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***Abstract* — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell's equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.**

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[5], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (*i.e.* h-refinement), of the interpolation order (*i.e.* p-enrichment) or of both of them (*i.e.* hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain Ω of $\mathbb{R}^3$. The electric permittivity tensor $\bar{\bar{\epsilon}}(x)$ and the magnetic permeability tensor $\bar{\bar{\mu}}(x)$ are varying in space and both symmetric positive definite. The electric field $\vec{E}$ and the magnetic field $\vec{H}$ verify: $\bar{\bar{\epsilon}}\partial_t\vec{E} = \mathrm{curl}\,\vec{H}$ and $\bar{\bar{\mu}}\partial_t\vec{H} = -\mathrm{curl}\,\vec{E}$. We consider a partition Ω_h of Ω into a set of tetrahedra τ_i. To each $\tau_i \in \Omega_h$ we assign an integer $p_i \geq 0$ (the local interpolation order) and we collect the p_i in the vector $p = \{p_i : \tau_i \in \Omega_h\}$. On the domain boundary $\partial\Omega$, we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition Ω_h and vector p, we seek approximate solutions to Maxwell's system in the finite dimensional subspace $V_p(\Omega_h) := \{v \in L^2(\Omega)^3 : v_{|\tau_i} \in \mathbb{P}_{p_i}(\tau_i)\ ,\ \forall\tau_i \in \Omega_h\}$, where $\mathbb{P}_{p_i}(\tau_i)$ denotes the space of nodal polynomials of total degree at most p_i inside the element τ_i. The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector $\mathbb{E}$ (resp. $\mathbb{H}$) of size d (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

$$\left\{\begin{array}{lcl} \mathbb{M}^{\epsilon}\partial_t\mathbb{E} & = & \mathbb{K}\mathbb{H} - \mathbb{A}\mathbb{H} - \mathbb{B}\mathbb{H}, \\ \mathbb{M}^{\mu}\partial_t\mathbb{H} & = & -\mathbb{K}\mathbb{E} + \mathbb{A}\mathbb{E} - \mathbb{B}\mathbb{E}, \end{array}\right. \tag{1}$$

where we have the following definitions and properties:

- $\mathbb{M}^{\epsilon}, \mathbb{M}^{\mu}$ and $\mathbb{K}$ are $d \times d$ block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore $\mathbb{M}^{\epsilon}$ and $\mathbb{M}^{\mu}$ are symmetric positive definite matrices, and $\mathbb{K}$ is a symmetric matrix.
- $\mathbb{A}$ is also a $d \times d$ block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then $\mathbb{A}$ is a symmetric matrix.
- $\mathbb{B}$ is a $d \times d$ block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then $\mathbb{B}$ is a skew-symmetric matrix.

The DGTD-$\mathbb{P}_{p_i}$ method using centered fluxes combined with Nth order leap-frog (LF_N) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix $\mathbb{S} = \mathbb{K} - \mathbb{A} - \mathbb{B}$, in the general form:

$$\left\{\begin{array}{lcl} \mathbb{M}^{\epsilon}\dfrac{\mathbb{E}^{n+1} - \mathbb{E}^{n}}{\Delta t} & = & \mathbb{S}_N\mathbb{H}^{n+\frac{1}{2}}, \\ \mathbb{M}^{\mu}\dfrac{\mathbb{H}^{n+\frac{3}{2}} - \mathbb{H}^{n+\frac{1}{2}}}{\Delta t} & = & -{}^t\mathbb{S}_N\mathbb{E}^{n+1}, \end{array}\right. \tag{2}$$

where the matrix $\mathbb{S}_N$ verifies:

$$\mathbb{S}_N = \left\{\begin{array}{ll} \mathbb{S} & \text{if} \quad N = 2, \\ \mathbb{S}(\mathbb{I} - \dfrac{\Delta t^2}{24}\mathbb{M}^{-\mu}\,{}^t\mathbb{S}\mathbb{M}^{-\epsilon}\mathbb{S}) & \text{if} \quad N = 4. \end{array}\right. \tag{3}$$

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy.

$$\mathcal{E}^n = \frac{1}{2}({}^t\mathbb{E}^n\mathbb{M}^{\epsilon}\mathbb{E}^n + {}^t\mathbb{H}^{n-\frac{1}{2}}\mathbb{M}^{\mu}\mathbb{H}^{n+\frac{1}{2}}). \tag{4}$$

Then we have the following Lemmas [2]:

Lemma 1 *Using the DGTD-$\mathbb{P}_{p_i}$ method (2)-(3), the total discrete electromagnetic energy $\mathcal{E}^n$ (4) is a positive definite quadratic form of all unknowns if:*

$$\Delta t \leq \frac{2}{d_N}, \ \textit{with}\ d_N = \|\mathbb{M}^{\frac{-\mu}{2}}\,{}^t\mathbb{S}_N\mathbb{M}^{\frac{-\epsilon}{2}}\|,$$

where $\|.\|$ denote the canonical norm of a matrix $(\forall X, \|AX\| \leq \|A\|\|X\|)$, and the matrix $\mathbb{M}^{\frac{-\sigma}{2}}$ (σ stands for ϵ or μ) is the inverse square root of $\mathbb{M}^{\sigma}$. Also, for a given mesh, the stability limit of the LF_4 scheme is roughly 2.85 times larger than that of the LF_2 scheme.

Lemma 2 *The convergence order in space and time of the DGTD-$\mathbb{P}_{p_i}$ method (2)-(3) is:*

$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^N),$$

where $\Delta t \in [0, T]$, h is the mesh size and the solution belongs to $H^s(\Omega)$ with $s > 1/2$ a regularity parameter.

IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (*i.e.* non refined) elements a high polynomial degree p_1 and to the refined region a low polynomial degree p_2. The resulting scheme is referred to as DGTD-$\mathbb{P}_{(p_1,p_2)}$. If $p_1 = p_2 = p$, the scheme is simply called DGTD-$\mathbb{P}_p$.

A. *eigenmode in a PEC cavity*

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with $\epsilon = \mu = 1$.

The 2D case: We compare the LF_2 and LF_4 time schemes using the DGTD-$\mathbb{P}_p$ method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall L^2 error of the DGTD-$\mathbb{P}_p$ method using the LF_2 and LF_4 schemes. Tab. I gives the L^2 error, the CPU time in minutes to reach 105 periods, and the convergence order "$\mathcal{O}$". It can be observed from Fig. 1 and Tab. I that the LF_4 is more accurate and requires less CPU time than the LF_2 scheme. Moreover, the convergence order is bounded by N which confirms the result of Lemma 2.

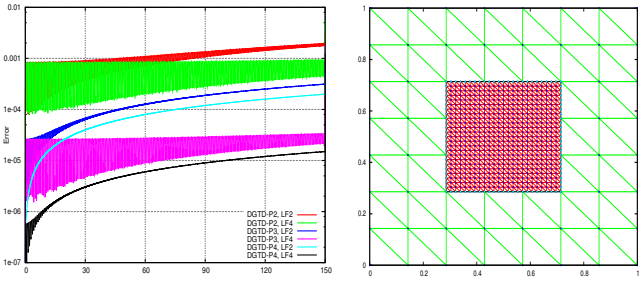

Fig. 1. Error of the DGTD-$\mathbb{P}_p$ method (left) on the non-conforming triangular mesh (right).

TABLE I. COMPARISON BETWEEN LF_2 AND LF_4 DGTD-$\mathbb{P}_p$ METHOD

	LF_2 scheme			LF_4 scheme		
p	L^2 error	CPU	$\mathcal{O}$	L^2 error	CPU	$\mathcal{O}$
2	1.8E-03	11	2.28	5.5E-04	8	2.32
3	3.1E-04	39	2.33	2.4E-05	28	2.97
4	1.9E-04	98	2.10	1.5E-05	70	3.99

The 3D case: Here we compare the DGTD-$\mathbb{P}_{(p_1,p_2)}$ with several DGTD-$\mathbb{P}_p$ methods using the LF_2 time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the L^2 error, the # DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree p_2 is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-$\mathbb{P}_{(p_1,p_2)}$ requires less CPU time than the DGTD-$\mathbb{P}_p$. Moreover, for a given CPU time, the DGTD-$\mathbb{P}_{(p_1,p_2)}$ method is roughly 8 times more efficient than the DGTD-$\mathbb{P}_p$ method.

B. *Scattering of a plane wave by a dielectric cylinder*

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius $r_0 = 0.6$ and bounds a material of relative permittivity $\epsilon_r = 2.25$. The computational domain is chosen

TABLE II. # DOF, L^2 ERRORS AND CPU TIME IN MINUTES

p	**0**	**1**	**2**	**3**
L^2 error	7.2E-01	2.0E-01	1.4E-02	8.0E-04
CPU	4	40	213	859
# DOF	4406	17624	44060	88120
(p_1,p_2)	**(2,0)**	**(2,1)**	**(3,1)**	**(3,2)**
L^2 error	3.6E-02	1.3E-02	1.0E-03	8.8E-04
CPU	35	106	260	499
# DOF	42908	43676	87096	86030

as a cylinder of radius 1.5 centered at $(0,0)$. At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-$\mathbb{P}_p$ method using a conforming mesh with the DGTD-$\mathbb{P}_{(p_1,p_2)}$ using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone $0.55 \leq r \leq 0.65$ of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (300 of them are hanging nodes). Tab. III shows the relative L^2 error, the # DOF and the CPU time in minutes to reach $t = 5$. As expected, the gain in CPU time between the DGTD-$\mathbb{P}_{(p_1,p_2)}$ and the conforming DGTD-$\mathbb{P}_p$ methods is considerable. For instance, to achieve an error level of 5%, the DGTD-$\mathbb{P}_{(2,0)}$ scheme requires 3 times less DOF and 21 times less CPU time than the DGTD-$\mathbb{P}_2$ scheme.

TABLE III. # DOF, L^2 ERRORS AND CPU TIME IN MINUTES

DGTD-$\mathbb{P}_p$: Conforming triangular mesh

p	**0**	**1**	**2**	**3**
L^2 error	13.6%	7.15%	5.20%	5.22%
CPU	20	178	542	1817
# DOF	11920	35760	71520	119200

DGTD-$\mathbb{P}_{(p_1,p_2)}$: Non-conforming triangular mesh

(p_1,p_2)	**(1,0)**	**(2,0)**	**(2,1)**	**(3,2)**
L^2 error	11.6%	5.36%	5.39%	5.37%
CPU	9	25	33	179
# DOF	11450	19700	26100	46700

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