The Cauchy problem at a node with buffer

Mauro Garavello, Paola Goatin

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Abstract

We consider the Lighthill-Whitham-Richards traffic flow model on a network composed by an arbitrary number of incoming and outgoing arcs connected together by a node with a buffer. Similar to [15], we define the solution to the Riemann problem at the node and we prove existence and well posedness of solutions to the Cauchy problem, by using the wave-front tracking technique and the generalized tangent vectors.

1 Introduction

Fluid dynamic models were developed in the literature in order to describe the macroscopic evolution of vehicular traffic in roads and in networks. In the network setting, different kinds of solutions at the intersections were recently proposed; see [6, 7, 8, 9, 14, 15, 16, 17, 20] and the references therein. The interest in this field was also motivated by other applications: data networks [8], supply chains [13], air traffic management [22], gas pipelines [1].

In this paper we consider the scalar Lighthill-Whitham-Richards model (see [19, 21]) on a network composed by a single junction with a buffer with finite size and capacity. Nodes with buffers have been introduced in the case of supply chains in [14] and also for car traffic in [12, 15]. These kinds of intersections take into account some dynamics inside the junction, described by

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ordinary differential equations depending on the difference between incoming and outgoing fluxes.

In the following sections, we prove existence and well posedness of solutions at the node with buffer and with an arbitrary number of incoming and outgoing roads. The results are obtained by means of the wave-front tracking method [3, 18] and on the generalized tangent vectors [2, 5]. In our case, the wave-front tracking method consists in producing piecewise constants approximate solutions both for the density of cars and for the load of the buffer and in proving uniform estimates for the approximate solutions in order to obtain compactness and so existence of solutions. Instead, the Lipschitz continuous dependence of the solution with respect to the initial condition is proved by viewing the vector space L^1 as a Finsler manifold and by considering the evolution in time of generalized tangent vectors along wave-front tracking approximate solutions. We remark that the results contained in [14] do not apply in our situation, while the papers [12, 15] describe only special cases of Riemann problems.

The paper is organized as follows. Section 2 contains some preliminary notations and definitions, while Section 3 describes in details the solution of Riemann problems at the node. Sections 4 and 5 deal respectively with the existence of solution and with the continuous dependence of the solution with respect to the initial condition. Finally, we recall in the appendix, for reader's convenience, some technical results of [11].

2 Basic Definitions and Notations

Consider a node J with n incoming arcs $I_1, \ldots, I_n$ and m outgoing arcs $I_{n+1}, \ldots, I_{n+m}$. We model each incoming arc I_i ($i \in \{1, \ldots, n\}$) of the node with the real interval $I_i =]-\infty, 0]$. Similarly, we model each outgoing arc I_j ($j \in \{n+1, \ldots, n+m\}$) of the node with the real interval $I_j = [0, +\infty[$. On each arc I_l ($l \in \{1, \ldots, n+m\}$), we consider the partial differential equation

$$\partial_t \rho_l + \partial_x f(\rho_l) = 0, \tag{1}$$

where $\rho_l = \rho_l(t, x) \in [0, \rho_{max}]$, is the *density* of cars, $v_l = v_l(\rho_l)$ is the *mean velocity* of cars and $f(\rho_l) = v_l(\rho_l)\,\rho_l$ is the *flux*. Moreover the real valued function $r(t) \in [0, r_{max}]$ denotes the total number of cars in the buffer inside the node J at time t.

We make the following assumptions on the flux f:

($\mathcal{F}$) $f : [0, 1] \to \mathbb{R}$ is a Lipschitz continuous and concave function satisfying

1. $f(0) = f(1) = 0$;

2. there exists a unique $\sigma \in]0,1[$ such that f is strictly increasing in $[0,\sigma[$ and strictly decreasing in $]\sigma,1]$.

The definitions of entropic solutions on arcs and weak solutions at the node are as follows.

Definition 1 *A function $\rho_l \in C([0,+\infty[; L^1_{loc}(I_l))$ is an entropy-admissible solution to (1) in the arc I_l if, for every $k \in [0,1]$ and every $\tilde{\varphi} : [0,+\infty[\times I_l \to \mathbb{R}$ smooth, positive and with compact support in $]0,+\infty[\times (I_l \setminus \{0\})$, it holds*

$$\int_0^{+\infty}\int_{I_l}\Big(|\rho_l - k|\partial_t\tilde{\varphi} + \operatorname{sgn}(\rho_l - k)(f(\rho_l) - f(k))\partial_x\tilde{\varphi}\Big)dxdt \geq 0. \tag{2}$$

Definition 2 *A collection of functions $\rho_l \in C([0,+\infty[; L^1_{loc}(I_l))$ ($l \in \{1,\ldots,n+m\}$) and $r \in W^{1,\infty}$ is a weak solution at J if*

1. *for every $l \in \{1,\ldots,n+m\}$, the function ρ_l is an entropy-admissible solution to (1) in I_l;*
2. *for every $l \in \{1,\ldots,n+m\}$ and for a.e. $t > 0$, the function $x \mapsto \rho_l(t,x)$ has a version with bounded variation;*
3. *for a.e. $t > 0$*

$$r'(t) = \sum_{i=1}^{n} f(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t,0+)),$$

where ρ_l stands for the version with bounded variation (see point 2).

For a collection of functions $\rho_l \in C([0,+\infty[; L^1_{loc}(I_l))$, $l \in \{1,\ldots,n+m\}$, such that, for every $l \in \{1,\ldots,n+m\}$ and a.e. $t > 0$, the map $x \mapsto \rho_l(t,x)$ has a version with bounded total variation, we define the functional

$$\Upsilon(t) := \mathrm{TV}_f(t) + Q(t), \tag{3}$$

where

$$\mathrm{TV}_f(t) := \sum_{l=1}^{n+m} \mathrm{TV}\left(f\left(\rho_l(t,\cdot)\right)\right) \tag{4}$$

and

$$Q(t) := \left|\sum_{i=1}^{n} f(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t,0+))\right|. \tag{5}$$

For later use, define the set

$$\Theta = \left\{ \boldsymbol{\theta} = (\theta_1,\ldots,\theta_{n+m}) \in \mathbb{R}^{n+m} : \begin{array}{c} \theta_1 > 0, \cdots, \theta_{n+m} > 0, \\ \sum_{i=1}^{n}\theta_i = \sum_{j=n+1}^{n+m}\theta_j = 1 \end{array} \right\}. \tag{6}$$

3 The Riemann Problem with buffer

Consider a node J with a buffer, whose demand and supply are equal to a constant $\mu \in]0, \max\{n, m\} f(\sigma)[$. Fix $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$, $r_0 \in [0, r_{max}]$ and consider the Riemann problem at J

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0, \\ \rho_l(0, \cdot) = \rho_{l,0}, \\ r'(t) = \displaystyle\sum_{i=1}^{n} f(\rho_i(t, 0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t, 0+)), \\ r(0) = r_0, \end{cases} \qquad l \in \{1, \ldots, n+m\}. \tag{7}$$

A solution to the Riemann problem at J is defined in the following way.

Definition 3 *A solution to the Riemann problem* (7) *is a weak solution at J, in the sense of Definition 2, such that $\rho_l(0, x) = \rho_{l,0}$ for every $l \in \{1, \ldots, n+m\}$ and for a.e. $x \in I_l$ and such that $r(0) = r_0$.*

We introduce the concept of Riemann solver at J.

Definition 4 *A Riemann solver $\mathcal{RS}$ is a function*

$$\begin{array}{rccc} \mathcal{RS}: & [0,1]^{n+m} & \longrightarrow & [0,1]^{n+m} \\ & (\rho_{1,0}, \ldots, \rho_{n+m,0}) & \longmapsto & (\bar\rho_1, \ldots, \bar\rho_{n+m}) \end{array}$$

satisfying the following

1. *for every $i \in \{1, \ldots, n\}$, the classical Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \begin{cases} \rho_{i,0}, & \text{if } x < 0, \\ \bar\rho_i, & \text{if } x > 0, \end{cases} \end{cases}$$

   is solved with waves with negative speed;

2. *for every $j \in \{n+1, \ldots, n+m\}$, the classical Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \begin{cases} \bar\rho_j, & \text{if } x < 0, \\ \rho_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$

   is solved with waves with positive speed.

Introduce the following sets

1. for every $i \in \{1, \ldots, n\}$ define

$$\mathcal{O}_i = \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \le \rho_{i,0} \le 1; \end{cases} \tag{8}$$

2. for every $j \in \{n+1, \ldots, n+m\}$ define

$$\mathcal{O}_j = \begin{cases} [0, f(\sigma)], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \le \rho_{j,0} \le 1; \end{cases} \tag{9}$$

3. for every $s \in [0, \sum_{i=1}^{n} \max \mathcal{O}_i]$ define

$$I_s = \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} \mathcal{O}_i \colon \sum_{i=1}^{n} \gamma_i = s \right\}; \tag{10}$$

4. for every $s \in \left[0, \sum_{j=n+1}^{n+m} \max \mathcal{O}_j\right]$ define

$$J_s = \left\{ (\gamma_{n+1}, \ldots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} \mathcal{O}_j \colon \sum_{j=n+1}^{n+m} \gamma_j = s \right\}. \tag{11}$$

In [15], the authors proposed to solve the Riemann problem (7) in the following way.

1. Fix $\boldsymbol{\theta} \in \Theta$ and define

$$\Gamma^1_{inc} = \sum_{i=1}^{n} \max \mathcal{O}_i, \quad \Gamma^1_{out} = \sum_{j=n+1}^{n+m} \max \mathcal{O}_j.$$

2. Define

$$\Gamma_{inc} = \begin{cases} \min \left\{\Gamma^1_{inc}, \mu\right\}, & \text{if } 0 \le r_0 < r_{max}, \\ \min \left\{\Gamma^1_{inc}, \Gamma^1_{out}, \mu\right\}, & \text{if } r_0 = r_{max}, \end{cases} \tag{12}$$

and

$$\Gamma_{out} = \begin{cases} \min \left\{\Gamma^1_{out}, \mu\right\}, & \text{if } 0 < r_0 \le r_{max}, \\ \min \left\{\Gamma^1_{out}, \Gamma^1_{inc}, \mu\right\}, & \text{if } r_0 = 0. \end{cases} \tag{13}$$

3. If $\Gamma_{inc} > \Gamma_{out}$, then

$$r(t) = \begin{cases} r_0 + (\Gamma_{inc} - \Gamma_{out})\, t, & \text{if } 0 < t < \frac{r_{max} - r_0}{\Gamma_{inc} - \Gamma_{out}}, \\ r_{max}, & \text{if } t > \frac{r_{max} - r_0}{\Gamma_{inc} - \Gamma_{out}}. \end{cases} \tag{14}$$

If $\Gamma_{inc} < \Gamma_{out}$, then

$$r(t) = \begin{cases} r_0 + (\Gamma_{inc} - \Gamma_{out})\, t, & \text{if } 0 < t < -\frac{r_0}{\Gamma_{inc} - \Gamma_{out}}, \\ 0, & \text{if } t > -\frac{r_0}{\Gamma_{inc} - \Gamma_{out}}. \end{cases} \tag{15}$$

If $\Gamma_{inc} = \Gamma_{out}$, then

$$r(t) = r_0 \tag{16}$$

for every $t > 0$.

4. If $\Gamma_{inc} \geq \Gamma_{out}$, then define

$$\begin{aligned} (\bar{\gamma}_1, \ldots, \bar{\gamma}_n) &= \mathrm{Proj}_{I_{\Gamma_{inc}}} \left(\Gamma_{inc}(\theta_1, \ldots, \theta_n)\right) \\ (\bar{\bar{\gamma}}_1, \ldots, \bar{\bar{\gamma}}_n) &= \mathrm{Proj}_{I_{\Gamma_{out}}} \left(\Gamma_{out}(\theta_1, \ldots, \theta_n)\right) \\ (\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) &= \mathrm{Proj}_{J_{\Gamma_{out}}} \left(\Gamma_{out}(\theta_{n+1}, \ldots, \theta_{n+m})\right) \\ (\bar{\bar{\gamma}}_{n+1}, \ldots, \bar{\bar{\gamma}}_{n+m}) &= (\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}); \end{aligned}$$

if $\Gamma_{inc} < \Gamma_{out}$, then define

$$\begin{aligned} (\bar{\gamma}_1, \ldots, \bar{\gamma}_n) &= \mathrm{Proj}_{I_{\Gamma_{inc}}} \left(\Gamma_{inc}(\theta_1, \ldots, \theta_n)\right) \\ (\bar{\bar{\gamma}}_1, \ldots, \bar{\bar{\gamma}}_n) &= (\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \\ (\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) &= \mathrm{Proj}_{J_{\Gamma_{out}}} \left(\Gamma_{out}(\theta_{n+1}, \ldots, \theta_{n+m})\right) \\ (\bar{\bar{\gamma}}_{n+1}, \ldots, \bar{\bar{\gamma}}_{n+m}) &= \mathrm{Proj}_{J_{\Gamma_{inc}}} \left(\Gamma_{inc}(\theta_{n+1}, \ldots, \theta_{n+m})\right), \end{aligned}$$

where Proj_I denotes the orthogonal projection on the convex set I.

5. For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Define $\mathcal{RS}_{r_0} : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_{r_0}(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}). \tag{17}$$

6. For every $i \in \{1, \ldots, n\}$, define $\bar{\bar{\rho}}_i$ either by $\bar{\rho}_i$ if $\bar{\bar{\gamma}}_i = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\bar{\gamma}}_i$ such that $\bar{\bar{\rho}}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\bar{\rho}}_j$ either by $\bar{\rho}_j$ if $\bar{\bar{\gamma}}_j = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\bar{\gamma}}_j$ such that $\bar{\bar{\rho}}_j \leq \sigma$.

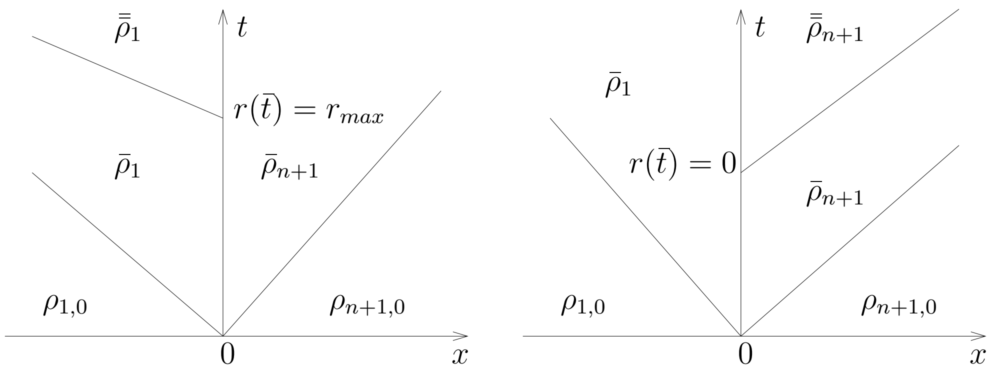

Figure 1: The solution to the Riemann problem (7): the case $\Gamma_{inc} > \Gamma_{out}$ on the left, the case $\Gamma_{inc} < \Gamma_{out}$ on the right.

7. The solution for the Riemann problem (7) is given by the unique weak solution at J $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x), r(t))$, in the sense of Definition 3, such that for a.e. $t > 0$, it holds

$$(\rho_1(t,0), \ldots, \rho_{n+m}(t,0)) = \mathcal{RS}_{r(t)}(\rho_1(t,0), \ldots, \rho_{n+m}(t,0)).$$

Remark 1 *Note that, for every $r_0 \in [0, r_{max}]$, the Riemann solver $\mathcal{RS}_{r_0}$ satisfies the consistency condition*

$$\mathcal{RS}_{r_0}(\mathcal{RS}_{r_0}(\rho_{1,0}, \ldots, \rho_{n+m,0})) = \mathcal{RS}_{r_0}(\rho_{1,0}, \ldots, \rho_{n+m,0})$$

for every $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

Remark 2 *The solution $(\rho_1, \ldots, \rho_{n+m}, r)$ to the Riemann problem (7), constructed by the previous procedure, satisfies the condition:*

$$\mathcal{RS}_{r(t)}(\rho_1(t,0), \ldots, \rho_{n+m}(t,0)) = (\rho_1(t,0), \ldots, \rho_{n+m}(t,0))$$

for a.e. $t > 0$.

For future use, we need some additional definitions.

Definition 5 *Given $r_0 \in [0, r_{max}]$, we say that $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ is an equilibrium for the Riemann solver $\mathcal{RS}_{r_0}$ if*

$$\mathcal{RS}_{r_0}(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\rho_{1,0}, \ldots, \rho_{n+m,0}).$$

Definition 6 *We say that a datum $\rho_i \in [0,1]$ in an incoming arc is a good datum if $\rho_i \in [\sigma, 1]$ and it is a bad datum otherwise.*

We say that a datum $\rho_j \in [0,1]$ in an outgoing arc is a good datum if $\rho_i \in [0, \sigma]$ and it is a bad datum otherwise.

4 The Cauchy Problem

In this section, we deal with the Cauchy problem at the node J. Fix n initial data for incoming arcs $\rho_{1,0},\dots,\rho_{n,0} \in BV(]-\infty,0];[0,1])$, m initial data for outgoing arcs $\rho_{n+1,0},\dots,\rho_{n+m,0} \in BV([0,+\infty[;[0,1])$ and $r_0 \in [0, r_{max}]$. Consider the Cauchy problem at J:

$$\begin{cases} \partial_t \rho_l(t,x) + \partial_x f(\rho_l(t,x)) = 0, \\ \rho_l(0,x) = \rho_{0,l}(x), \\ r'(t) = \displaystyle\sum_{i=1}^{n} f(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t,0+)), \\ r(0) = r_0, \end{cases} \tag{18}$$

for $x \in I_l \setminus \{0\},\ t>0,\ l \in \{1,\dots,n+m\}$.

Theorem 1 *For every $T>0$, the Cauchy problem (18) admits a weak solution at J $(\rho_1,\dots,\rho_{n+m},r)$ such that*

1. *for every $l \in \{1,\dots,n+m\}$, ρ_l is a weak entropic solution of*

$$\partial_t \rho_l + \partial_x f(\rho_l) = 0$$

in $[0,T]\times I_l$;

2. *for every $l \in \{1,\dots,n+m\}$, $\rho_l(0,x) = \rho_{0,l}(x)$ for a.e. $x \in I_l$;*

3. *for a.e. $t \in [0,T]$*

$$\mathcal{RS}_{r(t)}(\rho_1(t,0-),\dots,\rho_{n+m}(t,0+)) = (\rho_1(t,0-),\dots,\rho_{n+m}(t,0+));$$

4. *for a.e. $t \in [0,T]$*

$$r'(t) = \sum_{i=1}^{n} f(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t,0+)).$$

The proof of the theorem is constructed in the next subsections.

4.1 Wave-front tracking

Since solutions to Riemann problems are given, we are able to construct piecewise constant approximations via the wave-front tracking algorithm; see [3] for the general theory and [10] in the case of networks.

Definition 7 *Given $\varepsilon > 0$, we say that the maps $\rho_\varepsilon = (\rho_{1,\varepsilon}, \dots, \rho_{n+m,\varepsilon})$ and r_ε are an ε-approximate wave-front tracking solution to (18) if the following conditions hold.*

1. *For every $l \in \{1, \dots, n+m\}$, $\rho_{l,\varepsilon} \in C([0, +\infty[; L^1_{loc}(I_l; [0,1]))$.*

2. *$r_\varepsilon \in W^{1,\infty}([0, +\infty[; [0, r_{max}])$ and $r_\varepsilon(0) = r_0$.*

3. *For every $l \in \{1, \dots, n+m\}$, $\rho_{l,\varepsilon}(t,x)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in the (t,x)-plane. Moreover, jumps of $\rho_{l,\varepsilon}(t,x)$ can be shocks or (approximate) rarefactions and are indexed by $\mathcal{J}_l(t) = \mathcal{S}_l(t) \cup \mathcal{R}_l(t)$.*

4. *For every $l \in \{1, \dots, n+m\}$, along each shock $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{S}_l(t)$, we have*
$$\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-) < \rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+).$$
Moreover
$$\left| \dot{x}_{l,\alpha}(t) - \frac{f(\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-)) - f(\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+))}{\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-) - \rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+)} \right| \le \varepsilon.$$

5. *For every $l \in \{1, \dots, n+m\}$, along each rarefaction front $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{R}_l(t)$, we have*
$$\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+) < \rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-) < \rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+) + \varepsilon.$$
Moreover
$$\dot{x}_{l,\alpha}(t) \in \left[f'(\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-)), f'(\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+))\right].$$

6. *For every $l \in \{1, \dots, n+m\}$,*
$$\|\rho_{l,\varepsilon}(0, \cdot) - \rho_{0,l}(\cdot)\|_{L^1(I_l)} < \varepsilon.$$

7. *For a.e. $t > 0$*
$$\mathcal{RS}_{r_\varepsilon(t)}(\rho_{1,\varepsilon}(t, 0-), \dots, \rho_{n+m,\varepsilon}(t, 0+)) = (\rho_{1,\varepsilon}(t, 0-), \dots, \rho_{n+m,\varepsilon}(t, 0+)).$$

8. *For a.e. $t > 0$*
$$r'_\varepsilon(t) = \sum_{i=1}^{n} f(\rho_{i,\varepsilon}(t, 0-)) - \sum_{j=n+1}^{n+m} f(\rho_{j,\varepsilon}(t, 0+)).$$

For every $l \in \{1, \ldots, n+m\}$, consider a sequence $\rho_{0,l,\nu}$ of piecewise constant functions defined on I_l such that $\rho_{0,l,\nu}$ has a finite number of discontinuities and $\lim_{\nu \to +\infty} \rho_{0,l,\nu} = \rho_{0,l}$ in $L^1_{loc}(I_l; [0,1])$. For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve the Riemann problem at J (according to $\mathcal{RS}_{r_0}$) and all Riemann problems in each arc. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ travelling with the Rankine-Hugoniot speed. Moreover, if σ is in the range of a rarefaction shock, then its speed is zero. We repeat the previous construction at every time at which interactions between waves or of waves with J happen and at the times when the buffer becomes empty or full.

Remark 3 *By slightly modifying the speed of waves, we may assume that, at every positive time t, at most one interaction happens. Moreover, at every interaction time $\bar{t}$, exactly one of the following possibilities is verified.*

1. *Two waves interact in an arc.*
2. *A wave reaches the node J and*
$$r_\varepsilon(\bar{t}) \cdot (r_{max} - r_\varepsilon(\bar{t})) \neq 0 \quad \textit{or} \quad \lim_{t \to \bar{t}^-} r'_\varepsilon(t) = 0.$$
3. *Some waves exit the node J, i.e.*
$$r_\varepsilon(\bar{t}) \cdot (r_{max} - r_\varepsilon(\bar{t})) = 0 \quad \textit{and} \quad \lim_{t \to \bar{t}^-} r'_\varepsilon(t) \neq 0.$$

Remark 4 *For interactions in arcs, we split rarefaction waves into rarefaction fans just at time $t = 0$. At the node J, instead, we allow the formation of rarefaction fans at every positive time.*

Let us introduce the notions of generation order for waves, of big shocks and of waves with increasing or decreasing flux. We need these definitions in the proof of existence of a wave-front tracking approximate solution and of an uniform bound for the total variation of the flux.

Definition 8 *A wave of ρ_ε, generated at time $t = 0$, is said an original wave or a wave with generation order* 1.

If a wave with generation order $k \geq 1$ interacts with J, then the produced waves are said of generation $k + 1$.

If a wave with generation order $k \geq 1$ interacts in an arc with a wave with generation order $k' \geq 1$, then the produced wave is said of generation $\min\{k, k'\}$.

If a wave exits the node J at time $\bar{t} > 0$ and

$$r_\varepsilon(\bar{t}) \cdot (r_{max} - r_\varepsilon(\bar{t})) = 0, \qquad \lim_{t \to \bar{t}^-} r'_\varepsilon(t) \neq 0,$$

then it has generation order 2 if in the time interval $[0, \bar{t}[$ no wave interacts with J, otherwise it has generation order $k+1$, where k is the generation order of a wave, which interacts with J at time $\tilde{t} < \bar{t}$, and in the time interval $]\tilde{t}, \bar{t}[$ no wave interacts with J.

Definition 9 *We say that a wave (ρ_l, ρ_r) in an arc is a big shock if $\rho_l < \sigma < \rho_r$.*

Definition 10 *We say that a wave (ρ_l, ρ_r) interacting with J from an incoming arc has decreasing flux (resp. increasing flux) if $f(\rho_l) < f(\rho_r)$ (resp. $f(\rho_l) > f(\rho_r)$).*

We say that a wave (ρ_l, ρ_r) interacting with J from an outgoing arc has decreasing flux (resp. increasing flux) if $f(\rho_l) > f(\rho_r)$ (resp. $f(\rho_l) < f(\rho_r)$).

4.2 Bound on the total flux variation

Fix a wave-front tracking approximate solution for the Cauchy problem (18). We prove in Corollary 1, that the total variation of the flux is uniformly bounded by a constant which depends on the initial data. We need some preliminary results.

Lemma 2 *Assume that there exists $\bar{t} > 0$ such that*

$$\lim_{t \to \bar{t}-} r_\varepsilon(t)\,(r_\varepsilon(t) - r_{max}) = 0 \qquad \text{and} \qquad \lim_{t \to \bar{t}-} r'_\varepsilon(t) \neq 0. \tag{19}$$

Then exactly one of the following possibilities holds.

1. *If $r_\varepsilon(\bar{t}-) = 0$, then some waves are generated at J at time $\bar{t}$ only in the outgoing arcs.*

2. *If $r_\varepsilon(\bar{t}-) = r_{max}$, then some waves are generated at J at time $\bar{t}$ only in the incoming arcs.*

Moreover, in both cases we have $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$ and $TV_f(\bar{t}+) \geq TV_f(\bar{t}-)$.

Proof. Define by $\Gamma^{1\pm}_{inc}$, $\Gamma^{1\pm}_{out}$, $\Gamma^{\pm}_{inc}$ and $\Gamma^{\pm}_{out}$ the values, at $\bar{t}-$ and $\bar{t}+$, of the quantities introduced in Section 3. Finally with Γ^1_{inc}, Γ^1_{out}, Γ_{inc} and Γ_{out} we denote the values at time $\bar{t}$ of the quantities introduced in Section 3. By

Remark 3, at time $\bar{t}$ no wave interacts with J. Hypothesis (19) implies that either $r_\varepsilon(\bar{t}) = 0$ or $r_\varepsilon(\bar{t}) = r_{max}$.

If $r_\varepsilon(\bar{t}-) = 0$, it means that $\lim_{t\to\bar{t}^-} r'_\varepsilon(t) < 0$ and so $\Gamma^-_{inc} < \Gamma^-_{out}$, $Q(\bar{t}-) = \left|\Gamma^-_{inc} - \Gamma^-_{out}\right| = \Gamma^-_{out} - \Gamma^-_{inc}$. We have also $\Gamma^+_{inc} = \Gamma_{inc} = \Gamma^-_{inc} = \Gamma^+_{out}$ and so $Q(\bar{t}+) = 0$. Moreover, no waves exit from the incoming arcs, while by (13) and the fact that $r_\varepsilon(t) \neq 0$ for t in a left neighborhood of $\bar{t}$, $\Gamma^-_{out} > \Gamma_{out}$ and so some waves exit from the outgoing arcs. By Lemma 4 in [11], we deduce that all the waves generated at time $\bar{t}$ have decreasing flux. This implies that

$$\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) = \left|\Gamma^+_{out} - \Gamma^-_{out}\right| = \Gamma^-_{out} - \Gamma^+_{out}$$

and so

$$\Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) = 0.$$

Suppose now $r_\varepsilon(\bar{t}) = r_{max}$. It means that $\lim_{t\to\bar{t}^-} r'_\varepsilon(t) > 0$ and so $\Gamma^-_{inc} > \Gamma^-_{out}$, $Q(\bar{t}-) = \left|\Gamma^-_{inc} - \Gamma^-_{out}\right| = \Gamma^-_{inc} - \Gamma^-_{out}$. We have also $\Gamma^+_{inc} = \Gamma^-_{out} = \Gamma_{out} = \Gamma^+_{out}$ and so $Q(\bar{t}+) = 0$. Moreover, no waves exit from the outgoing arcs, while by (12) and the fact that $r_\varepsilon(t) \neq 0$ for t in a left neighborhood of $\bar{t}$, $\Gamma^-_{inc} > \Gamma_{inc}$ and so some waves are generated in the incoming arcs. By Lemma 4 in [11], we deduce that all the waves generated at time $\bar{t}$ have decreasing flux. This implies that

$$\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) = \left|\Gamma^+_{inc} - \Gamma^-_{inc}\right| = \Gamma^-_{inc} - \Gamma^+_{inc}$$

and so

$$\Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) = 0.$$

This concludes the proof. □

Lemma 3 *Assume that a wave (ρ_l, ρ_r) interacts with J at time $\bar{t}$ and suppose that $r_\varepsilon(\bar{t}) = 0$. Then $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$ and $TV_f(\bar{t}+) \leq TV_f(\bar{t}-)$.*

Proof. We denote by $(\rho^-_1, \ldots, \rho^-_{n+m})$ and $(\rho^+_1, \ldots, \rho^+_{n+m})$ the states at J respectively before and after the interaction. Define also by $\Gamma^{1\pm}_{inc}$, $\Gamma^{1\pm}_{out}$, $\Gamma^\pm_{inc}$ and $\Gamma^\pm_{out}$ the values, at $\bar{t}-$ and $\bar{t}+$, of the quantities introduced in Section 3.

By Remark 3, point 2, we deduce that $\lim_{t\to\bar{t}^-} r'_\varepsilon(t) = 0$. Since $r'_\varepsilon(t) = 0$ in a left neighborhood of $\bar{t}$, we have that

$$\Gamma^-_{inc} = \min\left\{\Gamma^{1-}_{inc}, \mu\right\} = \Gamma^-_{out} = \min\left\{\Gamma^{1-}_{out}, \Gamma^-_{inc}\right\},$$

i.e. $Q(\bar{t}-) = 0$.

First, let us assume that the wave (ρ_l, ρ_r) interacts with J from an incoming arc, say I_1, and so $\rho_l \leq \sigma$ and $\rho_r = \rho^-_1$. There are three different possibilities.

1. $\Gamma_{inc}^+ < \Gamma_{inc}^-$. In this case $\Gamma_{inc}^{1+} < \mu$ and $\Gamma_{inc}^+ = \Gamma_{inc}^{1+}$. Therefore no wave is generated in I_1 and the waves generated in the other incoming arcs have increasing flux. Moreover $\Gamma_{inc}^+ = \Gamma_{out}^+ < \Gamma_{out}^-$. By [11, Lemma 4], the waves generated in the outgoing arcs have decreasing flux and so

$$\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{i=2}^{n} \left| f(\rho_i^+) - f(\rho_i^-) \right| \\
&+ \sum_{j=n+1}^{n+m} \left| f(\rho_j^+) - f(\rho_j^-) \right| - |f(\rho_l) - f(\rho_r)| \\
= \sum_{i=2}^{n} \left[f(\rho_i^+) - f(\rho_i^-) \right] &+ \sum_{j=n+1}^{n+m} \left[f(\rho_j^-) - f(\rho_j^+) \right] + f(\rho_l) - f(\rho_r) \\
= \Gamma_{inc}^+ - \Gamma_{inc}^- &- f(\rho_1^+) + f(\rho_1^-) + \Gamma_{out}^- - \Gamma_{out}^+ + f(\rho_l) - f(\rho_r) \\
&= \Gamma_{inc}^+ - \Gamma_{inc}^- + \Gamma_{out}^- - \Gamma_{out}^+ = 0
\end{aligned}$$

and the conclusion follows, since $Q(\bar{t}+) = 0$.

2. $\Gamma_{inc}^+ = \Gamma_{inc}^-$. Since $\rho_l \le \sigma$, then $\mu \le \min\left\{\Gamma_{inc}^{1-}, \Gamma_{inc}^{1+}\right\}$ and $\Gamma_{out}^- = \Gamma_{out}^+$. Therefore no waves are generated in the outgoing arcs and in I_1. By [11, Lemma 5], the waves produced in the other incoming arcs have increasing fluxes if $f(\rho_l) < f(\rho_r)$, and decreasing fluxes if $f(\rho_l) > f(\rho_r)$. Thus

$$\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{i=2}^{n} \left| f(\rho_i^+) - f(\rho_i^-) \right| - |f(\rho_l) - f(\rho_r)| \\
= \operatorname{sgn}(f(\rho_l) - f(\rho_r)) &\left[\sum_{i=2}^{n} \left(f(\rho_i^-) - f(\rho_i^+) \right) - f(\rho_l) + f(\rho_r) \right] \\
= \operatorname{sgn}(f(\rho_l) - f(\rho_r)) &\left[\Gamma_{inc}^- - \Gamma_{out}^+ + f(\rho_1^+) - f(\rho_1^-) - f(\rho_l) + f(\rho_r) \right] \\
&= \operatorname{sgn}(f(\rho_l) - f(\rho_r)) \left[\Gamma_{inc}^- - \Gamma_{out}^+ \right] = 0
\end{aligned}$$

and we conclude, since $Q(\bar{t}+) = 0$.

3. $\Gamma_{inc}^+ > \Gamma_{inc}^-$. In this case we have that $\Gamma_{inc}^- < \mu$ and so $\rho_i^- \le \sigma$ for every $i \in \{1, \ldots, n\}$. Since the wave (ρ_l, ρ_r) has positive speed, then $\rho_1^- = \rho_r < \rho_l \le \sigma$. Therefore, in the incoming arcs, either no waves are produced (in the case $\Gamma_{inc}^{1+} \le \mu$) or waves with decreasing flux are generated (in the case $\Gamma_{inc}^{1+} > \mu$).
In the outgoing arcs, by (13) we easily deduce that $\Gamma_{out}^+ \ge \Gamma_{out}^-$, and so either no waves are created or waves with increasing flux are generated,

see [11, Lemma 4].
Hence we have

$$
\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) = & \sum_{i=2}^{n} \left| f(\rho_i^+) - f(\rho_i^-) \right| + \left| f(\rho_l) - f(\rho_1^+) \right| \\
& - \left| f(\rho_l) - f(\rho_r) \right| + \sum_{j=n+1}^{n+m} \left| f(\rho_j^+) - f(\rho_j^-) \right| \\
& = \sum_{i=2}^{n} \left[f(\rho_i^-) - f(\rho_i^+) \right] + f(\rho_l) - f(\rho_1^+) \\
& - f(\rho_l) + f(\rho_r) + \sum_{j=n+1}^{n+m} \left[f(\rho_j^+) - f(\rho_j^-) \right] \\
& = \Gamma_{inc}^- - \Gamma_{inc}^+ + \Gamma_{out}^+ - \Gamma_{out}^- = \Gamma_{out}^+ - \Gamma_{inc}^+ \le 0.
\end{aligned}
$$

Moreover,

$$
Q(\bar{t}+) = \left| \Gamma_{out}^+ - \Gamma_{inc}^+ \right| = \Gamma_{inc}^+ - \Gamma_{out}^+,
$$

and so the conclusion follows.

Finally, suppose that the wave (ρ_l, ρ_r) interacts with J from an outgoing arc, say I_{n+1}, and so $\rho_r \ge \sigma$ and $\rho_l = \rho_{n+1}^-$. In this case $\Gamma_{inc}^- = \Gamma_{inc}^+$ and so no waves are produced in the incoming arcs. There are three different possibilities.

1. $\Gamma_{out}^+ < \Gamma_{out}^-$. In this case $\Gamma_{out}^+ = \Gamma_{out}^{1+} < \Gamma_{inc}^+$ and so no wave is generated in I_{n+1}, while in the other outgoing arcs at most $m-1$ waves are generated and they have increasing flux. Therefore

$$
\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) = & \sum_{j=n+2}^{n+m} \left| f(\rho_j^+) - f(\rho_j^-) \right| - \left| f(\rho_l) - f(\rho_r) \right| \\
= & \sum_{j=n+2}^{n+m} \left[f(\rho_j^+) - f(\rho_j^-) \right] - f(\rho_l) + f(\rho_r) = \Gamma_{out}^+ - \Gamma_{out}^- \le 0.
\end{aligned}
$$

Moreover

$$
Q(\bar{t}+) = \left| \Gamma_{inc}^+ - \Gamma_{out}^+ \right| = \left| \Gamma_{inc}^- - \Gamma_{out}^+ \right| = \left| \Gamma_{out}^- - \Gamma_{out}^+ \right| = \Gamma_{out}^- - \Gamma_{out}^+
$$

and so $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.

2. $\Gamma_{out}^+ = \Gamma_{out}^-$. In this case $\Gamma_{out}^+ = \Gamma_{out}^- = \Gamma_{inc}^+$ and no wave is generated in I_{n+1}. In the other outgoing arcs, at most $m-1$ waves are generated.

By [11, Lemma 5], these waves have increasing flux if $f(\rho_r) < f(\rho_l)$, while they have decreasing flux if $f(\rho_r) > f(\rho_l)$; hence

$$\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{j=n+2}^{n+m} \left| f(\rho_j^+) - f(\rho_j^-) \right| - |f(\rho_l) - f(\rho_r)| \\
&= \mathrm{sgn}\left(f(\rho_r) - f(\rho_l)\right) \left[\sum_{j=n+2}^{n+m} \left(f(\rho_j^-) - f(\rho_j^+) \right) - \left(f(\rho_r) - f(\rho_l) \right) \right] \\
&= \mathrm{sgn}\left(f(\rho_r) - f(\rho_l)\right) \left[\sum_{j=n+2}^{n+m} \left(f(\rho_j^-) - f(\rho_j^+) \right) - \left(f(\rho_{n+1}^+) - f(\rho_{n+1}^-) \right) \right] \\
&= \mathrm{sgn}\left(f(\rho_r) - f(\rho_l)\right) \left[\Gamma_{out}^- - \Gamma_{out}^+ \right] = 0.
\end{aligned}$$

Moreover,

$$Q(\bar{t}+) = \left| \Gamma_{inc}^+ - \Gamma_{out}^+ \right| = 0$$

and so $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.

3. $\Gamma_{out}^+ > \Gamma_{out}^-$. In this case $\Gamma_{out}^- = \Gamma_{out}^{1-}$; so $\rho_j^- \geq \sigma$ for every $j \in \{n+1, \ldots, n+m\}$ and $\sigma \leq \rho_r < \rho_l = \rho_{n+1}$. If $\Gamma_{out}^{1+} \leq \Gamma_{inc}^+$ no wave is generated in outgoing arcs; if $\Gamma_{out}^{1+} > \Gamma_{inc}^+$, at most m waves with decreasing flux are created. Thus

$$\begin{aligned}
\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{j=n+2}^{n+m} \left| f(\rho_j^+) - f(\rho_j^-) \right| \\
&\quad + \left| f(\rho_{n+1}^+) - f(\rho_r^-) \right| - |f(\rho_l) - f(\rho_r)| \\
&= \sum_{j=n+2}^{n+m} \left[f(\rho_j^-) - f(\rho_j^+) \right] + f(\rho_r) - f(\rho_{n+1}^+) - f(\rho_r) + f(\rho_{n+1}^-) \\
&= \Gamma_{out}^- - \Gamma_{out}^+ < 0.
\end{aligned}$$

Moreover

$$Q(\bar{t}+) = \left| \Gamma_{inc}^+ - \Gamma_{out}^+ \right| = \left| \Gamma_{inc}^- - \Gamma_{out}^+ \right| = \left| \Gamma_{out}^- - \Gamma_{out}^+ \right| = \Gamma_{out}^+ - \Gamma_{out}^-$$

and so $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.

The proof is finished. □

Lemma 4 *Assume that a wave (ρ_l, ρ_r) interacts with J at time $\bar{t}$ and suppose that $r_\varepsilon(\bar{t}) = r_{max}$. Then $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$ and $TV_f(\bar{t}+) \leq TV_f(\bar{t}-)$.*

The proof is similar to that of Lemma 3 and so we omit it.

Lemma 5 *Assume that a wave (ρ_l, ρ_r) interacts with J at time $\bar{t}$ and suppose that $0 < r_\varepsilon(\bar{t}) < r_{max}$. Then $\Upsilon(\bar{t}+) \le \Upsilon(\bar{t}-)$ and $TV_f(\bar{t}+) \le TV_f(\bar{t}-)$.*

Proof. We denote by $(\rho_1^-, \dots, \rho_{n+m}^-)$ and by $(\rho_1^+, \dots, \rho_{n+m}^+)$ the states at J respectively before and after the interaction. Define also by $\Gamma_{inc}^{1\pm}$, $\Gamma_{out}^{1\pm}$, $\Gamma_{inc}^{\pm}$ and $\Gamma_{out}^{\pm}$ the values, at $\bar{t}-$ and $\bar{t}+$, of the quantities introduced in Section 3.

Since $0 < r_\varepsilon(t) < r_{max}$ in a left neighborhood of $\bar{t}$, we have that

$$\Gamma_{inc}^- = \min\left\{\Gamma_{inc}^{1-}, \mu\right\} \qquad \text{and} \qquad \Gamma_{out}^- = \min\left\{\Gamma_{out}^{1-}, \mu\right\}.$$

Assume that the wave (ρ_l, ρ_r) interacts with J from an incoming arc; say I_1. Thus $\rho_l \le \sigma$ and $\rho_r = \rho_1^-$. Moreover $\Gamma_{out}^+ = \Gamma_{out}^-$ and so no wave is produced in the outgoing arcs. We have three possibilities.

1. $\Gamma_{inc}^+ = \Gamma_{inc}^-$. Since $\rho_l \le \sigma$ and $f(\rho_l) \ne f(\rho_r)$, then $\Gamma_{inc}^{1+} \ne \Gamma_{inc}^{1-}$ and so $\Gamma_{inc}^+ = \Gamma_{inc}^- = \mu$. In this case at most n waves are generated in the incoming arcs.
   If $f(\rho_l) < f(\rho_r)$, then $\rho_1^+ = \rho_l$ and so no wave is produced in I_1, while the waves generated in the other incoming arcs have increasing flux, by [11, Lemma 5]. If $f(\rho_l) > f(\rho_r)$, then $f(\rho_r) \le f(\rho_1^+) \le f(\rho_l)$ and the waves generated in $I_2, \dots, I_n$ have decreasing flux, by [11, Lemma 5]. Thus
$$\begin{aligned}\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{i=2}^{n} \left|f(\rho_i^+) - f(\rho_i^-)\right| \\ &\quad + \left|f(\rho_1^+) - f(\rho_l)\right| - \left|f(\rho_l) - f(\rho_1^-)\right| \\ &= \mathrm{sgn}\left(f(\rho_l) - f(\rho_r)\right) \left[\sum_{i=2}^{n} (f(\rho_i^-) - f(\rho_i^+)) - (f(\rho_l) - f(\rho_1^-))\right] \\ &\quad + f(\rho_l) - f(\rho_1^+) \\ &= \mathrm{sgn}\left(f(\rho_l) - f(\rho_r)\right) \left[\Gamma_{inc}^- - \Gamma_{inc}^+ + f(\rho_1^-) - f(\rho_l)\right] + f(\rho_l) - f(\rho_1^+) \\ &= \mathrm{sgn}\left(f(\rho_l) - f(\rho_r)\right) \left[f(\rho_1^-) - f(\rho_l)\right] + \left|f(\rho_1^+) - f(\rho_l)\right| \le 0\end{aligned}$$
   by the previous considerations. Moreover $Q(\bar{t}-) = Q(\bar{t}+)$ and so we conclude that $\Upsilon(\bar{t}-) \ge \Upsilon(\bar{t}+)$.

2. $\Gamma_{inc}^+ < \Gamma_{inc}^-$. In this case we have that $\Gamma_{inc}^{1+} < \min\{\Gamma_{inc}^{1-}, \mu\}$ and so $\rho_1^+ = \rho_l$, $f(\rho_l) < f(\rho_r)$ and $f(\rho_i^+) \ge f(\rho_i^-)$ for every $i \in \{2, \dots, n\}$.

Hence

$$\begin{aligned}\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) &= \sum_{i=2}^{n} \left(f(\rho_i^+) - f(\rho_i^-)\right) - (f(\rho_r) - f(\rho_l)) \\ &= \Gamma_{inc}^+ - \Gamma_{inc}^-.\end{aligned}$$

Moreover

$$Q(\bar{t}+) - Q(\bar{t}-) = \left|\Gamma_{inc}^+ - \Gamma_{out}^+\right| - \left|\Gamma_{inc}^- - \Gamma_{out}^-\right|$$

and we easily conclude that $\Upsilon(\bar{t}+) \leq \Upsilon(\bar{t}-)$.

3. $\Gamma_{inc}^+ > \Gamma_{inc}^-$. In this case we have that $\Gamma_{inc}^{1-} < \min\{\Gamma_{inc}^{1+}, \mu\}$; so $\rho_i^- \leq \sigma$ for every $i \in \{1, \ldots, n\}$ and $f(\rho_l) > f(\rho_r)$. Moreover, the waves produced in the incoming arcs have decreasing flux; hence

$$\begin{aligned}\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-) = \sum_{i=2}^{n} \left(f(\rho_i^-) - f(\rho_i^+)\right) + \left(f(\rho_l) - f(\rho_1^+)\right) \\ - (f(\rho_l) - f(\rho_r)) = \Gamma_{inc}^- - \Gamma_{inc}^+.\end{aligned}$$

Moreover,

$$Q(\bar{t}+) - Q(\bar{t}-) = \left|\Gamma_{inc}^+ - \Gamma_{out}^+\right| - \left|\Gamma_{inc}^- - \Gamma_{out}^-\right|$$

and $\Upsilon(\bar{t}+) \leq \Upsilon(\bar{t}-)$ as before.

The case of the wave (ρ_l, ρ_r) interacting with J from an outgoing arc is similar to the previous one. □

Lemmas 2-5 imply that the functional Υ is decreasing, as stated in the following Proposition.

Proposition 1 *For a.e. $t > 0$, we have that*

$$\Upsilon(t) \leq \Upsilon(0). \tag{20}$$

Proof. The functional Υ is piecewise constant in time and it can vary only when two waves interact inside an arc or when a wave hits or exits from the node. If two waves interact in an arc, then TV_f is non-increasing and Q remains constant; hence Υ is non-increasing.

Consider therefore the case of a wave interacting or exiting from the node at time $\bar{t}$. For simplicity, we denote by $\Gamma_{inc}^{1\pm}$, $\Gamma_{out}^{1\pm}$, $\Gamma_{inc}^{\pm}$ and $\Gamma_{out}^{\pm}$ the values, at $\bar{t}-$ and $\bar{t}+$, of the quantities introduced in Section 3. At the node, we have the following two cases.

- A wave (ρ_l, ρ_r) hits the node at a certain time $\bar{t}$. We have three different possibilities.

    1. $r_\varepsilon(\bar{t}) = 0$: by Lemma 3, we conclude that $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.
    2. $r_\varepsilon(\bar{t}) = r_{max}$: by Lemma 4, we conclude that $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.
    3. $0 < r_\varepsilon(\bar{t}) < r_{max}$: by Lemma 5, we conclude that $\Upsilon(\bar{t}+) \leq \Upsilon(\bar{t}-)$.

- A wave exits the node at a certain time $\bar{t}$. In this case Lemma 2 states that $\Upsilon(\bar{t}+) = \Upsilon(\bar{t}-)$.

The proof is so finished. □

Corollary 1 *For every $t > 0$, we have that*

$$TV_f(t) \leq TV_f(0+) + (n + m)f(\sigma). \tag{21}$$

PROOF. By Proposition 1, we deduce that

$$\mathrm{TV}_f(t) = \Upsilon(t) - Q(t) \leq \Upsilon(0+) - Q(t) = \mathrm{TV}_f(0+) + Q(0+) - Q(t)$$

for every $t > 0$. The conclusion follows by the fact that $0 \leq Q(t) \leq (n + m)f(\sigma)$ for every $t \geq 0$. □

4.3 Existence of a wave-front tracking solution

In this subsection, we prove the existence of a wave-front tracking approximate solution. We have the following proposition, whose proof is very similar to that of [11, Proposition 10]. Here we give the proof for completeness.

Proposition 2 *For every $\nu \in \mathbb{N} \setminus \{0\}$ the construction in Subsection 4.1 can be done for every positive time, producing an $\frac{1}{\nu}$-approximate wave-front tracking solution to (18) with respect to the Riemann solver described in Definition 4.*

PROOF. For every $l \in \{1, \ldots, n + m\}$ and every $\nu \in \mathbb{N} \setminus \{0\}$, call $\rho_{l,\nu}$ the function built by the previous procedure. Moreover, for every $l \in \{1, \ldots, n + m\}$, $\nu \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N} \setminus \{0\}$ and for every time $t \geq 0$, define the functions $N_{l,\nu}(t)$ and $M_{l,k,\nu}(t)$, which count respectively the number of discontinuities of $\rho_{l,\nu}(t, \cdot)$ and the number of waves with generation order k of $\rho_{l,\nu}(t, \cdot)$.

Assume by contradiction that there exist $\bar{\nu} \in \mathbb{N} \setminus \{0\}$ and $T > 0$ such that

$$\sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) < +\infty$$

for every $t \in [0, T[$, and

$$\limsup_{t \to T^-} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) = +\infty. \tag{22}$$

Note that, for every time t,

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t) \leq \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) < +\infty.$$

Indeed, $\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t)$ is locally constant and can vary only at interaction times in the following way:

1. if at $\bar{t} > 0$ a wave with generation order 1 reaches the node J, then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$

2. if at $\bar{t} > 0$ two waves with generation order 1 interact in an arc, then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$

3. if at $\bar{t} > 0$ a wave with generation order k_1 interacts with a wave of order k_2 in an arc with $k_1 + k_2 \geq 3$ (so that we are not in the case $k_1 = k_2 = 1$), then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-).$$

Moreover, for every $l \in \{1, \ldots, n+m\}$ and for every $k \geq 0$, the function $M_{l,k,\bar{\nu}}(\cdot)$ is decreasing inside the arcs (i.e. it can increase only because of waves produced at the junction). For every $k \in \mathbb{N} \setminus \{0\}$ and for every time $t > 0$, we have

$$\sum_{l=1}^{n+m} M_{l,k,\bar{\nu}}(t) \leq (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) = (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(0+) < +\infty,$$

where $K_{\bar{\nu}} = 2(n+m)\bar{\nu}$. This bound is due to the fact that each wave with generation order k can interact with J and produce at most $\bar{\nu}$ waves with generation order $k+1$ in each arc (in the case of rarefactions) and the same can happen at a second time, when the function r_ε reaches 0 or r_{max}.

Now, there exists $0 < \eta < T$ such that no wave with generation order 1 interacts with J in the time interval $(T-\eta, T)$. Equation (22) implies also that in $(T-\eta, T)$ there is an infinite number of interactions of waves with J. Since waves of generation order 1 do not interact in $(T-\eta, T)$, the only possibility is that a wave with generation order $k \geq 2$ comes back to J producing waves of order $k+1$, some of which come back to J producing waves of order $k+2$ and so on. Moreover, by Lemma 4.3.7 of [10] (see the Appendix), if a wave of generation order $k \geq 2$ interacts with J from an arc in $(T-\eta, T)$, then, after the interaction, the datum in that arc is bad, since the wave can not interact with waves of generation order 1 and come back to J. In an arc a bad datum at J can change only in the following cases:

1. an original wave interacts with J from the arc;
2. a wave, which is a big shock, is originated at J on the arc and the new datum at J is good.

Obviously, in the time interval $(T-\eta, T)$ the first possibility can not happen; so only the second possibility may happen. Assume that there exist $t_1, t_2 \in (T-\eta, T)$ with $t_1 < t_2$ such that a big shock is originated at J at time t_1 in an arc and comes back to J at time t_2. In this arc, the datum before t_1 is bad, since a big shock is originated at time t_1. Moreover the big shock comes back to J at time t_2, and so an original wave cannot interact with the big shock; hence the bad datum of the big shock does not change. Therefore, in that arc after the time t_2, the datum is bad and is the same as the datum before t_1. Thus every arc I_l may take only a precise bad value $\bar{\rho}_l$, otherwise good values. The key point is that, at every time $t \in (T-\eta, T)$, there are finitely many possible combinations of bad data at the node J (obtained choosing the arcs which present a bad datum at J, the precise value being fixed). Since the Riemann solvers $\mathcal{RS}_{r_\varepsilon(t)}$ are indeed at most three ($\mathcal{RS}_0$, $\mathcal{RS}_{r_{max}}$ and $\mathcal{RS}_{\tilde{r}}$ with $\tilde{r} \in]0, r_{max}[$ arbitrary) and since each of them satisfies the property (P1) of [11] (i.e. the image of a Riemann solver depends only on bad data, for a proof see [11, Section 4.2]) we deduce that, for $t \in (T-\eta, T)$, $\rho_{\bar{\nu}}(t)$ at J may take only a finite number of values, thus waves produced by J have a finite set of possible velocities.

Denote with $\mathcal{G}$ the set of all $l \in \{1, \ldots, n+m\}$ such that $\rho_{\bar{\nu},l}(t,0)$ is a good datum for every time t in a left neighborhood of T. Consider $\bar{l} \in \mathcal{G}$. We claim that there exists a constant $C_{\bar{l}} > 0$ such that $N_{\bar{l},\bar{\nu}}(t) \leq C_{\bar{l}}$ for every time t in

a left neighborhood of T. Indeed, the number of different states, which can be produced at J, is finite by the previous considerations. Since all states are good, there is a minimal size of a flux jump along a discontinuity. Then the total number of discontinuities is necessary bounded by Corollary 1.
Consider now $\bar{l} \in \{1, \ldots, n+m\} \setminus \mathcal{G}$. If $\rho_{\bar{\nu},\bar{l}}(t, 0)$ is a bad datum for every time t in a left neighborhood of T, then clearly $N_{\bar{l},\bar{\nu}}(t)$ is uniformly bounded in the same time interval. The other case is that a big shock is originated in the arc $I_{\bar{l}}$ and comes back to J infinitely many times. We claim that there exists a constant $C_{\bar{l}} > 0$ such that $N_{\bar{l},\bar{\nu}}(t) \leq C_{\bar{l}}$ for every time $t \in [\tilde{t}_1, \tilde{t}_2]$, where $\tilde{t}_1$ and $\tilde{t}_2$ are the times, at which a big shock respectively is originated at J in $I_{\bar{l}}$ and comes back to J. In fact, in the time interval $]\tilde{t}_1, \tilde{t}_2[$, the datum $\rho_{\bar{\nu},\bar{l}}(t, 0)$ is good and the number of possible different states between J and the big shock is finite. Therefore, as before, if the number of discontinuity can not be bounded by a constant, then also the total variation of the flux can not be bounded and this is not true, by Corollary 1.

This concludes the proof by contradiction. □

4.4 Existence of a solution

This part deals with the proof of Theorem 1.
Proof of Theorem 1. Fix an ε-approximate wave-front tracking solution $(\rho_\varepsilon, r_\varepsilon)$ to (18), in the sense of Definition 7. By Corollary 1, we deduce that

$$\mathrm{TV}_f(t) \leq \mathrm{TV}_f(0+) + (n+m) f(\sigma)$$

for a.e. $t > 0$ and we derive the convergence of ρ_ε to a function $\rho = (\rho_1, \ldots, \rho_{n+m})$, such that ρ_l is an entropy-admissible solution to (1) in the arc I_l, as in [11, Theorem 8] .

Concerning r_ε, Ascoli-Arzelà Theorem guarantees the uniform convergence of a subsequence $r_{\varepsilon_k} \to r$. Moreover, Dunford-Pettis Theorem implies the weak compactness of $\{r'_\varepsilon\}_\varepsilon$ in $L^1([0, T])$, thus, up to a subsequence, $r'_{\varepsilon_k} \rightharpoonup s$ weakly in $L^1([0, T])$ and $r' = s$ in the weak sense. Thus, passing to the limit in the wave-front tracking approximations, we obtain that (ρ, r) satisfies points 3. and 4. of Theorem 1. □

5 Dependence of solutions on initial data

In this section we prove that, for every type of nodes, the solution to (18) depends in a Lipschitz continuous way with respect to the initial condition.

We use the technique of generalized tangent vectors, introduced in [4, 5] for hyperbolic systems of conservation laws. A complete description, in the case of scalar conservation laws on networks, is in [11]. Here we just analyze the estimates on the shifts of waves along wave-front tracking approximate solutions at the node. We recall the definition of shift of wave.

Definition 11 *Fix $\xi \in \mathbb{R}$ and a wave (ρ_l, ρ_r) of an ε-approximate wave-front tracking solution to (18). We say that ξ forms a shift for the wave (ρ_l, ρ_r) if we consider the same ε-approximate wave-front tracking solution, except for the position of the wave (ρ_l, ρ_r), which is translated by the quantity ξ in the x-direction.*

The proof of the continuous dependence is based on the following lemmas.

Lemma 6 *Let $(\rho_\varepsilon, r_\varepsilon)$ be an ε-approximate wave-front tracking solution to the Cauchy problem (18). Assume that a wave in an arc I_k ($k \in \{1, \dots, n+m\}$) interacts with J at time $\bar{t}$. Denote by $(\rho_1^-, \dots, \rho_{n+m}^-)$ and $(\rho_1^+, \dots, \rho_{n+m}^+)$ respectively the states at J before and after $\bar{t}$ and let $\hat{\rho}_k \neq \rho_k^-$ be the other side of the interacting wave. If the interacting wave in I_k is shifted by ξ_k^-, then all the produced waves at J are shifted by ξ_l^+ ($l \in \{1, \dots, n+m\}$), which satisfy the relations*

$$\left| \xi_k^- \frac{\hat{\rho}_k - \rho_k^-}{f(\hat{\rho}_k) - f(\rho_k^-)} \right| = \left| \xi_k^+ \frac{\hat{\rho}_k - \rho_k^+}{f(\hat{\rho}_k) - f(\rho_k^+)} \right| = \left| \xi_l^+ \frac{\rho_l^+ - \rho_l^-}{f(\rho_l^+) - f(\rho_l^-)} \right| \tag{23}$$

for every $l \in \{1, \dots, n+m\}$, $l \neq k$.

For a proof, see [11, Lemma 16].

Lemma 7 *Let $(\rho_\varepsilon, r_\varepsilon)$ be an ε-approximate wave-front tracking solution to the Cauchy problem (18). Assume that a wave in an arc I_k ($k \in \{1, \dots, n+m\}$) interacts with J at time t_1. Assume that there exists a time $t_2 > t_1$ such that no interactions at J happen in (t_1, t_2) and some waves exit J at time t_2. Define $(\rho_1^-, \dots, \rho_{n+m}^-)$, $(\rho_1^+, \dots, \rho_{n+m}^+)$ and $(\tilde{\rho}_1^+, \dots, \tilde{\rho}_{n+m}^+)$ respectively the states at J before t_1, in (t_1, t_2) and after t_2. Moreover, denote by $\hat{\rho}_k \neq \rho_k^-$ the other side of the interacting wave. If the interacting wave in I_k is shifted by ξ_k^-, then all the waves exiting J at t_2 are shifted by $\tilde{\xi}_l^+$ ($l \in \{1, \dots, n+m\}$), which satisfy the relations*

$$\left| \xi_k^- \frac{\hat{\rho}_k - \rho_k^-}{f(\hat{\rho}_k) - f(\rho_k^-)} \right| = \frac{|v_2 - v_1|}{|v_2|} \left| \tilde{\xi}_l^+ \frac{\tilde{\rho}_l^+ - \rho_l^+}{f(\tilde{\rho}_l^+) - f(\rho_l^+)} \right| \tag{24}$$

for every $l \in \{1, \dots, n+m\}$*, where*

$$\begin{cases} v_1 = \sum_{i=1}^{n} f(\rho_i^-) - \sum_{j=n+1}^{n+m} f(\rho_j^-), \\ v_2 = \sum_{i=1}^{n} f(\rho_i^+) - \sum_{j=n+1}^{n+m} f(\rho_j^+). \end{cases} \tag{25}$$

PROOF. Fix $\eta > 0$ such that in the time intervals $(t_1 - \eta, t_1)$ and $(t_2, t_2 + \eta)$ no wave interacts with J and no wave exits J. Therefore in $(t_1 - \eta, t_2)$ we have

$$r_\varepsilon(t) = \begin{cases} \bar{r} + v_1 t, & \text{if } t \in (t_1 - \eta, t_1], \\ \bar{r} + v_1 t_1 + v_2(t - t_1), & \text{if } t \in (t_1, t_2), \end{cases}$$

where $\bar{r} = \lim_{t \to (t_1 - \eta)^+} r_\varepsilon(t)$ and v_1 and v_2 are defined in (25).

If ξ_k^- is a shift in the wave defined by the states $\hat{\rho}_k$, ρ_k^-, then the function r_ε becomes

$$r_\varepsilon^h(t) = \begin{cases} \bar{r} + v_1 t, & \text{if } t \in (t_1 - \eta, t_1 + h], \\ \bar{r} + v_1(t_1 + h) + v_2(t - t_1 - h), & \text{if } t > t_1 + h, \end{cases}$$

where h satisfies

$$|h| = \left|\xi_k^-\right| \left| \frac{\rho_k^- - \hat{\rho}_k}{f(\rho_k^-) - f(\hat{\rho}_k)} \right| .$$

Let $\tilde{t} > t_1 + h$ be the first time at which $r_\varepsilon^h(\tilde{t}) = 0$ or $r_\varepsilon^h(\tilde{t}) = r_{max}$. Thus the waves $(\rho_l^+, \tilde{\rho}_l^+)$ are shifted in time by $\tilde{t} - t_2 = \frac{v_2 - v_1}{v_2} h$. This permits to conclude. □

Theorem 8 *Fix* $\boldsymbol{\theta} \in \Theta$ *and consider the Cauchy problem* (18)*. The solution* $(\rho_1, \dots, \rho_{n+m}, r)$ *constructed in Theorem 1 depends on the initial condition* $(\rho_{0,1}, \dots, \rho_{0,n+m}, r_0)$*, belonging to the space* $\left(\prod_{i=1}^{n} BV(]-\infty, 0]; [0,1])\right) \times \left(\prod_{j=n+1}^{n+m} BV([0, +\infty[; [0,1])\right) \times [0, r_{max}]$*, in a Lipschitz continuous way with respect to the strong topology of the cartesian product* $\left(\prod_{i=1}^{n} L^1(-\infty, 0)\right) \times \left(\prod_{j=n+1}^{n+m} L^1(0, \infty)\right) \times \mathbb{R}$ *(with Lipschitz constant* $L = 1$*).*

PROOF. First consider variations in the ρ component of the initial condition. As in the proof of Theorem 17 of [11], we can restrict the study to the evolution of shifts.

Fix a time $\bar{t} > 0$; we have the following possibilities.

a) No interaction of waves takes place in any arc at $\bar{t}$ and no wave interacts with J. In this case the shifts are constant.

b) Two waves interact at $\bar{t}$ on an arc and no other interaction takes place. In this case the norms of the tangent vectors are decreasing by Lemma 2.7.2 of [10].

c) A wave interacts with J at a time $\bar{t}$ from the arc I_k and no other interaction takes place. Denote by $\hat{\rho}_k \neq \rho_k^-$ the other side of the interacting wave. Using Lemma 6 and its notations, we deduce

$$\begin{aligned}\|(v,\xi)(\bar{t}+)\| - \|(v,\xi)(\bar{t}-)\| = & \sum_{l=1,\, l\neq k}^{n+m} \left|\xi_l^+\right| \left|\rho_l^+ - \rho_l^-\right| \\ & + \left|\xi_k^+\right| \left|\rho_k^+ - \hat{\rho}_k\right| - \left|\xi_k^-\right| \left|\hat{\rho}_k - \rho_k^-\right| \\ = \left[\sum_{l=1,\, l\neq k}^{n+m} \left|\frac{f(\rho_l^+) - f(\rho_l^-)}{f(\hat{\rho}_k) - f(\rho_k^-)}\right| + \left|\frac{f(\hat{\rho}_k) - f(\rho_k^+)}{f(\hat{\rho}_k) - f(\rho_k^-)}\right| - 1\right] & \left|\xi_k^-\right| \left|\hat{\rho}_k - \rho_k^-\right| \\ = \left(\mathrm{TV}_f(\bar{t}+) - \mathrm{TV}_f(\bar{t}-)\right) & \frac{\left|\xi_k^-\right| \left|\hat{\rho}_k - \rho_k^-\right|}{\left|f(\hat{\rho}_k) - f(\rho_k^-)\right|} \leq 0,\end{aligned}$$

by Lemmas 3, 4, 5.

d) Waves exit J at a time $\bar{t} > 0$ and no other interaction takes place. Define $\tilde{t} \in [0, \bar{t}[$ in the following way: $\tilde{t} = 0$ if no interaction at J happens in the time interval $(0, \bar{t})$, otherwise $\tilde{t}$ is the time at which a wave reaches J and no other interaction at J happens in the time interval $(\tilde{t}, \bar{t})$.
If $\tilde{t} = 0$ and since no variation in r_0 occurs, then no shift appears.

Assume now $\tilde{t} > 0$. Denote by $(\rho_1^-, \ldots, \rho_{n+m}^-)$, $(\rho_1^+, \ldots, \rho_{n+m}^+)$ and $(\tilde{\rho}_1^+, \ldots, \tilde{\rho}_{n+m}^+)$ respectively the states at J before $\tilde{t}$, in $(\tilde{t}, \bar{t})$ and after $\bar{t}$. Without loss of generality, we assume that the interacting wave $(\hat{\rho}_k, \rho_k^-)$ comes from an incoming arc I_k, $k \leq n$. Define $\Gamma_{inc}^- = \sum_{i=1}^n f(\rho_i^-)$, $\Gamma_{inc} = \sum_{i=1}^n f(\rho_i^+)$, $\Gamma_{out}^- = \sum_{j=n+1}^{n+m} f(\rho_j^-)$ and $\Gamma_{out} = \sum_{j=n+1}^{n+m} f(\rho_j^+)$.

By the definition of $\tilde{t}$ and by the point a) and b), we deduce that

$$\begin{aligned}\|(v,\xi)(\bar{t}+)\| - \left\|(v,\xi)(\tilde{t}-)\right\| \leq & \sum_{l=1,\, l\neq k}^{n+m} \left|\xi_l^+\right| \left|\rho_l^+ - \rho_l^-\right| + \left|\xi_k^+\right| \left|\rho_k^+ - \hat{\rho}_k\right| \\ & + \sum_{l=1}^{n+m} \left|\tilde{\xi}_l^+\right| \left|\rho_l^+ - \tilde{\rho}_l^+\right| - \left|\xi_k^-\right| \left|\hat{\rho}_k - \rho_k^-\right|.\end{aligned}$$

By Lemma 6 and Lemma 7, we get that

$$\|(v,\xi)(\bar{t}+)\| - \left\|(v,\xi)(\tilde{t}-)\right\| \leq \left|\xi_k^-\right| \frac{\left|\hat{\rho}_k - \rho_k^-\right|}{\left|f(\hat{\rho}_k) - f(\rho_k^-)\right|} I_1 \tag{26}$$

where

$$
\begin{aligned}
I_1 &= \sum_{l=1,\, l\neq k}^{n+m} \left|f(\rho_l^-) - f(\rho_l^+)\right| + \left|f(\hat{\rho}_k) - f(\rho_k^+)\right| \\
&\quad + \frac{|v_2 - v_1|}{|v_2|} \sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| - \left|f(\hat{\rho}_k) - f(\rho_k^-)\right| \qquad (27)
\end{aligned}
$$

and, by (25), $v_1 = \Gamma_{inc}^- - \Gamma_{out}^-$ and $v_2 = \Gamma_{inc} - \Gamma_{out}$. We claim that $I_1 = 0$. By Lemma 2, we have that

$$
\sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| = \sum_{i=1}^{n} \left|f(\tilde{\rho}_i^+) - f(\rho_i^+)\right| \qquad (28)
$$

or

$$
\sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| = \sum_{j=n+1}^{n+m} \left|f(\tilde{\rho}_j^+) - f(\rho_j^+)\right|. \qquad (29)
$$

If (28) holds, then, by [11, Lemma 4], we deduce that

$$
\begin{aligned}
\sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| &= \sum_{i=1}^{n} \left[f(\rho_i^+) - f(\tilde{\rho}_i^+)\right] \\
&= \sum_{i=1}^{n} f(\rho_i^+) - \sum_{j=n+1}^{n+m} f(\tilde{\rho}_j^+) \\
&= \sum_{i=1}^{n} f(\rho_i^+) - \sum_{j=n+1}^{n+m} f(\rho_j^+) = |v_2|.
\end{aligned}
$$

If (29) holds, then, by [11, Lemma 4], we deduce that

$$
\begin{aligned}
\sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| &= \sum_{j=n+1}^{n+m} \left[f(\rho_j^+) - f(\tilde{\rho}_j^+)\right] \\
&= \sum_{j=n+1}^{n+m} f(\rho_j^+) - \sum_{i=1}^{n} f(\tilde{\rho}_i^+) \\
&= \sum_{j=n+1}^{n+m} f(\rho_j^+) - \sum_{i=1}^{n} f(\rho_i^+) = |v_2|.
\end{aligned}
$$

Therefore

$$
\frac{|v_2 - v_1|}{|v_2|} \sum_{l=1}^{n+m} \left|f(\tilde{\rho}_l^+) - f(\rho_l^+)\right| = |v_2 - v_1|. \qquad (30)
$$

Moreover, by [11, Lemma 4 and Lemma 5] and by the fact that $\Gamma_{out}^{-} - \Gamma_{out}$ has the same sign of $f(\rho_k^-) - f(\hat{\rho}_k)$,

$$\sum_{l=1,\, l\neq k}^{n+m} \left|f(\rho_l^-) - f(\rho_l^+)\right| = \sum_{i=1,\, i\neq k}^{n} \left|f(\rho_i^-) - f(\rho_i^+)\right| + \left|\Gamma_{out}^{-} - \Gamma_{out}\right|$$
$$= \operatorname{sgn}\left(f(\hat{\rho}_k) - f(\rho_k^-)\right)\left[\Gamma_{inc}^{-} - \Gamma_{inc} - f(\rho_k^-) + f(\rho_k^+) + \Gamma_{out} - \Gamma_{out}^{-}\right].$$

Hence we deduce that

$$\begin{aligned} I_1 \;=\; & \operatorname{sgn}\left(f(\hat{\rho}_k) - f(\rho_k^-)\right)\left[\Gamma_{inc}^{-} - \Gamma_{inc} - f(\hat{\rho}_k) + f(\rho_k^+) + \Gamma_{out} - \Gamma_{out}^{-}\right] \\ & + f(\hat{\rho}_k) - f(\rho_k^+) + \left|\Gamma_{inc} - \Gamma_{inc}^{-} + \Gamma_{out}^{-} - \Gamma_{out}\right|. \end{aligned} \tag{31}$$

First, let us assume that $f(\hat{\rho}_k) < f(\rho_k^-)$. In this situation, we deduce that $f(\hat{\rho}_k) = f(\rho_k^+)$, $\Gamma_{inc} \le \Gamma_{inc}^{-}$, $\Gamma_{out} \le \Gamma_{out}^{-}$ and

$$I_1 = \Gamma_{inc} - \Gamma_{inc}^{-} + \Gamma_{out}^{-} - \Gamma_{out} + \left|\Gamma_{inc} - \Gamma_{inc}^{-} + \Gamma_{out}^{-} - \Gamma_{out}\right|.$$

If $\Gamma_{out} = \Gamma_{out}^{-}$, then

$$I_1 = \Gamma_{inc} - \Gamma_{inc}^{-} + \left|\Gamma_{inc} - \Gamma_{inc}^{-}\right| = 0.$$

If $\Gamma_{out} < \Gamma_{out}^{-}$, then, by (12) and (13), we deduce that $r_\varepsilon(\tilde{t}-) = 0$, $r_\varepsilon'(\tilde{t}-) = 0$ and so $\Gamma_{out} = \Gamma_{inc}$ and $\Gamma_{out}^{-} \le \Gamma_{inc}^{-}$; hence

$$I_1 = \Gamma_{out}^{-} - \Gamma_{inc}^{-} + \left|\Gamma_{inc}^{-} - \Gamma_{out}^{-}\right| = 0.$$

Assume now that $f(\hat{\rho}_k) > f(\rho_k^-)$. In this situation, we deduce that $f(\hat{\rho}_k) \ge f(\rho_k^+) \ge f(\rho_k^-)$, $\Gamma_{inc} \ge \Gamma_{inc}^{-}$, $\Gamma_{out} \ge \Gamma_{out}^{-}$ and

$$I_1 = \Gamma_{inc}^{-} - \Gamma_{inc} + \Gamma_{out} - \Gamma_{out}^{-} + \left|\Gamma_{inc} - \Gamma_{inc}^{-} + \Gamma_{out}^{-} - \Gamma_{out}\right|.$$

If $\Gamma_{out} = \Gamma_{out}^{-}$, then

$$I_1 = \Gamma_{inc}^{-} - \Gamma_{inc} + \left|\Gamma_{inc} - \Gamma_{inc}^{-}\right| = 0.$$

If $\Gamma_{out} > \Gamma_{out}^{-}$, then, by (12) and (13), we deduce that $r_\varepsilon(\tilde{t}-) = 0$, $r_\varepsilon'(\tilde{t}-) = 0$ and so $\Gamma_{out}^{-} = \Gamma_{inc}^{-}$ and $\Gamma_{out} \le \Gamma_{inc}$; hence

$$I_1 = \Gamma_{out} - \Gamma_{inc} + \left|\Gamma_{inc} - \Gamma_{out}\right| = 0.$$

Therefore the claim $I_1 = 0$ is proved; hence we conclude, by (26), that

$$\left\|(v,\xi)(\bar{t}+)\right\| \le \left\|(v,\xi)(\tilde{t}-)\right\|.$$

Consider now a variation on the initial condition of the buffer of the type $r_0 + h$, with $h \in \mathbb{R}$ small enough, let $\bar{t} > 0$ the time at which the first interaction at J takes place. Without loss of generality we can assume $\bar{t} < +\infty$. If at $\bar{t}$ a wave reaches J from an arc, then no shift is produced in the ρ component and the same variation h remains in the buffer after the interaction. Assume therefore that some waves exit J at $\bar{t}$, with speed λ_k, $k \in \{1, \ldots, n+m\}$. We easily deduce that the shifts in the ρ component produced by the perturbation are given by $\xi_k = h\lambda_k / r'(0+)$. Notice that the above shifts are produced only if $r'(0+) \neq 0$. Moreover, $r(\bar{t}+) = 0$ or $r(\bar{t}+) = r_{max}$ and $r'(\bar{t}+) = 0$, so r is no more affected by the perturbation. The conclusion follows by the previous analysis. □

Appendix

In this appendix, we recall, for reader's convenience, the statements of Lemmas 4 and 5 of [11].

Lemma 4 of [11]. *Fix $N \in \mathbb{N} \setminus \{0\}$, a set $\mathcal{P} = \prod_{l=1}^{N} [0, a_l]$, where $a_l > 0$ for every $l \in \{1, \ldots, N\}$, and an N-dimensional vector $(\vartheta_1, \ldots, \vartheta_N)$ such that $\vartheta_l > 0$ for every $l \in \{1, \ldots, N\}$ and $\sum_{l=1}^{N} \vartheta_l = 1$. For $0 \leq \Lambda \leq \sum_{l=1}^{N} a_l$, denote with $(\zeta_1, \ldots, \zeta_N) = P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ the orthogonal projection of $(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ on the set*

$$\mathcal{I} = \left\{ (\gamma_1, \ldots, \gamma_N) \in \mathcal{P} : \sum_{l=1}^{N} \gamma_l = \Lambda \right\}.$$

Then the value ζ_l ($l \in \{1, \ldots, N\}$) depends on Λ in a continuous way. Moreover, for all but a finite number of $0 < \Lambda < \sum_{l=1}^{N} a_l$, the derivative of ζ_l with respect to Λ exists and satisfies $\frac{\partial}{\partial \Lambda} \zeta_l \geq 0$.

Lemma 5 of [11]. *Under the same assumptions as Lemma 4 of [11] the value ζ_l, for $l \in \{1, \ldots, N\}$, depends in a continuous way on a_h for $h \in \{1, \ldots, N\}$. Moreover, if $l \neq h$, then for all but a finite number of a_h it is differentiable and it holds $\frac{\partial \zeta_l}{\partial a_h} \leq 0$.*

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