Gaia astrometric accuracy in the past

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Abstract

The use of the Gaia astrometric catalogue in the past needs some care to propagate the positions and derive their accuracy from the published values at the Catalogue epoch. The basic propagation model is presented and the accuracy in the past is provided as a function of magnitude and elapsed time.

1 Introduction

The ESA space astrometry mission, due for launch in late 2013, will survey the sky down to the 20th magnitude with an unprecedented astrometric accuracy of 25 muas at 15 mag, carrying out simultaneously multi-epoch photometry and spectroscopy. The mission is optimised to observe stellar sources to produce a stereoscopic and kinematic census of about one billion stars in our Galaxy enabling to probe the formation and evolution of the Milky Way. The expected astrometric accuracy is shown in Fig. 1 as a function of the G magnitude (very similar to R band for most stars). More precisely this gives the end-of-mission accuracy in astrometric parameters applicable to single stars at the mean epoch of the Catalogue, typically around 2016. More complex astrometric models will be applied for multiple systems and the accuracy should degrade a little, depending on the complexity of the dynamical model. Thanks to the simultaneous observation of several 10,000s quasars, the Gaia frame will be kinematically non rotating and aligned to the best version of the ICRF available at Gaia completion.

2 Propagation of Gaia Astrometry

To use the Gaia catalogue to reprocess old observations would require that a Gaia position at mean epoch is propagated in the past, over a timespan that could reach one century.

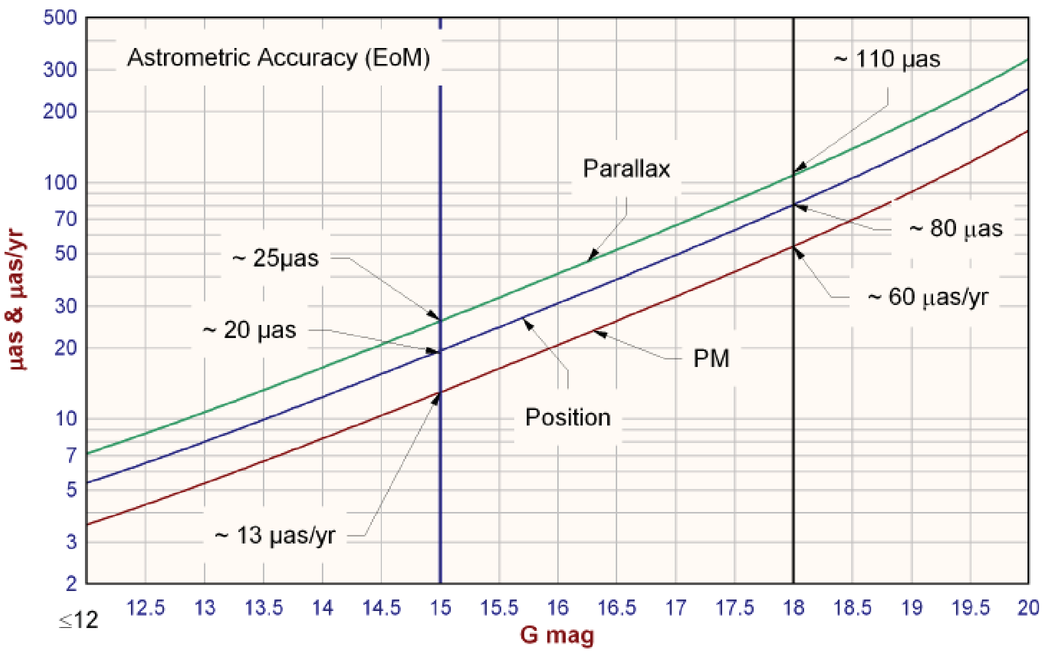

Figure 1: Gaia expected final astrometric accuracy at mean epoch as a function of the magnitude.

Since proper motions components are available this seems to be a trivial exercice to get the barycentric position with a linear transformation like,

$$\big(\alpha(t) - \alpha(t_0)\big)\cos\delta_0 = \mu_\alpha(t - t_0) \tag{1a}$$

$$\delta(t) - \delta(t_0) = \mu_\delta(t - t_0) \tag{1b}$$

which amounts to identifying the time derivatives of the coordinates, or better the components of the velocity on the plane of the sky, to the first differences.

But there are modelling errors in Eqs. 1 due to the neglect of terms of higher order in μ_α, μ_δ in the right-hand-side. Given the accuracy of Gaia, it can be shown that, depending on the star distance, they are not negligible when the propagation extends over several 10s years. A more general model is applicable to single stars for which the 3D motion can be considered rectilinear with constant velocity for several centuries. While this is true for the space motion, this is not for the projected displacement on the plane of the sky which is far from uniform at the Gaia accuracy.

2.1 The vectorial propagation equation

Based on this rectilinear motion we have the very simple propagation model,

$$\mathbf{OM}(t) = \mathbf{OM}_0 + \mathbf{V}\,t \tag{2}$$

relating the initial star position vector at the initial epoch (here $t_0 = 0$) to the position at any time t. Putting

$$|\mathbf{OM}_0| = r \tag{3}$$

for the distance at the first epoch and using respectively $\mathbf{u}_0$ and $\mathbf{u}$ for the unit vectors in the direction of the star at $t_0 = 0$ and t, one has

$$\frac{\mathbf{OM}}{r} = \mathbf{u}_0 + \frac{\mathbf{V}}{r}\,t \tag{4}$$

and then with $\mathbf{W} = \mathbf{V}/r$, to express the velocity in angular unit,

$$\frac{\mathbf{OM}}{r} = \mathbf{u}_0 + \mathbf{W}\,t \tag{5}$$

and finally with $W = |\mathbf{W}|$,

$$\begin{aligned} \mathbf{u}(t) &= \frac{\mathbf{u}_0 + \mathbf{W}\,t}{|\mathbf{u}_0 + \mathbf{W}\,t|} \\ &= (\mathbf{u}_0 + \mathbf{W}\,t)\left(1 + 2\mathbf{u}_0 \cdot \mathbf{W}\,t + W^2\,t^2\right)^{-1/2} \end{aligned} \tag{6}$$

Eq. (6) is the exact form of the propagation model for a single star giving the unit direction vector at time t without approximation, beyond the fact that the 3-D motion is uniform on a straight line. One can see that it depends on the three components of the velocity scaled by the distance, that is to say on the three components of the angular proper motion, including that in the radial direction. Numerically all the computations can be done in rectangular coordinates and transformed at the end in spherical coordinates, namely the right-ascension and declination at epoch. One must notice that the radial velocity v_r is required above the first order approximation.

2.2 The coordinate propagation equations

Although Eq. 6 solves the practical numerical problem, it is useful to express the result directly in spherical coordinates to extend the validity of Eq. 1 beyond the linear approximation. There are several methods to derive the more exact expressions, for example by expanding Eq. 6 in power of t and projecting on the plane of the sky. I propose below an alternative and more direct method based on the covariant derivative of the velocity.

Consider again the basic kinematical model with a uniform rectilinear motion in the Euclidean space. It is characterized by the fact that the velocity vector is parallel transported along the geodesic of the underlying space, i.e. along a straight line. This means that the 3-D acceleration $d^2\mathbf{r}/dt^2 = 0$ and more generally in any moving coordinate system that the covariant derivative is identically zero,

$$D^2\mathbf{r}/Dt^2 = 0 \tag{7}$$

Expressing this fact in curvilinear (here spherical) coordinates y^i one has

$$\gamma^i = \frac{d^2y^i}{dt^2} + \Gamma^i_{jk}\,\frac{dy^j}{dt}\,\frac{dy^k}{dt} \tag{8}$$

where the Christoffel symbols Γ^i_{jk} are computed from the metric of R_3 in spherical coordinates,

$$ds^2 = dr^2 + r^2\,\cos^2\delta\,d\alpha^2 + r^2\,d\delta^2 \tag{9}$$

using $y^1 = r$, $y^2 = \alpha$, $y^3 = \delta$, which gives in the local frame (normalised with unit vectors) $\mathbf{e}_r, \mathbf{e}_\alpha, \mathbf{e}_\delta$ the non-zero Christoffel symbols,

$$\Gamma^1_{22} = -r\,\cos^2\delta \qquad \Gamma^1_{33} = -r \qquad \Gamma^2_{12} = 1/r \tag{10a}$$

$$\Gamma^2_{23} = -\tan\delta \qquad \Gamma^3_{13} = 1/r \qquad \Gamma^3_{22} = \sin\delta\,\cos\delta \tag{10b}$$

Then with $\gamma^i = 0$ for a uniform rectilinear motion one has,

$$\ddot{r} = r\cos^2\delta\;\dot{\alpha}^2 + r\,\dot{\delta}^2 \tag{11a}$$

$$\ddot{\alpha} = -\frac{2}{r}\,\dot{r}\,\dot{\alpha} + 2\tan\delta\,\dot{\alpha}\dot{\delta} \tag{11b}$$

$$\ddot{\delta} = -\frac{2}{r}\,\dot{r}\dot{\delta} - \sin\delta\cos\delta\,\dot{\alpha}^2 \tag{11c}$$

Eqs. 11b-11c are the general expressions for the second derivatives of the right ascension and declination of a star moving on a straight line at constant speed in space. There are supplementary terms of the same order of magnitude which are different from zero even in the case of a purely tangential motion.

To write down the model to third order one needs also the third time derivatives of α, δ, which after a some algebra reduce to,

$$\frac{d^3y^i}{dt^3} = \left[-\frac{\partial\Gamma^i_{jk}}{\partial y^l} + 2\Gamma^i_{mj}\,\Gamma^m_{kl}\right]\,\dot{y}^j\,\dot{y}^k\,\dot{y}^l \tag{12}$$

and with the substitution of (10) one gets,

$$\frac{d^3\alpha}{dt^3} = 6\,\frac{\dot{r}^2}{r^2}\,\dot{\alpha} - 12\tan\delta\,\frac{\dot{r}}{r}\dot{\alpha}\dot{\delta} + 6\tan^2\delta\,\dot{\alpha}\dot{\delta}^2 - 2\dot{\alpha}^3 \tag{13a}$$

$$\frac{d^3\delta}{dt^3} = 6\,\frac{\dot{r}^2}{r^2}\,\dot{\delta} + 3\,\sin 2\delta\,\frac{\dot{r}}{r}\dot{\alpha}^2 - 3\,\dot{\alpha}^2\,\dot{\delta} - 2\dot{\delta}^3 \tag{13b}$$

Now one can write the propagation model as,

$$\alpha(t) = \alpha_0 + \dot{\alpha}\, t + \ddot{\alpha}\, \frac{t^2}{2} + \dddot{\alpha}\, \frac{t^3}{6} \tag{14a}$$

$$\delta(t) = \delta_0 + \dot{\delta}\, t + \ddot{\delta}\, \frac{t^2}{2} + \dddot{\delta}\, \frac{t^3}{6} \tag{14b}$$

or with Equations (11, 13, 14) and denoting $\mu_\alpha = \dot{\alpha} \cos\delta_0$, $\mu_\delta = \dot{\delta}, \mu_r = v_r/r$, where r is the distance of the star at initial epoch,

$$\begin{aligned} \Delta\alpha \cos\delta_0 = {} & \mu_\alpha\, t - [\mu_r\, \mu_\alpha - \tan\delta_0\, \mu_\alpha\, \mu_\delta]\ t^2 \\ & + \left[\mu_r^2\, \mu_\alpha - 2\tan\delta_0\, \mu_r\, \mu_\alpha\, \mu_\delta + \tan^2\delta\, \mu_\alpha\, \mu_\delta^2 - \frac{\mu_\alpha^3}{3\ \cos^2\delta_0}\right]\ t^3 \end{aligned} \tag{15}$$

$$\begin{aligned} \Delta\delta = {} & \mu_\delta\, t - \left[\mu_r\, \mu_\delta + \frac{\tan\delta_0}{2}\, \mu_\alpha^2\right]\ t^2 \\ & + \left[\mu_r^2\, \mu_\delta + \tan\delta_0\, \mu_r\, \mu_\alpha^2 - \frac{\mu_\alpha^2\, \mu_\delta}{2\ \cos^2\delta_0} - \frac{\mu_\delta^3}{3}\right]\ t^3 \end{aligned} \tag{16}$$

Equations (15)-(16) give the propagation model to third order of the proper motions. One sees the presence of the radial velocity in second and third order terms.

Solving now this system for μ_α and μ_δ, or equivalently inverting the series to the same order gives the proper motion components from the positions at two epochs. The process leads to the final transformation,

$$\mu_\alpha\, t = a(1 + \mu_r\, t) - \tan\delta_0\, a\, d + \frac{3\cos^2\delta_0 - 1}{6\ \cos^2\delta_0}\, a^3 - \tan\delta_0\, a\, d\, \mu_r t \tag{17a}$$

$$\mu_\delta\, t = d\,(1 + \mu_r\, t) + \frac{1}{2}\tan\delta_0\, a^2 + \frac{2\cos^2\delta_0 - 1}{2\ \cos^2\delta_0}\, a^2\, d + \frac{1}{2}\tan\delta_0\, a^2\, \mu_r t + \frac{d^3}{3} \tag{17b}$$

where $a = \Delta\alpha \cos\delta_0$, and $d = \Delta\delta$.

2.3 The propagation of the covariance matrix

For each star, the Gaia catalogue will provide simultaneously the 5 astrometric parameters (position, proper motion components and parallax) together with their covariance matrix in equatorial coordinates. The diagonal terms will give the standard deviation at epoch, that is to say an estimate of the uncertainty. This is the most compact statistical synthesis of the noisy observations based on the single star astrometric model. As we have propagated the position in the past, one must do the same for the estimated uncertainty. Basically if we start from the linear propagation model given in Eq. 1, one has essentially something close to,

$$\sigma_\alpha^2(t) \approx \sigma_{\alpha_0}^2 + \sigma_{\mu_\alpha}^2 (t - t_0)^2 \tag{18a}$$

$$\sigma_\delta^2(t) \approx \sigma_{\delta_0}^2 + \sigma_{\mu_\delta}^2 (t - t_0)^2 \tag{18b}$$

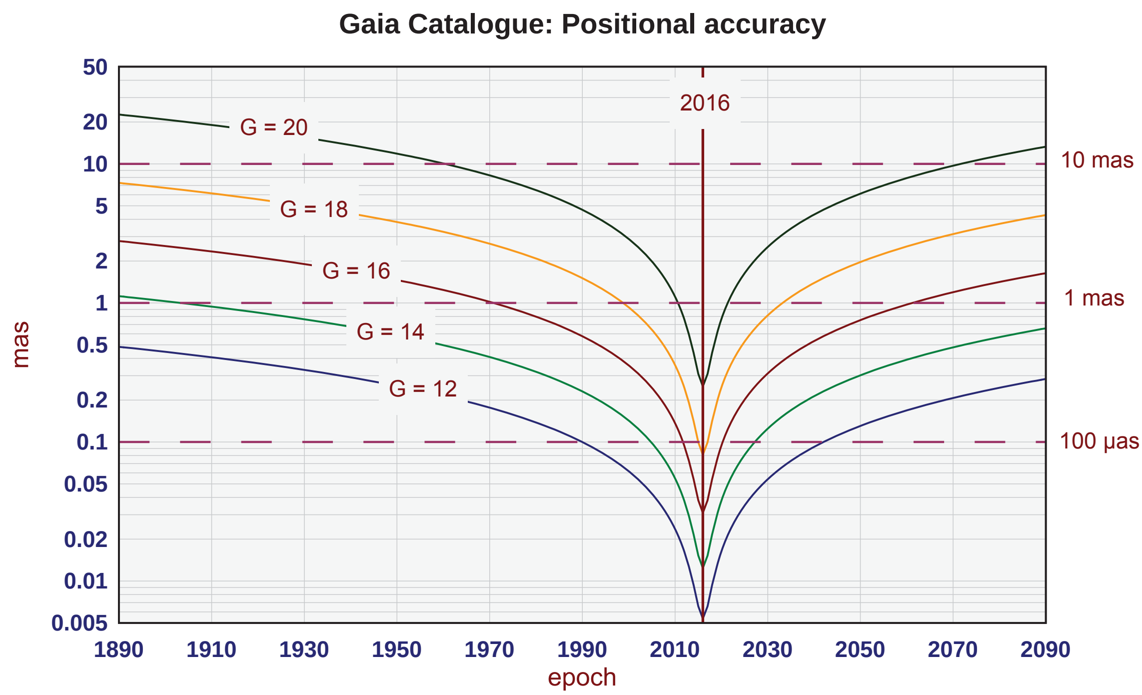

Figure 2: Evolution of the accuracy of the Gaia positional stellar catalogue with time due to the uncertainty in the annual proper motions of stars.

where correlations at t_0 have been neglected and a simple linear propagation model has been used. One should however note that α, μ_α and δ, μ_δ become quickly highly correlated and in applications using both positions and proper motions, extreme care must be exercised in the statistical inference. The propagation of the covariance matrix is much more complex and involves the Jacobian matrix of the general transformation between the astrometric parameters at epoch t_0 and t. More precisely the covariance matrices at two epochs are related by,

$$\mathbf{C}(t) = \mathbf{J}\mathbf{C}(t_0)\mathbf{J}^T \tag{19}$$

For the position the transformation is the generalisation of Eq. 1 given by Eqs. 15-16 and by their derivatives for the proper motion components, while Eq. 19 generalises Eqs. 18. Without entering into the details, this has been implemented in a computer program and allowed expressing the accuracy of the Gaia propagated astrometry in the past (and future) as a function the magnitude, using the expected accuracy at the Catalogue Epoch. The results are shown in Fig. 2 for the sky-averaged accuracy between 1890-2090. One must note the $\sim$ mas accuracy for the faint stars of the Carte du Ciel around the beginning of the program.