View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by HAL-UNICE

Virtual Roots of a Real Polynomial and Fractional Derivatives

Daniel Bembe, André Galligo

To cite this version:

Daniel Bembe, André Galligo. Virtual Roots of a Real Polynomial and Fractional Derivatives. Leykin, A. International Symposium on Symbolic and Algebraic Computation (ISSAC), Jun 2011, San Jose, United States. ACM Press, pp.27-34, 2011, <10.1145/1993886.1993897>. <hal-00559950>

HAL Id: hal-00559950

https://hal.archives-ouvertes.fr/hal-00559950

Submitted on 26 Jan 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Virtual Roots of a Real Polynomial and Fractional Derivatives

Daniel Bembé
Mathematisches Institut der Universität München
Theresienstr. 39, D-80333 München, Germany
bembe@math.lmu.de

André Galligo*
Universite de Nice-Sophia Antipolis,
Mathematiques
Parc Valrose 06108 Nice cedex 02, France
galligo@unice.fr

ABSTRACT

After the works of Gonzales-Vega, Lombardi, Mahé,[11] and Coste, Lajous, Lombardi, Roy [6], we consider the virtual roots of a univariate polynomial f with real coefficients. Using fractional derivatives, we associate to f a bivariate polynomial $P_f(x,t)$ depending on the choice of an origin a, then two type of plan curves we call the FDcurve and stem of f. We show, in the generic case, how to locate the virtual roots of f on the Budan table and on each of these curves. The paper is illustrated with examples and pictures computed with the computer algebra system Maple. .

Key words: virtual roots; real univariate polynomial; Budan table; fractional derivatives; FDcurve; stem.

1. INTRODUCTION

In [13], Rahman and Schmeisser note that rules of signs for calculating the roots of a polynomial are older than calculus. Nowadays subdivision methods, heirs of these rules, are widely applied for calculating good approximations of solutions of polynomial equations or intersections of surfaces in Computer Aided Geometric Design. The geometric dictionary in complex algebraic geometry between invariants readable on equations and features of varieties is ultimately based on the fact that a polynomial of degree n admits n roots. This is not the case for real roots, and make real algebraic geometry more complicated. A natural strategy for studying properties of real algebraic varieties is to consider simultaneously roots of iterated derivatives of the input. An important progress was achieved by Gonzales-Vega, Lombardi, Mahé when generalizing the real roots, they introduced in [11] the notion of virtual roots of a polynomial. The n virtual roots of a degree n polynomial provide a good substitute to the n complex roots.

Tables containing the signs of all the derivatives of a polynomial f are called in this paper Budan tables. They were used by various mathematicians including R. Thom for separating and labeling the different real roots of a polynomial, see [4]. Relying on Rolle theorem, we analyze the different admissible configurations of successive rows in such a table. Restricting to the generic case where all roots of all derivatives are two by two distinct, we identify the table with an infinite rectangle separated into positive and negative blocs. We study the topology of the positive (resp. negative) blocs components, and characterize the virtuals roots using connected blocs components.
We also view these connected bloc components inside the Budan table as plane surfaces delimited by discretized curves. To furtherexplore this analogy, we consider derivatives with non-integers orders called fractional derivatives. We point out that fractional derivatives of a polynomial admit a bivariate polynomial factor. This bivariate factor is used to construct two kinds of real planes curves attached to f: FDcurves and stem. The roots of all derivatives of f lie on each curve. These curve naturally realize a partition of the plane, hence can be used to geometrically determine the sign of a derivative at any point. We discuss and illustrate with examples, the possibility of using these curves to ease the location of the virtual roots in a Budan table.

The paper is organized as follows. Section 2 presents the virtual roots and give a quick proof of their characterization by jumps in the sign variations, followed by the definition of virtual multiplicity. Then admissible configurations for a table to be a Budan table are identified. Section 3 examinates what happens in the generic case and establishes our main connexity result. Section 4 is devoted to fractional derivatives and its applications to our setting, FDcurves are introduced and illustrated. Section 5 describe intersections of FDcurves (resp. stems) with a Budan table and their use for the location of virtual roots.

2. VIRTUAL ROOTS

In this section let $\mathbf{R}$ be the field of real numbers (more generally a real closed field). In [11] the virtual roots of a monic degree-n polynomial $f \in \mathbf{R}[X]$ were introduced. They provide n root functions $\rho_{n,k}$ $(1 \leq k \leq n)$ on the space of all monic monic degree-n polynomials. In particular they have the following properties:

1. For every k the $\rho_{n,k} : \mathbf{R}^n \to \mathbf{R}$ are continuous functions of the n coefficients $(a_0, \ldots, a_{n-1}) \in \mathbf{R}^n$ of the monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0$.

*and INRIA Mediterrannée, Galaad project team.

2. if $f(a) = 0$ then $a = \rho_{n,k}$ for at least one k,

3. for every k we have $\rho_{n,k} \leq \rho_{n-1,k} \leq \rho_{n,k+1}$, where $\rho_{n-1,k}$ denotes the k-th virtual root of f'.

From an approximate computational point of view, the advantage is that the coefficients need not been known with infinite precision in order to compute the virtual roots with finite precision.
We summarize some of the results of [11] and [2]. [6] shows that the Budan-Fourier count always gives the number of virtual roots (with multiplicities) on an interval. The authors of [6] present some of our results in the more general context of **f**-derivatives. At the end of this section we present the Budan table and some claim about the virtual multiplicity.

2.1 Definition

DEFINITION 2.1 (VIRTUAL ROOTS). *Let $f \in \mathbf{R}[X]$ monic of degree n and $f^{(i)}$ denote its i-th derivative. For $0 \leq j \leq n$ the j virtual roots of $f^{(n-j)}$, $\rho_{j,1} \leq \cdots \leq \rho_{j,j}$, are defined inductively:*

1. *Let $\rho_{j,0} = -\infty$ and $\rho_{j,j+1} = \infty$ for $0 \leq j \leq n$.*
2. *For fixed $1 \leq j \leq n$ let be the $\rho_{j-1,k}$ defined such that for $1 \leq k \leq j$*
$$f^{(n-j+1)}(x) f^{(n-j+1)}(y) \geq 0$$
for all $x, y \in \mathbf{R}_{j-1,k} = [\rho_{j-1,k-1}, \rho_{j-1,k}]$ (resp. the half-open interval if $k \in \{1, j\}$).
3. *Then for every $1 \leq k \leq j$ the $\rho_{j,k} \in \mathbf{R}_{j-1,k}$ is defined by the inequality*
$$|f^{(n-j)}(\rho_{j,k})| \leq |f^{(n-j)}(x)|$$
for all $x \in \mathbf{R}_{j-1,k}$. This is well-defined since $f^{(n-j)}$ is strictly monotone on $\mathbf{R}_{j-1,k}$.
We must distinguish three cases:
   - *(a) $\rho_{j-1,k-1} = \rho_{j-1,k}$,*
   - *(b) $f^{(n-j)}$ admits a real root in $\mathbf{R}_{j-1,k}$ then $\rho_{j,k}$ is at this root,*
   - *(c) $f^{(n-j)}$ admits no real root in $\mathbf{R}_{j-1,k}$ then $\rho_{j,k}$ is the point with the least absolute value under $f^{(n-j)}$ (see Figure 1). Hence it is either $\rho_{j-1,k-1}$ or $\rho_{j-1,k}$.*
4. *We get for $1 \leq k \leq j+1$*
$$f^{(n-j)}(x) f^{(n-j)}(y) \geq 0$$
for all $x, y \in [\rho_{j,k-1}, \rho_{j,k}]$.

Before we state a theorem which enables us to determine virtual roots, we consider the simple example with $n = 3$, $f := (x-2)^2 - 3(x-2) + 4$. See Figure 1.
Its virtual roots are: $\rho_{3,1} \approx -0.19$ (case b); $\rho_{3,2} = 3$ (case c); $\rho_{3,3} = 3$ (case c). We also have (case b), $\rho_{2,1} = 1$, $\rho_{2,2} = 3$. and $\rho_{1,1} = 2$.

DEFINITION 2.2. 1. *Let $f \in \mathbf{R}[X]$ and $a \in \mathbf{R}$.*

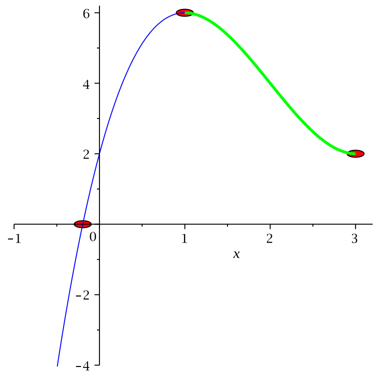

Figure 1: $\rho_{3,1} = -0.19$, $\rho_{2,1} = 1$, $\rho_{3,2} = \rho_{2,2} = 3$.

(a) *The* real multiplicity $\mathrm{rmult}_f(a)$ *denotes the number $m \geq 0$, for which $(X-a)^m$ divides $f(X)$ and $(X-a)^{m+1}$ does not. If $rmult_f(a) \geq 1$ we say, that a is a real root of f.*

(b) *If $\rho_{n,k} = a$ for a k we say, that a is a virtual root of f, and the* virtual multiplicity $\mathrm{vmult}_f(a)$ *denotes the biggest number $m \geq 1$, for which $\rho_{n,l+1} = a = \rho_{n,l+m}$ for any l.*
Otherwise $\mathrm{vmult}_f(a) = 0$.

2. (a) *For a sequence $(a_0, \ldots, a_n) \in (\mathbf{R} \setminus \{0\})^{n+1}$ the* number of sign changes $\mathbf{V}(a_0, \ldots, a_n)$ *is defined inductively in the following way:*
$$\mathbf{V}(a_0) := 0; \qquad \mathbf{V}(a_0, \ldots, a_i) :=$$
$$\begin{cases} \mathbf{V}(a_0, \ldots, a_{i-1}) & \textit{if } a_{i-1}a_i > 0, \\ \mathbf{V}(a_0, \ldots, a_{i-1}) + 1 & \textit{if } a_{i-1}a_i < 0. \end{cases}$$

(b) *To determine the number of sign changes of a sequence $(a_0, \ldots, a_n) \in \mathbf{R}^{n+1}$ delete the zeros in $(a_0, \ldots, a_n)$ and apply case 2a. ($\mathbf{V}$ of the empty sequence equals 0).*

THEOREM 2.3. *Let $f \in \mathbf{R}[X]$ monic of degree n, $\rho_{n,1} \leq \cdots \leq \rho_{n,n}$ its virtual roots and $\rho_{n,0} = -\infty$, $\rho_{n,n+1} = \infty$. Then we have for $1 \leq k \leq n+1$ with $\rho_{n,k-1} \neq \rho_{n,k}$*
$$x \in [\rho_{n,k-1}, \rho_{n,k}[\Longleftrightarrow$$
$$\mathbf{V}(f(x), f'(x), \ldots, f^{(n)}(x)) = n + 1 - k$$
(resp. for $k = 1$ the interval $x \in]-\infty, \rho_{n,1}[$).

PROOF. By induction on the degree j of $f^{(n-j)}$. Let $\rho_{j,1} \leq \cdots \leq \rho_{j,j}$ denote the virtual roots of $f^{(n-j)}$ and $\rho_{j,0} = -\infty$, $\rho_{j,j+1} = \infty$.
Let $j = 0$. Then $]\rho_{0,0}, \rho_{0,1}[= \mathbf{R}$ and $\mathbf{V}(f^{(n)}(x)) = 0$ for all $x \in \mathbf{R}$.
Let $j > 0$ and the claim be true for $j - 1$. Let $1 \leq k \leq j+1$ with $\rho_{j-1,k-1} \neq \rho_{j-1,k}$ and consider $x \in [\rho_{j-1,k-1}, \rho_{j-1,k}[$.

In case b) of the definition of the virtual roots we get

$$f^{(n-j+i)}(x)f^{(n-j)}(x) < 0 \text{ for } \rho_{j-1,k-1} = x$$
$$f^{(n-j+1)}(x)f^{(n-j)}(x) < 0 \text{ for } \rho_{j-1,k-1} < x < \rho_{j,k},$$
$$f^{(n-j)}(x) = 0 \text{ for } \rho_{j,k} = x$$
$$f^{(n-j+1)}(x)f^{(n-j)}(x) > 0 \text{ for } \rho_{j,k} < x < \rho_{j-1,k},$$

for the smallest $i \geq 1$ with $f^{(n-j+i)}(\rho_{j-1,k-1}) \neq 0$. In case c) the same argument holds. □

Corollary 2.4. *1. For every k the $\rho_{n,k} : \mathbf{R}^n \to \mathbf{R}$ are continuous functions of $(a_0, \ldots, a_{n-1})$ in $\mathbf{R}^n$, the n coefficients of the monic polynomial f.*

2. For every $a \in \mathbf{R}$ we have

$$\text{rmult}_f(a) \leq \text{vmult}_f(a).$$

3. For every $a \in \mathbf{R}$ we have

$$\text{vmult}_f(a) - \text{rmult}_f(a) \text{ is even.}$$

4. (Budan's theorem) For $x, y \in \mathbf{R}$ with $x < y$ we get

$$0 \leq \sum_{a \in]x,y]} \text{rmult}_f(a)$$
$$\leq \mathbf{V}(f(y) \ldots, f^{(n)}(y)) - \mathbf{V}(f(x) \ldots, f^{(n)}(x)).$$

Proof. 1. Let be $a := \rho_{n,k}(f)$ the k-th virtual root of f and $\epsilon \in \mathbf{R}$ be > 0 such that $f^{(i)}(a-\epsilon)f^{(i)}(a+\epsilon) \neq 0$ for $i \leq n$. Now change the coefficients of f in such a minimnal way that the following holds and denote the new polynomial by $\tilde{f}$. $f^{(i)}(a-\epsilon)\tilde{f}^{(i)}(a-\epsilon) > 0$ and $f^{(i)}(a+\epsilon)\tilde{f}^{(i)}(a+\epsilon) > 0$ for $i \leq n$. From theorem 2.3 we get $\rho_{n,k}(\tilde{f}) \in]a-\epsilon, a+\epsilon]$.

2. This follows from the following fact, which can be derived from the mean value theorem, applied inductively on f and its derivatives: Let $\deg(f) \geq 1$. For every $a \in \mathbf{R}$ exists an $\epsilon > 0$ such that

$$(-1)^{\text{rmult}_f(a)} f(x)f(y) > 0 \quad (1)$$
$$f(y)f'(y) > 0 \quad (2)$$

for every $x \in]a-\epsilon, a[$ and $y \in]a, a+\epsilon[$.

3. This follows from (1) and $f^{(n)}(x)f^{(n)}(y) > 0$.

4. This follows from 2. as

$$\mathbf{V}(f(y) \ldots, f^{(n)}(y)) - \mathbf{V}(f(x) \ldots, f^{(n)}(x))$$
$$= \sum_{a \in]x,y]} \text{vmult}_f(a),$$

as desired.

□

Remark 2.5 (About Budan's theorem). *Budan's theorem is stated in the appendix of [5]. According to [1], it was published for the first time in 1807, while Fourier published the equivalent result in 1820 ("Le Bulletin des Sciences par la Société Philomatique de Paris"). In fact, Budan's counting of roots is today known as "Budan-Fourier count".*
Budan proved the non-negativity of the difference by the equivalent claim: For $y > 0$, $f(X) = \sum a_i X^i$ and $f(X+y) = \sum b_i X^i$ we get $\mathbf{V}(a_0, \ldots, a_n) \geq \mathbf{V}(b_0, \ldots, b_n)$. While Budan does not use the sequence of derivatives, it is introduced by Fourier ("Analyse des Équations"), as mentioned in [14]. At the same text an elegant proof for this equivalence by Taylor series is presented.

A different proof for the continuity is given in [11]. The following important property is proved.

Theorem 2.6 ([11]). *The $\rho_{n,k}$ with $1 \leq k \leq n$ are continuous functions of the n coefficients, $(a_0, \ldots, a_{n-1}) \in \mathbf{R}^n$, of the monic polynomial f. Moreover they are semi algebraic continuous functions defined over $\mathbf{Q}$ and integral over the polynomials.*

2.2 Budan table and multiplicities

In the Budan table we present the roots and signs of $f(x)$ and its derivatives for all $x \in \mathbf{R}$ as an infinite rectangle, formed by $n+1$ bands (also called rows) $\mathbf{R} \times [j-0.5, j+0.5[$ with $0 \leq j \leq n$. A root a of $f^{(n-j)}$ is represented by a bar | positioned at a in the j-th band. between the bars | the sign of $f^{(n-j)}$ is fixed, if it is $-$ the bloc is colored, if it is $+$ it remains white. In the picture we often put a small disk at the roots to point them out, sometimes the colors distinguish the real roots from the non real ones. Consider figure 2, which shows the Budan table of a degree-6-polynnomial f without real roots. The black disks show the tree pairs of virtual roots of f.
The following arguments make it easy to determine the virtual roots in a given Budan table. First, we characterize the behavior of $\text{vmult}_f(a)$ and $\text{rmult}_f(a)$ when integrating f':

Proposition 2.7. *Let $f \in \mathbf{R}[X]$ monic, $a \in \mathbf{R}$. Provided as well $\text{vmult}_f(a) - \text{rmult}_f(a)$ as $\text{vmult}_{f'}(a) - \text{rmult}_{f'}(a)$ being even the following cases and only them can appear:*

1. $\text{rmult}_f(a) = 0 = \text{rmult}_{f'}(a)$ *and* $\text{vmult}_f(a) = \text{vmult}_{f'}(a)$*;*
2. $\text{rmult}_f(a) = \text{rmult}_{f'}(a) + 1$ *and* $\text{vmult}_f(a) = \text{vmult}_{f'}(a) + 1$*;*
3. $\text{rmult}_f(a) = 0 < \text{rmult}_{f'}(a)$ *and* $\text{vmult}_f(a) - \text{vmult}_{f'}(a) \in \{-1, 0, 1\}$.

Proof. This follows form the definitons and corresponding examples. □

This leads to the following way to determine if a real root of a derivative of f is a pair of virtual roots of f:

Proposition 2.8. *Let $f \in \mathbf{R}[X]$ monic, $a \in \mathbf{R}$. Let m be the number of $0 < i < n$ for which the following holds:*

$f^{(i)}(a) = 0$ *and it exists an* $\epsilon > 0$ *such that*

$$f^{(i-1)}(y) f^{(i)}(y) > 0$$
$$f^{(i)}(x) f^{(i)}(y) < 0$$
$$f^{(i+1)}(y) f^{(i)}(y) > 0$$

for every $x \in]a-\epsilon, a[$ *and* $y \in]a, a+\epsilon[$.
Then

$$\mathrm{vmult}_f(a) = \begin{cases} 2m & \text{if } f(a) \neq 0, \\ 2m+1 & \text{if } f(a) = 0. \end{cases}$$

PROOF. This follows by induction on the degree and proposition 2.7. □

3. GENERICITY AND RANDOMNESS

To simplify our analysis, we now on restrict to generic cases.

Genericity is a concept used in algebraic geometry. Often in computer algebra, to choose a generic element we rely on the random function rand(), which produces numbers uniformly distributed in an interval. However, the two notions should not be confused.

3.1 Genericity

The set of degree n polynomials form a real vector space endowed with two natural topologies. The usual inherited form that of **R** and the Zariski topology. In the second one a basis of closed sets is formed by algebraic hypersurfaces defined as the zeros of multivariate polynomials. A property is then said generic if it is satisfied by a Zariski-dense subset of polynomials.

In practice, we try to concentrate all the "bad" behaviors that we want to avoid into an algebraic hypersurface (which need not be explicitly computed) and then just say "generically". For instance all roots of the iterated derivatives of a generically given polynomial are two by two distinct.

PROPOSITION 1. *For a generic polynomial, all virtual not real roots are double.*

PROOF. As all roots of its iterated derivatives are 2 by 2 distinct, near such a root y of a derivative $f^{(i)}$ there is small positive number e and an interval $[y-e, y+e]$ where all the other derivatives keep a constant sign. So the only possibility for a sign variation between $y-e$ and $y+e$ is 0 or 2. □

For a generic polynomial f, we can use Maple to pointplot the roots of the derivatives together with vertical lines passing by them, as illustrated in Figure 2 with a polynomial of degree 6 having no real root. So we expect 3 double virtual roots. In order to locate these 3 virtual roots, we need to evaluate the signs of the derivatives on each row. We know that all the signs are + at ∞ and alternated + and − at $-\infty$. Since the signs change at each root, the signs in the Budan table can be easily completed. Therefore, we can apply the discussion we made in section 2 of the characterization of the patterns appearing in Budan table at a virtual

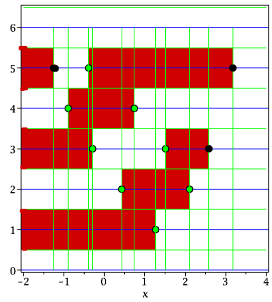

Figure 2: Blocs and roots

root. Then, a FDcurve or a stem of f (see below in the next section) can be used to express the propagation of the signs of the derivatives in a 2D picture. Let's now state our main connexity result.

THEOREM 3.1. *Let f be generic monic univariate polynomial of degree n. The Budan table of f is represented by $n+1$ bands of height one $\mathbb{R} \times [i-1/2, i+1/2[$ for i from 0 to n. A rectangle (possibly infinite) corresponding to negative values of a derivative is colored, while one corresponding to positive values remain white. The first band is white. Then:*

- *In this table, the number of connected colored components bounded on the right plus the number of connected white components bounded on the right equal the number of pairs of virtual non real roots plus the number of real roots.*
- *The rightest blocs of such a connected (bounded on the right) components, not on the n-th band, characterize the virtual non real roots. The row of such a bloc indicates the degree of a derivative vanishing at the corresponding virtual root.*
- *Replacing f by $-f$, exchange the colors of the blocs, and rightest by leftest.*

PROOF. By genericity, we have $n = 2m + p$ where p is the number of real roots and m the number of (double) virtual non real roots. On the n-th band there are $q := \lceil p/2 \rceil$ (negative) colored blocs bounded on the right and $p - q$ (positive) white blocs bounded on the right. These p blocs are connected to one of the n infinite left, colored or white, ends (corresponding to the signs of the derivatives at $-\infty$).

By Rolle theorem and genericity, there are an odd number of roots on a row L_{i-1} between two successive roots x_1 and x_2 on L_i. Two colored blocs on successive bands are connected either on the right or on the left. By the discussion we made in section 2, only connection on the left is allowed when there is a single root on L_{i-1} between x_1 and x_2, and this configuration does not give rise to a virtual root. While when there are more than $2r+1$ roots between x_1 and x_2, r pairs of virtual roots appear. The corresponding blocs are the end of two blocs components connected to the left (down) side of the table. Notice that such a virtual root corresponds to the right side of a colored bloc (resp white bloc) but is surrounded by two white (resp. colored) blocs components coming from $-\infty$. Hence $2m$ among the n ends at $-\infty$ arrive at such points. The count is complete. The claim follows from this description. $\square$

Figure 2 illustrates an example of degree 6 with 3 virtual non real roots of multiplicity 2. There are 3 "connected colored components bounded on the right" and 0 "connected white components bounded on the right". They characterize 2 virtual roots on the 5-th row and a virtual root on the 3-rd row. See also the example at the end of the paper, where the situation is less simple.

3.2 Randomness

We call random polynomial, a polynomial whose coefficients are obtained by a random distribution, in general image of a classical law (Normal, uniform, Bernouilli). Two kinds of random polynomials have been extensively studied. First, those obtained by choosing a basis of degree n polynomials (x^i or $\sqrt{1/i!}x^i$, etc..) and taking linear combination with random independent coefficients distributed with a classical law. Second, the characteristic polynomials of random matrices whose entries are distributed with a classical law.

A random polynomial f is, with a good probability, generic in the previous algebraic sens; but it is more specific. Indeed, its virtual roots inherit other statistical properties from the distribution of the coefficients of f; when the degree n tends to infinity, some properties are asymptotically almost sure.

For instance, generically the n complex roots of a polynomial are 2 by 2 distinct. But if we consider the characteristic polynomial of a dense random $(128, 128)$ matrix, whose entries are instances of independent centered normal variables with variance v, its complex roots (the eigenvalues) are almost uniformly distributed in a disk of radius $\sqrt{nv}$. This behavior is obviously not generic.

For large degree n (say 100), colored Budan tables look like discretized shapes, exploring this interpretation, it seems worthwile to also consider the derivation orders as discretized values, hence consider fractional derivatives.

4. FRACTIONAL DERIVATIVES

The idea to introduce and compute with derivatives or antiderivatives of non-integer orders goes back to Leibnitz. In the book [12], the authors relate the history of this concept from 1695 to 1975, the progression is illustrated by historical notes and they included more than a hundred enlightening citations from papers of several great mathematicians: Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, and many more.

In 1832 Liouville expanded functions in series of exponentials and defined q-th derivatives of such a series by operating term-by-term for q a real number, although Riemann proposed another approach via a definite integral. They give rise to an integral of fractional order called Riemann-Liouville integral for $q < 0$, which depends upon an origin a and generalizes the classical formula for iterated integrations:

$$[\frac{d^q f}{[d(x-a)]^q}]_{R-L} := \frac{1}{\Gamma(-q)} \int_a^x [x-y]^{-q-1} f(y) dy.$$

Then for positive order and any sufficiently derivable function f, one relies on a composition property with $\frac{d^m}{d(x-a)^m}$, for an integer m. So, for any real number $m+q$, one obtains:

$$[\frac{d^{m+q} f}{[d(x-a)]^{m+q}}]_{R-L} := \frac{d^m}{d(x-a)^m}[\frac{d^q f}{[d(x-a)]^q}]_{R-L}].$$

Thanks to properties of the Γ function, this definition is coherent when a change of (q, m) keeping $m+q$ constant. The generalization of this definition to other functions f. is discussed in the book [12]. The traditional adjective "fractional" corresponding to the order of derivation is misleading, since it need not be rational.

Let us emphasize that nowadays in mathematics, fractional derivatives are mostly used for the study of PDE in functional analysis. They are presented via Fourier or Laplace transforms. Fractional derivatives are seldom encountered in polynomial algebra or in computer algebra. The second author learned this concept and its history working on [10], then used the following very simple formula, with $q > 0$ and n an integer, attributed to Peacock.

$$\frac{d^q}{[d(x-a)]^q}(x-a)^n := \frac{n!}{(n-q)!}(x-a)^{n-q}.$$

We illustrate it with the monomials of a polynomial, $q = \frac{1}{2}$ and $a = 0$:

$$\frac{d^{1/2}}{[dx]^{1/2}}(x^2 - 2x + 3, x) = (\frac{8}{3}x^2 - 4x + 3)x^{-1/2}\frac{1}{\sqrt{\pi}}.$$

4.1 A bivariate polynomial

LEMMA 1. *Let $f(x)$ be a polynomial of degree n, then*

$$(x-a)^q \Gamma(-q) \frac{d^q f}{[d(x-a)]^q}$$

is a polynomial in x and q.

To interpolate the non vanishing roots of the successive derivatives of a polynomial f, only fractional derivatives, up to a power of $(x-a)$ are needed. We introduce the following notations for a family of univariate polynomials in q and another in $t := n - q$, indexed by their degrees.

Notations: For $i = 0, ..., n-1$,

$$l_0 := 1;\ l_{n-i}(q) := \prod_{j=i+1}^{n} 1 - \frac{q}{j};\ \lambda_0 := n!;$$

$$\lambda_{n-i}(t) := n! l_{n-i}(n-t) = i! t(t-1)...(t+i+1-n).$$

DEFINITION 1. *Let $f = \sum a_i(x-a)^i$ be a degree n polynomial. We call monic polynomial factor of a fractional derivative of order q, with respect to the origin a, of f,the*

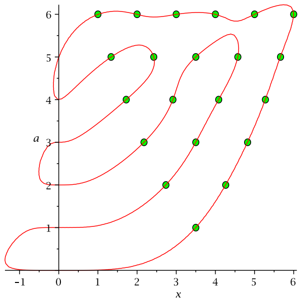

Figure 3: A simple FDcurve

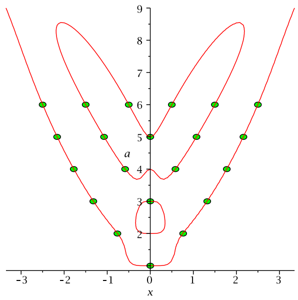

Figure 4: Changing the origin a

bivariate polynomial $(x-a)^q \frac{(n-q)!}{n!} \frac{d^q f}{[d(x-a)]^q}$. It is a polynomial of total degree n in $(x-a)$ and q which writes

$$\sum_{i=0}^{n-1} a_i (x-a)^i l_{n-i}(q).$$

It will be convenient to let $t = n - q$, and consider the polynomial obtained with this substitution:

$$P_f(x,t) := \frac{1}{n!} \sum_{i=0}^{n} a_i (x-a)^i \lambda_{n-i}(t).$$

For all $k = 0, ..., n$, we have $P_f(x, n-k) = \frac{(n-k)!}{n!}(x-a)^k f^{(k)}$.

It also holds $(x-a)\frac{\partial}{\partial x} P_f = P_{(x-a)f'}$.

The previous bivariate polynomial realizes an homotopy between the graphs of $f(x)$ and $(x-a)f'(x)$ when q varies between 0 and 1.

4.2 FDcurve

Definition 2. *We call FDcurve, with origin a, of a polynomial f of degree n, the real algebraic curve defined by the bivariate equation $P_f(x,t) = 0$.*

Notice that instead of taking the origin at a, we can fix the origin at 0, perform a substitution $x := x - a$ on f and then translate the obtained curve.

Figure 3 shows a simple example with $f := (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$, $n = 4, a = 0$, an hyperbolic polynomial, hence all its derivatives are hyperbolic. The roots of f and its derivatives are represented by small green disks. In Figure 4 we first performed a substitution with $a = 3.5$. The two curves are quite different, the second has 3 connected components and infinite branches, but both pass through all the roots. The FDcurve corresponding to other values a may have singularities (e.g. double points). So the topologies of the FDcurve can change with a.

In many examples all the connected components cut the axis $x = a$, but it is not always the case: Figure 5 shows the small lonesome component of the example, with $a = 0$,
$f = x^6 + 10.4x^5 + 34.55x^4 + 41.20x^3 + 29.85x^2 - 15.00x - 0.37$.
However no root lies on this small component, we do not know if it is always so. In this example f has two real roots,

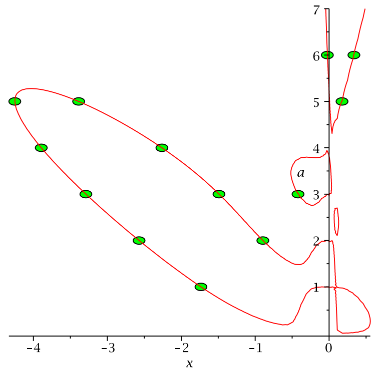

Figure 5: A lonesome component

it has also two virtual double roots, their location will be studied in the next section.

In [10], another curve (an algebraic C^0 spline), called the stem of f, is associated to a degree n polynomial f. It is defined as the union of the real curves formed by the roots of all the monic polynomial factors of the derivatives $f^{(i)}$ of f, for i from 0 to $n-1$ and $0 \leq q < 1$. Stems were designed to study the roots of the derivatives of random polynomials of high degrees and exploit their symmetries. To illustrate the differences between these two constructions, Figure 6 shows the stem corresponding to the previous FDcurve with the lonesome component: it is less curved.

5. LOCATION OF VIRTUAL ROOTS

For f a generic monic univariate polynomial of degree n, in this section, we consider partitions of the infinite rectangle R. R is the union of $n+1$ bands of height one $\mathbb{R} \times [i - 1/2, i + 1/2[$ for i from 0 to n. In the previous section, we have seen the partition of R corresponding to the Budan table: the rectangles (possibly infinite) corresponding to negative values of a derivative are colored while the ones corresponding to positive values remain white. Theorem 3.1 shows that this partition allows to locate the virtual roots of f. Here, we aim to rely on the ovals of FDcurves or stems to transmit "quickly" the sign information needed for the partition of the Budan table.

For this purpose, let us consider an example where all the roots of the derivatives of f are positive, and choose $a = 0$.

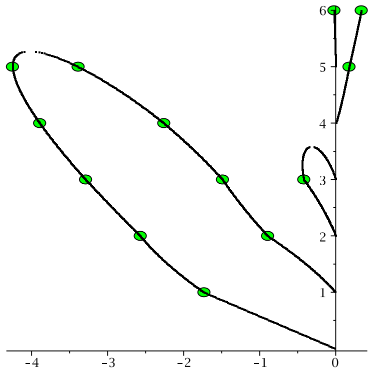

Figure 6: Stem of the previous curve

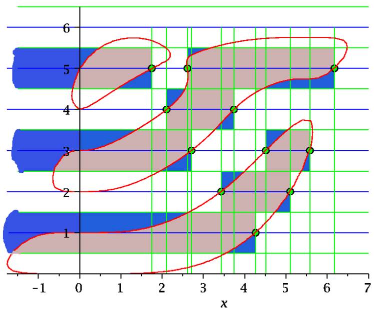

Figure 7: Budan inside and around an FDcurve

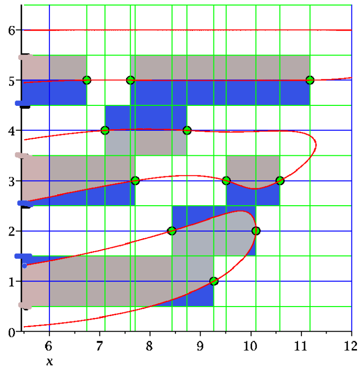

Figure 8: Connecting the components

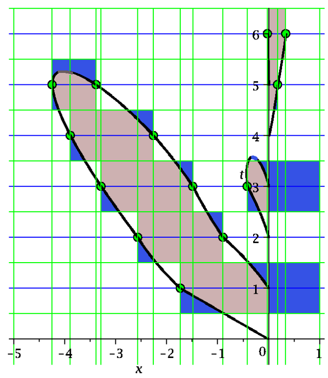

Figure 9: With a stem curve

This is always possible up to a translation on x. We take the intersection of the negative part of the Budan table and the negative locus of P_f (delimited by the components of the FDcurve). In Figure 7 the intersection zones are colored in grey. These intersection zones are helpful to see that some blocs are connected but not sufficient to guaranty that other blocs are disconnected. So, we also consider the zones colored in blue, shaped as curved triangles in the picture. Two blue zones attached to two separated connected components of the FDcurve may intersect, this happens in Figure 8 with the same example where we changed the origin a, hence the FDcurve. Let's do the same constructions with the stem of Figure 6. In that case, the interiors of the ovals correspond to positive values of an implicit function, so it is better to color the positive blocs. Now, the virtual roots correspond to the leftest blocs. This is illustrated in Figure 9: the 2 virtual roots are immediately located at the leftest roots on the two left ovals.

As a conclusion, we can say that depending on the shape of the stem of f or of an FDcurve, the location of the virtual roots may become very fast. But this possibility should be studied case by case.

5.1 An example of medium degree

We consider a randomly generated polynomial of degree $n = 16$, taking a random linear combination of the so-called

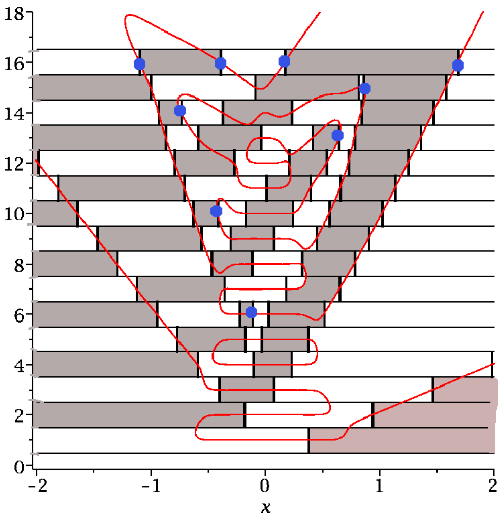

Figure 10: Collapsing blocs

Bernstein polynomials, used in Computer Aided Design.

It has 6 real roots. In Figure 10, we truncated the picture, and we see only 4 of them. So it remains 5 double virtual roots. In the picture real roots and virtual roots are represented by blue disks. We colored in grey the positive blocs. Among the 5 virtual roots, 4 correspond to grey blocs components and 1 to a white blocs component.

Notice that the FDcurve is helpful for locating the positive virtual roots (at the end of the ear shaped curves), but not for the negative virtual roots. Therefore it is useful to reduce to positive values and simultaneously consider the polynomial obtained by changing $f(x)$ into $(-1)^n f(-x)$.

6. CONCLUSION

We characterized the possible patterns between successive rows in a Budan table corresponding to a virtual roots. Restricting to the generic case we gave a global characterization (using connectivity of connected components) of the location of virtual roots in a Budan table. In addition, we used fractional derivatives to associate a bivariate polynomial to f, and introduced two types of plane curve associated to f, which help geometrically see the signs taken by the iterated derivatives of f hence locate, in many cases, a virtual roots near one of their critical points. We suggest three directions for future researches:

- Investigate what happens when we relax the genericity hypothesis (i.e. specialization to more degenerated cases),
- Study the relationship beteen virtual roots in an interval and pairs of conjugate complex roots which lie in a sector close to this interval counted by Obreschkoff theorem, see [13], chapter 10.
- generalize to other families of functions beyond the polynomials, as initiated in [6].

Acknowledgments

We are grateful to Henri Lombardi for valuable discussions. André Galligo was partially supported by the European ITN Marie Curie network SAGA. Daniel Bembé was partially supported by the Deutsch-Französische-Hochschule École-Franco-Allemande.

7. REFERENCES

[1] Akritas Alkiviadis G., *Reflections on an pair of theorems by Budan and Fourier*, University of Cansas **22**.

[2] Bembé, D: Budan's theorem and virtual roots of real polynomials Preprint.(2009)

[3] Bharucha-Reid, A. T. and Sambandham, M.: Random Polynomials. Academic Press, N.Y. (1986).

[4] Bochnack, J. and Coste, M. and Roy, M-F.: Real Algebraic Geometry. Springer (1998).

[5] Budan de Boislaurent, *Nouvelle méthode pour la résolution des équations numériques d'un degré quelconque.* Paris (1822). Contains in the appendix a proof of Budan's theorem edited by the Académie des Sciences (1811).

[6] Coste, M and Lajous, T and Lombardi, H and Roy, M-F : Generalized Budan-Fourier theorem and virtual roots. Journal of Complexity, 21, 478-486 (2005).

[7] Edelman,A and Kostlan,E: How many zeros of a random polynomial are real? Bulletin AMS, 32(1):137, (1995).

[8] Emiris, I and Galligo, A and Tsigaridas, E: Random polynomials and expected complexity of bissection methods for real solving. *Proceedings of the ISSAC'2010 conference,* pp 235-242, ACM NY, (2010).

[9] Farahmand,K: Topics in random polynomials. Pitman research notes in mathematics series 393, Addison Wesley, (1998).

[10] Galligo, A: Intriguing Patterns in the Roots of the Derivatives of some Random Polynomials. Submitted (2010).

[11] Gonzales-Vega, L and Lombardi, H and Mahé, L : Virtual roots of real polynomials. J. Pure Appl. Algebra,124, pp 147-166,(1998).

[12] Oldham, K and Spanier, J: The fractional calculus. Academic Press Inc. NY, (1974).

[13] Rahman, Q.I and Schmeisser,G: Analytic theory of polynomials, Oxford Univ. press. (2002).

[14] Vincent M,. *Sur la résolution des équations numéricques*, Journal de mathématicques pures et appliquées **44** (1836) 235–372.