Existence and regularity for critical anisotropic equations with critical directions

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EXISTENCE AND REGULARITY FOR CRITICAL ANISOTROPIC EQUATIONS WITH CRITICAL DIRECTIONS

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Abstract. We establish existence and regularity results for doubly critical anisotropic equations in domains of the Euclidean space. In particular, we answer a question posed by Fragalà–Gazzola–Kawohl [24] when the maximum of the anisotropic configuration coincides with the critical Sobolev exponent.

1. Introduction

In this paper, we investigate existence and regularity for doubly critical anisotropic equations. In dimension $n \geq 2$, we provide ourselves with an anisotropic configuration $\overrightarrow{p} = (p_1, \dots, p_n)$ with $p_i > 1$ for all $i = 1, \dots, n$. We let $D^{1,\overrightarrow{p}}(\Omega)$ be the anisotropic Sobolev space defined as the completion of the vector space of all smooth functions with compact support in Ω with respect to the norm $\|u\|_{D^{1,\overrightarrow{p}}(\Omega)} = \sum_{i=1}^n \|\partial u/\partial x_i\|_{L^{p_i}(\Omega)}$. We are concerned with the following anisotropic problem of critical growth

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda |u|^{p^*-2} u \quad \text{in } \Omega\,, \\ u \in D^{1,\overrightarrow{p}}(\Omega)\,, \end{cases} \tag{1.1}$$

on domains Ω in the Euclidean space $\mathbb{R}^n$, where λ is a positive real number, p^* is the critical Sobolev exponent (see (1.3) below), and $\Delta_{\overrightarrow{p}}$ is the anisotropic Laplace operator defined by

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \nabla_{x_i}^{p_i} u\,, \tag{1.2}$$

where $\nabla_{x_i}^{p_i} u = |\partial u/\partial x_i|^{p_i-2} \partial u/\partial x_i$ for all $i = 1, \dots, n$. As one can check, $\Delta_{\overrightarrow{p}}$ involves directional derivatives with distinct weights. Anisotropic operators appear in several places in the literature. Recent references can be found in physics [3, 7], in biology [11], and in image processing [46].

We consider in this paper the doubly critical situation $p_+ = p^*$, where $p_+ = \max(p_1, \dots, p_n)$ is the maximum value of the anisotropic configuration and p^* is the critical Sobolev exponent for the embeddings of the anisotropic Sobolev space $D^{1,\overrightarrow{p}}(\Omega)$ into Lebesgue spaces. In this setting, not only the nonlinearity has critical growth, but the operator itself has critical growth in particular directions of the Euclidean space. As a remark, the notion of critical direction is a pure anisotropic notion which does not exist when dealing with the Laplace operator or the p-Laplace operator. Given $i = 1, \dots, n$, the i-th direction is said to be critical if $p_i = p^*$, resp. subcritical if $p_i < p^*$. Critical directions induce a failure in the rescaling invariance rule associated with (1.1).

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Given an anisotropic configuration $\overrightarrow{p}$ satisfying $\sum_{i=1}^n 1/p_i > 1$ and $p_j \le n/\big(\sum_{i=1}^n \frac{1}{p_i} - 1\big)$ for all $j = 1, \dots, n$, the critical Sobolev exponent is equal to

$$p^* = \frac{n}{\sum_{i=1}^n \frac{1}{p_i} - 1}\,. \tag{1.3}$$

In this paper, we consider weak solutions of problem (1.1). We say that a function u in $D^{1,\overrightarrow{p}}(\Omega)$ is a weak solution of problem (1.1) if there holds

$$\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx = \int_\Omega |u|^{p^*-2} u\varphi dx$$

for all smooth functions φ with compact support in Ω.

In this paper, we prove an existence result and a regularity result for problem (1.1). The regularity result, stated in Theorem 1.2 below, is established on arbitrary domains (bounded or not), and is motivated in particular by a question posed by Fragalà–Gazzola–Kawohl [24, Section 8.3, Problem 1]. The existence result, stated in Theorem 1.1 below, is established on cylindric domains. Problem (1.1) on cylindric domains is involved in the description of the asymptotic behavior of Palais–Smale sequences for critical anisotropic problems (see Vétois [44]). The rescaling phenomenon is described in Section 3. Our existence result states as follows.

Theorem 1.1. *Let $n \ge 3$, $1 \le n_+ < n$, and $\overrightarrow{p} = (p_1, \dots, p_n)$, and assume that $\sum_{i=1}^n 1/p_i > 1$, $p_+ = p^*$, $p_{n-n_++1} = \dots = p_n = p_+$, and $p_i < p_+$ for all $i \le n - n_+$. Let V be a nonempty, bounded, open subset of $\mathbb{R}^{n_+}$, and assume that $\Omega = \mathbb{R}^{n-n_+} \times V$. Then there exists a positive real number λ such that problem* (1.1) *admits at least one nonnegative, nontrivial solution.*

Theorem 1.1 is concerned with cylindric domains. Theorem 1.2 below holds true for arbitrary domains Ω, including Ω bounded. This result, which answers the question of the regularity associated to (1.1), is stated as follows.

Theorem 1.2. *Let $n \ge 3$ and $\overrightarrow{p} = (p_1, \dots, p_n)$, and assume that $\sum_{i=1}^n 1/p_i > 1$ and $p_+ = p^*$. Let Ω be a nonempty, open subset of $\mathbb{R}^n$, and λ be a positive real number. Then any solution of problem* (1.1) *belongs to $L^\infty(\Omega)$.*

Theorem 1.2 is established on arbitrary domains. In case of bounded domains Ω, Theorem 1.2 answers a question posed by Fragalà–Gazzola–Kawohl [24, Section 8.3, Problem 1]. The boundedness of nonnegative weak solutions of problem (1.1) was established in case $p_+ < p^*$ by Fragalà–Gazzola–Kawohl [24]. It was suggested in [24] that the result should remain true in case $p_+ \ge p^*$ for solutions of the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda u^{p_+-1} \quad \text{in } \Omega\,, \\ u \in D^{1,\overrightarrow{p}}(\Omega) \cap L^{p_+}(\Omega)\,. \end{cases} \tag{1.4}$$

Theorem 1.2 answers positively to this question in case $p_+ = p^*$. On the other hand, we point toward a negative answer when $p_+ > p^*$. More precisely, we prove (by using Proposition 2.1, see Section 2) that for particular anisotropic configurations $\overrightarrow{p}$ satisfying $p_+ > p^*$, for instance when $p_1 = \dots = p_{n_-} = 2$ and $p_{n_-+1} = \dots = p_n = p_+$ with $p_+ > 2^*$, $2^* = 2n_-/(n_- - 2)$, and $2 < n_- < n$, if we assume the existence of nonnegative, unbounded solutions of the isotropic, supercritical problem

$$\begin{cases} -\Delta u = u^{p_+-1} \quad \text{in } \Omega'\,, \\ u \in D^{1,2}(\Omega') \cap L^{p_+}(\Omega')\,, \end{cases}$$

for some domain Ω' in $\mathbb{R}^{n_-}$, where $\Delta = \operatorname{div}(\nabla u)$ is the classical Laplace operator, then the anisotropic problem (1.4) with $\Omega = \Omega' \times \Omega''$ admits nonnegative, unbounded solutions for all domains Ω'' in $\mathbb{R}^{n-n_-}$, including Ω'' bounded. As is well-known, problems with supercritical growth may admit unbounded solutions (see, for instance, Benguria–Dolbeault–Esteban [8], Farina [22], and also Fragalà–Gazzola–Kawohl [24]).

In case $p_+ < p^*$, namely when all directions are subcritical, anisotropic equations with critical nonlinearities have been investigated by Alves–El Hamidi [2], El Hamidi–Rakotoson [19, 20], El Hamidi–Vétois [21], Fragalà–Gazzola–Kawohl [24], Fragalà–Gazzola–Lieberman [25], and Vétois [43]. Other recent references on anisotropic problems like (1.1) are Antontsev–Shmarev [4, 5], Bendahmane–Karlsen [9, 10], Bendahmane–Langlais–Saad [11], Cianchi [13], D'Ambrosio [14], Di Castro [17], Di Castro–Montefusco [18], García-Melián–Rossi–Sabina de Lis [27], Li [30], Lieberman [31, 32], Mihăilescu–Pucci–Rădulescu [35], Mihăilescu–Rădulescu–Tersian [36], Namlyeyeva–Shishkov–Skrypnik [37], Skrypnik [39], Tersenov–Tersenov [40], and Vétois [42, 44, 45].

In the isotropic configuration where $p_i = p$ for all $i = 1, \dots, n$, there holds $p < p^*$ and all directions are subcritical. In this particular situation, the operator (1.2) is comparable, though slightly different, to the p-Laplace operator $\Delta_p = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$. Possible references on critical p-Laplace equations are Alves–Ding [1], Arioli–Gazzola [6], Demengel–Hebey [15, 16], Filippucci–Pucci–Robert [23], Gazzola [28], and Guedda–Veron [29]. Needless to say, the above list does not pretend to exhaustivity.

We illustrate our results with examples in Section 2, we prove Theorem 1.1 in Section 3, and we prove Theorem 1.2 in Section 4.

2. Examples of solutions

In this section, we are concerned with the situation where the anisotropic configuration $\overrightarrow{p}$ consists in two distinct exponents p_- and p_+. In other words, we assume that there exist two indices $n_- \geq 2$ and $n_+ \geq 1$ such that $n = n_- + n_+$, $p_1 = \dots = p_{n_-} = p_-$, and $p_{n_-+1} = \dots = p_n = p_+$. Proposition 2.1 below is the basic tool in our construction. It relies on a direct computation.

Proposition 2.1. *Let $n_- \geq 2$, $n_+ \geq 1$, $n = n_- + n_+$, and $\overrightarrow{p} = (p_1, \dots, p_n)$, and assume that $p_1 = \dots = p_{n_-} = p_-$ and $p_{n_-+1} = \dots = p_n = p_+$. Let λ be a positive real number. Let Ω_1 be a nonempty open subset of $\mathbb{R}^{n_-}$ and Ω_2 be a nonempty open subset of $\mathbb{R}^{n_+}$. Let v be a solution of the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{n_-} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_--2} \frac{\partial v}{\partial x_i}\right) = |v|^{p_+-2} v \quad \text{in } \Omega_1\,, \\ v \in D^{1,p_-}(\Omega_1) \cap L^{p_+}(\Omega_1)\,, \end{cases} \tag{2.1}$$

and let w be a solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{n_+} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p_+-2} \frac{\partial w}{\partial x_i}\right) = |w|^{p_+-2} w - |w|^{p_--2} w \quad \text{in } \Omega_2\,, \\ w \in D^{1,p_+}(\Omega_2) \cap L^{p_-}(\Omega_2)\,. \end{cases} \tag{2.2}$$

Then the function u defined on $\Omega_1 \times \lambda^{\frac{-1}{p_+}} \Omega_2$ by

$$u\left(x_1, \dots, x_n\right) = \lambda^{\frac{-1}{p_+ - p_-}} v\left(x_1, \dots, x_{n_-}\right) w\Big(\lambda^{\frac{1}{p_+}} x_{n_-+1}, \dots, \lambda^{\frac{1}{p_+}} x_n\Big) \tag{2.3}$$

is a solution of the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda \left|u\right|^{p_+ - 2} u & \text{in } \Omega_1 \times \lambda^{\frac{-1}{p_+}} \Omega_2\,, \\ u \in D^{1,\overrightarrow{p}}\big(\Omega_1 \times \lambda^{\frac{-1}{p_+}} \Omega_2\big) \cap L^{p_+}\big(\Omega_1 \times \lambda^{\frac{-1}{p_+}} \Omega_2\big), \end{cases} \tag{2.4}$$

where $\Delta_{\overrightarrow{p}}$ is as in (1.2).

Proof. A direct computation provides the result. □

If $p_+ = p^*$, then a solution of equation (2.1) is given by

$$\mathcal{V}_{n_-,p_-}\left(x_1, \dots, x_{n_-}\right) = C_{n_-,p_-} \left(\frac{1}{1 + \sum_{i=1}^{n_-} \left|x_i\right|^{\frac{p_-}{p_- - 1}}}\right)^{\frac{n_- - p_-}{p_-}}, \tag{2.5}$$

where

$$C_{n_-,p_-} = \left(\frac{n_-\left(n_- - p_-\right)^{p_- - 1}}{\left(p_- - 1\right)^{p_- - 1}}\right)^{\frac{n_- - p_-}{p_-^2}}.$$

On the other hand, we search for solutions of equation (2.2) of the form

$$w\left(x_{n_-+1}, \dots, x_n\right) = \mathcal{W}\left(r\right) \quad \text{with} \quad r = \left(\sum_{i=n_-+1}^{n} \left|x_i\right|^{\frac{p_+}{p_+ - 1}}\right)^{\frac{p_+ - 1}{p_+}}.$$

As one can check, equation (2.2) then rewrites as

$$-r^{1-n_+}\left(r^{n_+ - 1}\left|\mathcal{W}'\right|^{p_+ - 2} \mathcal{W}'\right)' = \left|\mathcal{W}\right|^{p_+ - 2} \mathcal{W} - \left|\mathcal{W}\right|^{p_- - 2} \mathcal{W} \quad \text{in } \mathbb{R}_+\,. \tag{2.6}$$

In case $n_+ = 1$, the unique nonnegative, nontrivial C^1-solution of (2.6) is given by

$$\mathcal{W}\left(r\right) = \begin{cases} F^{-1}\left(F\left(\mathcal{W}_0\right) - r\right) & \text{if } r < F\left(\mathcal{W}_0\right), \\ 0 & \text{if } r \geq F\left(\mathcal{W}_0\right), \end{cases}$$

where

$$\mathcal{W}_0 = \left(\frac{p_+}{p_-}\right)^{\frac{1}{p_+ - p_-}} \quad \text{and} \quad F\left(t\right) = \left(\frac{p_+ - 1}{p_+}\right)^{\frac{1}{p_+}} \int_0^t \left(\frac{s^{p_-}}{p_-} - \frac{s^{p_+}}{p_+}\right)^{-\frac{1}{p_+}} ds\,.$$

In particular, there hold $\mathcal{W}\left(0\right) = \mathcal{W}_0$, $\mathcal{W}'\left(0\right) = 0$, $\mathcal{W} > 0$ and $\mathcal{W}' < 0$ in $\left(0, F\left(\mathcal{W}_0\right)\right)$, and $\mathcal{W} = 0$ in $\left[F\left(\mathcal{W}_0\right), +\infty\right)$. In case $n_+ \geq 2$, by Franchi–Lanconelli–Serrin [26], we get that equation (2.6) admits at least one nonnegative C^1-solution which satisfies $\mathcal{W}'\left(0\right) = 0$, $\mathcal{W} > 0$ and $\mathcal{W}' < 0$ in $\left(0, R\right)$, and $\mathcal{W} = 0$ in $\left[R, +\infty\right)$ for some positive real number R. Summarizing, we can state the following corollary of Proposition 2.1.

Corollary 2.1. *Let $n_- \geq 2$, $n_+ \geq 1$, $n = n_- + n_+$, and $\overrightarrow{p} = (p_1, \dots, p_n)$, and assume that $p_1 = \dots = p_{n_-} = p_-$, $p_{n_-+1} = \dots = p_n = p_+$, and $p_+ = p^*$. For any point $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$ and for any positive real numbers μ and λ, there exists a nonnegative solution $\mathcal{U}_{a,\mu,\lambda}$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ of equation (2.4) of the form*

$$\mathcal{U}_{a,\mu,\lambda}(x_1, \dots, x_n) = \mu^{-1} \lambda^{\frac{-1}{p_+ - p_-}} \mathcal{U}\Big(\mu^{\frac{p_- - p_+}{p_-}}(x_1 - a_1), \dots, \mu^{\frac{p_- - p_+}{p_-}}(x_{n_-} - a_{n_-}), \lambda^{\frac{1}{p_+}}(x_{n_-+1} - a_{n_-+1}), \dots, \lambda^{\frac{1}{p_+}}(x_n - a_n)\Big),$$

where

$$\mathcal{U}(x_1, \dots, x_n) = \mathcal{V}_{n_-, p_-}(x_1, \dots, x_{n_-}) \mathcal{W}\left(\left(\sum_{i=n_-+1}^{n} |x_i|^{\frac{p_+}{p_+ - 1}}\right)^{\frac{p_+ - 1}{p_+}}\right),$$

where $\mathcal{V}_{n_-, p_-}$ is as in (2.5) and where $\mathcal{W}$ is such that $\mathcal{W} > 0$ and $\mathcal{W}' < 0$ in $(0, R)$, and $\mathcal{W} = 0$ in $[R, +\infty)$ for some positive real number R.

Since the function $\mathcal{W}$ has compact support, Corollary 2.1 provides a class of solutions of problem (1.1) on cylindric domains $\Omega = \mathbb{R}^{n_-} \times V$ for all nonempty, open subsets V of $\mathbb{R}^{n_+}$. These solutions illustrate the general existence result stated in Theorem 1.1 in the particular case where the anisotropic configuration $\overrightarrow{p}$ consists in two distinct exponents p_- and p_+.

In the supercritical case $p_+ > p^*$, suppose there exists a nonnegative, unbounded solution of problem (2.1) for some domain Ω_1 in $\mathbb{R}^{n_-}$. Then we easily get with Proposition 2.1 that problem (1.4) with $\Omega = \Omega_1 \times \Omega_2$ admits nonnegative, unbounded solutions for all domains Ω_2 in $\mathbb{R}^{n_+}$, including Ω_2 bounded. Indeed, since the above function $\mathcal{W}$ has compact support, by rescaling $\mathcal{W}$, we get a nonnegative solution of the problem (2.2) on the domain Ω_2. Then Proposition 2.1 provides the existence of a nonnegative, unbounded solution of the form (2.3) of the problem (1.4) with $\Omega = \Omega_1 \times \Omega_2$.

3. The existence result

This section is devoted to the proof of Theorem 1.1. We let $n \geq 3$ and $\overrightarrow{p} = (p_1, \dots, p_n)$. We assume that $\sum_{i=1}^n 1/p_i > 1$, $p_+ = p^*$, and that there exists an index n_+ such that $p_{n-n_++1} = \dots = p_n = p_+$, and $p_i < p_+$ for all $i \leq n - n_+$. Moreover, we assume that $\Omega = \mathbb{R}^{n-n_+} \times V$, where V is a nonempty, bounded, open subset of $\mathbb{R}^{n_+}$. Without loss of generality, we may assume that the point 0 belongs to V.

The proof of Theorem 1.1 is based on concentration-compactness arguments. Let us first set some notations. For any function u in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ and any subset D of $\mathbb{R}^n$, we let the energy $\mathcal{E}(u, D)$ of u on D be defined by

$$\mathcal{E}(u, D) = \int_D u^{p_+} dx\,. \tag{3.1}$$

For any positive real number μ and any point $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$, we define the affine transformation $\tau_{\mu,a}^{\overrightarrow{p}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tau_{\mu,a}^{\overrightarrow{p}}(x_1, \dots, x_n) = \Big(\mu^{\frac{p_1 - p_+}{p_1}}(x_1 - a_1), \dots, \mu^{\frac{p_n - p_+}{p_n}}(x_n - a_n)\Big). \tag{3.2}$$

As is easily checked, (3.2) provides a general rescaling invariance rule associated with equation (1.1). Moreover for any subset D of $\mathbb{R}^n$, we get $\mathcal{E}(u, D) = \mathcal{E}\big(\mu u \circ \big(\tau_{\mu,a}^{\overrightarrow{p}}\big)^{-1}, \tau_{\mu,a}^{\overrightarrow{p}}(D)\big)$, where

$$\big(\tau_{\mu,a}^{\overrightarrow{p}}\big)^{-1}(x_1, \dots, x_n) = \Big(\mu^{\frac{p_+ - p_1}{p_1}} x_1 + a_1, \dots, \mu^{\frac{p_+ - p_n}{p_n}} x_n + a_n\Big).$$

Of importance in our critical setting is that the set D is only rescaled with respect to noncritical directions. Therefore, we observe a concentration phenomenon on affine subspaces of $\mathbb{R}^n$ spanned by critical directions. Figure 1 below illustrates the rescaling in case D is a three-dimensional ball, the first two directions being noncritical, the third one being critical. In case of the p-Laplace operator, the ball would have been rescaled to the whole euclidean space.

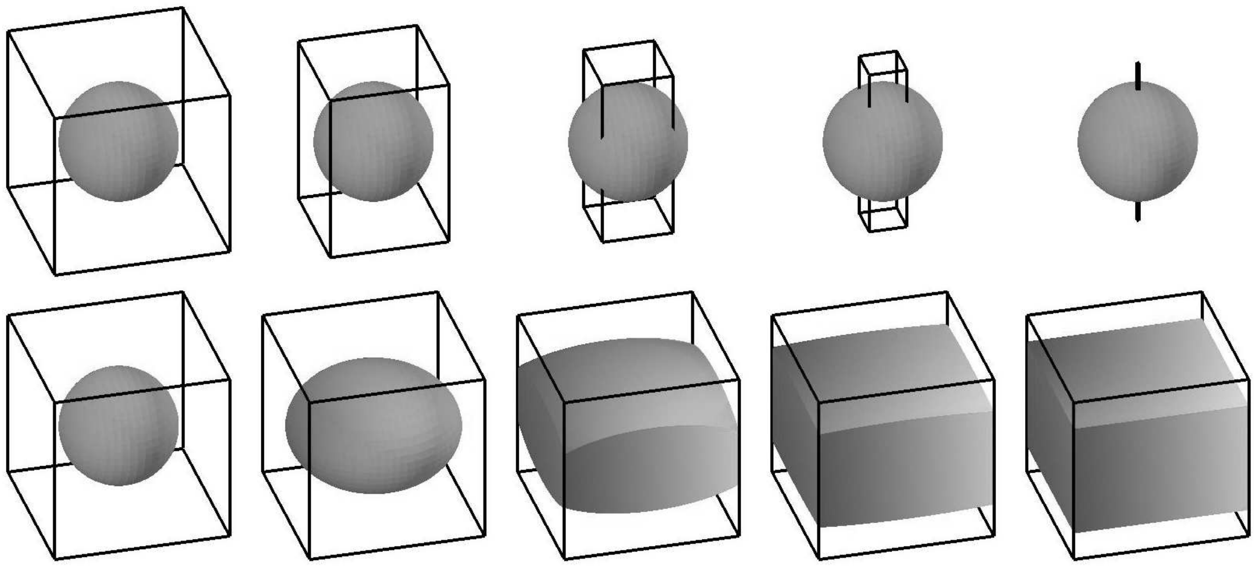

FIGURE 1. Rescaling of a ball ($n = 3$, $p_1 = p_2 = 1.5$, $p_3 = 6$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

We begin the proof of Theorem 1.1. We let $(u_\alpha)_\alpha$ be a sequence of functions in $D^{1,\overrightarrow{p}}(\Omega)$ such that

$$\int_\Omega |u_\alpha|^{p_+} dx = 1 \quad \text{and} \quad \lim_{\alpha\to+\infty} \sum_{i=1}^n \frac{1}{p_i} \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx = \inf_{\substack{u\in D^{1,\overrightarrow{p}}(\Omega)\\ \int_\Omega |u|^{p_+} dx = 1}} \sum_{i=1}^n \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\,. \quad (3.3)$$

Taking the absolute value, we may assume that for any α, the function u_α is nonnegative. Clearly, the sequence $(u_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(\Omega)$.

Step 3.1 below is the first step in the proof of Theorem 1.1. We say that a sequence $(v_\alpha)_\alpha$ in $D^{1,\overrightarrow{p}}(\Omega)$ is Palais–Smale for the functional I_λ defined in (3.4) if there hold $|I_\lambda(v_\alpha)| \le C$ for some positive constant C independent of α, and $\|DI_\lambda(v_\alpha)\|_{D^{1,\overrightarrow{p}}(\Omega)'} \to 0$ as $\alpha \to +\infty$.

Step 3.1. *Up to a subsequence, $(u_\alpha)_\alpha$ is a Palais–Smale sequence for the functional*

$$I_\lambda(u) = \sum_{i=1}^n \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \frac{\lambda}{p_+} \int_\Omega |u|^{p_+} dx\,, \quad (3.4)$$

where

$$\lambda = \lim_{\alpha\to+\infty} \sum_{i=1}^n \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx\,. \quad (3.5)$$

Proof. It easily follows from (3.3) that there holds $|I_\lambda(u_\alpha)| \le C$ for some positive contant C independent of α. We then prove that for any bounded sequence $(\varphi_\alpha)_\alpha$ in $D^{1,\overrightarrow{p}}(\Omega)$, there holds $DI_\lambda(u_\alpha).\varphi_\alpha \to 0$ as $\alpha \to +\infty$. By (3.3), we get that there exists a sequence $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to 0 such that for any real number t, there holds

$$\begin{aligned}\sum_{i=1}^n \frac{1}{p_i}\int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx - \varepsilon_\alpha &\le \sum_{i=1}^n \frac{1}{p_i}\int_\Omega \left|\frac{\partial}{\partial x_i}\left(\frac{u_\alpha + t\varphi_\alpha}{\left(\int_\Omega |u_\alpha + t\varphi_\alpha|^{p_+} dx\right)^{\frac{1}{p_+}}}\right)\right|^{p_i} dx\\ &= \sum_{i=1}^n \frac{1}{p_i}\left(\int_\Omega |u_\alpha + t\varphi_\alpha|^{p_+} dx\right)^{-\frac{p_i}{p_+}} \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i} + t\frac{\partial \varphi_\alpha}{\partial x_i}\right|^{p_i} dx\,. \quad (3.6)\end{aligned}$$

As is easily checked, there exists a positive real number C such that for any $i = 1, \dots, n$ and for any real numbers x and y, there holds

$$\left||x+y|^{p_i} - |x|^{p_i} - p_i |x|^{p_i-2} xy\right| \le C \begin{cases} |y|^{p_i} & \text{if } p_i \le 2\\ |y|^2\left(|x|^{p_i-2} + |y|^{p_i-2}\right) & \text{if } p_i > 2.\end{cases} \quad (3.7)$$

Since $(u_\alpha)_\alpha$ and $(\varphi_\alpha)_\alpha$ are bounded in $D^{1,\overrightarrow{p}}(\Omega)$, by (3.7) and Hölder's inequality, we get

$$\begin{aligned}&\left|\int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i} + t\frac{\partial \varphi_\alpha}{\partial x_i}\right|^{p_i} dx - \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx - p_i t\int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\alpha}{\partial x_i}\frac{\partial \varphi_\alpha}{\partial x_i} dx\right|\\ &\le C\begin{cases} t^{p_i}\int_\Omega \left|\frac{\partial \varphi_\alpha}{\partial x_i}\right|^{p_i} dx & \text{if } p_i \le 2\\ t^2\left(\int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx\right)^{\frac{2}{p_i}}\left(\int_\Omega \left|\frac{\partial \varphi_\alpha}{\partial x_i}\right|^{p_i} dx\right)^{\frac{p_i-2}{p_i}} + t^{p_i}\int_\Omega \left|\frac{\partial \varphi_\alpha}{\partial x_i}\right|^{p_i} dx & \text{if } p_i > 2.\end{cases}\\ &\le C'\begin{cases} t^{p_i} & \text{if } p_i \le 2\\ t^2\left(1 + t^{p_i-2}\right) & \text{if } p_i > 2.\end{cases} \quad (3.8)\end{aligned}$$

for all $i = 1, \dots, n$, and

$$\begin{aligned}&\left|\int_\Omega |u_\alpha + t\varphi_\alpha|^{p_+} dx - \int_\Omega u_\alpha^{p_+} dx - p_+ t\int_\Omega u_\alpha^{p_+-1}\varphi_\alpha dx\right|\\ &\le C\begin{cases} t^{p_+}\int_\Omega |\varphi_\alpha|^{p_+} dx & \text{if } p_+ \le 2\\ t^2\left(\int_\Omega |u_\alpha|^{p_+} dx\right)^{\frac{2}{p_+}}\left(\int_\Omega |\varphi_\alpha|^{p_+} dx\right)^{\frac{p_+-2}{p_+}} + t^{p_+}\int_\Omega |\varphi_\alpha|^{p_+} dx & \text{if } p_+ > 2.\end{cases}\\ &\le C'\begin{cases} t^{p_+} & \text{if } p_+ \le 2\\ t^2\left(1 + t^{p_+-2}\right) & \text{if } p_+ > 2.\end{cases} \quad (3.9)\end{aligned}$$

for some positive constants C and C' independent of α and t. By (3.6), (3.8), (3.9), we get

$$\begin{aligned}-\varepsilon_\alpha &\le t\left(\sum_{i=1}^n \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\alpha}{\partial x_i}\frac{\partial \varphi_\alpha}{\partial x_i} dx - \left(\sum_{i=1}^n \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx\right)\int_\Omega u_\alpha^{p_+-1}\varphi_\alpha dx\right) + \mathrm{o}(t)\\ &\le t\left(DI_\lambda(u_\alpha).\varphi_\alpha + \left(\lambda - \sum_{i=1}^n \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx\right)\int_\Omega u_\alpha^{p_+-1}\varphi_\alpha dx\right) + \mathrm{o}(t)\end{aligned}$$

as $t \to 0$ uniformly with respect to α, where λ is as in (3.5). Passing to the limit as $\alpha \to +\infty$, we get

$$0 \le \limsup_{\alpha \to +\infty} \left(tDI_\lambda\left(u_\alpha\right).\varphi_\alpha\right) + \mathrm{o}\left(t\right)$$

as $t \to 0$. Since the real number t takes either positive or negative values, it follows that $DI_\lambda\left(u_\alpha\right).\varphi_\alpha \to 0$ as $\alpha \to +\infty$. Since this holds true for all bounded sequences $\left(\varphi_\alpha\right)_\alpha$ in $D^{1,\overrightarrow{p}}\left(\Omega\right)$, we get $\left\|DI_\lambda\left(u_\alpha\right)\right\|_{D^{1,\overrightarrow{p}}(\Omega)'} \to 0$ as $\alpha \to +\infty$. This ends the proof of Step 3.1. □

Now, for any α, we define the concentration function $Q_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$Q_\alpha\left(s\right) = \max_{y \in \overline{\Omega}} \mathcal{E}\big(u_\alpha, \mathcal{P}_y^{\overrightarrow{p}}\left(s\right)\big)\,,$$

where the energy functional $\mathcal{E}$ is as in (3.1) and

$$\mathcal{P}_y^{\overrightarrow{p}}\left(s\right) = \Big\{\left(x_1, \dots, x_n\right) \in \Omega;\ \left|x_i - y_i\right| < s^{\frac{p_+ - p_i}{p_i}} \quad \forall i \in \left\{1, \dots, n - n_+\right\}\Big\} \tag{3.10}$$

for all positive real number s and for all point $y = \left(y_1, \dots, y_n\right)$ in $\overline{\Omega}$. By the continuity of the functions Q_α and by (3.3), given a real number δ_0 in $(0, 1)$, we get the existence of a sequence $\left(\mu_\alpha\right)_\alpha$ of positive real numbers such that there holds $Q_\alpha\left(\mu_\alpha\right) = \delta_0$ for all α. We let x_α be a point in $\overline{\Omega}$ for which $Q_\alpha\left(\mu_\alpha\right)$ is reached, so that there holds

$$\max_{y \in \overline{\Omega}} \mathcal{E}\big(u_\alpha, \mathcal{P}_y^{\overrightarrow{p}}\left(\mu_\alpha\right)\big) = \mathcal{E}\big(u_\alpha, \mathcal{P}_{x_\alpha}^{\overrightarrow{p}}\left(\mu_\alpha\right)\big) = \delta_0 \tag{3.11}$$

for all α. By definition of $\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}\left(\mu_\alpha\right)$, see (3.10), we may assume that the n_+ last coordinates of the point x_α are equal to 0. For any α, we then define the function $\widetilde{u}_\alpha$ by

$$\widetilde{u}_\alpha = \mu_\alpha u_\alpha \circ \big(\tau_{\mu_\alpha, x_\alpha}^{\overrightarrow{p}}\big)^{-1}\,,$$

where $\tau_{\mu_\alpha, x_\alpha}^{\overrightarrow{p}}$ is as in (3.2). Since $\Omega = \mathbb{R}^{n-n_+} \times V$, $p_{n-n_++1} = \dots = p_n = p_+$, and $p_+ = p^*$, we get $\tau_{\mu_\alpha, x_\alpha}^{\overrightarrow{p}}\left(\Omega\right) = \Omega$ for all α. As well as $\left(u_\alpha\right)_\alpha$, we get that $\left(\widetilde{u}_\alpha\right)_\alpha$ is a Palais–Smale sequence for the functional I_λ defined in (3.4). Moreover, there holds $\left\|\widetilde{u}_\alpha\right\|_{D^{1,\overrightarrow{p}}(\Omega)} = \left\|u_\alpha\right\|_{D^{1,\overrightarrow{p}}(\Omega)}$ for all α. In particular, the sequence $\left(\widetilde{u}_\alpha\right)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}\left(\Omega\right)$. Passing if necessary to a subsequence, we may assume that $\left(\widetilde{u}_\alpha\right)_\alpha$ converges weakly to a nonnegative function u_∞ in $D^{1,\overrightarrow{p}}\left(\Omega\right)$ and that $\left(\widetilde{u}_\alpha\right)_\alpha$ converges to u_∞ almost everywhere in Ω. The second step in the proof of Theorem 1.1 is as follows.

Step 3.2. *If the constant δ_0 is small enough, then $\left(\widetilde{u}_\alpha\right)_\alpha$ converges, up to a subsequence, to u_∞ in $L^{p_+}_{\mathrm{loc}}\left(\mathbb{R}^n\right)$.*

Proof. We fix a positive real number R, and we let $B_0\left(R\right)$ be the $(n - n_+)$-dimensional ball of center 0 and radius R. We show that the sequence $\left(\widetilde{u}_\alpha\right)_\alpha$ converges to u_∞ in $L^{p_+}\left(B_0\left(R\right)\right)$. For any α, we let $v_\alpha = \widetilde{u}_\alpha - u_\infty$. By Banach–Alaoglu theorem, since the sequence $\left(v_\alpha\right)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}\left(\Omega\right)$ and since $\Omega = \mathbb{R}^{n-n_+} \times V$, where V is bounded, passing if necessary to a subsequence, we may assume that there exist nonnegative, finite measures μ and $\nu_1, \dots, \nu_n$ on $\overline{B_0\left(2R\right)} \times \mathbb{R}^{n_+}$ such that $\left|v_\alpha\right|^{p_+} \rightharpoonup \mu$ and $\left|\partial v_\alpha / \partial x_i\right|^{p_i} \rightharpoonup \nu_i$ as $\alpha \to +\infty$ in the sense of measures on $\overline{B_0\left(2R\right)} \times \mathbb{R}^{n_+}$, for all $i = 1, \dots, n$. Moreover, the supports of the measures μ and $\nu_1, \dots, \nu_n$ are included in $\overline{B_0\left(2R\right) \times V}$. Now, we borrow some ideas in Lions [33,34] with the tricky difference here that the concentration holds on n_+-dimensional affine subspaces of $\mathbb{R}^n$. Since $p_+ = p^*$, by the anisotropic Sobolev inequality in Troisi [41], there exists a positive constant $\Lambda = \Lambda(\overrightarrow{p})$ such that for any α and any smooth function φ with compact support in $B_0(2R) \times \mathbb{R}^{n_+}$, there holds

$$\begin{aligned}\int_\Omega |v_\alpha \varphi|^{p_+} dx &\le \Lambda \prod_{i=1}^n \left(\int_\Omega \left| \frac{\partial (v_\alpha \varphi)}{\partial x_i} \right|^{p_i} dx \right)^{\frac{p_+}{np_i}} \\ &\le \Lambda \prod_{i=1}^n \left(\left(\int_\Omega \left| v_\alpha \frac{\partial \varphi}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} + \left(\int_\Omega \left| \frac{\partial v_\alpha}{\partial x_i} \varphi \right|^{p_i} dx \right)^{\frac{1}{p_i}} \right)^{\frac{p_+}{n}} . \end{aligned} \tag{3.12}$$

For $i = 1, \dots, n - n_+$, by the compact embeddings in Rákosník [38], we get that $(v_\alpha)_\alpha$ converges to 0 in $L^{p_i}(\operatorname{Supp} \varphi)$. Passing to the limit as $\alpha \to +\infty$ into (3.12) gives

$$\begin{aligned}\int_{\overline{B_0(R) \times V}} |\varphi|^{p_+} d\mu &\le \Lambda \prod_{i=1}^{n-n_+} \left(\int_{\overline{B_0(R) \times V}} |\varphi|^{p_i} d\nu_i \right)^{\frac{p_+}{np_i}} \\ &\quad \times \prod_{i=n-n_++1}^{n} \left(\left(\int_{\overline{B_0(R) \times V}} \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_+} d\mu \right)^{\frac{1}{p_+}} + \left(\int_{\overline{B_0(R) \times V}} |\varphi|^{p_+} d\nu_i \right)^{\frac{1}{p_+}} \right)^{\frac{p_+}{n}} . \end{aligned}$$

By an easy density argument, it follows that for any bounded measurable function φ on $\overline{B_0(R) \times V}$ which does not depend on the variables $x_{n-n_++1}, \dots, x_n$, there holds

$$\int_{\overline{B_0(R) \times V}} |\varphi|^{p_+} d\mu \le \Lambda \prod_{i=1}^n \left(\int_{\overline{B_0(R) \times V}} |\varphi|^{p_i} d\nu_i \right)^{\frac{p_+}{np_i}} . \tag{3.13}$$

In particular, for any Borelian set A in $\overline{B_0(R)}$, taking $\varphi = \mathbf{1}_{A \times \overline{V}}$, we get

$$\mu\left(A \times \overline{V}\right) \le \Lambda \prod_{i=1}^n \nu_i\left(A \times \overline{V}\right)^{\frac{p_+}{np_i}} . \tag{3.14}$$

Letting $\nu = \sum_{i=1}^n \nu_i$, since $\sum_{i=1}^n \frac{1}{p_i} = \frac{n+p_+}{p_+}$, it follows that

$$\mu\left(A \times \overline{V}\right) \le \Lambda \nu\left(A \times \overline{V}\right)^{\frac{n+p_+}{n}} . \tag{3.15}$$

We let $\widetilde{\mu}$ and $\widetilde{\nu_1}, \dots, \widetilde{\nu_n}$ be the measures defined on $\overline{B_0(R)}$ by $\widetilde{\mu}(A) = \mu\left(A \times \overline{V}\right)$ and $\widetilde{\nu_i}(A) = \nu_i\left(A \times \overline{V}\right)$ for all $i = 1, \dots, n$. We let $\widetilde{\nu} = \sum_{i=1}^n \widetilde{\nu_i}$. By the Lebesgue decomposition of $\widetilde{\nu}$ with respect to $\widetilde{\mu}$, there exist a nonnegative function f in $L^1\big(\overline{B_0(R)}, d\widetilde{\mu}\big)$ and a nonnegative bounded measure σ on $\overline{B_0(R)}$ such that there holds $\widetilde{\nu} = f\widetilde{\mu} + \sigma$ and such that σ is singular with respect to $\widetilde{\mu}$. We may assume in addition that the function f is identically zero on the support of the measure σ. By (3.15), we get $\widetilde{\mu}\big(\{x \in \overline{B_0(R)};\ f(x) = 0\}\big) = 0$. For any natural number β, any real number $q \ge 1$, and any Borelian set A in $\overline{B_0(R)}$, by (3.13) with $\varphi = f^q \mathbf{1}_{A_\beta}$, where $A_\beta = \{x \in A\,; \quad f(x) \le \beta\}$, we get

$$\begin{aligned}\int_{A_\beta} f^{qp_+} d\widetilde{\mu} &\le \Lambda \prod_{i=1}^{n} \left(\int_{A_\beta} f^{qp_i} d\widetilde{\nu}_i \right)^{\frac{p_+}{np_i}} \\ &\le \Lambda \prod_{i=1}^{n} \left(\int_{A_\beta} f^{qp_i+1} d\widetilde{\mu} \right)^{\frac{p_+}{np_i}} \\ &\le \Lambda \prod_{i=1}^{n-n_+} \left(\int_{A_\beta} f^{qp_i+1} d\widetilde{\mu} \right)^{\frac{p_+}{np_i}} \left(\beta \int_{A_\beta} f^{qp_+} d\widetilde{\mu} \right)^{\frac{n_+}{n}} .\end{aligned}$$

Choosing q large enough so that $q > 1/(p_+ - p_i)$ for all $i = 1, \dots, n - n_+$, by Hölder's inequality, it follows that

$$\int_{A_\beta} f^{qp_+} d\widetilde{\mu} \le \beta^{\frac{n_+}{n}} \Lambda \nu \left(\overline{B_0(R) \times V} \right)^{\frac{p_+ q - 1}{nq} \sum_{i=1}^{n-n_+} \frac{1}{p_i} - \frac{n-n_+}{n}} \left(\int_{A_\beta} f^{qp_+} d\widetilde{\mu} \right)^{\frac{1}{nq} \sum_{i=1}^{n-n_+} \frac{1}{p_i} + 1} .$$

We then get that either $\int_{A_\beta} f^{qp_+} d\widetilde{\mu} = 0$ or $\int_{A_\beta} f^{qp_+} d\widetilde{\mu} > C_\beta$, for some positive constant C_β independent of A. It follows that for any β, the measure $A \to \int_{A_\beta} f^{qp_+} d\widetilde{\mu}$ is a finite linear combination of Dirac masses. Since $\widetilde{\mu}\big(\{x \in \overline{B_0(R)};\, f(x) = 0\}\big) = 0$, it follows that for any β, the measure $A \to \widetilde{\mu}(A_\beta)$ is a finite linear combination of Dirac masses. Passing to the limit as $\beta \to +\infty$, we get that there exists an at most countable index set J of distinct points $y_j = \big(y_1^j, \dots, y_{n-n_+}^j\big)$ in $\overline{B_0(R)}$, $j \in J$, such that $\operatorname{Supp} \widetilde{\mu} = \{y_j\,; \quad j \in J\}$. It follows that

$$\operatorname{Supp} \mu \cap \overline{B_0(R) \times V} \subset \bigcup_{j \in J} \overline{V}_{y_j} , \tag{3.16}$$

where

$$\overline{V}_{y_j} = \big\{ \big(y_1^j, \dots, y_{n-n_+}^j\big) \big\} \times \overline{V} . \tag{3.17}$$

We end the proof of Theorem 1.1 by using Palais–Smale properties of the sequence $(\widetilde{u}_\alpha)_\alpha$. For any smooth function ϕ with compact support in Ω, we get

$$\sum_{i=1}^{n} \int_\Omega \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx = \lambda \int_\Omega \widetilde{u}_\alpha^{p_+ - 1} \phi dx + \mathrm{o}(1) \tag{3.18}$$

as $\alpha \to +\infty$. The functions $\widetilde{u}_\alpha^{p_+-1}$ keep bounded in $L^{p_+/(p_+-1)}(\Omega)$ and converge, up to a subsequence, almost everywhere to $u_\infty^{p_+-1}$ in Ω as $\alpha \to +\infty$. By standard integration theory, it follows that the functions $\widetilde{u}_\alpha^{p_+-1}$ converge weakly to $u_\infty^{p_+-1}$ in $L^{p_+/(p_+-1)}(\Omega)$. On the other hand, for any $i = 1, \dots, n$, the functions $|\partial \widetilde{u}_\alpha / \partial x_i|^{p_i-2} \partial \widetilde{u}_\alpha / \partial x_i$ keep bounded in $L^{p_i/(p_i-1)}(\Omega)$, and thus converge, up to a subsequence, weakly to a function ψ_i in $L^{p_i/(p_i-1)}(\Omega)$ as $\alpha \to +\infty$. Passing to the limit into (3.18) as $\alpha \to +\infty$, we get

$$\sum_{i=1}^{n} \int_\Omega \psi_i \frac{\partial \phi}{\partial x_i} dx = \lambda \int_\Omega u_\infty^{p_+-1} \phi dx . \tag{3.19}$$

By an easy density argument, (3.19) holds true for all functions ϕ in $D^{1,\overrightarrow{p}}(\Omega)$. Now, we let φ be a nonnegative, smooth function with support in $B_0(2R) \times \mathbb{R}^{n_+}$. Since the sequence $(\widetilde{u}_\alpha)_\alpha$

is Palais–Smale for the functional I_λ, we get

$$\sum_{i=1}^{n}\left(\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i}\varphi dx+\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\widetilde{u}_\alpha\frac{\partial\varphi}{\partial x_i}dx\right)=\lambda\int_\Omega\widetilde{u}_\alpha^{p_+}\varphi dx+DI_\lambda\left(\widetilde{u}_\alpha\right).\left(\widetilde{u}_\alpha\varphi\right)$$
$$\le\lambda\int_\Omega\widetilde{u}_\alpha^{p_+}\varphi dx+\mathrm{o}\left(1\right)\tag{3.20}$$

as $\alpha\to+\infty$. For any $i=1,\dots,n-n_+$, by the compact embeddings in Rákosník [38], we get that the sequence $\left(\widetilde{u}_\alpha\right)_\alpha$ converges to u_∞ in $L^{p_i}\left(\operatorname{Supp}\varphi\right)$, and thus that

$$\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\widetilde{u}_\alpha\frac{\partial\varphi}{\partial x_i}dx\longrightarrow\int_\Omega\psi_iu_\infty\frac{\partial\varphi}{\partial x_i}dx\tag{3.21}$$

as $\alpha\to+\infty$. For any α and any $i=n-n_++1,\dots,n$, we get

$$\left|\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_+-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\widetilde{u}_\alpha\frac{\partial\varphi}{\partial x_i}dx\right|\le\left\|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right\|_{L^{p_+}(\Omega)}^{p_+-1}\left\|\widetilde{u}_\alpha\right\|_{L^{p_+}(\Omega)}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)}.\tag{3.22}$$

Since the sequence $\left(\widetilde{u}_\alpha\right)_\alpha$ is bounded in $L^{p_+}\left(\Omega\right)$ and converges to u_∞ almost everywhere in Ω, by Brezis–Lieb [12], we get

$$\int_\Omega\widetilde{u}_\alpha^{p_+}\varphi dx\longrightarrow\int_\Omega u_\infty^{p_+}\varphi dx+\int_{\overline{B_0(2R)\times V}}\varphi d\mu\,.\tag{3.23}$$

Since there holds $|\partial\widetilde{u}_\alpha/\partial x_i|^{p_i}\ge|\partial v_\alpha/\partial x_i|^{p_i}-|\partial u_\infty/\partial x_i|^{p_i}$, where $v_\alpha=\widetilde{u}_\alpha-u_\infty$ for all α and $i=1,\dots,n$, we get

$$\liminf_{\alpha\to+\infty}\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i}\varphi dx\ge\int_{\overline{B_0(2R)\times V}}\varphi d\nu_i-\int_\Omega\left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i}\varphi dx\tag{3.24}$$

as $\alpha\to+\infty$. By (3.21), (3.22), (3.23), and (3.24), passing to the limit into (3.20) as $\alpha\to+\infty$, we get

$$\sum_{i=1}^{n}\int_{\overline{B_0(2R)\times V}}\varphi d\nu_i-\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i}\varphi dx+\sum_{i=1}^{n-n_+}\int_\Omega\psi_iu_\infty\frac{\partial\varphi}{\partial x_i}dx$$
$$\le\lambda\left(\int_\Omega u_\infty^{p_+}\varphi dx+\int_{\overline{B_0(2R)\times V}}\varphi d\mu\right)+C\sum_{i=n-n_++1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)}\tag{3.25}$$

for some positive constant C independent of φ. Increasing if necessary the constant C, it follows from (3.19) and (3.25) that

$$\sum_{i=1}^{n}\int_{\overline{B_0(2R)\times V}}\varphi d\nu_i-\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i}\varphi dx-\sum_{i=1}^{n}\int_\Omega\psi_i\frac{\partial u_\infty}{\partial x_i}\varphi dx$$
$$\le\lambda\int_{\overline{B_0(2R)\times V}}\varphi d\mu+C\sum_{i=n-n_++1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)}.\tag{3.26}$$

We let η be a smooth cutoff function on $\mathbb{R}^{n-n_+}$ such that $\eta=1$ in $B_0\left(1\right)$, $0\le\eta\le1$ in $B_0\left(2\right)\backslash B_0\left(1\right)$, and $\eta=0$ in $\mathbb{R}^{n-n_+}\backslash B_0\left(2\right)$. For any point $y=\left(y_1,\dots,y_{n-n_+}\right)$ in $\overline{B_0\left(R\right)}$ and for any positive real number ε, we let $\varphi_{\varepsilon,y}$ be the function defined on $\mathbb{R}^n$ by

$$\varphi_{\varepsilon,y}\left(x_1,\dots,x_n\right)=\eta\left(\frac{1}{\varepsilon}\left(x_1-y_1\right),\dots,\frac{1}{\varepsilon}\left(x_{n-n_+}-y_{n-n_+}\right)\right).$$

Plugging $\varphi=\varphi_{\varepsilon,y}$ into (3.26), and passing to the limit as $\varepsilon\to 0$, we get

$$\sum_{i=1}^{n}\nu_i\big(\overline{V}_y\big)\le\lambda\mu\big(\overline{V}_y\big),\tag{3.27}$$

where $\overline{V}_y$ is as in (3.17). By (3.14) and (3.27), we get that there holds either

$$\mu\big(\overline{V}_y\big)=0\quad\text{or}\quad\lambda\mu\big(\overline{V}_y\big)^{\frac{p_+}{n+p_+}}\ge\Lambda^{\frac{-n}{n+p_+}}\tag{3.28}$$

for all points y in $\mathbb{R}^{n-n_+}$. On the other hand, by (3.11) and by an easy change of variable, for any α, we get

$$\mathcal{E}\big(\widetilde{u}_\alpha,\mathcal{P}_y^{\overrightarrow{p}}(1)\big)\le\delta_0\,,\tag{3.29}$$

where the energy functional $\mathcal{E}$ is as in (3.1) and $\mathcal{P}_y^{\overrightarrow{p}}(1)$ is as in (3.10). By (3.23) and since there holds $\overline{V}_y\subset\mathcal{P}_y^{\overrightarrow{p}}(1)$, passing to the limit into (3.29) as $\alpha\to+\infty$, it follows that

$$\mathcal{E}\big(u_\infty,\mathcal{P}_y^{\overrightarrow{p}}(1)\big)\le\delta_0\,.\tag{3.30}$$

Choosing δ_0 small enough so that $\delta_0<\Lambda^{-\frac{n}{p_+}}\lambda^{-\frac{n+p_+}{p_+}}$, it follows from (3.28) and (3.30) that there holds $\mu\big(\overline{V}_y\big)=0$ for all points y in $\overline{B_0(R)}$. By (3.16), we then get that the measure μ is identically zero on $\overline{B_0(R)}$. It follows that $|v_\alpha|^{p_+}\rightharpoonup 0$ as $\alpha\to+\infty$, where $v_\alpha=\widetilde{u}_\alpha-u_\infty$, and thus that the sequence $(\widetilde{u}_\alpha)_\alpha$ converges to u_∞ in $L^{p_+}_{\mathrm{loc}}(B_0(R))$. This ends the proof of Step 3.2. ☐

The next step in the proof of Theorem 1.1 is as follows.

Step 3.3. *If the constant δ_0 is small enough, then $\nabla\widetilde{u}_\alpha$ converges, up to a subsequence, to ∇u_∞ almost everywhere in Ω.*

Proof. We let φ be a smooth function with compact support in $\mathbb{R}^n$. Since the sequence $(\widetilde{u}_\alpha)_\alpha$ is Palais–Smale for the functional I_λ, there holds $DI_\lambda(\widetilde{u}_\alpha).((\widetilde{u}_\alpha-u_\infty)\varphi)\to 0$ as $\alpha\to+\infty$, and thus

$$\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\left(\frac{\partial\widetilde{u}_\alpha}{\partial x_i}-\frac{\partial u_\infty}{\partial x_i}\right)\varphi dx+\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}(\widetilde{u}_\alpha-u_\infty)\frac{\partial\varphi}{\partial x_i}dx$$
$$-\lambda\int_\Omega\widetilde{u}_\alpha^{p_+-1}(\widetilde{u}_\alpha-u_\infty)\varphi dx\longrightarrow 0\tag{3.31}$$

as $\alpha\to+\infty$. By Hölder's inequality and by Step 3.2, we get

$$\left|\int_\Omega\widetilde{u}_\alpha^{p_+-1}(\widetilde{u}_\alpha-u_\infty)\varphi dx\right|\le\|\varphi\|_{L^\infty(\Omega)}\|\widetilde{u}_\alpha\|_{L^{p_+}(\Omega)}^{p_+-1}\|\widetilde{u}_\alpha-u_\infty\|_{L^{p_+}(\operatorname{Supp}\varphi)}\longrightarrow 0\tag{3.32}$$

and

$$\left|\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}(\widetilde{u}_\alpha-u_\infty)\frac{\partial\varphi}{\partial x_i}dx\right|$$
$$\le|\operatorname{Supp}\varphi|^{\frac{p_+-p_i}{p_+p_i}}\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^\infty(\Omega)}\left\|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i-1}\|\widetilde{u}_\alpha-u_\infty\|_{L^{p_+}(\operatorname{Supp}\varphi)}\longrightarrow 0\tag{3.33}$$

as $\alpha\to+\infty$ for all $i=1,\dots,n$. By (3.31), (3.32), and (3.33), we get

$$\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\left(\frac{\partial\widetilde{u}_\alpha}{\partial x_i}-\frac{\partial u_\infty}{\partial x_i}\right)\varphi dx\longrightarrow 0\tag{3.34}$$

as $\alpha \to +\infty$. On the other hand, since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges weakly to the function u_∞ in $D^{1,\overrightarrow{p}}(\Omega)$, we get

$$\int_\Omega \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\infty}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\varphi dx \longrightarrow \int_\Omega \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i} \varphi dx \tag{3.35}$$

as $\alpha \to +\infty$ for all $i = 1, \dots, n$. By (3.34) and (3.35), we get

$$\sum_{i=1}^n \int_\Omega \left(\left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\infty}{\partial x_i}\right)\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial u_\infty}{\partial x_i}\right)\varphi dx \longrightarrow 0 \tag{3.36}$$

as $\alpha \to +\infty$. Since (3.36) holds true for all smooth functions φ with compact support in $\mathbb{R}^n$, we then get that for any $i = 1, \dots, n$ and any bounded domain Ω' of $\mathbb{R}^n$, there holds

$$\int_{\Omega'} \left(\left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\infty}{\partial x_i}\right)\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial u_\infty}{\partial x_i}\right) dx \longrightarrow 0$$

as $\alpha \to +\infty$. In particular, up to a subsequence, there holds

$$\left(\left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\infty}{\partial x_i}\right)\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial u_\infty}{\partial x_i}\right) \longrightarrow 0 \quad \text{a.e. in } \Omega$$

as $\alpha \to +\infty$. It easily follows that the functions $\partial\widetilde{u}_\alpha/\partial x_i$ converge, up to a subsequence, almost everywhere to $\partial u_\infty/\partial x_i$ in Ω as $\alpha \to +\infty$. This ends the proof of Step 3.3. □

The final step in the proof of Theorem 1.1 is as follows.

Step 3.4. *The function u_∞ is a nontrivial, nonnegative solution of the problem* (1.1).

Proof. We let φ be a smooth function with compact support in Ω. Since the sequence $(\widetilde{u}_\alpha)_\alpha$ is Palais–Smale for the functional I_λ, we get

$$\sum_{i=1}^n \int_\Omega \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx - \lambda \int_\Omega \widetilde{u}_\alpha^{p_+-1}\varphi dx \longrightarrow 0 \tag{3.37}$$

as $\alpha \to +\infty$. By Step 3.3 (resp. Step 3.2), the functions $\partial\widetilde{u}_\alpha/\partial x_i$ (resp. $\widetilde{u}_\alpha$) converge almost everywhere to $\partial u_\infty/\partial x_i$ (resp. u_∞) in Ω as $\alpha \to +\infty$. Moreover, $|\partial\widetilde{u}_\alpha/\partial x_i|^{p_i-2}\,\partial\widetilde{u}_\alpha/\partial x_i$ (resp. $\widetilde{u}_\alpha^{p_+-1}$) keep bounded in $L^{p_i/(p_i-1)}(\Omega)$ (resp. $L^{p_+/(p_+-1)}(\Omega)$). By standard integration theory, it follows that

$$\int_\Omega \widetilde{u}_\alpha^{p_+-1}\varphi dx \longrightarrow \int_\Omega u_\infty^{p_+-1}\varphi dx \tag{3.38}$$

and

$$\int_\Omega \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx \longrightarrow \int_\Omega \left|\frac{\partial u_\infty}{\partial x_i}\right|^{p_i-2} \frac{\partial u_\infty}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx \tag{3.39}$$

as $\alpha \to +\infty$ for all $i = 1, \dots, n$. By (3.37), (3.38), and (3.39), we get that u_∞ is a solution of problem (1.1). Moreover, u_∞ is nonnegative since the functions $\widetilde{u}_\alpha$ are nonnegative. We finally claim that u_∞ is not identically zero. Indeed, by (3.11) and by an easy change of variable, for any α, we get

$$\mathcal{E}\big(\widetilde{u}_\alpha, \mathcal{P}_0^{\overrightarrow{p}}(1)\big) = \delta_0\,, \tag{3.40}$$

where the energy functional $\mathcal{E}$ is as in (3.1) and $\mathcal{P}_0^{\overrightarrow{p}}(1)$ is as in (3.10). By Step 3.2, passing to the limit into (3.40) as $\alpha \to +\infty$, we get

$$\mathcal{E}\big(u_\infty, \mathcal{P}_0^{\overrightarrow{p}}(1)\big) = \delta_0\,.$$

In particular, u_∞ is not identically zero. This ends the proof of Step 3.4. □

Step 3.4 ends the proof of Theorem 1.1

4. The regularity result

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may assume that there exists an index n_+ such that $p_{n-n_++1} = \dots = p_n = p_+$ and $p_i < p_+$ for all $i \le n - n_+$. We let u be a solution of problem (1.1). We begin with proving that u belongs to $L^q(\Omega)$ for all real numbers $q > p_+$. We let

$$\varphi_\alpha = \min\left(|u|^{\frac{q-p_+}{p_+}}, \alpha\right)$$

for all positive real numbers α. For any $j = 1, \dots, n$, multiplying equation (1.1) by $u\varphi_\alpha^{p_j}$ and integrating by parts on Ω, since $u = 0$ on $\partial\Omega$, we get

$$\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_j} dx \le \lambda \int_\Omega |u|^{p_+} \varphi_\alpha^{p_j} dx\,. \tag{4.1}$$

Moreover, for any positive real number β, we get

$$\int_\Omega |u|^{p_+} \varphi_\alpha^{p_j} dx \le \beta^{\frac{(q-p_+)p_j}{p_+}} \int_\Omega |u|^{p_+} dx + \int_{W_\beta} |u|^{p_+} \varphi_\alpha^{p_j} dx\,, \tag{4.2}$$

where

$$W_\beta = \{x \in \Omega\,; \quad |u(x)| > \beta\}\,. \tag{4.3}$$

By Hölder's inequality, we get

$$\int_{W_\beta} |u|^{p_+} \varphi_\alpha^{p_j} dx \le \left(\int_{W_\beta} |u|^{p_+} dx\right)^{\frac{p_+-p_j}{p_+}} \left(\int_\Omega |u|^{p_+} \varphi_\alpha^{p_+} dx\right)^{\frac{p_j}{p_+}}. \tag{4.4}$$

Since $p_+ = p^*$, by the anisotropic Sobolev inequality in Troisi [41], we get

$$\int_\Omega |u|^{p_+} \varphi_\alpha^{p_+} dx \le \Lambda \prod_{i=1}^n \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{p_+}{np_i}} \tag{4.5}$$

for some positive constant Λ independent of α and u. By Young's inequality, it follows that for any $\varepsilon > 0$, there holds

$$\begin{aligned}\int_\Omega |u|^{p_+} \varphi_\alpha^{p_+} dx \le &\frac{\Lambda}{n}\Bigg(\varepsilon^{\frac{-n_+}{n-n_+}} \sum_{i=1}^{n-n_+} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{p_+}{p_i}} \\ &+ \varepsilon \sum_{i=n-n_++1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_+} \varphi_\alpha^{p_+} dx\Bigg).\end{aligned} \tag{4.6}$$

By (4.1)–(4.6), we get

$$\begin{aligned}\int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p_j} \varphi_\alpha^{p_j} dx \le \lambda\beta^{\frac{(q-p_+)p_j}{p_+}} \int_\Omega |u|^{p_+} dx + \lambda\left(\frac{\Lambda}{n}\right)^{\frac{p_j}{p_+}} \left(\int_{W_\beta} |u|^{p_+} dx\right)^{\frac{p_+-p_j}{p_+}} \\ \times \left(\sum_{i=1}^{n-n_+} \left(\varepsilon^{\frac{-n_+}{n-n_+}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{p_j}{p_i}} + \left(\varepsilon \sum_{i=n-n_++1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_+} \varphi_\alpha^{p_+} dx\right)^{\frac{p_j}{p_+}}\right).\end{aligned} \tag{4.7}$$

Choosing ε small enough so that $\varepsilon < n/(\lambda\Lambda)$, it follows that

$$\sum_{i=n-n_++1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_+} \varphi_\alpha^{p_+} dx \le \frac{n\lambda\beta^{q-p_+}}{n-\lambda\Lambda\varepsilon} \int_\Omega |u|^{p_+} dx + \frac{\lambda\Lambda}{n-\lambda\Lambda\varepsilon} \sum_{i=1}^{n-n_+} \left(\varepsilon^{\frac{-n_+}{n-n_+}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{p_+}{p_i}}. \tag{4.8}$$

It follows from (4.7) with $\varepsilon = 1$ and (4.8) with $\varepsilon < n/(\lambda\Lambda)$ that

$$\sum_{i=1}^{n-n_+} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{1}{p_i}} \le C\Bigg(\beta^{\frac{q-p_+}{p_+}} \sum_{i=1}^{n-n_+} \left(\int_\Omega |u|^{p_+} dx\right)^{\frac{1}{p_i}} \tag{4.9}$$
$$+ \left(\sum_{i=1}^{n-n_+} \left(\int_{W_\beta} |u|^{p_+} dx\right)^{\frac{p_+-p_i}{p_+p_i}}\right) \left(\beta^{\frac{q-p_+}{p_+}} \left(\int_\Omega |u|^{p_+} dx\right)^{\frac{1}{p_+}} + \sum_{i=1}^{n-n_+} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{1}{p_i}}\right)\Bigg)$$

for some positive constant C independent of α, β, and u. Since the function u belongs to $L^{p_+}(\Omega)$, increasing if necessary the constant C, it follows from (4.8) and (4.9) that for β large, there holds

$$\sum_{i=1}^{n} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi_\alpha^{p_i} dx\right)^{\frac{1}{p_i}} \le C\beta^{\frac{q-p_+}{p_+}}, \tag{4.10}$$

where C is independent of α, β, and u. Passing to the limit into (4.10) as $\alpha \to +\infty$, we get

$$\sum_{i=1}^{n} \left(\int_\Omega \left|\frac{\partial}{\partial x_i}\left(|u|^{\frac{q}{p_+}}\right)\right|^{p_i} dx\right)^{\frac{1}{p_i}} < +\infty.$$

By the continuity of the embedding of $D^{1,\overrightarrow{p}}(\Omega)$ into $L^{p_+}(\Omega)$, it follows that $|u|^{\frac{q}{p_+}}$ belongs to $L^{p_+}(\Omega)$, and thus that u belongs to $L^q(\Omega)$ for all real numbers $q > p_+$. Now, we prove that u belongs to $L^\infty(\Omega)$. For any positive real number t, we define the function $\varphi_t : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_t(s) = \begin{cases} s+t & \text{if } s \le -t, \\ 0 & \text{if } -t < s < t, \\ s-t & \text{if } s \ge t. \end{cases}$$

Multiplying equation (1.1) by $\varphi_t(u)$ and integrating by parts on Ω, since $u = 0$ on $\partial\Omega$, we get

$$\sum_{i=1}^{n} \int_{W_t} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = \lambda \int_{W_t} |u|^{p_+-2} u\varphi_t(u)\, dx,$$

where W_t is as in (4.3). For any real number $q > p_+$, by Hölder's inequality, it follows that

$$\sum_{i=1}^{n} \int_{W_t} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \le \lambda |W_t|^{\frac{(p_+-1)(q-p_+)}{p_+q}} \left(\int_{W_t} |u|^q dx\right)^{\frac{p_+-1}{q}} \left(\int_{W_t} |\varphi_t(u)|^{p_+} dx\right)^{\frac{1}{p_+}}. \tag{4.11}$$

Since $p_+ = p^*$, by the anisotropic Sobolev inequality in Troisi [41], and by Young's inequality, we get

$$\begin{aligned}\left(\int_{W_t} \left|\varphi_t\left(u\right)\right|^{p_+} dx\right)^{\frac{n}{n+p_+}} &\le \Lambda \prod_{i=1}^{n} \left(\int_{W_t} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{p_+}{(n+p_+)p_i}}\\ &\le \frac{p_+\Lambda}{n+p_+} \sum_{i=1}^{n} \frac{1}{p_i} \int_{W_t} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \end{aligned}\tag{4.12}$$

for some positive constant Λ independent of t and u. Since the function u belongs to $L^q\left(\Omega\right)$, it follows from (4.11) and (4.12) that

$$\left(\int_{W_t} \left|\varphi_t\left(u\right)\right|^{p_+} dx\right)^{\frac{1}{p_+}} \le C\left|W_t\right|^{\frac{(n+p_+)(p_+-1)(q-p_+)}{(np_+-n-p_+)p_+q}}$$

for some positive constant C independent of t and u. By Fubini's theorem and Hölder's inequality, we then get

$$\int_t^{+\infty} \left|W_s\right| ds = \int_t^{+\infty} \int_{W_t} \mathbf{1}_{W_s} dx ds = \int_{W_t} \left|\varphi_t\left(u\right)\right| dx \le C\left|W_t\right|^{\frac{(p_+-1)(nq-n-p_+)}{(np_+-n-p_+)q}}.$$

Choosing q large enough so that

$$\frac{\left(p_+-1\right)\left(nq-n-p_+\right)}{\left(np_+-n-p_+\right)q} > 1\,,$$

it easily follows that there holds $\left|W_t\right| = 0$ for t large, and thus that u belongs to $L^\infty\left(\Omega\right)$. $\square$

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