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OPTIMAL HEDGING IN DISCRETE TIME

BRUNO RÉMILLARD AND SYLVAIN RUBENTHALER

ABSTRACT. Building on the work of Schweizer (1995) and Černý and Kallsen (2007), we present discrete time formulas minimizing the mean square hedging error for multidimensional assets. In particular, we give explicit formulas when a regime-switching random walk or a GARCH-type process is utilized to model the returns. Monte Carlo simulations are used to compare the optimal and delta hedging methods.

Hedging, option pricing, GARCH, regime-switching

1. INTRODUCTION

In many applications, one is interested in finding a discrete-time dynamically traded portfolio so that its value at maturity is as close as possible to a target function of the underlying assets. When the target function is a payoff, this can be interpreted as option pricing and hedging. However, sometimes the target function is not a payoff as it happens when one tries to replicate hedge funds or create synthetic funds (Papageorgiou et al., 2008). Assuming that the error measure is the average quadratic hedging error, Schweizer (1995) solved the hedging problem for one risky asset. He showed that the initial value of the portfolio, which can be interpreted as the “value” of the option, is the average, under the “real probability measure”, of the discounted payoff, multiplied by a martingale, which is not necessarily positive. In the latter case, the martingale cannot be used as the density of an equivalent martingale measure.

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Even if the hedging problem has been solved quite generally by Schweizer (1995) in the one-dimensional case, it seems to have been ignored or forgotten, e.g., (Bouchaud and Potters, 2002) or Cornalba et al. (2002). More troubling, delta hedging, based on the Black-Scholes-Merton model, is still used in practice even if it has been shown that the geometric Brownian motion model is an inadequate model for the underlying assets (Kat and Palaro, 2005). Even when the geometric Brownian motion model is adequate, the hedging error in discrete time is not zero, converging only to zero as the number of hedging periods tends to infinity. See, e.g., Boyle and Emanuel (1980), Wilmott (2006, Chapters 46-47). In addition, when the market is not complete but there is no arbitrage, there are infinitely many equivalent martingale measures (EMM). One then has to choose the “best” martingale measure with respect to some utility criterion. There is a huge literature on this subject. Indeed, when a NGARCH process is utilized to model the log-returns, Duan (1995) proposes a solution to the EMM problem. Unfortunately, Duan also suggests a delta hedging strategy, which has been shown to be wrong by Garcia and Renault (1998).

Motivated by applications in hedge fund replication, Papageorgiou et al. (2008) proposed a locally optimal solution minimizing the average quadratic hedging error at each period for the general multidimensional asset case. They erroneously claimed that it was globally optimal, which is only true if the discounted underlying assets are martingales. A first motivation for the present paper is to correct that mistake and give explicit formulas for the results in Černý and Kallsen (2007) and generalizing those of Schweizer (1995). A second motivation is to show that, when regime-switching random walks and GARCH processes are used to model the returns, the optimal solution of the hedging problem yields superior results to those obtained by delta hedging.

The optimal solution of the discrete time hedging problem is described in Section 2, giving explicit expressions for the results of Černý and Kallsen (2007). It is worth

noting that when the asset value process is Markovian, or a component of a Markov process, the optimal solution can be implemented using approximation techniques of dynamic programming. Two such cases are considered. Finally, in Section 3, simulations are used to compare optimal hedging with delta hedging for geometric random walks and NGARCH models.

2. OPTIMAL HEDGING STRATEGY IN DISCRETE TIME

Denote the price process by S , i.e., S_k is the value of the d underlying assets at period k and let $\mathbb{F} = \{\mathcal{F}_k, k = 0, \dots, n\}$ be a filtration under which S is adapted. Assume that S is square integrable. Set $\Delta_k = \beta_k S_k - \beta_{k-1} S_{k-1}$, where the discounting factors β_k are predictable, i.e. β_k is \mathcal{F}_{k-1} -measurable for $k = 1, \dots, n$.

The aim of this section is to find an initial investment amount V_0 and a predictable investment strategy $\vec{\phi} = (\phi_k)_{k=1}^n$ such that $\phi_k^\top \Delta_k$ is square integrable and which minimizes the expected quadratic hedging error $E \left[\left\{ G \left(V_0, \vec{\phi} \right) \right\}^2 \right]$, where $G = G \left(V_0, \vec{\phi} \right) = \beta_n C - V_n$, and $V_k = V_0 + \sum_{j=1}^k \phi_j^\top \Delta_j$, $k = 0, \dots, n$.

The existence of an optimal solution was proven first in the univariate case by Schweizer (1995). Motoczyński (2000) considered a multivariate setting without furnishing explicit solutions. Finally, Černý and Kallsen (2007) treated a much more general case. However, in the discrete time case, it is faster and easier to find directly the explicit formulas than trying to recover them from their results.

2.1. Offline computations. Once a dynamic model is chosen for the asset prices, one must start with some computations that are necessary for the implementation. Set $P_{n+1} = 1$, $\gamma_{n+1} = 1$, and for $k = n, \dots, 1$, define $A_k = E \left(\Delta_k \Delta_k^\top P_{k+1} | \mathcal{F}_{k-1} \right)$, $\mu_k = E \left(\Delta_k P_{k+1} | \mathcal{F}_{k-1} \right)$, $b_k = A_k^{-1} \mu_k$, $P_k = \prod_{j=k}^n \left(1 - b_j^\top \Delta_j \right)$, and $\gamma_k = E \left(P_k | \mathcal{F}_{k-1} \right)$, provided these expressions exist. Under some extra assumptions given below, it can be shown that they are indeed well defined. The proof is given in Appendix A.1.

Lemma 2.1.1. *Suppose that $E(\gamma_{k+1}|\mathcal{F}_{k-1})A_k - \mu_k\mu_k^\top$ is invertible P -a.s., for every $k = n, \dots, 1$. Then $\gamma_k \in (0, 1]$ and A_k is invertible for all $k = 1, \dots, n$. In addition $(\gamma_{k+1})_{k=0}^n$ is a positive submartingale.*

Remark 2.1.2. *In the univariate case, Schweizer (1995) states sufficient conditions for the validity of the assumptions of Lemma 2.1.1. It is not obvious how they could be generalized to the multivariate case. Therefore, in most applications, one has to verify these conditions, often using brute force calculations.*

2.2. Optimal solution of the hedging problem.

Theorem 2.2.1. *Under the assumptions of Lemma 2.1.1, the solution $(V_0, \vec{\phi})$ of the minimization problem is $V_0 = E(\beta_n C P_1)/\gamma_1$, and $\phi_k = \alpha_k - V_{k-1}b_k$, where $\alpha_k = A_k^{-1}E(\beta_n C \Delta_k P_{k+1}|\mathcal{F}_{k-1})$, $k = n, \dots, 1$.*

The proof is given in Appendix A.2.

2.2.1. *Option value.* Let C_k be the optimal investment at period k , so that the value of the portfolio at period n is as close as possible to C , in terms of mean square error. One could then interpret C_k as the value of the option at period k , for any $k = 0, \dots, n$. It then follows from Theorem 2.2.1 that C_k is given by

$$(2.2.1) \quad \beta_k C_k = \frac{E(\beta_n C P_{k+1}|\mathcal{F}_k)}{E(P_{k+1}|\mathcal{F}_k)}, \quad k = 0, \dots, n,$$

so one can write

$$(2.2.2) \quad \beta_{k-1} C_{k-1} = \frac{1}{\gamma_k} E \{ \beta_k C_k (1 - b_k^\top \Delta_k) \gamma_{k+1} | \mathcal{F}_{k-1} \} = E(\beta_n C \mathcal{U}_k \cdots \mathcal{U}_n | \mathcal{F}_{k-1}),$$

where $\mathcal{U}_k = \frac{E(P_k|\mathcal{F}_k)}{E(P_k|\mathcal{F}_{k-1})}$, $k = 1, \dots, n+1$, while an alternative expression for α_k is

$$(2.2.3) \quad \alpha_k = A_k^{-1} E(\beta_k C_k \Delta_k \gamma_{k+1} | \mathcal{F}_{k-1}).$$

Remark 2.2.2. *Setting $\mathcal{Z}_0 = 1$ and $\mathcal{Z}_k = \prod_{j=1}^k \mathcal{U}_j$, $k = 1, \dots, n$, one obtains that $(\mathcal{Z}_k, \beta_k C_k \mathcal{Z}_k, \beta_k S_k \mathcal{Z}_k)_{k=0}^n$ are martingales. However, in most applications, \mathcal{Z} does not define a change of measure unless it takes only positive values.*

2.2.2. Implementation issues. If the process S is Markov and $C_n = C_n(S_n)$, then $C_k = C_k(S_k)$, $\alpha_k = \alpha_k(S_{k-1})$, and $b_k = b_k(S_{k-1})$. It follows that all these functions can be approximated using the methodology developed in Papageorgiou et al. (2008). Another interesting case is when S_k is not a Markov process but (S_k, h_k) is, even if h_k is not observable, as in GARCH and regime-switching models. In this case, if $C_n = C_n(S_n)$, then $C_k = C_k(S_k, h_k)$, $\alpha_k = \alpha_k(S_{k-1}, h_{k-1})$, $b_k = b_k(S_{k-1}, h_{k-1})$, and $\gamma_k = \gamma_k(S_{k-1}, h_{k-1})$, for $k = 1, \dots, n + 1$. All these functions can be approximated using the methodology developed in Rémillard et al. (2010) for the regime-switching case. Implementation of the hedging strategy then requires predicting h_t .

Remark 2.2.3. *One could suggest to use the smallest filtration to get rid of the unobservable process h but in this case, all conditional expectations based on \mathcal{F}_k would depend on all past values S_0, \dots, S_k , making it impossible to implement in practice.*

2.3. Verification of the assumptions of Lemma 2.1.1. In what follows, we consider some interesting models used in practice, for which it is possible to show that the assumptions of Lemma 2.1.1 hold true and that the optimal solution can be computed via a dynamic program.

2.3.1. Regime-switching geometric random walks. An interesting model, which includes geometric random walk models, is to consider a regime-switching geometric random walk. These models can display serial dependence in the log-returns and may account for changing volatility over time. For implementation issues, including estimation and goodness-of-fit tests, see, e.g., Rémillard et al. (2010).

To define the process, suppose that τ is a finite homogeneous Markov chain with transition matrix Q with values in $\{1, \dots, l\}$ representing the non-observable regimes and set $\beta_k S_k^i = S_0^i \prod_{t=1}^k \{1 + \xi_t^i\}$, $i = 1, \dots, d$, where, given $\tau_1 = i_1, \dots, \tau_n = i_n$, ξ_1, \dots, ξ_n are independent with $\xi_j \sim \mathbb{P}_{i_j}$, $j = 1, \dots, n$, $\mathbb{E}_i(\xi_j) = E(\xi_j | \tau_j = i) = m(i)$, and $\mathbb{E}_i(\xi_j \xi_j^\top) = B(i)$. The interpretation of the model is easy: At a given period t , a regime τ_t is chosen at random, according to the Markov chain model, and given $\tau_t = i$, ξ_t is chosen at random according to distribution \mathbb{P}_i . When there is only one regime, one obtains a geometric random walk where all ξ s are independent.

We assume that the $B(i) - m(i)m(i)^\top$ is invertible for any $i = 1, \dots, l$. Setting $X_k = \beta_k S_k$, one gets $\Delta_k = X_k - X_{k-1} = D(X_{k-1})\xi_k$, $k = 1, \dots, n$, where $D(s)$ is the diagonal matrix constructed from vector s . Note that S is not a Markov process in general but (S, τ) is a Markov process. The validity of the assumptions of Lemma 2.1.1 follows from the next result, proved in Appendix A.3.

Proposition 2.3.1. *For any $k = 1, \dots, n$ and $i = 1, \dots, l$, $\gamma_k = \gamma_k(\tau_{k-1})$, $\gamma_k(i) \in (0, 1]$, $A_k = A_k(S_{k-1}, \tau_{k-1})$ and $b_k = b_k(S_{k-1}, \tau_{k-1})$, where*

$$(2.3.1) \quad A_k(s, i) = \beta_{k-1}^2 D(s) \left\{ \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) B(j) \right\} D(s),$$

$$(2.3.2) \quad b_k(s, i) = D^{-1}(\beta_{k-1} s) \rho_{k+1}(i),$$

$$(2.3.3) \quad \gamma_k(i) = \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) \{1 - \rho_{k+1}(i)^\top m(j)\},$$

with $\rho_{k+1}(i) = \left\{ \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) B(j) \right\}^{-1} \left\{ \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) m(j) \right\}$.

If in addition $C = C(S_n)$, then $C_k = C_k(S_k, \tau_k)$ and $\alpha_k = \alpha_k(S_{k-1}, \tau_{k-1})$, where

$$\begin{aligned} C_{k-1}(s, i) &= \frac{\beta_k}{\beta_{k-1}} \sum_{j=1}^l Q_{ij} \frac{\gamma_{k+1}(j)}{\gamma_k(i)} \\ &\quad \times \int C_k \left\{ \frac{\beta_{k-1}}{\beta_k} D(s)(\mathbf{1} + y), j \right\} \{1 - \rho_{k+1}(i)^\top y\} \mathbb{P}_j(dy), \\ \alpha_k(s, i) &= \frac{\beta_k}{\beta_{k-1}} D^{-1}(s) \left\{ \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) B(j) \right\}^{-1} \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j) \\ &\quad \times \int C_k \left\{ \frac{\beta_{k-1}}{\beta_k} D(s)(\mathbf{1} + y), j \right\} y \mathbb{P}_j(dy). \end{aligned}$$

2.3.2. GARCH-type models. Here, one assumes that $\Delta_k = \beta_k S_k - \beta_{k-1} S_{k-1} = \beta_{k-1} S_{k-1} \xi_k$, with $\xi_k = \pi_1(h_{k-1}, \epsilon_k)$, and $h_k = \pi_2(h_{k-1}, \epsilon_k)$ with π_2 having values in some set \mathcal{H} , and where the innovations ϵ_k are independent and identically distributed with probability law ν . It is immediate that (S_k, h_k) is a Markov process. Furthermore, almost all known GARCH(1,1) models can be written in that way.

Suppose that for every given possible $h \in \mathcal{H}$, $\pi_1(h, y)$ is not constant ν -a.s. Using Proposition B.0.1 and reverse induction, as in the proof of Proposition 2.3.1, it is easy to show that the assumptions of Lemma 2.1.1 are met, and that for all $k = n, \dots, 1$, $\gamma_k = \gamma_k(h_{k-1})$ and $A_k(s, h) = \beta_{k-1}^2 s^2 B_k(h)$, $b_k(s, h) = \frac{m_k(h)}{s \beta_{k-1} B_k(h)}$, where $B_k(h) = \int \pi_1^2(h, y) \gamma_{k+1} \{ \pi_2(h, y) \} \nu(dy)$, $m_k(h) = \int \pi_1(h, y) \gamma_{k+1} \{ \pi_2(h, y) \} \nu(dy)$, and $\gamma_k(h) = \int \left\{ 1 - \frac{m_k(h)}{B_k(h)} \pi_1(h, y) \right\} \gamma_{k+1} \{ \pi_2(h, y) \} \nu(dy)$. Also, if $C = C_n(S_n)$, then

$$\begin{aligned} C_{k-1}(s, h) &= \frac{\beta_1}{\gamma_k(h)} \int C_k \left[\frac{s}{\beta_1} \{1 + \pi_1(h, y)\}, \pi_2(h, y) \right] \\ &\quad \times \gamma_{k+1} \{ \pi_2(h, y) \} \left\{ 1 - \frac{m_k(h)}{B_k(h)} \pi_1(h, y) \right\} \nu(dy), \end{aligned}$$

$$\alpha_k(s, h) = \frac{\beta_1}{sB_k(h)} \int C_k \left[\frac{s}{\beta_1} \{1 + \pi_1(h, y)\}, \pi_2(h, y) \right] \\ \times \gamma_{k+1} \{ \pi_2(h, y) \} \pi_1(h, y) \nu(dy).$$

Hence, the optimal solution can be written as a dynamic program.

3. EXAMPLES OF APPLICATION

In this section we consider pricing and hedging of European calls for two geometric random walk models, when the returns are i.i.d. Gaussian and i.i.d. differences of Laplace distributions, and for a NGARCH model. It follows from the previous sections that optimal hedging solutions exist for these cases, and the optimal solution can be written as a dynamic program associated with functions of a finite number of variables. For solving such dynamic programs, we discretize the state space into a finite grid and we compute approximations of expectations using Monte Carlo simulations at every point of the grid. Linear interpolations are used for points outside the grid at each time step. Since the expectations are always with respect to the same probability measure, only one sequence of random numbers may be used, using the ideas in Del Moral et al. (2006, 2012).

3.1. Geometric random walk models. Here we consider discretized versions of the Black-Scholes (BS) and Variance Gamma (VG) models for the underlying asset over 23 periods. In each case, the 22 periodic returns are i.i.d., so that the mean and volatility at maturity are respectively 9% and 6%. For the BS model, the returns are Gaussian, while for the VG model, the returns are differences of i.i.d. Gamma variates, so that the distribution at maturity is Laplace (double exponential). These models are particular cases of regime-switching models with only one regime. We do not consider regime-switching models since it has been done in Rémillard et al. (2010), where the daily log-returns of the S&P 500 are analyzed.

We are going to price and hedge a call with strike $K = 100$ and maturity 1 year, using 22 replication periods and a 2000 points discretization of the asset values over the interval $[80, 120]$. The annual rate is 5%. A sequence of 10000 random points were used for the computation of functions α_k and A_k , while 10000 paths were used to compute the hedging errors. Delta hedging is optimal in the continuous time limit for the BS model, but not for the VG model. As expected, according to Figures 1–2, the values of the call C_0 and initial investment strategy ϕ_1 , obtained from the optimal hedging, are close to those obtained using the Black-Scholes formula (even with 22 hedging periods), while they differ for the VG model. This is also reflected in the distribution of the hedging errors, as illustrated in Figure 3.

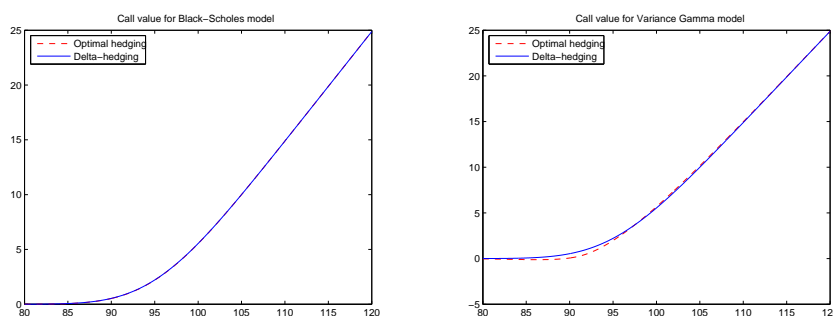


FIGURE 1. Call option value C_0 for the Black-Scholes (left panel) and Variance Gamma models (right panel) with 22 periods of hedging.

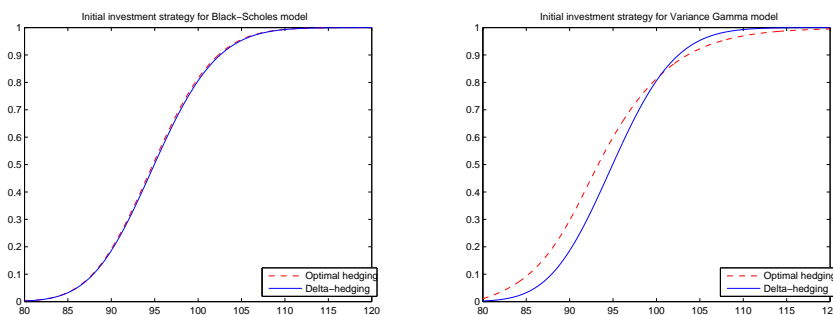


FIGURE 2. Initial investment strategy ϕ_1 in the underlying asset for the Black-Scholes (left panel) and Variance Gamma models (right panel) with 22 periods of hedging.

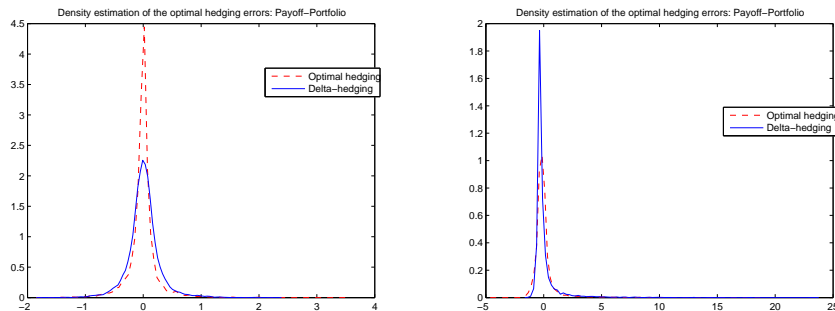


FIGURE 3. Estimated densities of the hedging error G for the Black-Scholes (left panel) and Variance Gamma models (right panel) with 22 periods of hedging.

Descriptive statistics of the hedging errors are given in Table 1. Simulations can also be used to show that as the number of hedging periods increases, the hedging error tends to zero for the BS model, while it is never 0 for the VG model. Note that the RMSE of the optimal hedging is always less than the one of the delta hedging.

TABLE 1. Statistics of hedging errors (Payoff-Portfolio) for the Black-Scholes and Variance Gamma models.

Stats	Black-Scholes		Variance Gamma	
	Optimal	Delta	Optimal	Delta
Average	-0.0065	0.0076	0.0151	0.0518
Median	-0.0014	0.0029	-0.1543	-0.2886
Volatility	0.2537	0.2774	1.0510	1.2529
Skewness	1.3781	0.5515	6.1479	6.0965
Kurtosis	22.0975	8.8391	63.3897	60.9747
Minimum	-1.7986	-1.7578	-4.4372	-1.5160
Maximum	3.5389	2.2978	18.5163	23.6422
VaR(99%)	0.8231	0.9244	4.6976	5.9450
VaR(99.9%)	1.9576	1.5779	11.7905	12.2222
RMSE	0.2538	0.2775	1.0511	1.2540

3.2. NGARCH model. As in Duan (1995), we consider the *NGARCH* model where $e^{\xi_k} - 1 = r + \lambda\sqrt{h_{k-1}} - \frac{1}{2}h_{k-1} + \sqrt{h_{k-1}}\varepsilon_k$, and $h_k = \alpha_0 + \alpha_1 h_{k-1}\varepsilon_k^2 + \beta_1 h_{k-1}$, with $\varepsilon_k \sim N(0, 1)$ and parameters $\alpha_0 = 1.524 \times 10^{-5}$, $\alpha_1 = 0.1883$, $\beta_1 = 0.7162$ and $\lambda = 7.452 \times 10^{-3}$. Under the EMM, we have $e^{\xi_k} - 1 = r + -\frac{1}{2}h_{k-1} + \sqrt{h_{k-1}}\varepsilon_k$ and $h_k = \alpha_0 + \alpha_1 h_{k-1}(\varepsilon_k - \lambda)^2 + \beta_1 h_{k-1}$. We price and hedge a call with strike

$K = 100$ and maturity 30 days using daily replication, using a grid of 500 points for the asset on $[60, 140]$, while the grid for the volatility consists in 90 points of the interval $[\.00005, \.0007]$. The annual rate is 0%. In what follows, B&S hedging means delta hedging using the B&S formulas, while Duan's methods consists in picking his suggested EMM and taking the delta of the option. The value of the option and the initial number of asset are displayed in Figure 4, while descriptive statistics of the 10000 hedging errors are given in Table 2 for the three hedging methodologies, showing that the errors are more concentrated about 0 for the optimal hedging.

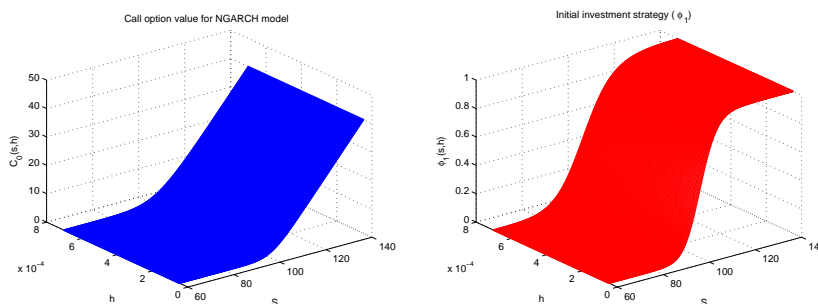


FIGURE 4. Optimal hedging call option value C_0 and initial investment strategy ϕ_1 for the NGARCH model with 30 periods.

TABLE 2. Statistics of hedging errors (Payoff-Portfolio) for the NGARCH model.

Stats	Optimal	Delta	Duan
Average	-0.0159	-0.0954	0.0085
Median	-0.1549	-0.2094	-0.1297
Volatility	0.8568	0.8951	0.9032
Skewness	1.7205	2.7558	2.9171
Kurtosis	10.5790	29.3709	29.9947
Minimum	-1.9966	-2.2302	-2.0789
Maximum	9.9114	17.4873	17.7232
VaR(99%)	2.7698	2.8613	3.0831
VaR(99.9%)	5.4893	6.3574	6.5070
RMSE	0.8569	0.9001	0.9033

4. CONCLUSION

In this paper we presented the optimal solution for a discrete time hedging portfolio. When the underlying process is Markov or a component of a Markov process, the optimal hedging strategy depends on deterministic functions that can be approximated. We also find explicit formulas for two interesting models. Finally, numerical simulations show that optimal hedging is preferable to delta hedging.

REFERENCES

- Bouchaud, J.-P. and Potters, M. (2002). Back to basics: historical option pricing revisited. *Philosophical Transactions: Mathematical, Physical & Engineering Sciences*, 357(1758):2019 – 2028.
- Boyle, P. P. and Emanuel, D. (1980). Discretely adjusted option hedges. *Journal of Financial Economics*, 8:259–282.
- Černý, A. and Kallsen, J. (2007). On the structure of general mean-variance hedging strategies. *Ann. Probab.*, 35(4):1479–1531.
- Cornalba, L., Bouchaud, J.-P., and Potters, M. (2002). Option pricing and hedging with temporal correlations. *Int. J. Theor. Appl. Finance*, 5(3):307–320.
- Del Moral, P., Rémillard, B., and Rubenthaler, S. (2006). Monte Carlo approximations of American options. Technical report, GERAD.
- Del Moral, P., Rémillard, B., and Rubenthaler, S. (2012). Monte Carlo Approximations of American Options that Preserve Monotonicity and Convexity. In *Numerical Methods in Finance*, pages 117–145. Springer.
- Duan, J.-C. (1995). The GARCH option pricing model. *Math. Finance*, 5(1):13–32.
- Garcia, R. and Renault, . (1998). A note on hedging in ARCH and stochastic volatility option pricing models. *Math. Finance*, 8(2):153–161.

- Kat, H. M. and Palaro, H. P. (2005). Who needs hedge funds? A copula-based approach to hedge fund return replication. Technical report, Cass Business School, City University.
- Motoczyński, M. (2000). Multidimensional variance-optimal hedging in discrete-time model—a general approach. *Math. Finance*, 10(2):243–257. INFORMS Applied Probability Conference (Ulm, 1999).
- Papageorgiou, N., Rémillard, B., and Hocquard, A. (2008). Replicating the properties of hedge fund returns. *Journal of Alternative Investments*, 11:8–38.
- Rémillard, B., Hocquard, A., and Papageorgiou, N. A. (2010). Option Pricing and Dynamic Discrete Time Hedging for Regime-Switching Geometric Random Walks Models. Technical report, SSRN Working Paper Series No. 1591146.
- Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.*, 20(1):1–32.
- Wilmott, P. (2006). *Paul Wilmott on Quantitative Finance*, volume 3. John Wiley & Sons, second edition.

APPENDIX A. PROOFS OF THE MAIN RESULTS

A.1. Proof of Lemma 2.1.1. First, we will show that $\gamma_k \in (0, 1]$ and A_k is invertible for all $k = 1, \dots, n$. By hypothesis, $E(\gamma_{k+1} | \mathcal{F}_{k-1}) A_k - \mu_k \mu_k^\top$ is invertible for all $k = 1, \dots, n$. In particular, it is true for $k = n$, yielding that $\Sigma_n = A_n - \mu_n \mu_n^\top$ is invertible, which is the conditional covariance matrix of Δ_n given \mathcal{F}_{n-1} . It then follows from Proposition B.0.1 that A_n is invertible. Without loss of generality, one may assume that A_n is diagonal. Otherwise, we diagonalize it in the form $A_n = M_n B_n M_n^\top$, with M_n, B_n \mathcal{F}_{n-1} -measurable, B_n is diagonal, $M_n^\top M_n = I$ and set $\tilde{\Delta}_n = M_n^\top \Delta_n$. Since $M_n M_n^\top = I$, it follows that M_n is bounded, so $\tilde{\Delta}_n$ is square integrable. Finally $B_n = E(\tilde{\Delta}_n \tilde{\Delta}_n^\top | \mathcal{F}_{n-1})$. A_n being diagonal, it then follows that $b_n^\top \Delta_n$ is square integrable and $\gamma_n = 1 - b_n^\top \mu_n = 1 - \mu_n^\top A_n^{-1} \mu_n$. It also follows from Proposition B.0.1 that

$\mu_n^\top A_n^{-1} \mu_n = \frac{\mu_n^\top \Sigma_n^{-1} \mu_n}{1 + \mu_n^\top \Sigma_n^{-1} \mu_n}$, so $\gamma_n = \frac{1}{1 + \mu_n^\top \Sigma_n^{-1} \mu_n} \in (0, 1]$. As a result, $\gamma_n \leq E(\gamma_{n+1} | \mathcal{F}_n) = 1$.

The rest of the proof follows easily by reverse induction, using Proposition B.0.1 with the mean and covariance matrix of Δ_k under the probability distribution Q_k , with $Q_k(O) = E(\mathbb{I}_O \gamma_{k+1} | \mathcal{F}_{k-1}) / E(\gamma_{k+1} | \mathcal{F}_{k-1})$, $O \in \mathcal{F}_k$, for $k = n - 1, \dots, 1$. \square

A.2. Proof of Theorem 2.2.1. Using the proof of Lemma 2.1.1, one can easily check that a_k , b_k and ϕ_k make sense and that $\phi_k^\top \Delta_k$ is square integrable. Next, it is easy to check that a necessary and sufficient condition for $(V_0, \vec{\phi})$ to minimize $E \left[\left\{ G(V_0, \vec{\phi}) \right\}^2 \right]$ is that $E \left\{ G(V_0, \vec{\phi}) \right\} = 0$ and $E \left\{ G(V_0, \vec{\phi}) \Delta_k | \mathcal{F}_{k-1} \right\} = 0$ for all $k = 1, \dots, n$. The necessity comes from the fact that for any event $O \in \mathcal{F}_{k-1}$, one must have $0 = \frac{d}{d\epsilon} \Big|_{\epsilon=0} E \left[\left\{ G(V_0, \vec{\phi}) - \epsilon \mathbb{I}_O \Delta_k \right\}^2 \right] = -2E \left\{ G(V_0, \vec{\phi}) \Delta_k \mathbb{I}_O \right\}$, which is equivalent to the condition $E \left\{ G(V_0, \vec{\phi}) \Delta_k | \mathcal{F}_{k-1} \right\} = 0$, while the condition $E \left\{ G(V_0, \vec{\phi}) \right\} = 0$ comes from the fact that for any θ , one must have

$$0 = \frac{d}{d\epsilon} \Big|_{\epsilon=0} E \left[\left\{ G(V_0 + \epsilon\theta, \vec{\phi}) \right\}^2 \right] = -2E \left\{ G(V_0, \vec{\phi}) \right\}.$$

To see that the conditions are sufficient, it suffices to check that

$$E \left[\left\{ G(V_0 + \theta_0, \vec{\phi} + \vec{\psi}) \right\}^2 \right] = E \left[\left\{ G(V_0, \vec{\phi}) \right\}^2 \right] + E \left\{ \left(\theta_0 + \sum_{k=1}^n \psi_k^\top \Delta_k \right)^2 \right\}.$$

The proof that $\vec{\phi}$ is the solution is based on the following equation, which can be easily proven by induction.

$$(A.2.1) \quad E(V_n | \mathcal{F}_k) = V_k E(P_{k+1} | \mathcal{F}_k) + E \{ \beta_n C (1 - P_{k+1}) | \mathcal{F}_k \}, \quad k = 1, \dots, n.$$

To complete the proof of theorem, note that from (A.2.1),

$$(A.2.2) \quad E \left\{ G(V_0, \vec{\phi}) | \mathcal{F}_k \right\} = E(\beta_n C P_{k+1} | \mathcal{F}_k) - V_k E(P_{k+1} | \mathcal{F}_k), \quad k = 0, \dots, n.$$

Using (A.2.2), one has $E \left\{ G \left(\left(V_0, \vec{\phi} \right) \Delta_k | \mathcal{F}_k \right) \right\} = E(\beta_n C \Delta_k P_{k+1} | \mathcal{F}_k) - E(V_k \Delta_k P_{k+1} | \mathcal{F}_k)$, so $E \left\{ G \left(\left(V_0, \vec{\phi} \right) \Delta_k | \mathcal{F}_{k-1} \right) \right\} = E(\beta_n C \Delta_k P_{k+1} | \mathcal{F}_{k-1}) - E(V_k \Delta_k P_{k+1} | \mathcal{F}_{k-1}) = A_k(a_k - V_{k-1}b_k - \phi_k) = 0$. Hence $E \left\{ G \left(\left(V_0, \vec{\phi} \right) \right) \right\} = E(\beta_n C P_1) - V_0 E(P_1) = 0$. \square

A.3. Proof of Proposition 2.3.1. The result is obviously true for $k = n + 1$. Suppose it is true for $k + 1$. For i given, set $\pi_j = Q_{ij} \gamma_{k+1}(j) / D$, where $D = \sum_{j=1}^l Q_{ij} \gamma_{k+1}(j)$. By hypothesis, π_1, \dots, π_l are probabilities adding to 1, so if $X \sim \mathbb{P}_j$ with probability π_j , then $\gamma_k(i) = D(1 - \mu^\top B^{-1} \mu)$, where $\mu = E(X)$ and $B = E(XX^\top)$. Let Σ be the covariance matrix of X . It is non singular since the covariance of X under \mathbb{P}_j is non singular. It then follows from Proposition B.0.1 that $1 - \mu^\top B^{-1} \mu = \frac{1}{1 + \mu^\top \Sigma^{-1} \mu} > 0$. Since $D > 0$ by hypothesis, one may conclude that $\gamma_k(i) > 0$. As a by-product we get that $\gamma_k(i) \leq 1$ if $\gamma_{k+1}(j) \leq 1$ for all $j = 1, \dots$. Since that is true for $\gamma_{n+1} \equiv 1$, one may conclude that for all $k = 1, \dots, n$, $\gamma_k(i) \leq 1$. The rest of the proof is easy. \square

APPENDIX B. AUXILIARY RESULTS

Proposition B.0.1. *Suppose $A = \Sigma + bb^\top$ where Σ is symmetric and invertible. Then A is invertible, and $A^{-1} = \Sigma^{-1} - \frac{\Sigma^{-1}bb^\top\Sigma^{-1}}{1+b^\top\Sigma^{-1}b}$. Moreover, $1 - b^\top A^{-1}b = \frac{1}{1+b^\top\Sigma^{-1}b} > 0$.*

Proof: Since $A \left(\Sigma^{-1} - \frac{\Sigma^{-1}bb^\top\Sigma^{-1}}{1+b^\top\Sigma^{-1}b} \right) = I$, A is invertible and $A^{-1} = \Sigma^{-1} - \frac{\Sigma^{-1}bb^\top\Sigma^{-1}}{1+b^\top\Sigma^{-1}b}$. Setting $c = b^\top \Sigma^{-1} b$, one gets $1 - b^\top A^{-1} b = 1 - c + \frac{c^2}{1+c} = \frac{1}{1+c} > 0$. \square

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