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(Circular) backbone colouring: tree backbones in planar graphs

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Abstract: Consider an undirected graph G and a subgraph H of G, on the same vertex set. The *q-backbone chromatic number* $\mathrm{BBC}_q(G,H)$ is the minimum k such that G can be properly coloured with colours from $\{1,\dots,k\}$, and moreover for each edge of H, the colours of its ends differ by at least q. In this paper we focus on the case when G is planar and H is a forest. We give a series of NP-hardness results as well as upper bounds for $\mathrm{BBC}_q(G,H)$, depending on the type of the forest (matching, galaxy, spanning tree). Eventually, we discuss a circular version of the problem.

Key-words: backbone colouring, planar graph, forest, NP-complete

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Coloration dorsale (circulaire) : arbres dorsaux dans les graphes planaires

Résumé : On considère un graphe G (non-orienté) et un sous-graphe H de G. Le *nombre chromatique q-dorsal* $\mathrm{BBC}_q(G,H)$ est le plus petit entier k tel que G puisse être coloré proprement avec les couleurs $\{1,\ldots,k\}$ de telle sorte qu'en plus pour toute arête de H, les couleurs de ses deux extrémités diffèrent d'au moins q. Dans ce rapport, nous étudions le cas où G est planaire et H est une forêt. Nous donnons une série de résultats de NP-complétude ainsi que des bornes supérieures pour $\mathrm{BBC}_q(G,H)$, suivant le type de forêt (couplage, galaxie, arbre couvrant). Nous abordons également une version circulaire du problème.

Mots-clés : coloration dorsale, graphe planaire, forêt, NP-complet

1 Introduction

All the graphs considered in this paper are simple. Let $G = (V,E)$ be a graph, and let $H = (V,E(H))$ be a spanning subgraph of G, called the *backbone*. A *k-colouring* of G is a mapping $f : V \to \{1,2,\dots,k\}$. Let f be a k-colouring of G. It is a *proper colouring* if $|f(u)-f(v)| \geq 1$. It is a *q-backbone colouring* for (G,H) if f is a proper colouring of G and $|f(u)-f(v)| \geq q$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k-colouring of G. The *q-backbone chromatic number* $\mathrm{BBC}_q(G,H)$ is the smallest integer k for which there exists a q-backbone k-colouring of (G,H).

If f is a proper k-colouring of G, then g defined by $g(v) = q \cdot f(v) - q + 1$ is a q-backbone colouring of (G,H) for any spanning subgraph H of G. Moreover it is well-known that if $G = H$, this q-backbone colouring of (G,H) is optimal. Therefore, since $\mathrm{BBC}_q(H,H) \leq \mathrm{BBC}_q(G,H) \leq \mathrm{BBC}_q(G,G)$, we have

$$q \cdot \chi(H) - q + 1 \leq \mathrm{BBC}_q(G,H) \leq q \cdot \chi(G) - q + 1. \tag{1}$$

If H is empty (i.e. $E(H) = \emptyset$), then $\mathrm{BBC}_q(G,H) = \chi(G)$. Hence for any $k \geq 3$, deciding if $\mathrm{BBC}_q(G,H) \leq k$ is NP-complete because deciding if a graph is k-colourable is NP-complete (See [7]). However, when we impose G or H to belong to certain graph classes, the problem sometimes become polynomial-time solvable. A trivial example is when we impose H to have chromatic number at least $r > (k+q-1)/q$. Then $\mathrm{BBC}_q(G,H) \geq rq - q + 1$, and so deciding if $BBC(G,H) \leq k$ can be done instantly by always returning 'no'. A less trivial example is when we impose H to have minimum degree 1. For such an H, deciding if $BBC(G,H) \leq q+1$ is also polynomial-time solvable, because $\mathrm{BBC}_q(G,H) = q+1$ if and only if G is bipartite. This simple observation was already made by Broersma et al. [5] when H is a 1-*factor* (a spanning subgraph in which every vertex has degree exactly 1). Furthermore, if we also impose H to be connected, we show in Theorem 17 that deciding if $BBC(G,H) \leq q+2$ can be done in polynomial time. In contrast, if the condition of H being connected is removed, then it is NP-complete (Theorem 18).

In this paper, we will focus on the particular case when G is a planar graph and H is a forest (i.e. an acyclic graph). Inequality (1) and the Four-Colour Theorem imply that for any planar graph G and spanning subgraph H, $\mathrm{BBC}(G,H) \leq 3q+1$. However, for $q = 2$, Broersma et al. [4] conjectured that this is not best possible if the backbone is a forest.

Conjecture 1. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_2(G,F) \leq 6$.

If true Conjecture 1 would be best possible. Broersma et al. [4] gave an example of a graph $\hat{G}$ with a forest $\hat{F}$ such that $\mathrm{BBC}_2(\hat{G},\hat{F}) = 6$. See Figure 1. It is then natural to ask how large $\mathrm{BBC}_q(G,F)$ could be when G is

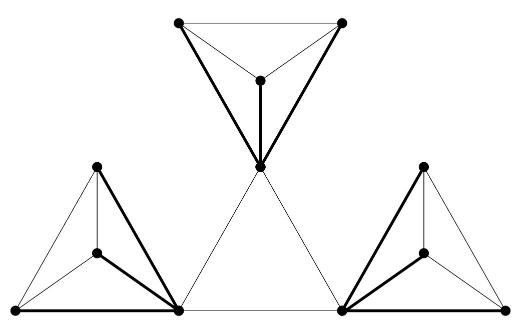

Figure 1: A planar graph $\hat{G}$ with a forest $\hat{F}$ (bold edges) such that $\mathrm{BBC}_q(\hat{G},\hat{F}) = q+4$.

planar and F is a forest for larger values of q. We prove the following.

Theorem 2. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_q(G,F) \leq q+6$.

In fact, we prove a more general result in Proposition 13 : for any pair (G,H) with H a subgraph of G,

$$\mathrm{BBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H)-q+2.$$

For $q \geq 4$, Theorem 2 is best possible. Indeed, we show a planar graph G^* together with a spanning tree T^* such that $\mathrm{BBC}_q(G^*,T^*) = q+6$ for all $q \geq 4$. See Figure 2 and Proposition 15. Furthermore, we show in Theorem 32,

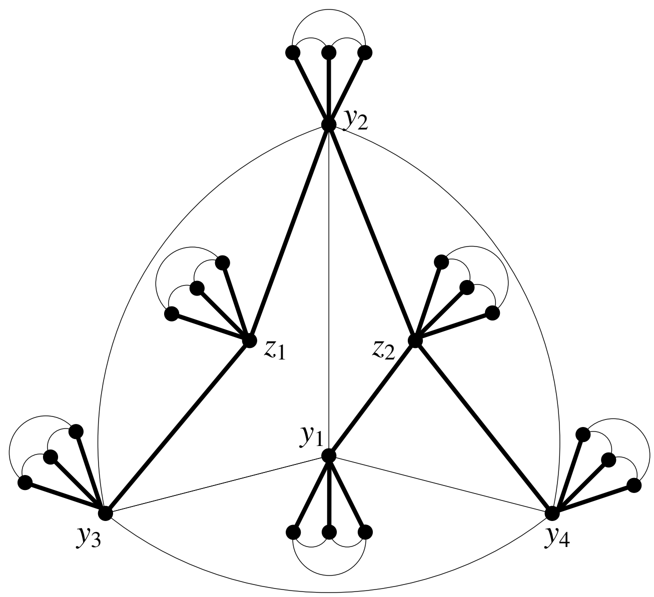

Figure 2: A planar graph G^* and a tree T^* (bold edges) such that $\mathrm{BBC}_q(G^*,T^*) = q+6$ for $q \geq 4$.

that for any fixed $q \geq 4$, given a planar graph G and a spanning tree T of G, it is NP-complete to decide if $\mathrm{BBC}_q(G,T) \leq q+5$.

On the other hand, we believe that if $q = 3$, Theorem 2 is not best possible.

Conjecture 3. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_3(G,F) \leq 8$.

If true, Conjecture 3 would be tight. The pair (G^*,F^*) of Figure 2 satisfies $\mathrm{BBC}_3(G^*,F^*) = 8$. We show in Proposition 16 that Conjecture 1 implies Conjecture 3.

A *star* is a tree in which a vertex v, called the *center* is adjacent to every other. A *galaxy* is a forest of stars. As evidence in support of Conjectures 1 and 3, Broersma et al. [5] showed that if F is a galaxy in a planar graph G, then $\mathrm{BBC}_q(G,F) \leq q+4$. This result is best possible even if F has maximum degree 3 as shown by the example of Figure 1. Furthermore, we show in Theorems 21 and 29 that, for any $q \geq 2$, it is NP-complete to decide if $\mathrm{BBC}_q(G,F) \leq q+3$ given a planar graph G and a galaxy of maximum degree 3.

However, if the backbone is a *matching*, i.e. a galaxy with maximum degree 1, then fewer colours are needed. Indeed, Broersma et al. [5] showed that if M is a matching in a planar graph G, then for any $q \geq 3$, $\mathrm{BBC}_q(G,M) \leq q+3$. They conjectured that the same holds for $q = 2$.

Conjecture 4 (Broersma et al. [5]). *If G is a planar graph G and M a matching in G, then* $\mathrm{BBC}_2(G,M) \leq 5$.

It is natural to ask the same question for galaxies with maximum degree at least 2. When $q = 2$, we answer in the negative by showing that there are pairs of planar graphs and spanning forests of maximum degree 2 whose 2-backbone chromatic number is 6. Furthermore, we show that given a planar graph G and a spanning forest F of maximum degree 2, it is NP-complete to decide whether $BBC_2(G,F) \leq 5$ (Theorem 23). We also show that given a planar graph G with a hamiltonian path P, it is NP-complete to decide whether $BBC_2(G,F) \leq 5$. This result refines a result of Broersma et al. [3, 4] who proved it for a general graph G.

For $q = 3$, the problem remains open.

Problem 5. *If G is a planar graph G and F a galaxy of maximum degree* 2*, is is true that* $\mathrm{BBC}_q(G,F) \leq q+3$*, for all* $q \geq 3$*?*

Broersma et al. [5] proved that deciding if $\mathrm{BBC}_q(G,M) \leq q+2$ for a given graph G and matching M is NP-complete. We prove in Subection 2.2 that it remains NP-complete even if we impose G to be planar. In contrast, we prove that deciding if $\mathrm{BBC}_q(G,T) \leq q+2$ for a given graph G and spanning tree T is polynomial-time solvable.

One can generalize the notion of backbone colouring by allowing a more complicated structure of the frequency space. The most natural one is to consider a circular metric. A *circular k-colouring* of G or $\mathbb{Z}_k$*-colouring* is a mapping $f : V \to \mathbb{Z}_k$. The notions of *circular q-backbone colouring* and *circular q-backbone chromatic number* are defined similarly to those of *q-backbone colouring* and *circular q-backbone chromatic number* by replacing colouring by circular colouring. The circular q-backbone chromatic number of a graph pair (G,H) is denoted $\mathrm{CBC}_q(G,H)$.

A circular q-backbone k-colouring is trivially a q-backbone k-colouring. On the other hand, a q-backbone k-colouring yields a circular q-backbone $(k+q-1)$-colouring. Hence for every graph pair (G,H) (where H is a spanning subgraph of G) we have

$$\mathrm{BBC}_q(G,H) \leq \mathrm{CBC}_q(G,H) \leq \mathrm{BBC}_q(G,H)+q-1. \tag{2}$$

Also,

$$q \cdot \chi(H) \leq \mathrm{CBC}(G,H) \leq q \cdot \chi(G). \tag{3}$$

Observe that if G is bipartite and H is non-empty, Equation (3) implies that $CBC(G,H) = 2q$. More generally, if $\chi(G) = H$, then $\mathrm{CBC}(G,H) = q \cdot \chi(G)$. However if $2 \leq \chi(H) < \chi(G)$, one can improve on the upper bound. We show in Proposition 33 that, for any pair (G,H) with H a subgraph of G,

$$\mathrm{CBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H). \tag{4}$$

Since $\mathrm{CBC}(G,H) = \chi(G)$ when H is empty and k-COLOURABILITY is NP-complete, for any fixed $k \geq 3$, given a graph G and a subgraph H it is NP-complete to decide if $\mathrm{CBC}(G,H) \leq k$. But if insist that H is not empty, then $\mathrm{CBC}_q(G,H) \geq 2q$ by Proposition 3. Hence deciding if $\mathrm{CBC}_q(G,H)$ is at most k with $k \leq 2q-1$ can be done instantly by always returning 'no'. Less trivially, Proposition 36 shows that if H is a connected spanning subgraph of G, then $\mathrm{CBC}_q(G,H) = 2q$ if and only if G is bipartite. Hence deciding deciding if $\mathrm{CBC}(G,H) = 2q$ can be done in polynomial time.

Inequality (4) implies that $\mathrm{CBC}_q(G,F) \leq 2q+4$ for any planar graph G and forest F in G. We believe that this upper bound can be reduced by at least one.

Conjecture 6. *If G is a planar graph and F a spanning forest of G, then* $\mathrm{CBC}_q(G,F) \leq 2q+3$.

A natural question is to ask whether conjecture would be best possible.

Problem 7. *For any* $q \geq 2$*, does there exist a planar graph* G_q *and a spanning forest* F_q *of* G_q *such that* $\mathrm{CBC}_q(G_q,F_q) = 2q+3$*?*

Conjecture 6 holds if the backbone F is a galaxy. It follows directly from (2) and the fact that $\mathrm{BBC}_q(G,F) \leq q+4$ in such a case, as mentioned earlier. We believe however that one can use one colour less.

Conjecture 8. *Let G be a planar graph and F a galaxy in G, then* $\mathrm{CBC}_q(G,F) \leq 2q+2$.

If true, this conjecture would be tight, since the circular q-backbone chromatic number of a K_4 with backbone $K_{1,3}$ is $2q+2$. As evidence in support of this conjecture 6, Broersma et al. [5] deduced from the Four-Colour Theorem that if G is a planar graph and M a matching in G then $\mathrm{CBC}_q(G,M) \leq 2q+2$.

Broersma et al. also give an example of a planar graph G and a matching M such that (G,M) has no 2-backbone $\mathbb{Z}_5$-colouring. We show in Theorems 37 and 41 that for any fixed $k \in \{4,5\}$, it is NP-complete to decide if $BBC_2(G,M) \leq k$ for given planar graph G and matching M. For larger values of q, the following questions are still open.

Problem 9. *Let G be a planar graph and let M be a matching M in G. For any $q \geq 3$, is it true that* $\mathrm{CBC}_q(G,M) \leq 2q+1$ *?*

Problem 10. *Is it NP-complete to decide if* $\mathrm{CBC}(G,F) \leq 6$ *for a planar graph G and spanning forest F ?*

Problem 11. *For any $g \geq 5$, is it NP-complete to decide if* $\mathrm{CBC}(G,M) \leq 4$ *for a planar graph G of girth at least g and matching M?*

We prove in Theorem 34 that if G has girth at least 5, then $\mathrm{CBC}(G,M) \leq 2q+1$. We wonder if the same holds for planar graph of girth 4.

Problem 12. *Let G be a planar graph of girth* 4 *and let M be a matching in G. Is it true that* $\mathrm{CBC}_q(G,M) \leq 2q+1$ *?*

2 Backbone colouring

2.1 About Conjectures 1 and 3

Proposition 13. *Let G be a graph and let H be a subgraph of G. Then* $\mathrm{BBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H)-q+2$.

Proof. Let g be a $\chi(G)$-colouring of G and h a $\chi(H)$-colouring of H. Let f be the colouring defined by:

$$f(v) = \begin{cases} (h(v)-1)(q-2+\chi(G))+g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v)-1)(q-2+\chi(G))+\chi(G)-g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

It is simple matter to check that f is a q-backbone $((\chi(G)+q-2)\chi(H)-q+2)$-colouring of (G,H). □

A *parachute on v* or a *parachute with harness v* is a complete graph on four vertices whose three edges incident to v are in the backbone.

Proposition 14. *(i) For $q \geq 2$, in a q-backbone $(q+3)$-colouring of a parachute, the harness is coloured in $\{1,q+3\}$.*

(ii) For $q \geq 3$, in a q-backbone $(q+4)$-colouring of a parachute, the harness is coloured in $\{1,2,q+3,q+4\}$.

(iii) For $q \geq 4$, in a q-backbone $(q+5)$-colouring of a parachute, the harness is coloured in $\{1,2,3,q+3,q+4,q+5\}$.

Proof. Let y be the harness.

(ii) If $3 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in \{1,2,q+3,q+4\}$.

(iii) If $4 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in \{1,2,3,q+3,q+4,q+5\}$. □

Proposition 15. *Let G^* and T^* be the graph and its spanning tree depicted Figure 2. For any $q \geq 4$, $BBC_q(G^*,T^*) \geq q+6$.*

Proof. Assume for a contradiction that there is a q-backbone $(q+5)$-colouring ϕ of (G^*,T^*). By Proposition 14-(iii), the vertices y_1,y_2,y_3,y_4,z_1,z_2 are coloured in $\{1,2,3,q+3,q+4,q+5\}$. Without loss of generaltiy, we may assume that $\phi(y_2)\in\{1,2,3\}$. But then $\phi(z_1)$ and $\phi(z_2)$ must be in $\{q+3,q+4,q+5\}$, because y_2z_1 and y_2z_2 are in $E(T^*)$. And $\phi(y_1)$, $\phi(y_2)$ and $\phi(y_3)$ are in $\{1,2,3\}$ because y_3z_1 and y_1z_2 and y_4z_2 are in $E(T^*)$. But $\{y_1,y_2,y_3,y_4\}$ is a clique in G^*, so they must all get different colours, a contradiction. □

Conjecture 3 is implied by Conjecture 1.

Proposition 16. *Conjecture 1 implies Conjecture 3.*

Proof. Assume that Conjecture 1 holds. Let G be a planar graph and F a forest in G. Then (G,F) admits a 2-backbone 6-colouring ϕ. Let ψ be defined by $\psi(v)=\phi(v)$ if $\phi(v)\in\{1,2\}$, $\psi(v)=\phi(v)+1$ if $\phi(v)\in\{3,4\}$, and $\psi(v)=\phi(v)+2$ if $\phi(v)\in\{5,6\}$. One easily check that ψ is a 3-backbone 8-colouring of (G,F). □

2.2 q-backbone $(q+2)$-colouring

Theorem 17. *Given a connected graph G and a spanning connected subgraph H, one can decide in polynomial time if* $\mathrm{BBC}_q(G,H)\leq q+2$.

Proof. Observe first that if H is not bipartite, then $\mathrm{BBC}(H,H)=2q+1$ by (1), and so $\mathrm{BBC}(G,H)\geq q+3$. So we first check if H is bipartite. If not, we return 'no'. If it is, we get a bipartition (A,B) of H.

Observe that if (G,H) has a q-backbone $(q+2)$-colouring, then (free to rename A and B) all the vertices of A are coloured in $\{1,2\}$ and all the vertices of B in $\{q+1,q+2\}$, because H is connected. We then can transform our instance into an instance $I(G,H)$ of 2SAT as follows. For each vertex v, we create a variable x_v. Intuitively, for a vertex $x\in A$ (resp. $x\in B$), the variable x_v will be true if and only if v is coloured 1 (resp. $q+2$) and false if and only if v is coloured 2 (resp. $q+1$). Now for each edge uv, we create the following clauses.

- If u and v are both in A or both in B, we create the clauses $x_u\vee x_v$ and $\bar{x}_u\vee\bar{x}_v$;
- if $u\in A$ and $v\in B$, we create the clause $x_u\vee x_v$.

It is easy to check that (G,H) has a q-backbone $(q+2)$-colouring if and only if $I(G,H)$ is satisfiable.

Since 2SAT is well-known to be polynomial-time solvable, we can decide in polynomial time if $\mathrm{BBC}_q(G,H)\leq q+2$. □

Theorem 18. *For any $q\geq 2$, the following problem is NP-complete problem.*
Input: A planar graph G and a 1*-factor F of G.*
Question: $\mathrm{BBC}_q(G,F)\leq q+2$*?*

Proof. The problem is trivially in NP since a $q+2$-backbone colouring of (G,F) is clearly a certificate.

Reduction from NOT-ALL-EQUAL 3SAT, which is defined as follows:
Input: A set of clauses each having three literals.
Question: Does there exists a *suitable* truth assignment, that is such that each clause has at least one true and at least one false literal?

This problem was shown NP-complete by Schaefer [11].

Let $\mathcal{C}=\{C_1,\dots,C_n\}$ be a collection of clauses of size three over a set U of variables. We will construct a graph pair (G,F) such that F is a 1-factor of G. Since $V(F)=V(G)$, we only precise which edges are in $E(F)$.

The following gadget will be useful. A *forcing gadget at v* or a *forcing gadget with head v* is the graph depicted Figure 3.

A key point in the reduction will be the following claim.

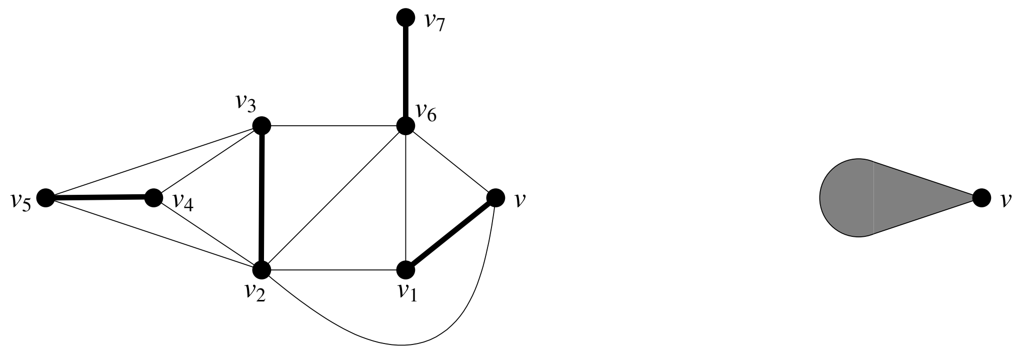

Figure 3: A forcing gadget with head v (left) and its symbol (right) (Edges of $E(F)$ are in bold.)

Claim 19. *In any q-backbone* $(q+2)$*-colouring of a forcing gadget, its head is coloured in* $\{1,q+2\}$.

Proof. Consider a forcing gadget, whose vertices are named as in Figure 3, and ϕ a q-backbone $(q+2)$-colouring of it. Since all the vertices are matched in F, there all must be coloured in $\{1,2,q+1,q+2\}$.

Assume for a contradiction that $\phi(v)=2$. Then $\phi(v_1)=q+2$. Thus $\phi(v_2)\in\{1,q+1\}$. Now if $\phi(v_2)=q+1$, then necessarily $\phi(v_3)=1$. Therefore, whatever the colouring may be, v_4 and v_5 are both adjacent to a vertex coloured 1. Hence $\{\phi(v_4),\phi(v_5)\}=\{2,q+2\}$. Therefore $\{\phi(v_2),\phi(v_3)\}=\{1,q+1\}$. But then v_6 cannot be coloured.

Similarly, we get a contradiction if $\phi(v)=q+1$. □

For every variable $u\in U$, create a *variable subgraph* P_u which is obtained from the path $(a_1(u),b_1(u),a_2(u),b_2(u),\dots,a_n(u),b_n(u))$ by adding a forcing gadget on each of its vertex.

For every clause $C_i=\ell_1\vee\ell_2\vee\ell_3$, create a clause gadget D_i as shown Figure 4.

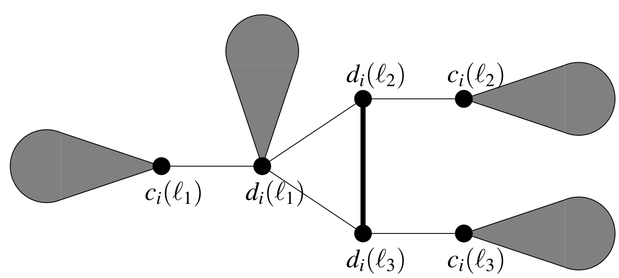

Figure 4: The clause D_i. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Then for each clause C_i and each literal ℓ of C_i, we add a path of length three $(c_i(l),c'_i(l),c''_i(l),a_i(u))$ if ℓ is the non-negated variable u, and $(c_i(l),c'_i(l),c''_i(l),b_i(u))$ if ℓ is the negated variable $\bar{u}$. We also add two forcing gadgets with heads $c'_i(l)$ and $c''_i(l)$.

It is easy to see that the resulting graph G' may be drawn in the plane such that the crossed edges are those of type $c'_i(\ell)c''_i(\ell)$ for some literal ℓ. In particular, the two endvertices of a crossed edge are heads of forcing gadgets.

As long as there is a crossing C between two edges $t(C)u(C)$ and $v(C)w(C)$, we replace these two edges by the crossing gadget $CG(C)$ depicted Figure 5, so that the only edges that are possibly crossed (if there were several crossings on tu or uv) are $t(C)t'(C)$, $u(C)u'(C)$, $v(C)v'(C)$ and $w(C)w'(C)$. After this process, there is no more crossing so the resulting graph G is planar.

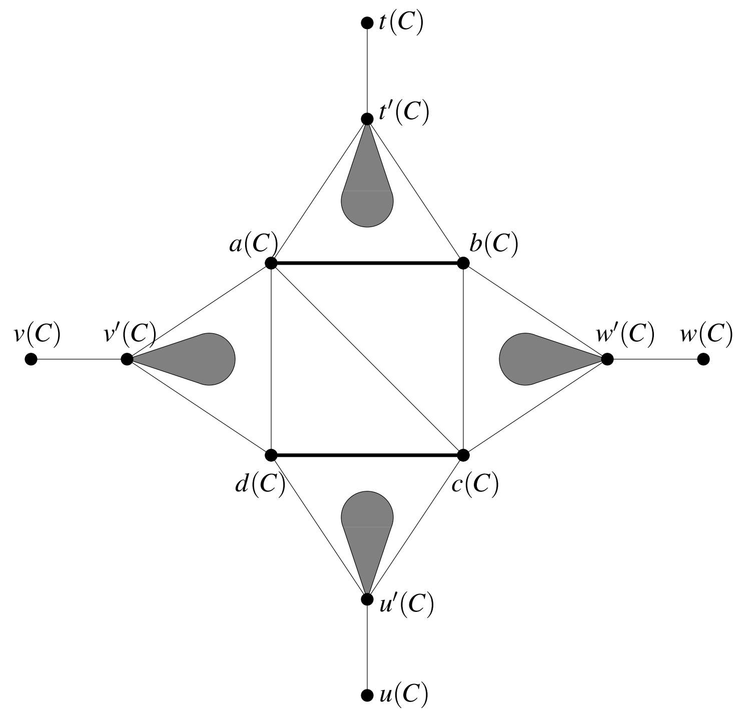

Figure 5: The crossing gadget $CG(C)$. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Let us now prove that (G,F) admits a q-backbone $(q+2)$-colouring if and only if $\mathcal{C}$ has a suitable truth assignment.

Assume first that (G,F) admits a q-backbone $(q+2)$-colouring ϕ. Let u be a variable. Since there are heads of forcing gadgets, by Claim 19, all the $a_i(u)$ and $b_i(u)$ are coloured in $\{1,q+2\}$. Moreover, since they form a path, all the $a_i(u)$ are coloured with the same colour and all the $b_i(u)$ are coloured with the other. Hence one can define the truth assignment ψ by $\psi(u) = true$ if $\phi(a_i(u)) = 1$ for $1 \le i \le n$, and $\psi(u) = false$ if $\phi(a_i(u)) = q+2$ for $1 \le i \le n$.

We shall prove that ψ is suitable.

Claim 20. *For all crossing C in G', we have $\{\phi(t(C)),\phi(u(C))\} = \{1,q+2\}$ and $\{\phi(v(C)),\phi(w(C))\} = \{1,q+2\}$.*

Subproof. By induction on the reverse order of creation of the crossing gadget.

By construction, $t(C)$, $u(C)$, $v(C)$, $w(C)$, $t'(C)$, $u'(C)$, $v'(C)$, and $w'(C)$ are heads of forcing gadgets. So they are coloured 1 or $q+2$. Without loss of generality, we may assume that $\phi(t(C)) = 1$.

If the edge $t(C)t'(C)$ was crossed and then replaced by a series of crossing gadget, by induction, $\phi(t'(C)) = q+2$. It is also trivially the case if $t(C)t'(C)$ still exists. Hence $\{\phi(a(C)),\phi(b(C))\} = \{1,q+1\}$.

Assume for a contradiction that $\phi(u(C)) \ne q+2$. Then, as above, $\{\phi(c(C)),\phi(d(C))\} = \{1,q+1\}$. This is a contradiction, because $a(C)c(C)$ and $a(C)d(C)$ are edges. Hence $\phi(u(C)) = q+2$, and so $\phi(u'(C)) = 1$ and $\{\phi(c(C)),\phi(d(C))\} = \{2,q+2\}$.

In particular, one vertex of $\{a(C),b(C),c(C),d(C)\}$ is coloured 1 and another is coloured $q+2$. Now assume for a contradiction that $\{\phi(v(C)),\phi(w(C))\} \ne \{1,q+2\}$. Then $v(C)$ and $w(C)$ are both coloured 1 or both coloured $q+2$, and so $v'(C)$ and $w'(C)$ are both coloured $q+2$ or both coloured 1, respectively. This is a contradiction, as all vertices of $\{a(C),b(C),c(C),d(C)\}$ are adjacent to some vertex in $\{v'(C),w'(C)\}$. $\diamondsuit$

Let $C_i = \ell_1 \vee \ell_2 \vee \ell_3$ be clause. Claim 20 implies that for $j \in \{1,2,3\}$, $\phi(c_i(\ell_j)) = 1$ if $\psi(u) = false$ and $\phi(c_i(\ell_j)) = q+2$ if $\psi(u) = true$. Now the three $c_i(\ell_j)$, $1 \le j \le 3$, cannot be all coloured 1 (resp. $q+2$), for otherwise $\{\phi(d_i(\ell_2)),\phi(d_i(\ell_3))\}$ must be $\{2,q+2\}$ (resp. $\{1,q+1\}$) and so $d_i(\ell_1)$ cannot be coloured, because it must be coloured in $\{1,q+2\}$ as head of a forcing gadget. Thus at least one of the $c_i(l_j)$ is coloured 1 and at least one is coloured $q+2$, and so C_i has at least one true and at least one false literal.

Hence ψ is suitable.

Reciprocally, assume that $\mathcal{C}$ has a suitable truth assignment ψ. For all $u \in U$ and all $1 \le i \le n$, let us define $\phi(a_i(u)) = 1$ and $\phi(b_i(u)) = q+2$ if $\psi(u) = true$, and $\phi(a_i(u)) = q+2$ and $\phi(b_i(U)) = 1$ if $\psi(u) = false$. Similarly, for every literal ℓ, we set $\phi(c_i(\ell)) = 1$, $\phi(c'_i(\ell)) = q+2$, $\phi(c''_i(\ell)) = 1$, if ℓ is false, and $\phi(c_i(\ell)) = q+2$, $\phi(c'_i(\ell)) = 1$, $\phi(c''_i(\ell)) = q+2$, if ℓ is true.

One can extend ϕ into a q-backbone $(q+2)$-colouring of (G,F). Indeed, it is sufficient to show that we can extend it to forcing, clause and crossing gadgets.

If v is the head of a forcing gadget and $\phi(v) = 1$, we can set $\phi(v_1) = q+2$, $\phi(v_2) = q+1$, $\phi(v_3) = 1$, $\phi(v_4) = q+2$, $\phi(v_5) = 2$, $\phi(v_6) = 2$, and $\phi(v_7) = q+2$. Similarly, we can extend the colouring to the forcing gadget if $\phi(v) = q+2$.

Consider a clause gadget D_i. Since C_i has at least one true and at least one false literal, at least one vertex of $c_i(\ell_1)$, $c_i(\ell_2)$ $c_i(\ell_3)$ is coloured 1 and at least one is coloured $q+2$. If $c_i(\ell_1)$ is coloured $q+2$, and $c_i(\ell_2)$ and $c_i(\ell_2)$ are assigned 1, then we can set $\phi(d_i(\ell_1)) = 1$, $\phi(d_i(\ell_2)) = 2$, and $\phi(d_i(\ell_3)) = q+2$. If $c_i(\ell_1)$ and $c_i(\ell_2)$ are coloured 1, and $c_i(\ell_3)$ is assigned $q+2$, then we can set $\phi(d_i(\ell_1)) = q+2$, $\phi(d_i(\ell_2)) = q+1$, and $\phi(d_i(\ell_3)) = 1$.

Finally consider a crossing gadget such that $\{\phi(t(C)),\phi(u(C))\} = \{\phi(v(C)),\phi(w(C))\} = \{1,q+2\}$. By symmetry, we may assume that $\phi(t(C)) = \phi(v(C)) = 1$ and $\phi(u(C)) = \phi(w(C)) = q+2$. Then we can set $\phi(t'(C)) = \phi(v'(C)) = q+2$, $\phi(u'(C)) = \phi(w'(C)) = 2$, $\phi(a(C)) = 1$, $\phi(b(C) = q+1$, $\phi(c(C)) = q+2$, and $\phi(d(C)) = 2$. $\square$

2.3 2-backbone 5-colouring

2.3.1 Galaxy backbone

Theorem 21. *The following problem is NP-complete.*
Input: A planar graph G and a galaxy F with maximum degree 3.
Question: Is $\mathrm{BBC}_2(G,F) \leq 5$*?*

Proof. Reduction from PLANAR 3-COLOURABILITY, which consists of deciding if a given connected planar graph is 3-colourable. This problem was shown to be NP-complete by Stockmeyer [13].

Let H be a 2-connected planar graph. We shall construct a planar graph G and a galaxy F with maximum degree 3 in G such that $\mathrm{BBC}_2(G,F) \leq 5$ if and only if H is 3-colourable.

As a forcing gadget at v, we will use the parachute with harness v. It is easy to see that in a 2-backbone 5-colouring of a parachute, its harness is coloured in $\{1,5\}$.

We consider any embedding of H. For each face $(x_1,x_2,\dots,x_k,x_1)$ of H, we put a cycle $(z_1,z_2,\dots,z_{2k}z_1)$, inside which we put parachutes on every vertex z_i for every $1 \leq i \leq 2k$. We then add the edges x_iz_{2i} x_iz_{2i+1} for all $1 \leq i \leq k$.

Assume that (G,F) has a 2-backbone 5-colouring ϕ, then, because of the parachutes, all the vertices in the cycles added inside faces must be coloured in $\{1,5\}$. Moreover consecutive vertices on one such cycles get different colours, so one is coloured 1 and the other is coloured 5. Hence all the vertices in H are coloured in $\{2,3,4\}$. Hence ϕ induces a proper 3-colouring on H with colours $\{2,3,4\}$.

Reciprocally, assume that H is 3-colourable. Then there exists a proper 3-colouring c of H into $\{2,3,4\}$. One can then colour all the cycles inside faces with 1 and 5. The colouring can then easily be extended into a 2-backbone 5-colouring of (G,F). □

Corollary 22. *The following problem is NP-complete problem.*
Input: A planar graph G and a spanning tree T of G.
Question: $\mathrm{BBC}_2(G,T) \leq 5$*?*

Proof. Reduction from the problem of Theorem 21. Given an instance (G,F) of this problem, one can find a set E_1 of edges of $G-F$ such that $F \cup E_1$ is a spanning tree of G. Then for every edge e of E_1 we add a path P_e of length 4 (with 3 new internal vertices) to get a graph G' and we let T' be the tree whose edge set is $E(F) \cup \bigcup_{e \in E_1} E(P_e)$. Since for any pair $(\alpha,\beta) \in \{1,2,3,4,5\}^2$, there is a 2-backbone 5-colouring of the path of length 4 such that the first vertex is coloured α and the last vertex is coloured β, it follows that (G,F) has a 2-backbone 5-colouring if and only if (G',T') has a 2-backbone 5-colouring. □

Theorem 23. *The following problem is NP-complete problem.*
Input: A planar graph G and a galaxy F with maximum degree 2.
Question: Is $\mathrm{BBC}(G,F) \leq 5$*?*

Proof. The proof is identical the one of Theorem 21. The only difference comes from the forcing gadget, which is more complicated because it cannot contains stars of degree 3 in F.

To construct the forcing gadget, we need an auxiliary gadget, called *no-3-gadget*. It is depicted Figure 6.

Claim 24. *In any* 2*-backbone* 5*-colouring of a no-3-gadget, its roof is not coloured in* 3.

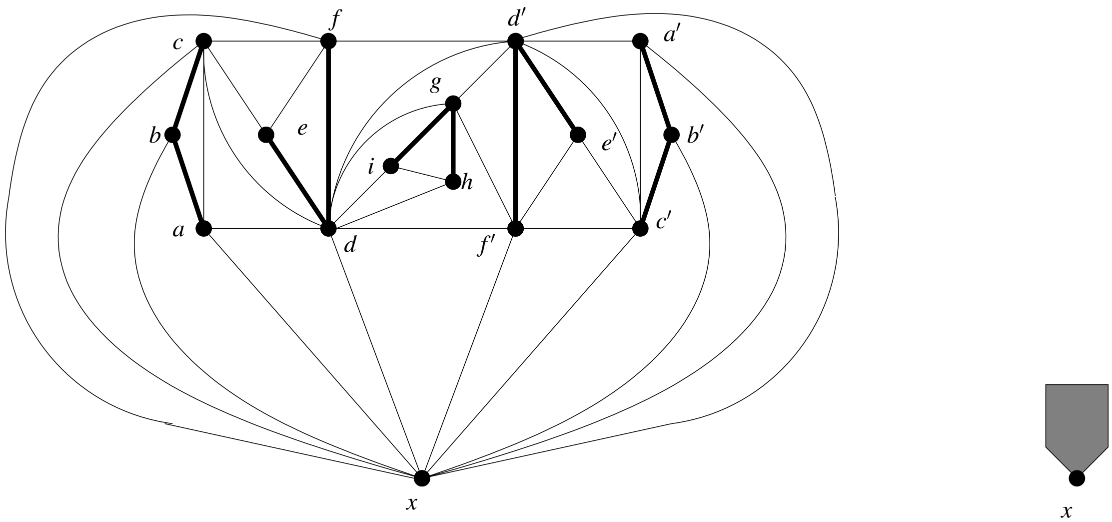

Figure 6: The no-3-gadget with roof x and its symbol.

Proof. We will denote the vertices of the no-3-gadget by their names in Figure 6. Assume for a contradiction that there is a 2-backbone 5-colouring ϕ of a no-3-gadget such that $\phi(x) = 3$.

Assume first that $\phi(a) \in \{4,5\}$, then $\phi(b) \in \{1,2\}$ and $\{\phi(a), \phi(c)\} = \{4,5\}$. Hence $\phi(d) \in \{1,2\}$ and so $\{\phi(f), \phi(c)\} = \{4,5\}$. Therefore $\phi(e) = 3$ and so $\phi(d) = 1$. Similarly, if $\phi(a) \in \{1,2\}$, we obtain that $\phi(d) = 5$. Hence, $\phi(d) \in \{1,5\}$.

Similarly, $\phi(d') \in \{1,5\}$. Free to consider $6-\phi$ instead of ϕ, we may assume that $\phi(d) = 1$ and $\phi(d') = 5$. Thus $\phi(f') = 2$.

Now $\phi(g) \in \{3,4\}$. If $\phi(g) = 3$, then $\{\phi(i), \phi(h)\} = \{1,5\}$, and if $\phi(g) = 4$, then $\{\phi(i), \phi(h)\} = \{1,2\}$. In both cases, one of h and i is coloured 1, which is impossible because $\phi(d) = 1$. □

The forcing gadget is the one depicted Figure 7.

Claim 25. *In any* 2*-backbone* 5*-colouring of a forcing gadget, its head is coloured in* $\{1,5\}$.

Proof. Consider a forcing gadget, whose vertices are named as in Figure 3, and ϕ a 2-backbone 5-colouring of it.

Let us prove that $\phi(w) = 3$ and so that $\phi(v) \in \{1,5\}$. Assume for a contradiction that $\phi(w) \neq 3$. Without loss of generality, we may assume that $\phi(w) \in \{1,2\}$.

Observe that the vertices x, y, z, x', y', z' are not assigned 3 because they are roofs of no-3-gadgets.

If $\phi(w) = 1$, then $(\phi(x), \phi(y), \phi(z))$ and $(\phi(x'), \phi(y'), \phi(z'))$ is either $(4,2,5)$ or $(5,2,4)$. Hence the vertices x, x' and z are all coloured in $\{4,5\}$, which is impossible, since they form a triangle.

If $\phi(w) = 2$, then $(\phi(x), \phi(y), \phi(z))$ and $(\phi(x'), \phi(y'), \phi(z'))$ is either $(4,1,5)$ or $(5,1,4)$. Hence the vertices x, x' and z are all coloured in $\{4,5\}$, which is impossible, since they form a triangle. □

To get the equivalence between the 3-colourability of the original graph H and the existence of a 2-backbone 5-colouring of (G,F), it remains to prove that for any $\alpha \in \{1,5\}$, there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured α.

We denote the vertices by their names in Figure 3. Set $\phi(w) = 3$, $\phi(x) = \phi(y') = 1$, $\phi(y) = \phi(z') = 5$, $\phi(z) = 2$ and $\phi(x') = 4$.

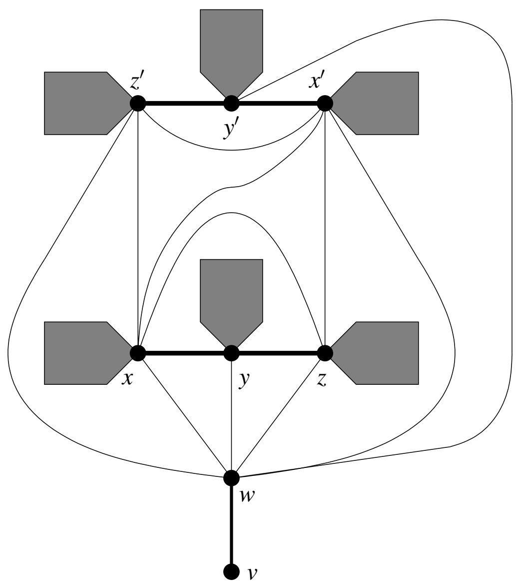

Figure 7: The forcing gadget with head v. (Edges of $E(F)$ are in bold, no-3-gadgets are represented by their symbols.)

Observe that no vertex in $\{x,y,z,x',y',z'\}$ has been coloured 3. Hence, it remains to prove that for any $\beta \in \{1,2,4,5\}$, there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured β. By symmetry, ϕ and $6-\phi$, it suffices to prove that one exists for $\beta \in \{1,2\}$. We denote the vertices by their names in Figure 6. Let us denote by $\bar{\beta}$ the colour of $\{1,2\} \setminus \{\beta\}$.

$\phi(a) = 3$, $\phi(b) = \bar{\beta}$, $\phi(c) = 5$, $\phi(d) = 4$, $\phi(e) = \beta$, $\phi(f) = \bar{\beta}$, $\phi(a') = 3$, $\phi(b') = 5$, $\phi(c') = \bar{\beta}$, $\phi(d') = 5$, $\phi(e') = \beta$, $\phi(f') = 3$, $\phi(g) = 1$, $\phi(h) = 5$, $\phi(i) = 3$. □

2.3.2 Hamiltonian-path backbone

Theorem 26. *The following problem is NP-complete problem.*
Input: A planar graph G with a hamiltonian path P.
Question: $\mathrm{BBC}_2(G,P) \leq 5$?

To prove this theorem, we shall use a reduction similar to the one of Theorem 21. However, we do not reduce directly from PLANAR 3-COLOURABILITY but use an intermediate problem whose NP-completeness is proven by reducing PLANAR 3-COLOURABILITY to it.

This intermediate problem is the following:

TRACEABLE PLANAR 3-COLOURABILITY
Input: A planar graph G with a hamiltonian path P.
Question: Is G 3-colourable?

Lemma 27. TRACEABLE PLANAR 3-COLOURABILITY *is NP-complete.*

Proof. Reduction from PLANAR 3-COLOURABILITY. Let H be a connected planar graph. We will construct a planar graph G having a hamiltonian path P such that $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

To do so, we shall construct a sequence of pairs (G_i,P_i) for $1 \leq i \leq |V(H)|$ such that P_i is a path in the planar connected graph G_i, $|V(P_i)| = |V(G_i)| - |V(H)| + i$, and $\chi(G_i) \leq 3$ if and only if $\chi(H) \leq 3$. Then the path $P := P_{V(H)}$ will be a hamiltonian path of $G := G_{V(H)}$ and $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

Let x be a vertex of H. We set $G_1 := H$ and $P_1 := (x)$. Trivially, (G_1,P_1) verifies the above property.

Assume now that $i \geq 1$ and let us construct (G_{i+1},P_{i+1}) from (G_i,P_i). Let $P_i = (v_1,v_2,\dots,v_\ell)$ be a path. Since G_i is connected, there exists j such that v_j is adjacent to a vertex y in $V(G_i) \setminus V(P_i)$. If $j = 1$, then let $P_{i+1} := (y,v_1,v_2,\dots,v_\ell)$, and $G_{i+1} := G_i$; if $j = p$, then let $P_{i+1} := (v_1,v_2,\dots,v_\ell,y)$, and $G_{i+1} := G_i$; if y is also incident to v_{j+1}, let $P_{i+1} := (v_1,\dots,v_j,y,v_{j+1},\dots,v_\ell))$. In those three cases, (G_{i+1},P_{i+1}) has trivially the desired property.

So we may assume that $1 < j < \ell$ and y is not adjacent to v_{j+1}. Let $y_1,y_2,\dots y_r$ be the neighbours of v_j in their order around it such that $v_{j+1} = y_r$, $y_k = y$ and $v_{j-1} = y_q$ for $q < r$.

Let G_{i+1} be the graph obtained from G_i as follows. For all $1 \leq s \leq k-1$, remove the edge v_jy_s, add three vertices a_s,b_s,c_s and the edges $a_sb_s,b_sc_s,c_sa_s,v_ja_s,v_jb_s,b_sy_s$; Add the edges c_sa_{s+1} for all $1 \leq s \leq k-2$, and $v_{j+1}a_1$. Finally add a vertex y' and the edges yy' and $y'c_{k-1}$. Let P_{i+1} be the path obtained from P_i by replacing the edge v_jv_{j+1} by the subpath $(v_j,y,c_{k-1},b_{k-1},a_{k-1},\dots,c_1,b_1,a_1,v_{j+1})$. See Figure 8, which illustrates the construction when $k = 5$.

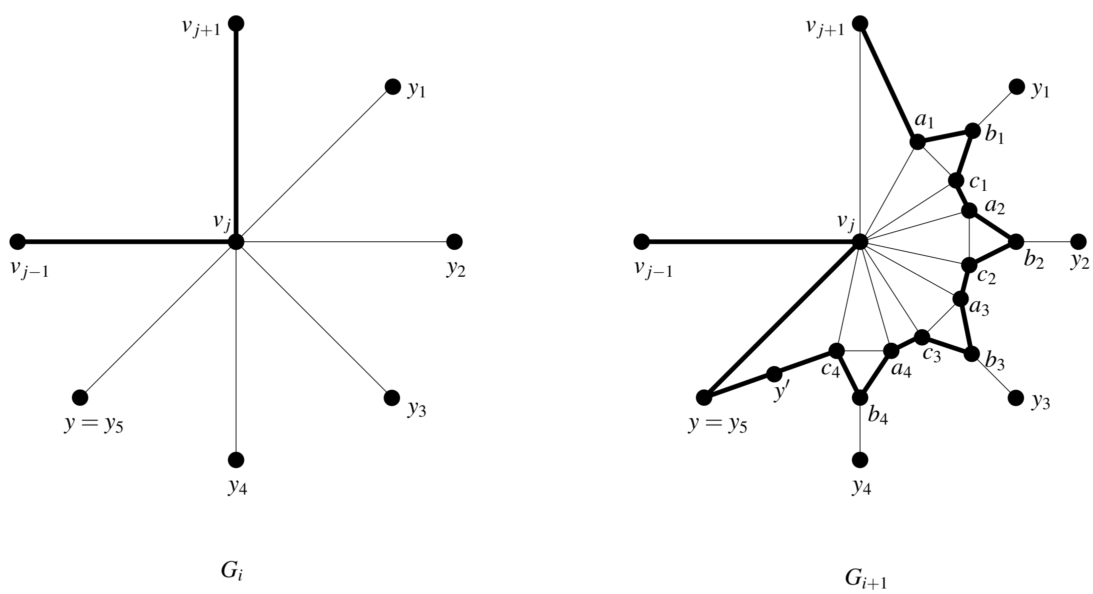

Figure 8: Constructing (G_{i+1},P_{i+1}) from (G_i,P_i) (Egdes of the paths are in bold.)

Clearly, the number of vertices not covered by P_{i+1} in G_{i+1} is one less than the number of vertices not covered by P_i in G_i. So, since $|V(P_i)| = |V(G_i)| - |V(H)| + i$, we have $|V(P_{i+1})| = |V(G_{i+1})| - |V(H)| + i + 1$.

It remains to prove that G_{i+1} is 3-colourable if and only if G_i is.

Assume first that G_{i+1} admits a proper 3-colouring ϕ in $\{1,2,3\}$. We claim that it also induces a proper 3-colouring of G_i. Indeed, without loss of generality, we may assume that $\phi(v_j) = 1$ and $\phi(v_{j+1}) = 2$. Then for all $1 \leq s \leq s-1$, $\phi(a_s) = 3$ and $\phi(c_s) = 2$, so $\phi(b_s) = 1$. Hence $\phi(y_s) \neq 1$. Therefore, for all $1 \leq s \leq s-1$, $\phi(y_s) \neq \phi(v_j)$. Since the v_jy_s, $1 \leq s \leq k-1$, are the only edges of G_i which are not in G_{i+1}, ϕ is a proper 3-colouring of G_i.

Conversely, assume that G_i admits a 3-colouring ϕ in $\{1,2,3\}$. It induces a partial proper 3-colouring of G, such that $\phi(v_j) \neq \phi(y_s)$ for all $1 \leq s \leq k-1$. Let us extend it. Without loss of generality, $\phi(v_j) = 1$ and

$\phi(v_{j+1}) = 2$. For all $1 \le s \le s-1$, set $\phi(a_s) = 3$, $\phi(b_s) = 1$, and $\phi(c_s) = 2$. Finally, colour y' with the colour in $\{1,2,3\} \setminus \{\phi(y), \phi(c_{k-1})\}$. This gives a proper 3-colouring of G_{i+1}. □

Proof of Theorem 26. Reduction from TRACEABLE PLANAR 3-COLOURABILITY. Let (H,Q) be an instance of this problem. We shall construct a graph G and a hamiltonian path P of G such that $\mathrm{BBC}(G,P) \le 5$ if and only if $\chi(H) \le 3$. To do so we start from H and for each edge xy of Q, we will plug in an edge gadget $E(xy)$ containing a hamiltonian path $P(xy)$ from x to y. The union of all the $P(xy)$, $xy \in E(Q)$, will then be a hamiltonian path P of the resulting graph G.

To construct the edge gadget, we use an auxiliary forcing gadget depicted Figure 9. The *head* of such a gadget is the vertex denoted by v in the figure. Its *fringes* are the vertices denoted by a and e.

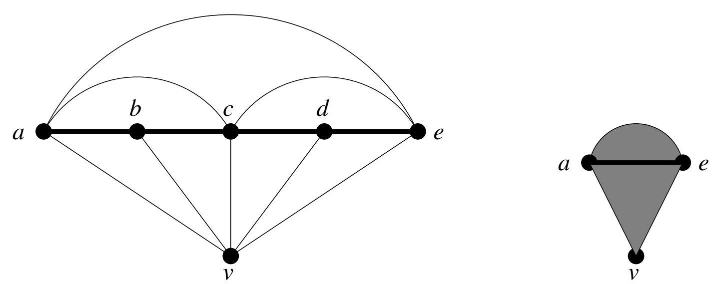

Figure 9: The forcing gadget with head v and fringes a and e (left) and its symbol (right)

Claim 28. *In any* 2*-backbone* 5*-colouring of a forcing gadget, the head is coloured in* $\{2,4\}$.

Proof. We denote the vertices by their names in Figure 9. Suppose for a contradiction that there is a 2-backbone 5-colouring ϕ such that $\phi(v) \notin \{2,4\}$. By the symmetry $\phi \to 6-\phi$, we may assume that $\phi(v) \in \{1,3\}$.

If $\phi(v) = 3$, then all the vertices a,b,c,d,e are coloured in $\{1,2,4,5\}$. On the path (a,b,c,d,e), vertices coloured $\{1,2\}$ alternate with vertices coloured $\{4,5\}$. Hence a, c, and e are all coloured in $\{1,2\}$, or all coloured in $\{4,5\}$, which is a contradiction as they form a clique.

If $\phi(v) = 1$, then all the vertices a,b,c,d,e are coloured in $\{2,3,4,5\}$. Now $\phi(b)$ is at distance 2 from the two distinct colours $\phi(a)$ and $\phi(c)$, hence $\phi(b) \in \{2,5\}$. Similarly, $\phi(d) \in \{2,5\}$. But $\phi(c)$ is at distance 2 from $\phi(b)$ and $\phi(d)$, so $\phi(b) = \phi(d)$. Then the three vertices a, c, and e are all coloured in $\{2,3,4,5\} \setminus \{\phi(b)-1, \phi(b), \phi(b)+1\}$, which has cardinality 2. This is a contradiction as those three vertices form a clique. □

Now the edge gadget is the one depicted Figure 10.

Let us now prove that $\mathrm{BBC}(G,P) \le 5$ if and only if $\chi(H) \le 3$.

Assume first that (G,P) admits a 2-backbone 5-colouring ϕ. Since H is a subgraph of G, ϕ induces a proper colouring on H. We shall prove that every vertex of H is coloured in $\{1,3,5\}$, thus proving that this proper colouring uses (at most) 3 colours.

Every vertex v of H is contained in an edge xy of Q, so it is contained in the edge gadget $E(xy)$ in G. So it is adjacent to two vertices (namely v_1 and v_2 if $v = x$, and v_2 and v_3 if $v = y$), which are heads of forcing gadgets and adjacent. Hence by Claim 40, one of these vertices is coloured 2 and the other is coloured 4. Hence v must be coloured in $\{1,3,5\}$.

Let us now assume that H is 3-colourable. Then there exists a proper colouring ϕ of H with $\{1,3,5\}$. Let us now extend into a 2-backbone 5-colouring of (G,P). It is sufficient to prove that we can extend it to every edge-gadget.

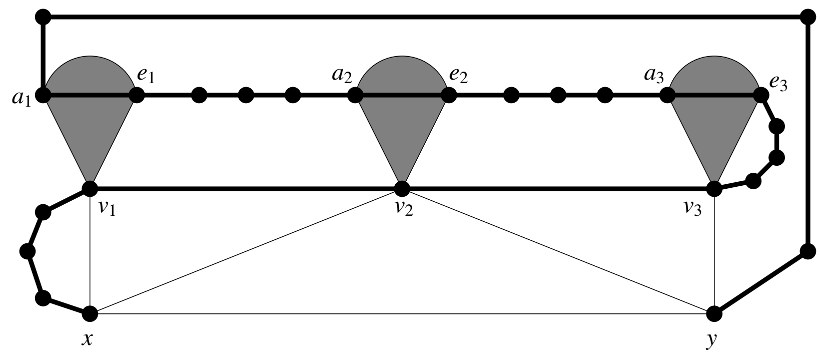

Figure 10: The edge gadget $E(xy)$ and its hamiltonian path $P(xy)$ in bold (Forcing gadgets are represented by their symbols.)

To extend it the edge-gadget $E(xy)$ (we use the names of Figure 10), set $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = 4$. Now, since for any pair $(\alpha, \beta) \in \{1,2,3,4,5\}^2$, there is a 2-backbone 5-colouring of the path of length 4 such that the first vertex is coloured α and the last vertex is coloured β, it suffices to prove that we can extend ϕ to the forcing gadget.

Consider such a forcing gagdet (with vertex names as in Figure 9). Then $\phi(v) \in \{2,4\}$. By the symmetry $\phi \to 6-\phi$, we may assume that $\phi(v) = 2$. Then setting $\phi(a) = 4$, $\phi(b) = \phi(d) = 1$, $\phi(c) = 3$ and $\phi(e) = 5$, we obtain the desired extension.

Hence, $\mathrm{BBC}(G,P) \leq 5$. □

2.4 q-backbone $(q+3)$-colouring for $q \geq 3$

Theorem 29. *For any $q \geq 3$, the following problem is NP-complete problem.*
Input: A planar graph G and a galaxy F with maximum degree 3.
Question: Is $\mathrm{BBC}_q(G,F) \leq q+3$?

Proof. Reduction from PLANAR 3-COLOURABILITY.

We shall need the graph, which we call a *kite*, depicted Figure 11. The vertex named t on the figure is the *tip* of the kite, the one named u its *corner*.

Claim 30. *If ϕ is a q-backbone $(q+3)$-colouring of a kite such that $\phi(t) \in \{1,2,3,q+1,q+2,q+3\}$, then either $\phi(t) \in \{1,2,3\}$ and $\phi(u) = q+3$, or $\phi(t) \in \{q+1,q+2,q+3\}$ and $\phi(u) = 1$.*

Proof. Observe that the vertices v, z_1, z_2, z_3 are harnesses of parachutes. This by Proposition 14-(i), they must be assigned 1 or $q+3$.

Assume that $\phi(v) = 1$, then $\phi(z_1) = \phi(z_2) = \phi(z_3) = q+3$. Thus $\{\phi(s_1), \phi(s_2)\} = \{q+1, q+2\}$ and so $\phi(u) = q+3$ and $\phi(t) \in \{1,2,3\}$.

Similarly if $\phi(u) = q+3$, we obtain $\phi(u) = 1$ and $\phi(t) \in \{q+1,q+2,q+3\}$. □

Let H be a planar graph. Let (G,F) be the graph pair obtained from H as follows. Firstly, for each face f of H, we create a parachute P_f with harness v_f, and for each vertex x incident to f, we create a kite $K_f(x)$ with tip x and corner $u_f(x)$. We then link the vertex v_f to all the $u_f(x)$. Secondly, for every vertex $x \in V(H)$, we add a vertex y_x and the edge xy_x in the backbone.

Clearly, the resulting graph G is planar and the resulting backbone F is a galaxy with maximum degree 3.

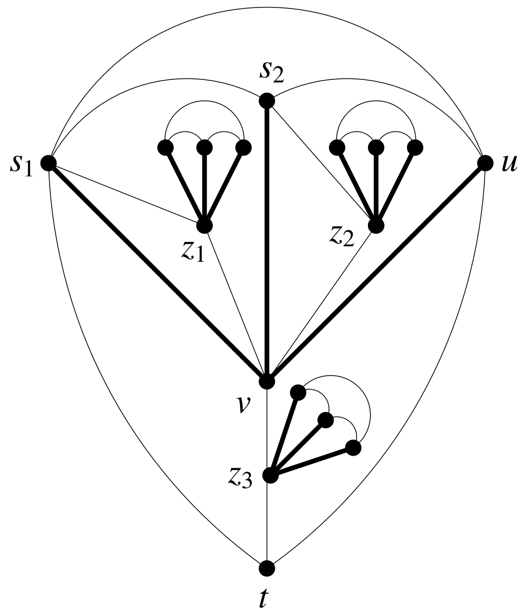

Figure 11: The kite

Let us now prove that $BBC_q(G,F) \leq q+3$ if and only if H is 3-colourable.

Assume first that (G,F) admits a q-backbone $(q+3)$-colouring ϕ. Observe that each vertex x in $V(H)$ is coloured in $\{1,2,3,q+1,q+2,q+3\}$, because it is adjacent to y_x in F.

Let x be a vertex in $V(H)$. Free to consider $q+4-\phi$, we may assume that $\phi(x) \in \{1,2,3\}$. Consider a face f incident to x in H. By Claim 30, the kite $K_f(x)$ has its corner coloured $q+3$. Together with Proposition 14-(i), this implies that $\phi(v_f) = 1$. Thus, the corner of the kites in f cannot be coloured 1, therefore there are coloured $q+3$ and so by Claim 30, all the vertices incident to f in H are all coloured in $\{1,2,3\}$. Applying this reasoning to each face of H, we obtain that all vertices of H are coloured in $\{1,2,3\}$. Hence, ϕ induces a proper 3-colouring on H.

Conversely, assume that H admits a proper 3-colouring c. One can extend into a q-backbone $(q+3)$-colouring of (G,F) as follows. For every $x \in V(H)$, we colour y_x with $q+3$; for every face f, we colour the vertex v_f with 1 and the corners of the kites by $q+3$. One can then extend the colouring to each kite (as in the proof of Claim 30) to obtain a q-backbone $(q+3)$-colouring of (G,F). □

The reduction above can be modified to have a spanning tree T for the backbone in place of the galaxy F. It suffices consider a spanning tree U of H and do the following: add a path of length two in the backbone along each edge of the tree U; for each kite, add tz_3 and vz_3 in the backbone and add paths of length two in the backbone along edges z_1v and z_2v. This will prove the following statement.

Theorem 31. *The following problem is NP-complete problem.*
Input: A planar graph G and a spanning tree T of G.
Question: Is $\mathrm{BBC}_2(G,T) \leq 5$*?*

2.5 q-backbone $(q+5)$-colouring

Theorem 32. *For any $q \geq 4$, the following problem is NP-complete problem.*
Input: A planar graph G and a spanning tree T of G.

Question: Is $\mathrm{BBC}_q(G,F) \leq q+5$*?*

Proof. Reduction from PLANAR 3-COLOURABILITY.

Let H be a planar graph. We shall construct a planar graph G together with a spanning tree T such that H is 3-colourable if and only if $BBC_q(G<T) \leq q+5$. Take U be a spanning tree of H.

We first construct a graph G' from H by adding for every edge $e=uv$ of U we add a vertex x_e linked to u and v. We let T' be the spanning tree of G' induced by the new edges. The pair (G,T) is then obtained from (G',T') by adding a parachute on every vertex. Clearly G is planar as for each edge $e=uv$ the path ux_ev can be drawn along the edge uv.

Suppose that (G,T) admits a q-backbone $(q+5)$-colouring. Then by Proposition 14-(iii), every vertex in G' is coloured in $\{1,2,3,q+3,q+4,q+5\}$. Now that the vertices of H form one of the part of the bipartition of T'. Hence, the colours of the vertices of H are either all in $\{1,2,3\}$ or all in $\{q+3,q+3,q+5\}$. In both cases, ϕ induces a proper 3-colouring on H.

Conversely, it is straightforward to extend a a proper 3-colouring of H into a q-backbone $(q+5)$-colouring of (G,T). □

3 Circular backbone colouring

The following Proposition is an analogue to Proposition 13 and its proof is similar.

Proposition 33. *Let G be a graph and let H be a subgraph of G such that $2 \leq \chi(H) < \chi(G)$. Then $\mathrm{CBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H)$.*

Proof. Let g be a $\chi(G)$-colouring of G and h a $\chi(H)$-colouring of H.

Assume first that $\chi(H)$ is even. Let f be the colouring defined by:

$$f(v) = \begin{cases} (h(v)-1)(q-2+\chi(G))+g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v)-1)(q-2+\chi(G))+\chi(G)-g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

One can easily check that f is a circular q-backbone $((\chi(G)+q-2)\chi(H))$-colouring of (G,H).

Assume now that $\chi(H)$ is odd. Let f be the colouring defined by:

$$f(v) = \begin{cases} 1, & \text{if } h(v)=1 \text{ and } g(v)=\chi(G), \\ g(v)+1, & \text{if } h(v)=1 \text{ and } g(v)<\chi(G), \\ \chi(G)+q-1, & \text{if } h(v)=2 \text{ and } g(v)=\chi(G)-1, \\ \chi(G)+q, & \text{if } h(v)=2 \text{ and } g(v)=\chi(G), \\ 2\chi(G)+q-2-g(v), & \text{if } h(v)=2 \text{ and } g(v)<\chi(G)-1, \\ (h(v)-1)(q-2+\chi(G))+g(v), & \text{if } h(v) \text{ is odd and } h(v)>2, \\ (h(v)-1)(q-2+\chi(G))+\chi(G)-g(v) & \text{if } h(v) \text{ is even and } h(v)>2. \end{cases}$$

One can easily check that f is a circular q-backbone $((\chi(G)+q-2)\chi(H))$-colouring of (G,H). □

3.1 Planar graphs of girth at least 5

Theorem 34. *Let G be a planar graph of girth at least 5 and M a matching in G. Then $\mathrm{CBC}_q(G,M) \leq 2q+1$.*

Proof. Our proof is based on a structural result of Borodin and Glebov [1]. See also [9].

Theorem 35 (Borodin and Glebov [1]). *The vertex set of every planar graph of girth at least 5 can be partitioned into an independent set and a set which induces a forest.*

Let (S,F) be a partition of $V(G)$ such that S is stable and F induces a forest. Let us colour every vertex of S with 1. Now since F is a forest, it has an ordering $v_1,\dots,v_p$ such that for every i, v_i has at most one neighbour in $\{v_1,\dots,v_{i-1}\}$. We colour the vertices of F according to this ordering as follows. If v_i has no neighbour in $\{v_1,\dots,v_{i-1}\}$, then colour it with $q+1$. If v_i has a neighbour u in $\{v_1,\dots,v_{i-1}\}$ and $uv_i \notin E(M)$, then colour it with a colour of $\{q+1,q+2\}$ not assigned to u. If v_i has a neighbour u in $\{v_1,\dots,v_{i-1}\}$ and $uv_i \in E(M)$, then assign $2q+1$ (resp. 2 to v_i) if u is coloured $q+1$ (resp. $q+2$). It is easy to check that the obtained colouring is a q-backbone $\mathbb{Z}_{2q+1}$-colouring of (G,M). □

3.2 Circular q-backone $2q$-colouring

Proposition 36. *Let G be a graph and H a spanning connected subgraph of G. Then $\mathrm{CBC}_q(G,H)=2q$ if and only if G is bipartite.*

Proof. If G is bipartite, then $\chi(G)=\chi(H)=2$. Thus, by Equation 3, $\mathrm{CBC}_q(G,H)=2q$.

Assume now that (G,H) admits a circular q-backbone $2q$-colouring f. Let v be a vertex of G. Without loss of generality, we may assume that $f(v)=1$. Then all the neighbours of v in H must be coloured $q+1$. And so on, by induction, all the vertices at even distance from v in H are coloured 1 and all the vertices at odd distance from v in H are coloured $q+1$. Since H is connected and spans G, it follows that all vertices are coloured 1 or $q+1$, so G is bipartite. □

Proposition 36 implies that given a graph G and a spanning connected subgraph H, deciding if $\mathrm{CBC}_q(G,H)=2q$ can be done in polynomial time. In contrast, if the condition of G be connected is removed, when $q=2$, the problem becomes NP-complete, as shown by the following theorem.

Theorem 37. *The following problem is NP-complete problem.*
Input: A planar graph G and a matching M in G.
Question: Is $\mathrm{CBC}_2(G,M)\le 4$?

Proof. The problem is trivially in NP since a circular 2-backbone 4-colouring of (G,F) is clearly a certificate.

To prove it is NP-complete, we give a reduction from NOT-ALL-EQUAL 3SAT.

Let $\mathcal{C}=\{C_1,\dots,C_n\}$ be a collection of clauses of size three over a set U of variables. We will construct a graph pair (G,M) such that M is a matching in G.

To do so we need some definitions and gadgets.

Colours 1 and 3 are said to be *twins* and so do the colour 2 and 4. Trivially two vertices joined by an edge of M receives distinct twin colours. Two colours are *siblings* if they are equal or twins.

A *link* with ends u and v and central edge w_1w_2 is a subgraph with vertex set $\{u,v,w_1,w_2\}$ and edge set $\{uw_1,uw_2,vw_1,vw_2\}$ with $w_1w_2\in M$. Two ends of a link are said to be *linked*.

Claim 38. *In a circular 2-backbone 4-colouring c, the colours of the ends of a link are siblings.*

Proof. The two vertices w_1 and w_2 are joined by an edge of M, so $\{c(w_1),c(w_2)\}\in\{\{1,3\};\{2,4\}\}$. Hence if u is coloured in $\{1,3\}$ (resp. $\{2,4\}$), then $\{c(w_1),c(w_2)\}$ is $\{2,4\}$ (resp. $\{1,3\}$), and so v is coloured in $\{1,3\}$ (resp. $\{2,4\}$). □

For each variable $u\in U$, we create a *variable gadget G^u* which is obtained from the distinct vertices $a^u_1,a^u_2,\dots,a^u_n$ by linking, from $1\le i\le n-1$, the vertices a^u_i and a^u_{i+1} by an link with central edge $b^u_ic^u_i$.

Claim 38 (and its proof) immediately implies the following.

Claim 39. *In a circular* 2*-backbone* 4*-colouring of* G^u*, all the* a^i_u *are coloured with two sibling colours and all the* b^i_u *with the two other colours (which are also siblings).*

For each clause $C_i = \ell^i_1 \vee \ell^i_2 \vee \ell^i_3$, we create a triangle $z^i_1 z^i_2 z^i_3$. Now for $j = 1,2,3$, if ℓ^i_j is the nonnegated literal u, we join z^i_j with a^u_i, and if ℓ^i_j is the negated literal $\bar{u}$, we join z^i_j with b^u_i. Such edges are said to be red. So far, the obtained graph H is not planar, but we can clearly draw it such that only red edges cross. We can now subdivide every red edge into a red path such that every edge is crossed at most once. We then replace the red edges which are not crossed by a link (with the same end) and two red edges uv and xy that cross each other by the crossing gadget depicted Figure 12. The resulting graph G is planar and it comes with a matching M.

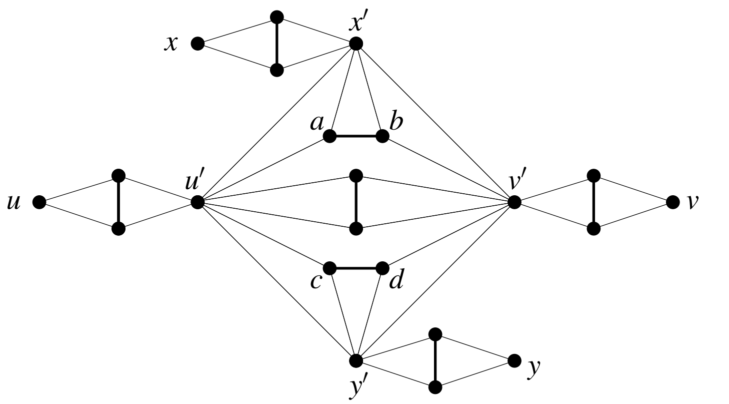

Figure 12: The crossing gadget

Claim 40. *In a circular* 2*-backbone* 4*-colouring of the forcing gadget, the colours of u and v are siblings and the colours of u and v are siblings. In addition, for any* 4*-tuple* $\{c_u, c_v, c_x, c_y\}$ *such that* c_u *and* c_v *are siblings and* c_x *and* c_y *are siblings, there is a circular* 2*-backbone* 4*-colouring c of the forcing gadget such that* $c(u) = c_u$*,* $c(v) = c_v$*,* $c(x) = c_x$*, and* $c(y) = c_y$*.*

Proof. Consider first a circular 2-backbone 4-colouring of the forcing gadget. u is linked to u', which is linked to v', which in turn is linked to v. Hence, by Claim 38, the colours of u and v are siblings.

Assume that x is coloured in $\{1,3\}$, then x' is also coloured in $\{1,3\}$, say 1. The vertices a and b are assigned twin colours, so one is coloured 2 and the other 4. We now distinguish two cases depending on the colour of u'.

1. Assume u' is coloured 3. Then v' must also be coloured 3. The vertices c and d are assigned twin colours, so one is coloured 2 and the other 4. Hence y' is coloured 1.

2. Assume u' is coloured in $\{2,4\}$. Without loss of generality, we may assume it is coloured 2. Then a is coloured 4 and b is coloured 2, so v' is coloured 4. Hence y' is coloured in $\{1,3\}$.

In both cases the colour of x and y' are siblings, and so, by Claim 38, the colours of x and y are siblings.

For any 4-tuple $\{c_u, c_v, c_x, c_y\}$ such that c_u and c_v are siblings and c_x and c_y are siblings, finding the desired circular 2-backbone 4-colouring is straightforward and left to the reader. □

We shall now prove that $\mathcal{C}$ admits a suitable truth assignment if and only if $\mathrm{CBC}(G,M) \leq 4$.

Assume first that (G,M) admits a circular 2-backbone 4-colouring. Let ϕ be the truth assignment defined by $\phi(u) = true$ if all the a^i_u are coloured in $\{1,3\}$, and $\phi(u) = false$ if all the a^i_u are coloured in $\{2,4\}$. Note that is well defined by Claim 39. Now by Claims 38 and 40, for each clause $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$, the vertex z_j^i is coloured in $\{1,3\}$ if and only if the literal ℓ_j^i is true. But since $z_1^i z_2^i z_3^i$ is a triangle, at least three colours must appear on these vertices, and so at least one from $\{1,3\}$ and at least one from $\{2,4\}$. Hence, at least one of the literals of C_i is true and at least one is false. Thus ϕ is suitable.

Reciprocally, assume that $\mathcal{C}$ admits a suitable truth assignment ϕ. If $\phi(u) = true$, then colour all the a_u^i with 1, all the b_u^i with 2 and all the c_u^i with 4. And if $\phi(u) = false$, then colour all the a_u^i with 2, all the b_u^i with 1 and all the c_u^i with 3. Now, for each clause $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$, some literal, say ℓ_1^i, is true and some literal, say ℓ_3^i, is false. Then assign 1 to z_1^i, 2 to z_3^i, and colour z_2^i with 3 if ℓ_2^i is true and 4 otherwise. By Claims 38 and 40, this partial colouring may be extended into a circular 2-backbone 4-colouring of (G,M). □

Theorem 41. *The following problem is NP-complete problem.*
Input: A planar graph G and a matching M in G.
Question: Is $\mathrm{CBC}_2(G,M) \leq 5$*?*

Proof. The reduction is from PLANAR C_5-COLOURING which is defined as follows:
Input: A planar graph G.
Question: Does G have a homomorphism onto C_5, the cycle of length 5?
This was proved to be NP-complete by MacGillivray and Siggers [10].

To make the reduction we need an *edge gagdet*. This gadget is built from the planar graph $H_1(u,v)$ together with the matching $M_1(u,v)$ depicted in Figure 13.

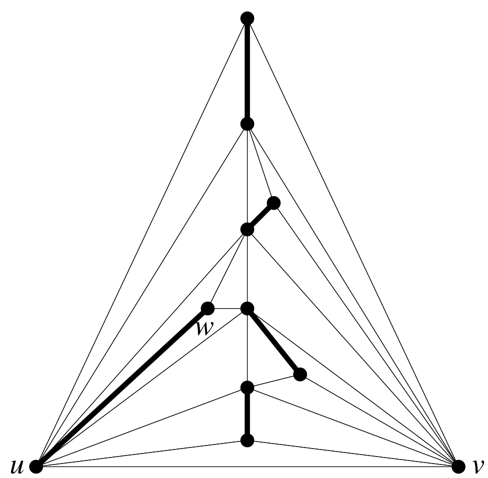

Figure 13: Graph $H_1(u,v)$ with matching $M_1(u,v)$ (in bold)

The graph $H_2(x,y)$ is obtained from $H_1(x,y)$ by replacing the edge uw by $H_1(w,u)$. The matching $M_2(x,y)$ is then the union of $M_1(x,y) \setminus \{uw\}$ and $M_1(w,u)$. Observe that x and y are incident to no edges of $M_2(x,y)$. The pair $(H_2(x,y), M_2(x,y))$ is the edge gadget.

Broersma et al. [5] proved that in any circular 2-backbone 5-colouring of $(H_1(u,v), M_1(u,v))$, vertices u and v receive colours which are cyclically 2 apart. In addition, its straightforward to see that any precolouring of u and v with colours that are cyclically 2 apart can be extended into a circular 2-backbone 5-colouring of $(H_1(u,v), M_1(u,v))$. These two facts imply the following claim.

Claim 42.

(i) In any circular 2*-backbone* 5*-colouring of* $(H_2(x,y),M_2(x,y))$*, vertices x and y receive colours which are cyclically 2 apart.*

(ii) Any precoloring of x and y with colours that are cyclically 2 apart can be extended into a circular 2*-backbone* 5*-colouring of* $(H_2(x,y),M_2(x,y))$.

Let H be an instance of PLANAR C_5-COLOURING. Replace each edge $xy \in E(G)$ an edge gadget $(H_2(x,y),M_2(x,y))$ to obtain a planar graph G and a matching M (the union of the $M_2(x,y)$). By Claim 42-(i), every circular 2-backbone 5-colouring of (G,M) induces a C_5-colouring of H (the vertices of the C_5 are the colours $(1,3,5,2,4)$). Conversely, by Claim 42-(ii), any C_5-colouring of H can be extended into a circular 2-backbone 5-colouring of (G,M). Hence H admits a C_5-colouring if and only if (G,M) admits a circular 2-backbone 5-colouring. □

Remark 43. Adding long paths along existing edges to transform the matching into a spanning tree, one derives that deciding if $\mathrm{CBC}(G,T) \leq 5$, given a planar graph G and a spanning tree T of G, is NP-complete.

4 Further research

Campos et al. [6] proved that if G is planar and T has diameter at most 3, then $\mathrm{BBC}_2(G,T) \leq 5$. Hence one can the find the 2-backbone chromatic number of such a pair in polynomial time. One can ask of the complexity for larger diameter.

Problem 44. *For a fixed* $d \geq 4$*, what is the complexity of finding the* 2*-backbone chromatic number of* (G,T)*, when G is planar and T a spanning tree of diameter d ?*

Since deciding if the 2-backbone chromatic number of (G,T) is at most k, for any fixed $k \leq 4$ can be done in polynomial in polynomial time, if Conjecture 1 holds, Problem 44 is equivalent to finding the complexity of deciding if $\mathrm{BBC}_2(G,T) \leq 5$.

If G is a triangle-free planar graph, then, by Grötzsch's Theorem [8], it is 3-colourable, and so $\mathrm{BBC}_q(G,H) \leq 2q+1$ and $\mathrm{CBC}(G,H) \leq 3q$ for any subgraph H of G. Hence both Conjectures 1 and 6 for $q=2$, when G is triangle-free. A natural first step would be to extend prove the two conjectures or at least improve on the above upper bounds for larger values of q.

Steinberg's Conjecture (1976) states that every planar graph without 4- and 5-cycles is 3-colourable. Towards this, Erdős (1991) proposed the following relaxation of Steinberg's Conjecture: Determine the smallest value of k, such that every planar graph without cycles of length from 4 to k is 3-colourable. The best known bound for such a k is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [2]. Hence, an evidence to both Conjecture 6 and Steinberg's Conjecture would be to prove the following:

Conjecture 45. *If G is a planar graph without 4- and 5-cycles and F a spanning forest of G, then* $\mathrm{CBC}(G,F) \leq 7$.

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