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DECOMPOSITION OF HOMOGENEOUS POLYNOMIALS WITH LOW RANK

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ABSTRACT: *Let F be a homogeneous polynomial of degree d in $m + 1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the s-th secant variety of the d-uple Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d-th powers of linear forms $M_1, \ldots, M_r$ is $F = M_1^d + \cdots + M_r^d$ with $r > s$. We show that if $s + r \leq 2d + 1$ then such a decomposition of F can be split in two parts: one of them is made by linear forms that can be written using only two variables, the other part is uniquely determined once one has fixed the first part. We also obtain a uniqueness theorem for the minimal decomposition of F if r is at most d and a mild condition is satisfied.*

INTRODUCTION

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem arising from classical Algebraic Geometry ([1], [14]), Computational Complexity ([15]) and Signal Processing ([20]). Any statement on homogeneous polynomials can be translated in an equivalent statement on symmetric tensors. In fact, if we indicate with V a vector space of dimension $m+1$ defined over a field K of characteristic 0, and with V^* its dual space, then, for any positive integer d, there is an obvious identification between the vector space of symmetric tensors $S^d V^* \subset (V^*)^{\otimes d}$ and the space of homogeneous polynomials $K[x_0, \ldots, x_m]_d$ of degree d defined over K. In this paper we will always work with an algebraically closed field K of characteristic 0. The requirement that a form (or a symmetric tensor) involves a minimum number of terms is a quite recent and very interesting problem coming from applications. Given a form $F \in K[x_0, \ldots, x_m]_d$ (or a symmetric tensor $T \in S^d V^*$), the minimum positive integer r for which there exist linear forms $L_1, \ldots, L_r \in K[x_0, \ldots, x_m]_1$ (vectors $v_1, \ldots, v_r \in V^*$ respectively) such that

$$(1) \qquad F = L_1^d + \cdots + L_r^d, \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d})$$

is called the *symmetric rank* $\mathrm{sr}(F)$ of F ($\mathrm{sr}(T)$ of T respectively). Computations of the symmetric rank for a given form (or a given symmetric tensor) are studied in [11], [3], [4] and [2]. First of all we focus our attention on those particular decompositions of a form $F \in K[x_0, \ldots, x_m]_d$ (or $T \in S^d V^*$) of the type (1) with $r = \mathrm{sr}(F)$ ($r = \mathrm{sr}(T)$ respectively). What about the possible uniqueness of the decomposition of such a form F (T respectively)? A general form, for example, can have a unique decomposition as in (1) only if $\frac{1}{n+1}\binom{n+d}{n} \in \mathbb{Z}$ (see [19], [17], [18], [9] also for further results on this normal form). If the polynomial is not general, very few things are known.

Let $X_{m,d} \subset \mathbb{P}^N$, with $m \geq 1$, $d \geq 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d-uple Veronese embedding $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$. The *s-th secant variety* $\sigma_s(X_{m,d})$ of the Veronese variety $X_{m,d}$ is the Zariski closure in $\mathbb{P}^N$ of the union of all linear spans $\langle P_1, \ldots, P_s \rangle$ with $P_1, \ldots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\mathrm{sbr}(P) = s$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P. By a famous theorem of J. Alexander and A. Hirschowitz all integers $\dim(\sigma_s(X_{m,d}))$ are known ([1], [8], [5]). Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \ldots, x_m]_1) \simeq \mathbb{P}(V^*)$, the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (a symmetric tensor) of type (1). Unfortunately, for a given $P \in \mathbb{P}^N$, we only have the inequality $\mathrm{sbr}(P) \leq \mathrm{sr}(P)$. For the forms F for which the decomposition (1) is not unique, it makes sense to study those different decompositions. There is a uniqueness theorem for general points with prescribed non-maximal symmetric border rank s using the notion of $(s-1)$-weakly non-defectivity introduced by C. Ciliberto and L. Chiantini ([7], [10], Proposition 1.5). In this paper we are interested in those particular decompositions of a given $F \in K[x_0, \ldots, x_m]_d$ of the type (1) with $r = \mathrm{sr}(F)$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$ ($T \in S^d V^*$ respectively). In many applications one would like to reduce the number of variables, at least for a part of the data. For such a particular choice of F, is it possible to find linear forms $L_1, L_2, M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$ and a binary form $Q \in K[L_1, L_2]_d$, such that a given polynomial $F \in K[x_0, \ldots, x_m]_d$ can be written as $F = Q + M_1^d + \cdots + M_t^d$? (On normal forms of homogeneous polynomials see also [16], [13], [14].) The main result of this paper is the following.

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d} - 1$. Suppose that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P) \text{ and}$$
$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a smoothable 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$ and $\deg \mathcal{Z} \leq \mathrm{sbr}(P)$. Let also $C_d \subset X_{m,d}$ be the unique rational normal curve that intersects $\mathcal{S} \cup \mathcal{Z}$ in degree at least $d + 2$. Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \quad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.
Moreover $\deg(\mathcal{Z}) = \mathrm{sbr}(P)$ and the scheme $\mathcal{S}_2$ is unique.

The existence of such a scheme $\mathcal{Z}$ was known from [3] and [6] (see Remark 1). The assumption " $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1$ " is sharp (see Example 1).

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let $F \in K[x_0, \ldots, x_m]_d$ be such that $\mathrm{sbr}(F) + \mathrm{sr}(F) \leq 2d + 1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $t \geq 0$, linear forms $L_1, L_2, M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \leq \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$. Moreover t, $M_1, \ldots, M_t$ and the linear span of L_1, L_2 are uniquely determined by F.*

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let $T \in S^d V^*$ be such that $\mathrm{sbr}(T) + \mathrm{sr}(T) \leq 2d + 1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $t \geq 0$, vectors $v_1, v_2, w_1, \ldots, w_t \in S^1 V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \leq \mathrm{sbr}(T) + \mathrm{sr}(T) - d - 2$, and $\mathrm{sr}(T) = \mathrm{sr}(v) + t$. Moreover t, $w_1, \ldots, w_t$ and $\langle v_1, v_2 \rangle$ are uniquely determined by T.*

Observe that the variables L_1, L_2 in Corollary 1 and the vectors v_1, v_2 in Corollary 2 correspond to the line $\ell \subset \mathbb{P}^m$ such that $C_d := \nu_d(\ell)$ is the rational normal curve introduced in Theorem 1. Moreover the integer t in Corollaries 1 and 2 is $\sharp(\mathcal{S}_2)$ where $\mathcal{S}_2$ is as in Theorem 1.

The decompositions $Q = R_1^d + \cdots + R_{r'}^d$ with $R_i \in K[L_1, L_2]_1$, are not unique (analogously the decompositions $v = u_1^{\otimes d} + \cdots + u_{r'}^{\otimes d}$ with $u_i \in \langle v_1, v_2 \rangle$), but one of them may be found using Sylvester's algorithm or any of the available algorithms ([11], [16], [3]). Unfortunately, given F as in Corollary 1 (T as in Corollary 2 respectively) we do not have any explicit algorithm to find $M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_d$ and hence $Q \in K[L_1, L_2]_d$ ($w_1, \ldots, w_t \in S^1 V^*$ and $v \in S^d(\langle v_1, v_2 \rangle)$) respectively).

Using Theorem 1 and a related lemma (Lemma 3) it is also possible to address the question on the uniqueness of the decomposition (1).

Theorem 2. *Assume $d \geq 5$. Fix a finite set $B \subset \mathbb{P}^m$ such that $\rho := \sharp(B) \leq d$ and no subset of it with cardinality $\lfloor (d+1)/2 \rfloor$ is collinear. Fix $P \in \langle \nu_d(B) \rangle$ such that $P \notin \langle \mathcal{E} \rangle$ for any $\mathcal{E} \subsetneq \nu_d(B)$. Then $\mathrm{sr}(P) = \mathrm{sbr}(P) = \rho$ and $\nu_d(B)$ is the only 0-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $\deg(\mathcal{Z}) \leq \rho$ and $P \in \langle \mathcal{Z} \rangle$.*

Unfortunately, for a given $P \in \mathbb{P}^N$ that satisfies the hypothesis of Theorem 2 we are not able to give explicitly the set B. Knowing the uniqueness of a decomposition is very interesting both from the applications and the pure mathematical point of view, but very few results are known. Theorem 2 is an extension of [6] with an additional assumption. It is worth noting that without some additional assumption [6], Theorem 1.2.6, cannot be extended (e.g., it is sharp when $m = 1$). We give an example showing that if $m = 2$, then Theorem 2 is sharp (see Example 2), even taking B in linearly general position.

1. Preliminaries

In this section we prove two auxiliary lemmas that will be crucial in the proof of the main result of this paper. Theorems 1 and 2 are well-known if $m = 1$ since Sylvester. Hence we may assume that $m \geq 2$.

Definition 1. We say that a smoothable 0-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ computes the symmetric border rank $\mathrm{sbr}(P)$ of $P \in \mathbb{P}^N$ if $\deg(\mathcal{Z}) = \mathrm{sbr}(P)$ and $P \in \langle \mathcal{Z} \rangle$. A reduced 0-dimensional scheme $\mathcal{S} \subset X_{m,d}$ computes the symmetric rank $\mathrm{sr}(P)$ of $P \in \mathbb{P}^N$ if $\sharp(\mathcal{S}) = \mathrm{sr}(P)$ and $P \in \langle \mathcal{S} \rangle$.

By the definition of symmetric rank, if $\mathcal{S}$ computes $\mathrm{sr}(P)$, then $P \notin \langle \mathcal{S}' \rangle$ for any reduced 0-dimensional scheme $\mathcal{S}' \subset X_{m,d}$ with $\deg(\mathcal{S}') < \deg(\mathcal{S})$. Hence $\mathcal{S}$ is linearly independent.

Lemma 1. *Fix any $P \in \mathbb{P}^r$ and two 0-dimensional subschemes A, B of $\mathbb{P}^r$ such that $A \neq B$, $P \in \langle A \rangle$, $P \in \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup B}(1)) > 0$.*

Proof. Since A and B are 0-dimensional, $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup B}(1)) \geq \max\{h^1(\mathbb{P}^r, \mathcal{I}_A(1)), h^1(\mathbb{P}^r, \mathcal{I}_B(1))\}$. Thus we may assume $h^1(\mathbb{P}^r, \mathcal{I}_A(1)) = h^1(\mathbb{P}^r, \mathcal{I}_B(1)) = 0$, i.e. $\dim(\langle A \rangle) = \deg(A) - 1$ and $\dim(\langle B \rangle) = \deg(B) - 1$. Set $D := A \cap B$ (scheme-theoretic intersection). Thus $\deg(A \cup B) = \deg(A) + \deg(B) - \deg(D)$. Since $D \subseteq A$ and A is linearly independent, we have $\dim(\langle D \rangle) = \deg(D) - 1$. Since $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup B}(1)) > 0$ if and only if $\dim(\langle A \cup B \rangle) \leq \deg(A \cup B) - 2$, we get $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup B}(1)) > 0$ if and only if $\langle D \rangle \subsetneq \langle A \rangle \cap \langle B \rangle$. Since $A \neq B$, then $D \subsetneq A$. Hence $P \notin \langle D \rangle$. Since $P \in \langle A \rangle \cap \langle B \rangle$, we are done. $\square$

The next observation shows the existence of the scheme $\mathcal{Z} \subset X_{m,d}$ that computes the symmetric border rank of a point $P \in \mathbb{P}^N$ that satisfies the conditions of Theorem 1.

Remark 1. Fix integers $m \geq 1$, $d \geq 2$ and $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) \leq d+1$. By [6], Lemma 2.1.5, or [3], Proposition 11, there is a smoothable 0-dimensional scheme $\mathcal{E} \subset X_{m,d}$ such that $\deg(\mathcal{E}) \leq \mathrm{sbr}(P)$ and $P \in \langle \mathcal{E} \rangle$. Moreover, $\mathrm{sbr}(P)$ is the minimal of the degrees of any such smoothable scheme $\mathcal{E}$.

In the statement of Theorem 1 we claimed the existence of a unique rational normal curve $C_d \subset X_{m,d}$ such that $\deg((\mathcal{S} \cup \mathcal{Z}) \cap C_d) \geq d + 2$. This will be a consequence of the following lemma where the line $\ell \subset \mathbb{P}^m$ and the scheme $W \subset \mathbb{P}^m$ will be used in the proof of Theorem 1 with $\nu_d(\ell) = C_d$, while as $\nu_d(W)$ we will take several different schemes associated to $\mathcal{S} \cup \mathcal{Z}$.

Lemma 2. *Fix an integer $x \geq 1$. Let $W \subset \mathbb{P}^m$, $m \geq 2$, be a 0-dimensional scheme of degree $\deg(W) \leq 2x + 1$ and such that $h^1(\mathbb{P}^m, \mathcal{I}_W(x)) > 0$. Then there is a unique line $\ell \subset \mathbb{P}^m$ such that $\deg(\ell \cap W) \geq x + 2$ and*

$$\deg(W \cap \ell) = x + 1 + h^1(\mathbb{P}^m, \mathcal{I}_W(x)).$$

Proof. For the existence of the line $\ell \subset \mathbb{P}^m$ see [3], Lemma 34.
Since $\deg(W) \leq 2x + 1$ and since the scheme-theoretic intersection of two different lines has length at most one and $\deg(W) \leq 2x + 2$, there is no line $R \neq \ell$ such that $\deg(R \cap W) \geq x + 2$. Thus ℓ is unique.
We prove the formula $\deg(W \cap \ell) = x + 1 + h^1(\mathcal{I}_W(x))$ by induction on m.

First assume $m = 2$. In this case ℓ is a Cartier divisor of $\mathbb{P}^m$. Hence the residual scheme $\mathrm{Res}_\ell(W)$ of W with respect to ℓ has degree $\deg(\mathrm{Res}_\ell(W)) = \deg(W) - \deg(W \cap \ell)$. The exact sequence that defines the residual scheme $\mathrm{Res}_\ell(W)$ is:

$$(2) \qquad 0 \to \mathcal{I}_{\mathrm{Res}_\ell(W)}(x - 1) \to \mathcal{I}_W(x) \to \mathcal{I}_{W \cap \ell, \ell}(x) \to 0.$$

Since $\dim(\mathrm{Res}_\ell(W)) \leq \dim(W) \leq 0$ and $x - 1 \geq -2$, we have $h^2(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_\ell(W)}(x - 1)) = 0$. Since $\deg(W \cap \ell) \geq x + 1$, we have $h^0(\ell, \mathcal{I}_{W \cap \ell}(x)) = 0$. Since $\deg(\mathrm{Res}_\ell(W)) = \deg(W) - \deg(W \cap \ell) \leq x$, we obviously have $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_\ell(W)}(x - 1)) = 0$ (this is also a particular case of [3], Lemma 34). Thus the cohomology exact sequence of (2) gives $h^1(\mathbb{P}^m, \mathcal{I}_W(x)) = \deg(W \cap \ell) - x - 1$. This proves the lemma for $m = 2$.

Now assume $m \geq 3$ and that the result is true for $\mathbb{P}^{m-1}$. Take a general hyperplane $H \subset \mathbb{P}^m$ containing ℓ and set $W' := W \cap \ell$. The inductive assumption gives $h^1(H, \mathcal{I}_{W'}(x)) = \deg(W' \cap \ell) - x - 1$. Since $\deg(\mathrm{Res}_H(W)) \leq x - 1$, we get, as above, $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_H(W)}(x - 1)) = 0$. Consider now the analogue exact sequence of (2) using H instead of ℓ:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(W)}(x - 1) \to \mathcal{I}_W(x) \to \mathcal{I}_{W \cap H, H}(x) \to 0.$$

Since $W \cap \ell = W' \cap \ell$, we get, as above, that $h^1(\mathbb{P}^m, \mathcal{I}_W(x)) = \deg(W \cap \ell) - x - 1$. $\square$

2. THE PROOFS

In this section we prove Theorems 1 and 2.

Proof of Theorem 1. The existence of the smoothable scheme $\mathcal{Z} \subset X_{m,d}$ that computes $\mathrm{sbr}(P)$ is assured by Remark 1. Any such smoothable scheme has degree $\mathrm{sbr}(P)$ (Remark 1). Let S (resp. Z) be the only subset (resp. subscheme) of $\mathbb{P}^m$ such that $\mathcal{S} = \nu_d(S)$ (resp. $\mathcal{Z} = \nu_d(Z)$). By hypothesis $\sharp(S) = \mathrm{sr}(P)$ and $\deg(Z) = \mathrm{sbr}(P)$. Set $W := S \cup Z$ and $\mathcal{W} := \nu_d(W)$. We have $\deg(W) = \mathrm{sr}(P) + \mathrm{sbr}(P) \leq 2d + 1$. Let $\mathcal{T}$ be a minimal subscheme of $\mathcal{Z}$ such that $P \in \langle \mathcal{T} \rangle$. Since $\deg(\mathcal{T}) \leq \deg(\mathcal{Z}) < \deg(\mathcal{S})$, we have $\mathcal{T} \neq \mathcal{S}$. Lemma 1 applied to $r := N$, $A := \mathcal{T}$ and $B := \mathcal{S}$ gives $h^1(\mathcal{I}_{\mathcal{T} \cup \mathcal{S}}(1)) > 0$. Thus $h^1(\mathcal{I}_{\mathcal{W}}(1)) > 0$. Thus $\dim(\langle \mathcal{W} \rangle) \leq \deg(\mathcal{W}) - 2$. Since $\deg(\mathcal{W}) \leq \deg(\mathcal{Z}) + \deg(\mathcal{S}) = \mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1$ and $h^1(\mathcal{I}_{\mathcal{W}}(1)) = h^1(\mathbb{P}^m, \mathcal{I}_W(d))$, there is a unique line $\ell \subset \mathbb{P}^m$ whose image $C_d := \nu_d(\ell)$ in $X_{m,d}$ contains a subscheme of $\mathcal{W}$ with length at least $d + 2$ (Lemma 2). Since $C_d = \langle C_d \rangle \cap X_{m,d}$ (scheme-theoretic intersection), we have $\mathcal{W} \cap C_d = \nu_d(W \cap \ell)$, $\mathcal{Z} \cap C_d = \nu_d(Z \cap \ell)$ and $\mathcal{S} \cap C_d = \nu_d(S \cap \ell)$.

(a) Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ be as defined in the statement and set $\mathcal{S}_3 := \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$. Let $S_3 \subset \mathbb{P}^m$ be the only subset such that $\mathcal{S}_3 = \nu_d(S_3)$. Set $W' := W \setminus S_3$ and $\mathcal{W}' := \nu_d(W') = \mathcal{W} \setminus \mathcal{S}_3$. Notice that W' is well-defined, because each point of S_3 is a connected component of the scheme W.

In this step we prove $S_3 = \emptyset$, i.e. $\mathcal{S}_3 = \emptyset$.

Assume that this is not the case and that $\sharp(\mathcal{S}_3) > 0$. Lemma 2 gives $h^1(\mathbb{P}^m, \mathcal{I}_{W \cap \ell}(d)) = h^1(\mathbb{P}^N, \mathcal{I}_{\mathcal{W}}(1))$ and $h^0(\mathcal{I}_{\mathcal{W}}(1)) = h^0(\mathcal{I}_{C_d \cap \mathcal{W}}(1)) - \deg(\mathcal{W}) + \deg(\mathcal{W} \cap C_d)$. Hence we get

$$\dim(\langle \mathcal{W} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \sharp(\mathcal{S}_3).$$

Now, by definition, we have that $\mathcal{S} \cap \mathcal{W}' = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{W} = \mathcal{W}' \sqcup \mathcal{S}_3$ and $\mathcal{Z} \cup \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{W}'$. Grassmann's formula gives $\dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \dim(\langle \mathcal{S} \rangle) - \dim(\langle \mathcal{W}' \cup \mathcal{S} \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$. Since $\mathcal{S}$ is linearly independent, we have $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$. Hence $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle)$; since $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle \subseteq \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$ we get $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle = \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$. Since $P \in \langle \mathcal{Z} \rangle \cap \langle \mathcal{S} \rangle \subseteq \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle = \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$, we get that $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$. Since we supposed that $\mathcal{S} \subset X_{m,d}$ is a set computing the symmetric rank of P, it is absurd that P belongs to the span of a proper subset of $\mathcal{S}$, then necessarily $\sharp(\mathcal{S}_3) = 0$, that is equivalent to the fact that $\mathcal{S}_3 = \emptyset$. Thus in this step we have just proved $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$.

In steps (b), (c) and (d) we will prove $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$ in a very similar way (using Z instead of S). In each of these steps we take a subscheme $W_2 \subset W$ such that $S \subset W_2$, $W_2 \cap \ell = W \cap \ell$ and $W_2 \cup Z = W$. Then we play with Lemma 2. In steps (b) (resp. (c), resp. (d)) we call $W_2 = W''$ (resp. $W_2 = W_Q$, resp. $W_2 = W_1$). Since $\deg(\nu_d(Z)) \le d + 1$, the scheme $\nu_d(Z)$ is linearly independent.

(b) Let $Z_4 \subset \mathbb{P}^n$ be the union of the connected components of Z which do not intersect $\ell \cup S_2$. Here we prove $Z_4 = \emptyset$. Set $W'' := W \setminus Z_4$. The scheme W'' is well-defined, because Z_4 is a union of some of the connected components of W. Lemma 2 gives $\dim(\langle \nu_d(W) \rangle) = \dim(\langle \nu_d(W'') \rangle) + \deg(Z_4)$. Since $W = W'' \cup Z$, Grassmann's formula gives $\dim(\langle \nu_d(W'' \cup Z) \rangle) = \dim(\langle \nu_d(W'') \rangle) + \dim(\langle \nu_d(Z) \rangle) - \dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle)$. Thus $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) + \deg(Z_4)$. Since $\nu_d(Z)$ is linearly independent and $Z = (Z \cap W'') \sqcup Z_4$, we get $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W'') \rangle) + \deg(Z_4)$. Thus $\dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W'') \rangle)$. Since $\nu_d(W'' \cap Z) \subseteq \langle \nu_d(W'') \cap \nu_d(Z) \rangle$, $\deg(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(W'' \cap Z) \rangle) + 1$, and $\nu_d(W'')$ is linearly independent, then the linear space $\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle$ is spanned by $\nu_d(W'' \cap Z)$. Since $S \subseteq W''$ and $P \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(S) \rangle$, we have $P \in \langle \nu_d(W'' \cap Z) \rangle$. Since $\nu_d(Z)$ computes $\text{sbr}(P)$, we get $W'' \cap Z = Z$, i.e. $Z_4 = \emptyset$.

(c) Here we prove that each point of S_2 is a connected component of Z. Fix $Q \in S_2$ and call Z_Q the connected component of Z such that $(Z_Q)_{red} = \{Q\}$. Set $Z[Q] := (Z \setminus Z_Q) \cup \{Q\}$ and $W_Q := (W \setminus Z_Q) \cup \{Q\}$. Since Z_Q is a connected component of W, the schemes $Z[Q]$ and W_Q are well-defined. Assume $Z_Q \ne \{Q\}$, i.e. $W_Q \ne W$, i.e. $Z[Q] \ne Z$. Since $W_Q \cap \ell = W \cap \ell$, Lemma 2 gives $\dim(\langle \nu_d(W) \rangle) - \dim(\langle \nu_d(W_Q) \rangle) = \deg(Z_Q) - 1 > 0$. Since $\nu_d(Z)$ is linearly independent, we have $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z[Q]) \rangle) + \deg(Z_Q) - 1$. Grassmann's formula gives $\dim(\langle \nu_d(Z[Q]) \rangle) = \dim(\langle \nu_d(W_Q) \rangle \cap \langle \nu_d(Z) \rangle)$. Since $\langle \nu_d(Z[Q]) \rangle \subseteq \langle \nu_d(W_Q) \rangle \cap \langle \nu_d(Z) \rangle$ and $Z[Q]$ is linearly independent, we get $\langle \nu_d(Z[Q]) \rangle = \langle \nu_d(W_Q) \rangle \cap \langle \nu_d(Z) \rangle$. Since $Q \in S_2 \subseteq S$, we have $S \subset W_Q$. Thus $P \in \langle \nu_d(W_Q) \rangle$. Thus $P \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(W_Q) \rangle = \langle \nu_d(Z[Q]) \rangle$. Since $\mathcal{Z}$ computes $\text{sbr}(P)$, $Z[Q] \subseteq Z$ and $P \in \langle \nu_d(Z[Q]) \rangle$, we get $Z[Q] = Z$. Thus each point of $\mathcal{S}_2$ is a connected component of $\mathcal{Z}$.

(d) To conclude that $Z = (Z \cap \ell) \sqcup S_2$ it is sufficient to prove that every connected component of Z whose support is a point of ℓ is contained in ℓ. Set $\eta := \deg(Z \cap \ell)$ and call μ the sum of the

degrees of the connected components of Z whose support is contained in ℓ.

Set $W_1 := (W \cap \ell) \cup S_2$. Notice that $\deg(W_1) = \deg(W) + \eta - \mu$. Lemma 2 gives $\dim(\langle \nu_d(W_1) \rangle) = \dim(\langle \nu_d(W) \rangle) + \eta - \mu$. Since $W = W_1 \cup Z$, Grassmann's formula gives $\dim(\langle \nu_d(W_1 \cup Z) \rangle) = \dim(\langle \nu_d(W_1) \rangle) + \dim(\langle \nu_d(Z) \rangle) - \dim(\langle \nu_d(W_1) \rangle \cap \langle \nu_d(Z) \rangle)$. Thus $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(W_1) \rangle \cap \langle \nu_d(Z) \rangle) + \mu - \eta$. Notice that $Z \cap W_1 = (Z \cap \ell) \sqcup S_2$, i.e. $\deg(Z \cap W_1) = \deg(Z) - \eta + \mu$. Since $\nu_d(Z)$ is linearly independent, we get $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W_1) \rangle) + \mu - \eta$. Thus $\dim(\langle \nu_d(W_1) \rangle \cap \langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W_1) \rangle)$, i.e. $\langle \nu_d(W_1) \rangle \cap \langle \nu_d(Z) \rangle$ is spanned by $\nu_d(W_1 \cap Z)$. Since $S \subset W_1$ and $P \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(S) \rangle$, we have $P \in \langle \nu_d(W_1 \cap Z) \rangle$. Since Z computes the symmetric border rank of P, we get $W_1 \cap Z = Z$, i.e. $\eta = \mu$. Together with steps (b) and (c) we get $Z = (Z \cap \ell) \sqcup S_2$. Thus from steps (b), (c) and (d) we get $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$.

(e) Here we prove the uniqueness of the rational normal curve C_d. Notice that ℓ and $C_d = \nu_d(\ell)$ are uniquely determined by the choice of a pair (Z, S) with $\nu_d(Z)$ computing $\mathrm{sbr}(P)$ and $\nu_d(S)$ computing $\mathrm{sr}(P)$. Fix another pair (Z', S') with $\nu_d(Z')$ computing $\mathrm{sbr}(P)$ and $\nu_d(S')$ computing $\mathrm{sr}(P)$. Let ℓ' be the line associated to $Z' \cup S'$. Assume $\ell' \neq \ell$. First assume $S' = S$. The part of Theorem 1 proved before gives $Z = Z_1 \sqcup S_2$, $Z' = Z_1' \sqcup S_2'$ and $S = S_1' \sqcup S_2'$ with $Z_1 = Z \cap \ell$, $Z_1' = Z' \cap \ell'$, $S_1 = S \cap \ell$ and $S_1' = S_1 \cap \ell'$. Now $\mathrm{sbr}(P) = \deg(Z_1) + \sharp(S_2) = \deg(Z_1') + \sharp(S_2')$, $\mathrm{sr}(P) = \deg(S_1) + \sharp(S_2) = \deg(S_1') + \sharp(S_2')$, $\deg(S_1) > \deg(Z_1)$, $\deg(S_1) + \deg(Z_1) \geq d + 2$ and $\deg(S_1') + \deg(Z_1') \geq d + 2$. Since $\ell' \neq \ell$, at most one of the points of S_1 may be contained in ℓ' and at most one of the points of S_1' may be contained in ℓ. Thus $\deg(S_1') - 1 \leq \sharp(S_2)$ and $\deg(S_1) - 1 \leq \sharp(S_2')$. Since $\deg(S_1) + \deg(Z_1) + 2(\sharp(S_2)) = \deg(S_1') + \deg(Z_1') + 2(\sharp(S_2')) \leq 2d + 1$, $\deg(S_1) + \deg(Z_1) \geq d + 2$ and $\deg(S_1') + \deg(Z_1') \geq d + 2$, we get $2(\sharp(S_2)) \leq d - 1$ and $2(\sharp(S_2')) \leq d - 1$. Since $\deg(S_1) + \deg(Z_1) \geq d + 2$ and $\deg(S_1) > \deg(Z_1)$, we have $\deg(S_1) \geq (d + 3)/2$. Hence $\deg(S_1) - 1 \geq (d + 1)/2 > (d - 1)/2 \geq \sharp(S_2')$, contradiction. Thus all pairs (Z', S) give the same line ℓ. Now assume $S' \neq S$. Call ℓ'' the line associated to the pair (Z, S'). The part of Theorem 1 proved in the previous steps gives that ℓ is the only line containing an unreduced connected component of Z. Thus $\ell'' = \ell$. Since we proved that the lines associated to (Z', S') and (Z, S') are the same, we are done.

(f) Here we prove the uniqueness of $\mathcal{S}_2$. Take any pair (Z', S') with $\nu_d(Z')$ computing $\mathrm{sbr}(P)$ and $\nu_d(S')$ computing $\mathrm{sr}(P)$. By step (e) the same line ℓ is associated to any pair (Z'', S'') as above. Hence the set $S_2' := S' \setminus (S' \cap \ell)$ associated to the pair (Z, S') is the union of the connected components of Z not contained in ℓ. Thus $S_2' = S \setminus S \cap \ell = S_2$. We apply the part of Theorem 1 proved in steps (a), (b), (c) and (d) to the pair (Z', S). We get that $S \setminus S \cap \ell$ is the union of the connected components of Z' not contained in ℓ. Applying the same part of Theorem 1 to the pair (Z', S') we get $S' \setminus S' \cap \ell = S \setminus S \cap \ell$, concluding the proof of the uniqueness of $\mathcal{S}_2$. $\square$

The following example shows that the assumption " $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1$ " in Theorem 1 is sharp.

Example 1. Fix integers $m \geq 2$ and $d \geq 4$. Let $C \subset \mathbb{P}^m$ be a smooth conic. Let $Z \subset C$ be any unreduced degree 3 subscheme. Set $\mathcal{Z} := \nu_d(Z)$. Since $d \geq 2$, then $\mathcal{Z}$ is linearly independent. Since $\mathcal{Z}$ is curvilinear, it has only finitely many degree 2 subschemes. Thus the plane $\langle \mathcal{Z} \rangle$ contains only finitely many lines spanned by a degree 2 subscheme of $\mathcal{Z}$. Fix any $P \in \langle \mathcal{Z} \rangle$ not contained in one of these lines. Remark 1 gives $\mathrm{sbr}(P) = 3$. The proof of [3], Theorem 4, gives $\mathrm{sr}(P) = 2d - 1$ and the existence of a set $S \subset C$ such that $\sharp(S) = 2d - 1$, $S \cap Z = \emptyset$ and $\nu_d(S)$ computes $\mathrm{sr}(P)$. We have $\mathrm{sbr}(P) + \mathrm{sr}(P) = 2d + 2$.

Lemma 3. *Fix $P \in \mathbb{P}^N$ such that $\rho := \mathrm{sbr}(P) = \mathrm{sr}(P) \leq d$. Let Ψ be the set of all 0-dimensional schemes $A \subset \mathbb{P}^m$ such that $\deg(A) = \rho$ and $P \in \langle \nu_d(A) \rangle$. Assume $\sharp(\Psi) \geq 2$. Fix any $A \in \Psi$. Then there is a line $\ell \subset \mathbb{P}^m$ such that $\deg(\ell \cap A) \geq (d+2)/2$.*

Proof. Since $\mathrm{sr}(P) = \rho$ and $\sharp(\Psi) \geq 2$, there is $B \in \Psi$ such that $B \neq A$ and at least one among the schemes A and B is reduced. Since $\deg(A \cup B) \leq 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{A \cup B}(d)) > 0$, there is a line $\ell \subset \mathbb{P}^m$ such that $\deg((A \cup B) \cap \ell) \geq d + 2$. We may repeat verbatim the proof of Theorem 1, because it does not use the inequality $\deg(A) < \deg(B)$, but only that $\deg(\mathcal{Z}) \leq \deg(\mathcal{S})$ and $\mathcal{Z} \neq \mathcal{S}$ (if $\mathcal{T} \neq \mathcal{Z}$, then $\deg(\mathcal{T}) < \deg(\mathcal{Z}) \leq \deg(\mathcal{S})$ and hence $\mathcal{T} \neq \mathcal{S}$). We get $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup A_2$ with A_2 reduced, $A_2 \cap \ell = \emptyset$ and $A_1 \cup B_1 \subset \ell$. Since $\deg(A) = \deg(B)$, we have $\deg(A_1) = \deg(B_1)$. Thus $\deg(A_1) \geq (d+2)/2$. $\qquad\qquad\square$

Proof of Theorem 2. Since $\mathrm{sbr}(P) \leq \rho \leq d$, the border rank is the minimal degree of a smoothable 0-dimensional scheme $\mathcal{A} \subset X_{m,d}$ such that $P \in \langle \mathcal{A} \rangle$ (Remark 1). Thus it is sufficient to prove the last assertion. Assume the existence of a 0-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $z := \deg(\mathcal{Z}) \leq \rho$ and $P \in \langle \mathcal{Z} \rangle$. If $z = \rho$ we also assume $\mathcal{Z} \neq \nu_d(B)$. Taking z minimal, we may also assume $z \leq \mathrm{sbr}(P)$. Let $Z \subset \mathbb{P}^m$ be the only scheme such that $\nu_d(Z) = \mathcal{Z}$. If $z < \rho$ we apply a small part of the proof of Theorem 1 to the pair $(\mathcal{Z}, \nu_d(B))$ (we just use or reprove that $\deg((Z \cup B) \cap \ell) \geq d + 2$ and that $\deg(B \cap \ell) = \deg(Z \cap \ell) + \rho - z \geq \deg(Z \cap \ell))$. We get a contradiction: indeed $B \cap \ell$ must have degree $\geq (d+1)/2$, contradiction. If $z = \rho$, then we use Lemma 3. $\qquad\qquad\square$

Example 2. Assume $m = 2$ and $d \geq 4$. Let $C \subset \mathbb{P}^2$ be a smooth conic. Fix sets $S, S' \subset C$ such that $\sharp(S) = \sharp(S') = d + 1$ and $S \cap S' = \emptyset$. Since no 3 points of C are collinear, the sets S, S' and $S \cup S'$ are in linearly general position. Since $h^0(C, \mathcal{O}_C(d)) = 2d + 1$ and C is projectively normal, we have $h^1(\mathbb{P}^2, \mathcal{I}_S(d)) = h^1(\mathbb{P}^2, \mathcal{I}_{S'}(d)) = 0$ and $h^1(\mathbb{P}^2, \mathcal{I}_{S \cup S'}(d)) = 1$. Thus $\nu_d(S)$ and $\nu_d(S')$ are linearly independent and $\langle \nu_d(S) \rangle \cap \langle \nu_d(S') \rangle$ is a unique point. Call P this point. Obviously $\mathrm{sr}(P) \leq d + 1$. In order to get the example claimed in the Introduction after the statement of Theorem 2, it is sufficient to prove that $\mathrm{sbr}(P) \geq d+1$. Assume $\mathrm{sbr}(P) \leq d$ and take Z computing $\mathrm{sbr}(P)$. We may apply a small part of the proof of Theorem 1 to P, S, Z (even if a priori S may not compute $\mathrm{sr}(P)$). We get the existence of a line ℓ such that $\deg(Z \cap \ell) < \sharp(S \cap \ell)$ and $\deg(Z \cap \ell) + \sharp(S \cap \ell) \geq d + 2$. Since $d \geq 4$, we get $\sharp(S \cap \ell) \geq 3$, that is a contradiction.

We do not have experimental evidence to raise the following question (see [3] for the cases with $\mathrm{sbr}(P) \leq 3$).

Question 1. Is it true that $\mathrm{sr}(P) \leq d(\mathrm{sbr}(P) - 1)$ for all $P \in \mathbb{P}^N$ and that equality holds if and only if $P \in TX_{m,d} \setminus X_{m,d}$ where $TX_{m,d} \subset \mathbb{P}^N$ is the tangential variety of the Veronese variety $X_{m,d}$?

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