

$$\lim_{t\to 0^+} \frac{L_T^{(m)}(r+t) - L_T^{(m)}(r)}{t} \leq \lim_{t\to 0^+} \frac{\tilde{L}(r+t) - \tilde{L}(r)}{t}$$

Since $\tilde{L}(r+t) = \int_{\mathbb{S}^r_{T^{(m)}}} \theta(r+t, v) dv$ and $\frac{\partial \theta}{\partial r} = h\, \theta$, we obtain, by differentiating this integral expression of $\tilde{L}$ (Note that $h\theta$ and $\psi_k \theta$ are integrable on the set $\mathbb{S}^r_{T^{(m)}}$ (which could be false for T and this is why we introduced the sets $T^{(m)}$): for any $t \in [0, \frac{1}{m-1} r[$, the closure of $(r+t).\mathbb{S}^r_{T^{(m)}}$ in T_xM is compact and belongs to $U_x \backslash \{0_x\}$ because the cut-radius is continuous on $\mathbb{S}_x^{(n-1)}$ (see [17]) and bounded below by $\frac{m}{m-1} r > r+t$ on $\mathbb{S}^r_{T^{(m)}}$; But, the function $h = \frac{1}{\theta}\frac{\partial \theta}{\partial r}$ is smooth on $U_x \backslash \{0_x\}$, and so uniformly bounded on every set $(r+t).\mathbb{S}_T^{(m)}(r)$),

$$\lim_{t\to 0^+} \frac{\tilde{L}(r+t) - \tilde{L}(r)}{t} = \int_{\mathbb{S}_x^{n-1}} h \mathbb{1}_{T^{(m)}} \theta\, dv \leq \int_{\mathbb{S}_x^{n-1}} (\psi_k + h_k) \mathbb{1}_{T^{(m)}} \theta\, dv$$

Combining the last two inequalities, we get:

$$\overline{\lim}_{t\to 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t} \leq h_k(r) L_{T^{(m)}}(r) + \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T^{(m)}} \psi_k \theta.$$

The case $\alpha = 1$ of (a) easily follows, noting that L_k has derivative $h_k L_k$:

$$\limsup_{t\to 0^+} \frac{\frac{L_{T^{(m)}}(r+t)}{L_k(r+t)} - \frac{L_{T^{(m)}}(r)}{L_k(r)}}{t} =$$

$$\limsup_{t\to 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t L_k(r)}$$

$$+ \lim_{t\to 0^+} \left[L_{T^{(m)}}(r+t) \frac{1}{t} \left(\frac{1}{L_k(r+t)} - \frac{1}{L_k(r)} \right) \right]$$

$$= \frac{1}{\operatorname{Vol} \mathbb{S}^{n-1} \theta_k(r)} \left[\limsup_{t\to 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t} - h_k(r) L_{T^{(m)}}(r) \right]$$

Let $B = \frac{1}{\operatorname{Vol} \mathbb{S}^{n-1}} \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{\mathbb{S}_T^{(m)}} \psi_k \frac{\theta}{\theta_k} \, dv$. For all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that for all $t \in]0, t_\epsilon[$, we have $\frac{L_T^{(m)}(r+t)}{L_k(r+t)} \leq \frac{L_T^{(m)}(r)}{L_k(r)} + t(B+\epsilon)$. Moreover, by concavity, we get:

$$\left(\frac{L_T^{(m)}(r)}{L_k(r)} + t(B+\epsilon) \right)^\alpha - \left(\frac{L_T^{(m)}(r)}{L_k(r)} \right)^\alpha \leq \alpha \left(\frac{L_T^{(m)}(r)}{L_k(r)} \right)^{\alpha-1} \eta(B+\epsilon)$$

It follows that $\limsup_{t\to 0^+} \frac{F(r+t) - F(r)}{t} \leq \alpha(B+\epsilon) \left(\frac{L_T^{(m)}(r)}{L_k(r)} \right)^{\alpha-1}$ for every $\epsilon > 0$ and we get inequality (b) for any $\alpha \in]0, 1]$ by letting ϵ tend to 0.

q.e.d.

3 Comparison lemma on mean curvature

The following lemma improves lemma 2.2 in [15] and theorem 2.1 in [14]. We provide a pointwise bound on ψ_k which, in case $k > 0$ admits a sharp polynomial blow-up when $r \to \frac{\pi}{\sqrt{k}}$; these both improvements are necessary for our proof of theorem 1.2 (see the proof of lemma 4.1).

Lemma 3.1 *Let $k \in \mathbb{R}$, and $p > n/2$ and $r > 0$; assume $r \leq \frac{\pi}{2\sqrt{k}}$ if $k > 0$. We have:*

$$\psi_k^{2p-1}(r, v) \, \theta(r, v) \leq (2p-1)^p \left(\frac{n-1}{2p-n} \right)^{p-1} \int_0^r \rho_k^p(t, v) \theta(t, v) \, dt.$$

Moreover if $k>0$ and $\frac{\pi}{2\sqrt{k}} < r < \frac{\pi}{\sqrt{k}}$, then

$$\sin^{4p-n-1}(\sqrt{k}r)\,\psi_k^{2p-1}(r,v)\,\theta(r,v) \leq (2p{-}1)^p\left(\frac{n{-}1}{2p{-}n}\right)^{p-1}\int_0^r \rho_k^p(t,v)\theta(t,v)\,dt$$

These two inequalities hold for all normal vector $v{\in}\mathbb{S}_x^{n-1}$, even if we replace θ everywhere by $\mathbb{1}_{[0,s_v[}\,\theta$ (for any $s_v \geq 0$).

Remark. — The bounds diverge when p tends to $n/2$ except in the case $n=2$ (which then yields a control of ψ_k by the L_1-norm of ρ_k).

Proof. — Let ϕ be a nonnegative, C^1 function on $U_x\backslash\{0\}$, bounded in the neighborhood of 0. By lemma 2.1, the function $r{\mapsto}\phi(r,v)\,\psi_k^{2p-1}(r,v)\,\theta(r,v)$ is continuous and right differentiable on I_v, and its derivative satisfies:

$$\frac{\partial}{\partial r}(\phi\,\psi_k^{2p-1}\,\theta) \leq (2p{-}1)\rho_k\,\phi\,\psi_k^{2p-2}\theta - \left(\frac{2p{-}n}{n{-}1}\right)\phi\,\psi_k^{2p}\,\theta + \Big(\frac{4p{-}n{-}1}{n{-}1}h_k - \frac{1}{\phi}\frac{\partial\phi}{\partial r}\Big)_-\phi\,\psi_k^{2p-1}\,\theta$$

where we used $\frac{\partial\theta}{\partial r} = h\theta \leq h_k\theta{+}\psi_k\theta$. Setting $X = \left(\int_0^r \phi\,\psi_k^{2p}\theta\,dt\right)$ and integrating, we get:

$$0 \leq \phi\psi_k^{2p-1}\theta(r) \leq (2p{-}1)\Big(\int_0^r \phi\rho_k^p\theta\,dt\Big)^{1/p}X^{1-\frac{1}{p}} - \Big(\frac{2p{-}n}{n{-}1}\Big)X + \Big[\int_0^r\Big(\frac{4p{-}n{-}1}{n{-}1}h_k{-}\frac{1}{\phi}\frac{\partial\phi}{\partial r}\Big)_-^{2p}\phi\,\theta\,dt\Big]^{1/2p}X^{1-\frac{1}{2p}} \qquad (*)$$

where we used $\lim\limits_{t\to 0}\phi(t,v)\,\psi_k^{2p-1}(t,v)\,\theta(t,v) = 0$. Dividing out by $X^{1-\frac{1}{p}}$, we obtain a quadratic polynomial that takes a non-negative value at $X^{\frac{1}{2p}}$ and we infer:

$$\Big(\int_0^r \phi\,\psi_k^{2p}\theta\,dt\Big)^{\frac{1}{2p}} \leq \sqrt{\frac{(n{-}1)(2p{-}1)}{2p{-}n}}\Big(\int_0^r \phi\,\rho_k^p\,\theta\,dt\Big)^{1/2p} + \frac{n{-}1}{2p{-}n}\left(\int_0^r\Big(h_k\frac{2p{-}1{+}(2p{-}n)}{n{-}1} - \frac{\partial\phi/\partial r}{\phi}\Big)_-^{2p}\phi\,\theta\,dt\right)^{1/2p}.$$

We prove the first inequality of lemma 3.1 by taking $\phi(r,v)=1$. Indeed then, the above inequality and the positivity of h_k yield:

$$\int_0^r \psi_k^{2p}\theta\,dt \leq \left(\frac{(2p{-}1)(n{-}1)}{2p{-}n}\right)^p\int_0^r \rho_k^p\theta\,dt.$$

Plugging this into the above inequality $(*)$, we obtain

$$\psi_k^{2p-1}\theta(r) \leq (2p-1)^p \Big(\frac{n-1}{2p-n}\Big)^{p-1} \left(\int_0^r \rho_k^p \theta\, dt\right).$$

For the second inequality, we set $\phi = \sin^{4p-n-1}(\sqrt{k}r)$ and observe that, in this case, the last term of inequality $(*)$ vanishes. So we get for all $r < \frac{\pi}{\sqrt{k}}$:

$$\sin^{4p-n-1}(\sqrt{k}r)\psi_k^{2p-1}\theta \leq (2p-1)^p \Big(\frac{n-1}{2p-n}\Big)^{p-1} \int_0^{\frac{\pi}{\sqrt{k}}} \rho_k^p \theta\, dt.$$

q.e.d.

4 Hyper-concentration of the measure

In this section we prove the first volume estimate required in our proof of theorem 1.2. It says that, if the Ricci curvature concentrates sufficiently above $n-1$ on a star-shaped subset T of M at x, then the Riemannian measure of T is almost contained in $B(x,\pi) \cap T$.

Lemma 4.1 *There exists an explicit constant $C(p,n)$ such that if (M^n, g) contains a subset T, star-shaped at a point x, on which:*

$$\epsilon = R_T^2 \Big[\frac{1}{\operatorname{Vol} T}\int_T \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p\Big]^{\frac{1}{p}} \leq \big(\frac{\pi}{6}\big)^{2-\frac{1}{p}},$$

where R_T is such that $T \subset B(x, R_T)$, then, for all radius $R_T \geq r \geq \pi$:

$$L_T(r) \leq \frac{C(p,n)}{r} \epsilon^{\frac{p(n-1)}{2p-1}} \operatorname{Vol} T.$$

Remark. — The same conclusion holds in case $n = 2$ and $p = 1$ by letting $n = 2$ and $p \to 1$ in the proof below.

Proof. — Lemma 2.2 (with $0 < t \leq r < \frac{\pi}{\sqrt{k}}$, $\alpha = \frac{1}{2p-1}$ and $k > 0$ fixed) yields:

$$\big(\frac{L_T(r)}{L_k(r)}\big)^{\frac{1}{2p-1}} - \big(\frac{L_T(t)}{L_k(t)}\big)^{\frac{1}{2p-1}} \leq \frac{1}{2p-1} \int_t^r \big(\frac{L_T}{L_k}\big)^{\frac{1}{2p-1}-1} \frac{1}{\operatorname{Vol} \mathbb{S}^{n-1}} \int_{T_x} \psi_k \frac{\theta}{\theta_k}.$$

As

$$\frac{\big(L_T/L_k\big)^{\frac{2(1-p)}{2p-1}}}{\operatorname{Vol} \mathbb{S}^{n-1}} \int_{T_x} \psi_k \frac{\theta}{\theta_k} \leq \frac{1}{\big(L_k\big)^{\frac{1}{2p-1}}} \Big(\int_{T_x} \psi_k^{2p-1}\theta\Big)^{\frac{1}{2p-1}},$$

we get:

$$\frac{1}{\operatorname{Vol}\mathbb{S}^{n-1}}\int_t^r\Big(\frac{L_T(s)}{L_k}\Big)^{\frac{2-2p}{2p-1}}\int_{T_x}\psi_k\frac{\theta}{\theta_k}\,dvds$$
$$\leq\int_t^r\frac{(\sqrt{k})^{\frac{n-1}{2p-1}}}{\sin^2(\sqrt{k}s)}\Big(\int_{T_x}\frac{\sin^{4p-n-1}(\sqrt{k}s)\psi_k^{2p-1}\theta}{\operatorname{Vol}\mathbb{S}^{n-1}}\,dv\Big)^{\frac{1}{2p-1}}ds.$$

Lemma 3.1 implies:

$$\Big(\frac{L_T(r)}{\sin^{n-1}(\sqrt{k}r)}\Big)^{\frac{1}{2p-1}}-\Big(\frac{L_T(t)}{\sin^{n-1}(\sqrt{k}t)}\Big)^{\frac{1}{2p-1}}$$
$$\leq\Big(\frac{(n-1)}{(2p-1)(2p-n)}\Big)^{\frac{p-1}{2p-1}}\Big(\int_{T\cap B(x,r)}\rho_k^p\Big)^{\frac{1}{2p-1}}\int_t^r\frac{1}{\sin^2(\sqrt{k}s)}\,ds$$

Setting $\epsilon'=\epsilon^{\frac{p}{2p-1}}$, $k=\frac{(\pi-\epsilon')^2}{r^2}$ and assuming $t\in[\frac{\pi}{2(\pi-\epsilon')}r,r]$, and since, by concavity of the sine function, $\int_t^r\frac{1}{\sin^2(\sqrt{k_r}s)}\,ds\leq\frac{\pi r}{2\epsilon'}$, we have:

$$\frac{L_T(r)^{\frac{1}{2p-1}}}{(\sin(\sqrt{k_r}r))^{\frac{n-1}{2p-1}}}-\frac{L_T(t)^{\frac{1}{2p-1}}}{(\sin(\sqrt{k_r}t))^{\frac{n-1}{2p-1}}}$$
$$\leq\frac{\pi}{2R_T^{\frac{1}{2p-1}}}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\operatorname{Vol}(T)^{\frac{1}{2p-1}}$$

Multiplying this inequality by $(\sin(r\sqrt{k_r}))^{\frac{n-1}{2p-1}}\leq(\epsilon')^{\frac{n-1}{2p-1}}$, we infer that for all $t\in[\frac{\pi}{2(\pi-\epsilon')}r,r]$,

$$L_T(r)^{\frac{1}{2p-1}}\leq L_T(t)^{\frac{1}{2p-1}}\left(\frac{\epsilon'}{\sin((\pi-\epsilon')\frac{t}{r})}\right)^{\frac{n-1}{2p-1}}$$
$$+\frac{\pi}{2R_T^{\frac{1}{2p-1}}}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\operatorname{Vol}T^{\frac{1}{2p-1}}\epsilon'^{\frac{n-1}{2p-1}}.$$

Using the inequality $(a+b)^\alpha\leq 2^{\alpha-1}(a^\alpha+b^\alpha)$ (for all $a,b\geq 0$), with $\alpha=2p-1$, and the fact that $\sin[(\pi-\epsilon')\frac{t}{r}]\geq\sin(\frac{\pi}{6})=\frac{1}{2}$, when $t\in[\frac{\pi}{2(\pi-\epsilon')}r,\frac{5\pi}{6(\pi-\epsilon')}r]$, we get:

$$L_T(r)\leq 2^{2p+n-3}\epsilon^{\frac{p(n-1)}{2p-1}}L_T(t)$$
$$+\frac{\pi^{2p-1}}{2R_T}\operatorname{Vol}(T)\epsilon^{\frac{p(n-1)}{2p-1}}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1},$$

for all $t\in[\frac{\pi}{2(\pi-\epsilon')}r,\frac{5\pi}{6(\pi-\epsilon')}r]$ (note that $\frac{5\pi}{6(\pi-\epsilon')}r\leq r$, hence $\frac{t}{r}\leq 1$).

By the mean value property, there exists $t\in[\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$ such that $L_T(t)$ is bounded above by $\frac{3(\pi-\epsilon')}{\pi r}\int_{\frac{\pi r}{2(\pi-\epsilon')}}^{\frac{5\pi r}{6(\pi-\epsilon')}} L_T(s)\,ds$ which is less than $\frac{3}{r}\int_0^{R_T} L = \frac{3}{r}\,\mathrm{Vol}(T)$. In summary, we conclude:

$$L_T(r) \leq \left[3.2^{2p+n-3}+\frac{\pi^{2p-2}}{2}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\right]\frac{\mathrm{Vol}(T)}{r}\epsilon^{\frac{p(n-1)}{2p-1}}.$$

q.e.d.

5 Lower Bound on the volume of geodesic balls

In this section, we bound from below the relative volume of the geodesic balls. It is the second step of the proof of theorem 1.2.

Lemma 5.1 *Let $n \geq 2$ be an integer and $p > n/2$ be a real. There exist (computable) constants $C(p,n) > 0$ and $B(p,n)$ such that when (M^n, g) contains a star-shaped subset T which satisfies*

$$\epsilon = R_T^2\Big[\frac{1}{\mathrm{Vol}\,T}\int_T (\underline{\mathrm{Ric}})_-^p\Big]^{\frac{1}{p}} \leq B(p,n), \quad \textit{then we have}$$

(i) for all $0 < r \leq R \leq R_T$, $\frac{A_T(r)}{A_T(R)} \geq \Big(1-C(p,n)\epsilon^{\frac{p}{2p-1}}\Big)\frac{r^n}{R^n}$.
(ii) if $T = B(x,R_0)$, $y\in T$ and $r \geq 0$ satisfy $d(x,y)+r \leq R_0$ then

$$\Big(\frac{\mathrm{Vol}\,B(y,r)}{\mathrm{Vol}\,B(x,R_0)}\Big)^{\frac{1}{2p'-1}} \geq$$
$$\Big(\frac{r}{R_0}\Big)^{\frac{n}{2p'-1}}\Big[\Big(\frac{2}{3}-C(p,n)\epsilon^{\frac{p'}{2p'-1}}\Big)\Big(\frac{r}{R_0}\Big)^{\frac{2n}{2p'-1}}-C(p,n)\epsilon^{\frac{p'}{2p'-1}}\Big],$$

where $p' = \max(n,p)$.

Proof. — Lemma 2.2 (with $k=0$ and $\alpha=1$) and the Hölder inequality yield, for all $t \leq r \leq R_T$:

$$\frac{L_T(r)}{r^{n-1}}-\frac{L_T(t)}{t^{n-1}} \leq \int_t^r \frac{L_T(s)^{1-\frac{1}{2p-1}}}{s^{n-1}}\Big(\int_{\mathbb{S}_x^{n-1}} 1\!\!1_{T_x}\psi_0^{2p-1}\theta\,dv\Big)^{\frac{1}{2p-1}}ds$$

Lemma 3.1 implies then

$$\frac{L_T(r)}{r^{n-1}}-\frac{L_T(t)}{t^{n-1}} \leq C(p,n)\int_t^r \frac{L_T(s)^{1-\frac{1}{2p-1}}}{s^{n-1}}\Big(\int_{B(x,s)\cap T}\rho_0^p\Big)^{\frac{1}{2p-1}}$$

$$\leq \frac{C(p,n)}{t^{n-1}} \left(\int_T \rho_0^p\right)^{\frac{1}{2p-1}} \int_t^r L_T^{1-\frac{1}{2p-1}}.$$

Multiplying this inequality by $nr^{n-1}t^{n-1}$, using the inequality $\int_t^r L_T^{1-\frac{1}{2p-1}} \leq (r-t)^{\frac{1}{2p-1}}\big(A_T(r)-A_T(t)\big)^{1-\frac{1}{2p-1}}$, and integrating the result with respect to t from 0 to r. We get

$$\frac{d}{dr}\Big(\frac{A_T}{r^n}\Big) \leq \Big(\frac{A_T(r)}{r^n}\Big)^{1-\frac{1}{2p-1}} C(p,n) \left(\int_T \rho_0^p\right)^{\frac{1}{2p-1}} nr^{\frac{1-n}{2p-1}}$$

(since $\frac{A_T(r)}{r^n}$ is right differentiable). Integrating once again yields

$$\Big[\frac{A_T(R)}{R^n}\Big]^{\frac{1}{2p-1}} - \Big[\frac{A_T(r)}{r^n}\Big]^{\frac{1}{2p-1}} \leq C(p,n)\Big(\int_T \rho_0^p\Big)^{\frac{1}{2p-1}} R^{\frac{2p-n}{2p-1}}. \quad (E_T^{r,R})$$

Inequality $\big(E_{T,R,R_T}\big)$ implies

$$\Big[\frac{A_T(R)}{A_T(R_T)R^n}\Big]^{\frac{1}{2p-1}} \geq R_T^{\frac{-n}{2p-1}}\big(1-C(p,n)\epsilon^{\frac{p}{2p-1}}\big) \geq \frac{1}{2}R_T^{\frac{-n}{2p-1}}$$

as soon as $B(p,n)$ is sufficiently small. This and $\big(E_T^{r,R}\big)$ imply (i).

To show (ii), we may assume, by the Hölder inequality, that $p\in]n/2,n]$. Let $y\in B(x,R_0)$ and (r,R) such that $0 < r \leq R \leq R_0-d(x,y)$. Multiplying $\big(E_{B(y,R),r,R}\big)$ by $\Big(\frac{1}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}$ and noting the inclusion $B(y,R) \subset B(x,R_0)$, we get

$$\Big(\frac{A_y(R)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}} \leq C(p,n)\Big(\frac{R^2\epsilon}{R_0^2}\Big)^{\frac{p}{2p-1}} + \Big(\frac{R}{r}\Big)^{\frac{n}{2p-1}}\Big(\frac{A_y(r)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}.$$

We will construct a sequence of decreasing balls $B_i = B(y_i,R_i)$ such that $B_1 = B(y,r)$, B_k is almost concentric to $B(x,R_0)$, and B_i contains a ball centered at y_{i+1} and of radius r_{i+1} close to R_i.

Let $\gamma : [0,d(x,y)] \to M$ be a minimizing geodesic from x to y and $\alpha = \alpha(p,n) < 1$ close enough to 1 such that we have $-\mathrm{Log}\,\alpha \leq 2\mathrm{Log}(2-\alpha)$ and $(2-\alpha)^{\frac{2p-n}{2p-1}}\alpha^{\frac{n}{2p-1}} < 1$. For all integers $1 \leq i \leq k = E\Big[1+\frac{\mathrm{Log}\big(\frac{\mathrm{d(x,x_0)+r}}{\mathrm{r}}\big)}{\mathrm{Log(2-\alpha)}}\Big]$, let

$$y_i = \gamma\big(d(x,x_0)+r-(2-\alpha)^{i-1}r\big), \qquad r_i = \alpha(2-\alpha)^{i-2}r,$$
$$R_i = (2-\alpha)^{i-1}r$$

Then $B(y_{i+1},r_{i+1}) \subset B(y_i,R_i) \subset B(x,R_0)$ and so, by the above inequality (in which we replace y by y_{i+1}, R by R_{i+1} and r by r_{i+1}), we get

$$\Big(\frac{A_{y_{i+1}}(R_{i+1})}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}$$

$$\leq C(p,n)\Big(\frac{r^2\epsilon}{R_0^2}\Big)^{\frac{p}{2p-1}}(2-\alpha)^{\frac{2pi}{2p-1}}+\Big(\frac{(2-\alpha)^n A_{y_i}(R_i)}{\alpha^n A_x(R_0)}\Big)^{\frac{1}{2p-1}},$$

hence also

$$\Big(\frac{A_{y_i}(R_i)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}$$
$$\leq \Big(\frac{2-\alpha}{\alpha}\Big)^{\frac{n(i-1)}{2p-1}}\left[\Big(\frac{A_y(r)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}+\frac{C(p,n)(\frac{r^2\epsilon}{R_0^2})^{\frac{p}{2p-1}}}{(\frac{(2-\alpha)^{n-2p}}{\alpha^n})^{\frac{1}{2p-1}}-1}\right]$$

For $i=k$, we have $d(x,y_k)\leq(1-\alpha)R_k$, so $B(y_k,R_k)\supset B(x,\alpha R_k)$. Inequality (i) thus yields

$$\Big(\frac{A_{y_k}(R_k)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}\geq\big(1-C(p,n)\epsilon^{\frac{p}{2p-1}}\big)\Big(\alpha\frac{R_k}{R_0}\Big)^{\frac{n}{2p-1}}$$
$$\geq\big(1-C(p,n)\epsilon^{\frac{p}{2p-1}}\big)\alpha^{\frac{n}{2p-1}}(2-\alpha)^{\frac{n(k-1)}{2p-1}}\Big(\frac{r}{R_0}\Big)^{\frac{n}{2p-1}}$$

These two estimates on $\frac{A_{y_k}(R_k)}{A_x(R_0)}$, and the fact that by assumption $\alpha^{\frac{n(k-1)}{2p-1}}\geq\Big(\frac{r}{r+d(x,y)}\Big)^{\frac{-n\mathrm{Log}\alpha}{(2p-1)\mathrm{Log}(2-\alpha)}}\geq\big(\frac{r}{R_0}\big)^{\frac{2n}{2p-1}}$, imply that there exist constants $C(p,n)>0$ and $B(p,n)>0$ such that when $\epsilon\leq B(p,n)$,

$$\Big(\frac{A_y(r)}{A_x(R_0)}\Big)^{\frac{1}{2p-1}}$$
$$\geq\big(\frac{r}{R_0}\big)^{\frac{n}{2p-1}}\left[\big(\frac{2}{3}-C(p,n)\epsilon^{\frac{p}{2p-1}}\big)\big(\frac{r}{R_0}\big)^{\frac{2n}{2p-1}}-C(p,n)\epsilon^{\frac{p}{2p-1}}\right],$$

where we have assumed $\alpha^{\frac{n}{2p-1}}\geq\frac{2}{3}$.

q.e.d.

In the case $(n,p)=(2,1)$, the following lemma holds

Lemma 5.2 *There exists constants $B>0$ and $C>0$ such that when a surface (S^2,g) contains a star-shaped subset T on which the sectional curvature K satisfies $\epsilon=\frac{R_T^2}{\mathrm{Vol}\,T}\int_T K^-\leq B$, then*

$$(i)\qquad \frac{A_T(r)}{\mathrm{Vol}\,T}\geq\big(\frac{r}{R_T}\big)^2\Big(1-\epsilon\mathrm{Log}\big(\frac{\mathrm{R_T}}{\mathrm{r}}\big)\Big),$$

for all $r\leq R_T$. If $T=B(x,R_0)$, $y\in T$ and $d(x,y)+r\leq R_0$, then

$$(ii)\qquad \frac{\mathrm{Vol}\,B(y,r)}{\mathrm{Vol}\,B(x,R_0)}\geq\big(\frac{r}{R_0}\big)^4\Big(1-3\epsilon\big(\frac{R_0}{r}\big)^2\Big)$$

PROOF. — An easy computation gives that the constant $C(p,n)$ involved in the differential inequality satisfied by $\frac{A_T}{r^n}$ in the above proof satisfies $C(p,n) = \frac{2p-1}{2p}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}$. In case $n = 2$ we may let p tend to 1 in that differential inequality and get $\frac{d}{dr}\left(\frac{A}{r^2}\right) \leq \frac{1}{r}\int_T K^-$ which, integrated, yields $\frac{A(R)}{R^2} - \frac{A(r)}{r^2} \leq (\mathrm{LogR}-\mathrm{Logr})\int_{\mathrm{T}} \mathrm{K}^-$, proving (i).

(ii) is proved as in lemma 5.1 (note that, in this case, we may let α tend to 1, which simplifies the final formula).

q.e.d.

6 Diameter bound

6.1 Proof of lemma 1.4

Note that if $B(p,n)$ is sufficiently small then lemma 5.1 implies $\frac{A_T(R)}{\mathrm{Vol}\, T} \geq \frac{R^n}{2R_T^n}$ hence we may assume that $T = B(x, R_0)$ and $\pi < R_0 \leq 2\pi$. Fix $\delta \in]0, \frac{R_0-\pi}{2}[$. If $y \in M$ is at a distance greater than $(\pi+\delta)$ from x, then we have $B(y,\delta) \subset B(x, \pi+2\delta) \backslash B(x,\pi)$. Lemma 4.1 now yields the bounds

$$\mathrm{Vol}\, B(y,\delta) \leq \int_\pi^{\pi+2\delta} L \leq 2C(p,n)A(R_0)\delta\epsilon^{\frac{p(n-1)}{(2p-1)}}$$

(where $A(R_0) = \mathrm{Vol}\, B(x, R_0)$). On the other hand, lemma 5.1 (ii) provides:

$$\mathrm{Vol}\, B(y,\delta) \geq \left(\frac{\delta}{2\pi}\right)^n \left[\frac{1}{2}\left(\frac{\delta}{2\pi}\right)^{\frac{2n}{2p'-1}} - C(p,n)\epsilon^{\frac{p'}{2p'-1}}\right]^{2p'-1} A(R_0)$$

by taking $B(p,n)$ small enough (still setting $p' = \max(p,n)$).
At this stage, we can distinguish two cases:

either $\left(\frac{\delta}{2\pi}\right)^{\frac{2n}{2p'-1}} \leq 4C(p,n)\epsilon^\beta$, where $\beta = \frac{2np(n-1)}{(2p-1)(2p'-1)(3n-1)}$,

or the above inequality becomes (since $\beta \leq \frac{p'}{2p'-1}$)

$$\mathrm{Vol}\, B(y,\delta) \geq C(p,n)\left(\frac{\delta}{2\pi}\right)^n A(R_0)\epsilon^{(2p'-1)\beta}$$

These two estimates on $\mathrm{Vol}\, B(y,\delta)$ imply a bound on δ:

$$\pi+\delta \leq \pi + C(p,n)\epsilon^{\beta\frac{2p'-1}{2n}} \leq \pi + C(p,n)\epsilon^{\frac{1}{10}} < R_0$$

We infer that $M \subset B(x, R_0)$. Let z be any point of M. We have $\rho^{(p)}_{z,R_0} \leq \left(\frac{\operatorname{Vol} B(x,R_0)}{\operatorname{Vol} B(z,R_0)}\right)^{\frac{1}{p}} \epsilon$. But $B(x, R_0 - \pi - C(p,n)\epsilon^{\frac{1}{10}}) \subset B(z, R_0)$ and so lemma 5.1 (i) implies:

$$\frac{\operatorname{Vol} B(z, R_0)}{\operatorname{Vol} B(x, R_0)} \geq \frac{\left(R_0 - \pi - C(p,n)\epsilon^{\frac{1}{10}}\right)^n}{2(2\pi)^n} \geq \frac{(R_0 - \pi)^n}{4(2\pi)^n}$$

What has done above for x can be done for any $z \in M$ (just replace ϵ by $\frac{4(2\pi)^{n/p}}{(R_0-\pi)^{n/p}}\epsilon$, for $\rho^{(p)}_{z,R_0} \leq \frac{4(2\pi)^{n/p}}{(R_0-\pi)^{n/p}}\epsilon$), which completes the proof.

6.2 Proof of the geometric inequalities of theorem 1.2

Let (M^n, g) be a complete manifold such that $\int_M \left(\underline{\operatorname{Ric}} - (n-1)\right)^p_-$ is finite and let $\left(B(x_i, 2\pi)\right)_{i \in I}$ be a maximal family of disjoint balls in M. The Dirichlet domains $T_i = \left\{y \,/\, d(x_i, y) < d(x_j, y),\ \forall j \neq i\right\}$ satisfy the three following classical facts:
1) $B(x_i, 4\pi) \supset T_i \supset B(x_i, 2\pi)$,
2) T_i is star-shaped at the x_i and
3) except for a set of zero measure, M is the disjoint union of the sets T_i.
Thus, setting $\alpha = \inf_{i \in I} \left[\frac{1}{\operatorname{Vol} T_i} \int_{T_i} \left(\underline{\operatorname{Ric}} - (n-1)\right)^p_-\right]^{\frac{1}{p}}$, we have

$$\int_M \left(\underline{\operatorname{Ric}} - (n-1)\right)^p_- = \sum_{i \in I} \int_{T_i} \left(\underline{\operatorname{Ric}} - (n-1)\right)^p_- \geq \alpha^p \sum_{i \in I} \operatorname{Vol} T_i = \alpha^p \operatorname{Vol} M$$

If $\alpha > \left[\frac{B(p,n)}{2^{101} 16\pi^2}\right]^p$ (where $B(p,n)$ is the constant of lemma 1.4), then $\operatorname{Vol} M \leq C(p,n)\rho^{(p)}(M)$ (where $C(p,n)$ is a universal constant). Elsewhere, there exists a star-shaped set T_i satisfying the assumptions of lemma 1.4. In the latter case (which is the only possible one under the stronger assumption $\rho^{(p)}_M \leq \frac{\operatorname{Vol} M}{C(p,n)}$, with $C(p,n)$ sufficiently large) we bound the diameter of M with Lemma 1.4 and the volume of M using lemma 5.1.

7 Fundamental group finiteness

To show the π_1-finiteness of the manifolds that satisfy $\frac{\rho_p}{\operatorname{Vol} M} \leq \frac{1}{C(p,n)}$, we just have to show their the universal covers are compact. We will apply lemma 1.4 to the universal Riemannian covering space $(\widetilde{M},\tilde{g})$, and so we have to construct a *good* star-subset subset in $\widetilde{M}$ (i.e. a star-shaped subset on which the pinching on the Ricci curvature is controlled by $\frac{\rho_p}{\operatorname{Vol} M}$).

The fundamental group acts freely and isometrically on the universal Riemannian cover. For all $\tilde{x}\in\widetilde{M}$ and any subset T of $\widetilde{M}$, we denote by $m_T(\tilde{x})$ the cardinality of $T\cap\pi_1.\tilde{x}$. Set $\tilde{x}_0\in\widetilde{M}$ and $\tilde{x}\in B(\tilde{x}_0,2\pi)$ that maximizes $m_{B(\tilde{x}_0,2\pi)}$. Since we may assume $\operatorname{Diam} M\leq 2\pi$, we have $1\leq m_{B(\tilde{x}_0,2\pi)}(y)\leq N$ and $m_{B(\tilde{x}_0,6\pi)}(y)\geq N$ for all $y\in B(\tilde{x}_0,2\pi)$ (where $N=m_{B(\tilde{x}_0,2\pi)}(\tilde{x})$). For all y in $B(\tilde{x}_0,2\pi)$, we choose N distincts points $y_1,\cdots,y_N$ in $\pi_1.y$ that are closer to $\tilde{x}_0$ than the other points of $\pi_1.y$, and let T be the union of these $\{y_1,\cdots,y_N\}$ for all $y\in B(\tilde{x}_0,2\pi)$. Hence $B(\tilde{x}_0,6\pi)\supset T\supset B(\tilde{x}_0,2\pi)$ and $m_T\equiv N$ on $\widetilde{M}$. We infer

$$\frac{1}{\operatorname{Vol} T}\int_T\big(\underline{\mathrm{Ric}}-(n-1)\big)_-^p\,dv_{\tilde{g}}=\frac{1}{\operatorname{Vol} M}\int_M\big(\underline{\mathrm{Ric}}-(n-1)\big)_-^p\,dv_g$$

It only remains to show that T is a star-shaped subset of $(\widetilde{M},\tilde{g})$. Set $y\in T$ and let γ be a minimizing geodesic from y to $\tilde{x}_0$. Assume there exists $z\in\gamma\setminus T$. Since $m_T(z)=N$, there exist $(\sigma_1,\cdots,\sigma_N)$ in $\pi_1(M)\setminus\{id\}$ such that $\sigma_i.z\in T$ for all $1\leq i\leq N$. But every element of $\pi_1(M)\setminus\{id\}$ acts without fixed points on $\widetilde{M}$, thus there exists $1\leq i_0\leq N$ such that $\sigma_{i_0}.y\notin T$. Since σ_{i_0} acts isometrically, we have

$$d(\tilde{x}_0,y)\leq d(\tilde{x}_0,\sigma_{i_0}.y),\quad d(\tilde{x}_0,z)\geq d(\tilde{x}_0,\sigma_{i_0}.z),$$
$$d(z,y)=d(\sigma_{i_0}.z,\sigma_{i_0}.y).$$

The relations above combined with $d(\tilde{x}_0,y)=d(\tilde{x}_0,z)+d(z,y)$ and the triangle inequality provide

$$d(\tilde{x}_0,y)=d(\tilde{x}_0,\sigma_{i_0}y)=d(\tilde{x}_0,\sigma_{i_0}z)+d(\sigma_{i_0}z,\sigma_{i_0}y).$$

We infer that there exists a minimizing geodesic segment from $\sigma_{i_0}.y$ to $\tilde{x}_0$ which contains $\sigma_{i_0}.z$. But $d(\sigma_{i_0}.z,\sigma_{i_0}.y)=d(z,y)<d(\tilde{x}_0,y)\leq d(\tilde{x}_0,\sigma_{i_0}.y)$, so there is only one geodesic minimizing the distance between $\sigma_{i_0}.z$ and $\sigma_{i_0}.y$, which implies that the geodesic $\sigma_{i_0}(\gamma)$ contains $\tilde{x}_0$. Since $d(z,\tilde{x}_0)=d(\sigma_{i_0}.z,\tilde{x}_0)$, we have $\sigma_{i_0}.x_0=x_0$, contradicting the fact that σ_{i_0} has no fixed point.

8 Spectral lower bounds

To prove proposition 1.5 we need bounds on some Sobolev constants. In [8], S. Gallot provides such bounds under the pinching $\operatorname{Diam}(M)^2\big(\frac{1}{\operatorname{Vol} M}\int_M(\underline{\mathrm{Ric}})_-^p\big)^{\frac{1}{p}}\leq\epsilon(p,n)$, where $p>n/2$ and

$\epsilon(p,n) > 0$ is a universal constant. Combined with theorem 1.2 this yields

Proposition 8.1 *Let (M^n,g) be a complete Riemannian manifold. If $\frac{\rho_M^{(p)}}{\mathrm{Vol}\, M} \leq \frac{1}{C(p,q,n)}$ (for $p>n/2$ and $q>n$), then we have*
(i) for all $u\in H^{1,2}(M)$, $\|u\|_{\frac{2q}{q-2}} \leq \mathrm{Diam}(M)C(p,q,n)\|du\|_2+\|u\|_2$.
(ii) for all $u\in H^{1,q}(M)$, $\sup u - \inf u \leq \mathrm{Diam}(M)C(p,q,n)\|du\|_q$.

We now prove proposition 1.5. Let α be a 1-form on M such that $\|\alpha\|_2^2 = 1$ and $\triangle\alpha = \lambda\alpha$. The Bochner formula (see [17]) yields

$$\int_M \frac{g(\triangle\alpha,\alpha)}{\mathrm{Vol}\, M} = \|D\alpha\|_2^2 + \int_M \frac{\big(\mathrm{Ric} - (n-1)\big)(\alpha,\alpha)}{\mathrm{Vol}\, M} + (n-1)$$

Combined with Hölder's inequality, this implies:

$$\lambda \geq \|D\alpha\|_2^2 - \Big(\frac{\rho_p}{\mathrm{Vol}\, M}\Big)^{\frac{1}{p}} \|\alpha\|^2_{\frac{2p}{p-1}} + (n-1)$$

Since we may assume $\mathrm{Diam}\, M \leq 2\pi$, proposition 8.1:

$$\|\alpha\|^2_{\frac{2p}{p-1}} \leq C(p,n)\|D\alpha\|_2^2 + 2\|\alpha\|_2^2 .$$

We infer $\quad \big(\lambda - (n-1) + 2\epsilon\big) \geq \Big(1 - C(p,n)\Big(\frac{\rho_p}{\mathrm{Vol}\, M}\Big)^{\frac{1}{p}}\Big)\|D\alpha\|_2^2 \quad (*)$.

Splitting orthogonally the 2-tensor $D\alpha$ into antisymmetric part $\frac{d\alpha}{2}$, traceless symmetric part and scalar part $-\frac{\delta\alpha}{n}g$, we obtain $\|D\alpha\|_2^2 \geq \frac{1}{n}\|\delta\alpha\|_2^2 + \frac{1}{2}\|d\alpha\|_2^2$. Combining the splitting with the inequality $(*)$ above and distinguishing the case $d\alpha = 0$ (where $\|\delta\alpha\|_2^2 = \lambda$) and the case $\delta\alpha = 0$ (where $\|d\alpha\|_2^2 = \lambda$), we easily get proposition 1.5.

9 $L^{\frac{n}{2}}$-pinching on the Ricci curvature

In the case $n=2$ and $p=1$, the π_1-finiteness follows readily from the Gauss-Bonnet theorem. The proofs of Theorems 1.1 and 1.2, Lemma 1.4, and Propositions 1.5 and 1.6 may be easily adapted. For instance, to prove Lemma 1.4 we just use Lemma 5.2 in place of Lemma 5.1. To prove Proposition 1.5, we may assume $\lambda \leq 2n$ and use the Sobolev inequality $\|u\|_4 \leq C\|du\|_2 + \|u\|_2$ to show by Moser's iteration that $\|\alpha\|_\infty \leq C'$; this implies that inequality $(*)$ still holds and then we finish the proof as in the case $p>1$.

We now focus on counter-examples or density results announced in the introduction. Let σ (resp. $\underline{\sigma}(x)$) stand for the sectional curvature (resp. the smallest sectional curvature of tangent planes at x).

Proposition 9.1 *Set $n \geq 3$. For any $p, \epsilon > 0$, the n-Riemannian manifolds with $\int_M |\sigma|^p \leq \epsilon$ and $\mathrm{Vol}(M) \leq \epsilon$ are dense in (pointed) Gromov-Hausdorff distance amongs all the (non compact) length spaces.*

Proof. — The $(n-1)$-Riemannian manifolds are obviousely GH-dense amongs all the finite graphs (by performing some connected sums of spheres $\mathbb{S}^{n-1}$ to get small slightely thickened graphs). Then, just take Riemannian product of these manifolds with a sufficiently small $\mathbb{S}^1$.

q.e.d.

The next density results are more interesting since we want to keep a control on the volume of our family of manifolds.

Proposition 9.2 *For any reals K and $V_0 > 0$, any integer $n \geq 3$ and real any $\epsilon > 0$ the compact Riemannian n-manifolds (M^n, g) that satisfy*

$$\int_M \left(\underline{\sigma} - K\right)_-^{\frac{n}{2}} < \epsilon \quad \textit{and} \quad \mathrm{Vol}\, M = V_0$$

are dense in (pointed) Gromov-Hausdorff distance amongs all the (non compact) length spaces.

We can also replace $\int_M \left(\underline{\sigma} - K\right)_-^{\frac{n}{2}}$ by $\int_M |\underline{\sigma}|^{\frac{n}{2}}$ or by $\int_M |\sigma|^p$ for any $p < n/2$.

With the same kind of glueing techniques, it is not difficult to construct complete, non compact n-manifolds with non finite volume and which satisfy $\rho_{n/2} \leq \epsilon$ (for any $n \geq 3$ and any $\epsilon > 0$).

Proposition 9.3 *Let (M^n, g) be any compact Riemannian n-manifold ($n \geq 3$). There exists a sequence of complete Riemannian metrics (g_m) that converge to g in the Gromov-Hausdorff distance and such that*

$$\frac{\rho_{n/2}(g_m)}{\mathrm{Vol}\, g_m} \to 0 \qquad \mathrm{Vol}(g_m) \to \infty \qquad \forall l \in \mathbb{N},\ \lambda_l(g_m) \to 0$$

where λ_l denote the l-th eigenvalue of the Laplacian on functions.

Proof. — We define the following five families of cylinders $I \times \mathbb{S}^{n-1}$ with warped-product metric $dt^2 + b(t)^2 g_{\mathbb{S}^{n-1}}$

- $\mathbf{C}_\nu^{-1} = [0, \sqrt{\nu}] \times \mathbb{S}^{n-1}$ with $b(t) = \eta(t^2 + \nu^2)^{\alpha/2}$, where $\alpha = 1 + \frac{1}{\sqrt{-Log(\nu)}}$ and $\eta = \frac{\sqrt{1+\nu}}{\alpha(\nu+\nu^2)^{\frac{\alpha-1}{2}}}$ for any $\nu > 0$.
- $\mathbf{F}_\nu = [\theta - \frac{\pi}{2}, 0] \times \mathbb{S}^{n-1}$ with $b(t) = \eta' \cos t$, $\theta = \tan^{-1}(\frac{\sqrt{\nu}}{\alpha}(1+\nu))$ and $\eta' = \frac{\sqrt{\alpha^2 + \nu(1+\nu)^2}}{\alpha} = \frac{1}{\cos\theta}$.

- $\overline{\mathbf{F}}_\nu = [0, \frac{\eta'\pi}{2}] \times \mathbb{S}^{n-1}$ with $b(t) = \eta' \cos\frac{t}{\eta'}$.
- $\mathbf{C}^0_\nu = [0, \frac{\sqrt{\nu}(1+\nu)}{2\alpha}] \times \mathbb{S}^{n-1}$ with $b(t) = t + \frac{\sqrt{\nu}(1+\nu)}{2\alpha}$.
- $\overline{\mathbf{C}}^0_{\nu,L} = [0, L] \times \mathbb{S}^{n-1}$ with $b(t) = \frac{\nu^{\frac{\alpha+1}{2}}}{\alpha(1+\nu)^{\frac{\alpha}{2}-1}}$.

If (X,Y) is an orthonormal family of tangent vectors to $\mathbb{S}^{n-1}$, then the sectional curvatures $\sigma(X,Y)$ of the manifolds $\mathbf{F}_\nu$, $\overline{\mathbf{F}}_\nu$, $\mathbf{C}^{-1}_\nu$ and $\mathbf{C}^0_\nu$ are equal to

$$\frac{1}{b^2} - \left(\frac{b'}{b}\right)^2 = \begin{cases} 0 \text{ on } \mathbf{C}^0_\nu \text{ or } \overline{\mathbf{C}}^0_{\nu,L}, \\ \frac{\nu^2\alpha^2}{(t^2+\nu^2)^2} - \frac{\alpha^2}{t^2+\nu^2}\Big(1 - \frac{1}{1+\nu}\big(\frac{\nu+\nu^2}{t^2+\nu^2}\big)^{\alpha-1}\Big) \text{ on } \mathbf{C}^{-1}_\nu, \\ 1 - \frac{\sin^2\theta}{\cos^2 t} \text{ on } \mathbf{F}_\nu, \\ \frac{1}{\eta'^2} \text{ on } \overline{\mathbf{F}}_\nu. \end{cases}$$

If X is a unit vector tangent to $\mathbb{S}^{n-1}$, then

$$\sigma(X, \frac{\partial}{\partial r}) = -\frac{b''}{b} = \begin{cases} 0 \text{ on } \mathbf{C}^0_\nu \text{ or } \overline{\mathbf{C}}^0_{\nu,L}, \\ -\frac{\alpha(2-\alpha)\nu^2}{(t^2+\nu^2)^2} - \frac{\alpha(\alpha-1)}{t^2+\nu^2} \text{ on } \mathbf{C}^{-1}_\nu, \\ 1 \text{ on } \mathbf{F}_\nu, \\ \frac{1}{\eta'^2} \text{ on } \overline{\mathbf{F}}_\nu. \end{cases}$$

We now obtain readily the following upper bounds ($\forall \nu \le \frac{1}{C(n)}$)

$$\int_{\mathbf{F}_\nu} (\underline{\sigma}-1)_-^{\frac{n}{2}} \le C(n) \int_0^{\frac{\pi}{2}-\theta} \frac{\sin^n\theta}{\cos t}\, dt \le C(n)\sin^n\theta \le \frac{C(n)}{\big(-\ln\nu\big)^{\frac{n-2}{4}}},$$

$$\int_{\overline{\mathbf{F}}_\nu} (\underline{\sigma}-1)_-^{\frac{n}{2}} \le C(n) \int_0^{\frac{\eta'\pi}{2}} \frac{\sin^n\theta}{\cos^n\theta} \cos^{n-1}\frac{t}{\eta'}\, dt$$
$$\le C(n)\sin^n\theta \le \frac{C(n)}{\big(-\ln\nu\big)^{\frac{n-2}{4}}},$$

$$\int_{\mathbf{C}^0_\nu} (\underline{\sigma}-1)_-^{\frac{n}{2}} \le C(n)\nu^{\frac{n}{2}} \le \frac{C(n)}{\big(-\ln\nu\big)^{\frac{n-2}{4}}}.$$

Concerning $\mathbf{C}^{-1}_\nu$, first note that $\sigma(X,Y)$ is decreasing on $[0,\sqrt{\nu}]$ and so $\sigma(X,Y) \ge 0$ for ν small enough. Hence, using $\sqrt{\frac{a}{2}}+\sqrt{\frac{b}{2}} \le \sqrt{a+b} \le \sqrt{a}+\sqrt{b}$, we have

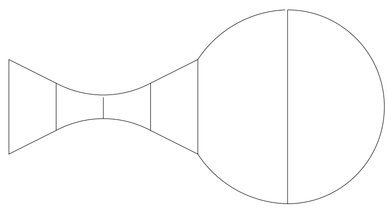
$$\int_{\mathbf{C}^{-1}_\nu} (\underline{\sigma}-1)_-^{\frac{n}{2}}$$
$$\le C(n)\eta^{n-1}\Big[\nu^n \int_0^{\sqrt{\nu}} (t^2+\nu^2)^{\frac{\alpha(n-1)}{2}-n}\, dt + \int_0^{\sqrt{\nu}} (t^2+\nu^2)^{\frac{\alpha(n-1)}{2}}\, dt$$

$$+(\alpha-1)^{n/2}\int_0^{\sqrt{\nu}}(t^2+\nu^2)^{\frac{\alpha(n-1)}{2}-\frac{n}{2}}\,dt\Big]$$

$$\leq C(n)\eta^{n-1}\Big[\nu^{(\alpha-1)(n-1)}+(\nu+\sqrt{\nu})^{(n-1)\alpha+1}$$

$$+(\alpha-1)^{\frac{n}{2}-1}(\nu+\sqrt{\nu})^{(\alpha-1)(n-1)}\Big]\leq\frac{C(n)}{(-\ln\nu)^{\frac{n-2}{4}}}.$$

The metrics of these cylinders are normalized to yield a C^1 metric when the small (resp. the large) connected component of the boundary of $\mathbf{F}_\nu$ is identified with the large connected component of the boundary of $\mathbf{C}^0_\nu$ (resp. with the boundary of $\overline{\mathbf{F}}_\nu$). Similarly, note that for any $\nu>0$ small enough, there exists $\beta<1$ such that we get a C^1 metric by identifying a connected component of the boundary of $\mathbf{C}^{-1}_{\beta\nu}$ with the small connected component of $\mathbf{C}^0_\nu$. We set $\overline{\mathbf{B}}_\nu$ for the manifold $\mathbf{C}^0_v\#\mathbf{C}^{-1}_{\beta\nu}\#\mathbf{C}^{-1}_{\beta\nu}\#\mathbf{C}^0_v\#\mathbf{F}_\nu\#\overline{\mathbf{F}}_\nu$:



We then have $\int_{\overline{\mathbf{B}}_\nu}(\underline{\sigma}-1)_-^{\frac{n}{2}}\leq\frac{C(n)}{(-\ln\nu)^{\frac{n-2}{4}}}$, also $\operatorname{Diam}\overline{\mathbf{B}}_\nu\leq 2\pi$ and $\operatorname{Vol}\overline{\mathbf{B}}_\nu\geq\frac{1}{C(n)}$ for any ν small enough. For all $N\in\mathbb{N}$, there exists a small ν' to have $\mathbf{C}^0_\nu$ containing at least N disjoint balls of radius $\frac{\sqrt{\nu'}(1+\nu')}{\alpha(\nu')}$. Excise these balls from one of the $\mathbf{C}^0_\nu$ part of $\overline{\mathbf{B}_\nu}$ and glue the resulting manifold to N manifolds $\overline{\mathbf{B}_{\nu'}}$ along the spheres of radius $\frac{\sqrt{\nu'}(1+\nu')}{\alpha(\nu')}$ of their boundaries. Taking $N=\big(-\ln\nu\big)^{\frac{n-2}{8}}$ and multiplying the metric by $\frac{1}{(-\ln\nu)^{\frac{n-2}{8n}}}$, we get a manifold B_ν which is diffeomorphic to B^n and satisfies $\operatorname{Diam}B_\nu\leq\frac{4\pi}{(-\ln\nu)^{\frac{n-2}{16n}}}$, $\operatorname{Vol}B_\nu\geq\frac{(-\ln\nu)^{\frac{n-2}{16}}}{C(n)}$ and $\int_{B_\nu}(\underline{\sigma}-1)_-^{\frac{n}{2}}\leq\frac{C(n)}{(-\ln\nu)^{\frac{n-2}{8}}}$.

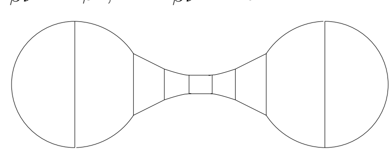
To prove proposition 9.3, fix a point x_0 in the compact manifold M. For any $m\in\mathbb{N}$, there exists a $r\in]0,\operatorname{inj}(M,g)[$ and a metric g' on M which is equal to g on $M\backslash B(x_0,2r)$, is flat on $B(x_0,r)$ and is at Gromov-Hausdorff distance from g bounded above by $\frac{1}{2m}$. For any $\nu>0$ such that $\frac{\sqrt{\nu}(1+\nu)}{\alpha}<r$ we obtain a new metric g'_ν on M by replacing the flat metric on $B(x_0,\frac{\sqrt{\nu}(1+\nu)}{\alpha})$ by the metric of B_ν. We can find ν_m small enough to have a Gromov-Hausdorff distance between g and g'_{ν_m} less than $\frac{1}{m}$, and also $\operatorname{Vol}(g'_{\nu_m})\geq mC(n)$ and $\frac{1}{\operatorname{Vol}g'_{\nu_m}}\int_{(M,g'_{\nu_m})}\big(\underline{\sigma}-1\big)_-^{\frac{n}{2}}\leq\frac{1}{m}$. We

then set $g_m = g'_{\nu_m}$. It only remains to show the collapsing of the eigenvalues of the metrics g_m. In that purpose, first consider on $\overline{\mathbf{B}}_\nu$ the continuous function f that is equal to 1 on the part $\mathbf{C}^0_\nu \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$, equal to 0 on the part $\mathbf{C}^0_\nu \# \mathbf{C}^{-1}_\nu$ and equal to $f(t) = \frac{t}{\sqrt{\beta\nu}}$ on the remaining part $\mathbf{C}^{-1}_{\beta\nu}$. For this functions f, we have

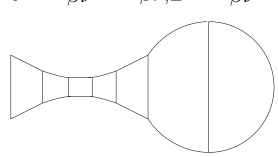
$$\frac{\int_{\overline{\mathbf{B}}_\nu} |\nabla f|^2}{\int_{\overline{\mathbf{B}}_\nu} |f|^2} \leq \frac{\eta^{n-1}}{C(n)\eta'^{n-1}} \int_0^{\sqrt{\nu}} \left|\frac{\partial f}{\partial t}\right|^2 (t^2+\nu^2)^{\frac{\alpha(n-1)}{2}} dt \leq C(n)\nu^{\frac{n-2}{2}}.$$

(M^n, g_m) contains $\left(-\ln \nu_m\right)^{\frac{n-2}{8}}$ manifolds $\overline{\mathbf{B}}_{\nu'_m}$ whose metric has been multiplied by $\frac{1}{(-\ln \nu_m)^{\frac{n-2}{8n}}}$. We extend to M by zero the function f corresponding to each $\overline{\mathbf{B}}_{\nu'_m}$ part of (M^n, g_m). Thus, we obtain $\left(-\ln \nu_m\right)^{\frac{n-2}{8}}$ L^2-orthogonal functions on (M^n, g_m), whose Rayleigh quotients are bounded above by $C(n)\nu_m^{\frac{n-2}{2}} \left(\ln \frac{1}{\nu_m}\right)^{\frac{n-2}{8n}}$. As we can suppose that ν_m tends to 0, the min-max principle implies the collapsing of all eigenvalues to 0 (this collapsing implies that the g_m do not tend to g in the C^0 sense and that the Sobolev constants are not bounded under $L^{\frac{n}{2}}$ pinching, otherwise the proof of Proposition 1.5 would hold).

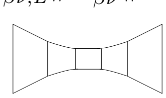
We now adapt the above construction to prove Proposition 9.2. Note that on $\mathbf{C}^{-1}_\nu$ we have $-\frac{\alpha(2-\alpha)\nu^2}{(t^2+\nu^2)^2} - \frac{\alpha(\alpha-1)}{t^2+\nu^2} \leq \sigma(t) \leq \frac{\nu^2\alpha^2}{(t^2+\nu^2)^2} + \frac{\alpha^2\nu^{\alpha-1}}{(1+\nu)^{2-\alpha}(t^2+\nu^2)^\alpha}$, and so we have, for any $p < n/2$, $\int_{\mathbf{C}^{-1}_\nu} |\sigma|^p \leq C(n,p)\nu^{\frac{n}{2}-p}$. There exists $\beta < 1$ such that a connected component of the boundary of $\mathbf{C}^{-1}_{\beta\nu}$ glue metrically in a C^1-way with the small connected component of $\mathbf{C}^0_\nu$. We set $\mathbf{B}^2_{\nu,L}$ the manifold $\overline{\mathbf{F}}_\nu \# \mathbf{F}_\nu \# \mathbf{C}^0_v \# \mathbf{C}^{-1}_{\beta\nu} \# \overline{\mathbf{C}}^0_{\beta\nu,L} \# \mathbf{C}^{-1}_{\beta\nu} \# \mathbf{C}^0_v \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$:



we set also $\mathbf{B}^1_{\nu,L} = \mathbf{C}^0_v \# \mathbf{C}^{-1}_{\beta\nu} \# \overline{\mathbf{C}}^0_{\beta\nu,L} \# \mathbf{C}^{-1}_{\beta\nu} \# \mathbf{C}^0_v \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$:



and $\mathbf{B}^0_{\nu,L} = \mathbf{C}^0_v \# \mathbf{C}^{-1}_{\beta\nu} \# \overline{\mathbf{C}}^0_{\beta\nu,L} \# \mathbf{C}^{-1}_{\beta\nu} \# \mathbf{C}^0_v$:



It is now easy to see that for any $L > 0$, $\epsilon > 0$ and $K \in \mathbb{R}$ we can choose two sequences (L_l) and λ_l such that the sequence $\overline{B}^{0,L}_{l,\epsilon} = \left(\lambda_n B^0_{1/l,L_l}\right)$ (resp. $\overline{B}^{1,L}_{n,\epsilon} = \left(\lambda_l B^1_{1/l,L_l}\right)$ or $\overline{B}^{2,L}_{l,\epsilon} = \left(\lambda_l B^2_{1/l,L_l}\right)$) are at Gromov-Hausdorff distance from the segment $[0,L]$ less than ϵ and the integrals $\int_{\overline{B}^{i,L}_{l,\epsilon}} \left(\underline{\sigma} - K\right)_-^{\frac{n}{2}}$ tend to 0 (resp. and the volume of $\overline{B}^{i,L}_{l,\epsilon}$ tends to any given real in $]0, C(\epsilon, K, L)]$). Note also that if we take m large enough we can glue a number as large as needed of manifolds $\overline{B}^{1,L}_{m,\epsilon}$ or $\overline{B}^{0,L}_{m,\epsilon}$ to one of the $\mathbf{C}^0$ part of $\overline{B}^{i,L}_{l,\epsilon}$. We deduce that, for any finite graph, we can glue a family $\overline{B}^{i_k,L_k}_{l_k,\epsilon^2}$ (with the n_p large enough) to get a manifold which is at Gromov Hausdorff distance from the graph less than $\epsilon/2$ and which satisfies $\int \left(\underline{\sigma} - K\right)_-^{\frac{n}{2}} \leq \epsilon/2$ and with volume less than $V_0/2$. To get a volume equal to V_0 we glue enough copies of $\overline{B}^{1,\epsilon^2}_{l,\epsilon^4}$ (for $K = \frac{1}{\epsilon^8}$): the small change on the distance to the graph does not depend on the number of these copies and that we can choose the volume of each copies of these $\overline{B}^{1,\epsilon^2}_{l,\epsilon^4}$ equal to any number in $]0, C(\epsilon^2, \frac{1}{\epsilon^8}, L)[$. Since the finite graph are dense in Gromov Hausdorff distance this ends the proof of theorem 9.2.

To prove the version of theorem 9.2 with the pinching on $\int_M |\underline{\sigma}|^{\frac{n}{2}}$ or $\int_M |\sigma|^p$ $(p < n/2)$ we just have to replace the parts $\mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$ in the above definition of the $\overline{B}^{i,L}_{l,\epsilon}$ by some small flat n-torus and remark that for the metrics contructed by this way we have $\underline{\sigma} \leq 0$.

q.e.d.

Note that in the proof of Proposition 9.3 above we only need that $\operatorname{Vol} M$ and $\int_M \left(\underline{\sigma} - 1\right)_-^{\frac{n}{2}}$ are finite. It is classical that any manifold supports a complete metric with finite volume but we do not know if both finitenesses above are always fulfilled for at least one complete metric on any (noncompact) manifold. Note also that the finiteness of $\int_M \left(\underline{\sigma} - 1\right)_-^{\frac{n}{2}}$ does not imply $\operatorname{Vol} M < \infty$ since, for any $\epsilon > 0$, we can start from $\mathbf{B}^2_{\nu,1}$ and then iteratively glue some $\mathbf{B}^1_{\nu_k,1}$ to the remaining free $\overline{\mathbf{C}}^0_{\beta\nu_{k-1}}$ element with a sequence ν_k chosen so as to get a complete manifold with infinite volume and $\int_M \left(\underline{\sigma} - 1\right)_-^{\frac{n}{2}} \leq \epsilon$.

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