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Victor Campos, Claudia Linhares Sales, Ana Karolinna Maia, Rudini Sampaio

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HAL Id: hal-00951135
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Submitted on 24 Feb 2014

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Maximization Coloring Problems on graphs with few $P_4$s

V. Campos*  C. Linhares Sales *  A. K. Maia†  R. Sampaio *

February 24, 2014

Abstract

Given a graph $G = (V, E)$, a greedy coloring of $G$ is a proper coloring such that, for each two colors $i < j$, every vertex of $V(G)$ colored $j$ has a neighbor with color $i$. The greatest $k$ such that $G$ has a greedy coloring with $k$ colors is the Grundy number of $G$. A $b$-coloring of $G$ is a proper coloring such that every color class contains a vertex which is adjacent to at least one vertex in every other color class. The greatest integer $k$ for which there exists a $b$-coloring of $G$ with $k$ colors is its $b$-chromatic number. Determining the Grundy number and the $b$-chromatic number of a graph are NP-hard problems in general.

For a fixed $q$, the $(q, q - 4)$-graphs are the graphs for which no set of at most $q$ vertices induces more than $q - 4$ distinct induced $P_3$s. In this paper, we obtain polynomial-time algorithms to determine the Grundy number and the $b$-chromatic number of $(q, q - 4)$-graphs, for a fixed $q$. They generalize previous results obtained for cographs and $P_4$-sparse graphs, classes strictly contained in the $(q, q - 4)$-graphs.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph, without loops or multiple edges. A $k$-coloring of $G$ is a surjective mapping $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The sets of vertices $S_1, \ldots, S_k$ with colors $1, 2, \ldots, k$, respectively, that form a partition of $V(G)$ in stable sets, are called color classes. The chromatic number $\chi(G)$ of $G$ is the smallest integer $k$ such that $G$ admits a $k$-coloring. It is well known that determining $\chi(G)$ is a NP-hard problem.

Hence lots of heuristics have been developed to color a graph. One of the most basic and used is the greedy algorithm. Given an order $v_1, v_2, \ldots, v_n$ of the vertices of $G$, the greedy algorithm colors the vertices of $G$ assigning to $v_i$ the minimum positive integer that was not already assigned to its neighbors in the set $\{v_1, \ldots, v_{i-1}\}$. Such a coloring is called a greedy coloring. The maximum number of colors of a greedy coloring of a graph $G$, over all possible orderings of the vertices of $V(G)$, is the Grundy number of $G$ and it is denoted by $\Gamma(G)$.

Zaker [1] showed that, for any fixed $k$, one can decide in polynomial time if a given graph has Grundy number at least $k$ (that is, deciding if $\Gamma(G) \geq k$ is fixed parameter tractable on $k$). However determining the Grundy number of a graph is NP-hard [1]. Moreover, in 2010, Havet and Sampaio [2] proved that it is NP-complete to decide if $\Gamma(G) = \Delta(G) + 1$. In addition, Asté et al. [3] showed that, for any constant $c \geq 1$, it is NP-complete to decide if $\Gamma(G) \leq c \cdot \chi(G)$.

Another alternative way of dealing with the coloring problem is to try to improve any coloring $c$ of the graph by applying some strategy, obtaining from $c$ a coloring with a smaller number of colors.

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*ParGO Research Group, Federal University of Ceará, Campus do Pici, Bloco 910, 60455-760 - Fortaleza - Brazil. Partially supported by CNPq/Brazil and FUNCAP/Brazil. (email: [campos, linhares, rudini]@lia.ufc.br)

†COATI Project, I3S (CNRS/UNSA) & INRIA, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis Cedex, France. Partially supported by ANR Blanc STINT and CAPES/Brazil. (email: karol.maia@inria.fr)
Observe that, if \( c \) has a color class \( S_i \) such that for every vertex \( v \in S_i \), there is at least one other color class \( S_j \) such that \( v \) does not have neighbors in \( S_j \), we could eliminate \( S_i \) by recoloring every vertex \( v \) from \( S_i \) with the color \( j \) that does not appear in its neighborhood. A vertex \( v \) from \( S_i \) is said to be dominant if \( v \) is adjacent to at least one vertex in \( S_j \) for all \( j \neq i \). It is easy to see that if every color class \( S_i \in c \) has a dominant vertex, then it is not possible to improve \( c \) by applying the above strategy.

A \( b \)-coloring of \( G \) is a coloring such that every color class contains a dominant vertex. The \( b \)-chromatic number \( \chi_b(G) \) of a graph \( G \) is the maximum number \( k \) such that there exists a \( b \)-coloring of \( G \) with \( k \) colors. Observe that the \( b \)-chromatic number of \( G \) measures the worst performance of the improvement strategy of a coloring described previously. This parameter has been introduced by R. W. Irving and D. F. Manlove [4]. They proved that determining the \( b \)-chromatic number is polynomial-time solvable for trees, but it is NP-hard for general graphs. In [5], Kratochvíl, Tuza and Voigt proved that computing the \( b \)-chromatic number is NP-hard even if \( G \) is a connected bipartite graph.

Let \( G = (V, E) \) be a graph. We say that \( G \) is a \( P_4 \) if \( V(G) = \{w, x, y, z\} \) and \( E(G) = \{wx, xy, yz\} \), that is, an induced path on four vertices. We say that \( w \) and \( z \) are the endpoints and \( x \) and \( y \) the midpoints of the \( P_4 \).

A cograph is a \( P_4 \)-free graph and a \( P_4 \)-sparse graph is a graph \( G \) such that each subset of \( G \) with five vertices induces at most one \( P_4 \). The \( P_4 \)-sparse graphs, introduced in [6], generalize cographs and can be recognized in linear time [7].

Many NP-hard problems were proved to be polynomial-time solvable on cographs and \( P_4 \)-sparse graphs. In particular, polynomial-time algorithms were presented to solve the problem of determining the Grundy number and the \( b \)-chromatic number for these graphs [8, 9, 10].

Babel and Olariu [11] defined a graph as \((q, q - 4)\)-graph if no set of at most \( q \) vertices induces more than \( q - 4 \) distinct \( P_4 \)s. For example, cographs and \( P_4 \)-sparse graphs are precisely \((4, 0)\)-graphs and \((5, 1)\)-graphs, respectively.

Our main result (Theorem 1) says that, for every fixed integer \( q > 0 \), there is a polynomial algorithm to obtain the Grundy number and the \( b \)-chromatic number of a \((q, q - 4)\)-graph.

**Theorem 1** (Main result). Let \( q > 0 \) be a fixed integer. The Grundy number and the \( b \)-chromatic number of a \((q, q - 4)\)-graph \( G \) can be computed in polynomial time.

This paper is organized as follows. Section 2 contains structural results for \((q, q - 4)\)-graphs. Section 3 presents the results used to calculate the Grundy number of these graphs and in Section 4 we show how to determine their \( b \)-chromatic number.

## 2 Decomposing \((q, q - 4)\)-graphs

A graph \( H \) is \( p \)-connected if, for every partition of \( V(G) \) into nonempty disjoint sets \( V_1 \) and \( V_2 \), there exists an \((V_1, V_2)\)-crossing \( P_4 \), that is, an induced \( P_4 \) containing vertices from both \( V_1 \) and \( V_2 \). A \( p \)-connected graph \( H \) is separable if there exists a partition of \( V(G) \) into nonempty disjoint subsets \( V_1 \) and \( V_2 \) such that each \((V_1, V_2)\)-crossing \( P_4 \) has its midpoints in \( V_1 \) and its endpoints in \( V_2 \). We say that \((V_1, V_2)\) is the separation of \( H \) and \( H_1 \) and \( H_2 \) are the graphs \( H[V_1] \) and \( H[V_2] \), respectively. A maximal \( p \)-connected induced subgraph is called a \( p \)-component. Vertices which are not contained in a nontrivial \( p \)-component are called weak.

A decomposition tree of a graph \( G \) is a tree \( T_G \), where the leaves are subsets of vertices of \( G \) and each non-leaf node \( v \) in \( T_G \), with children \( v_1, \ldots, v_l \), represents the subgraph of \( G \), denoted by \( G(v) \), induced by the leaves of the subtree of \( T_G \) rooted by \( v \). Moreover, \( v \) is labelled according to its relation with the graphs \( G(v_1), \ldots, G(v_l) \). Clearly, the intersection of the leaves must be empty and their union must be the set of vertices of \( G \). The root node of \( T_G \) represents the original graph \( G \).
In [12], Jamison and Olariu suggest a decomposition tree for general graphs, called primeval decomposition tree, which can be computed in linear time [12]. The leaves of its decomposition tree are \( p \)-connected graphs and its weak vertices, and its internal nodes are labelled union, join or \( p \)-component.

If the label of a node \( v \) is union, \( G(v) \) is the disjoint union of \( G(v_1), \ldots, G(v_l) \), that is, the set of vertices of \( G(v) \) is the union of the set of vertices of \( G(v_1), \ldots, G(v_l) \) and the set of edges of \( G(v) \) is the union of the set of edges of \( G(v_1), \ldots, G(v_l) \).

If the label of a node \( v \) is join, \( G(v) \) is the join of \( G(v_1), \ldots, G(v_l) \), that is, the set of vertices of \( G(v) \) is the union of the set of vertices of \( G(v_1), \ldots, G(v_l) \) and the set of edges of \( G(v) \) is the union of the set of edges of \( G(v_1), \ldots, G(v_l) \).

If \( v \) is labelled \( p \)-component, it has two children on the tree: a separable \( p \)-component \( H \), which is a leaf on the primeval decomposition tree and an internal node that represents the graph \( G(v) - H \). Moreover, every vertex from \( G(v) - H \) is adjacent to every vertex in \( H_1 \) and to no vertex in \( H_2 \).

A graph is a spider if its vertex set can be partitioned into three sets \( S, K \) and \( R \) in such a way that \( S \) is a stable set, \( K \) is a clique, all the vertices of \( R \) are adjacent to all the vertices of \( K \) and to none of the vertices of \( S \) and there exists a bijection \( f : S \to K \) such that, for all \( s \in S \), either the neighborhood of \( s N(s) = \{ f(s) \} \) (and it is a thin spider) or \( N(s) = K - \{ f(s) \} \) (and it is a thick spider). We say that the spider is without head if \( R = \emptyset \).

In [11], Babel and Olariu also proved that the primeval decomposition of a \((q,q-4)\)-graph has a special property: every node \( v \) on the tree labelled as \( p \)-component is such that its separable \( p \)-component \( H \) is a headless spider or it has less than \( q \) vertices. If \( H \) is the headless spider, it is easy to see that \( H_1 \) is the clique and \( H_2 \) is the stable set. Since every vertex from \( V(G(v) - H) \) is adjacent to every vertex in \( H_1 \) and non-adjacent to every vertex in \( H_2 \), we have that \( G(v) \) is itself a spider with head, where \( G(v) - H \) is the head.

In this paper, we calculate the Grundy number and the \( b \)-chromatic number of \((q,q-4)\)-graphs through bottom-up traversal on the their primeval decomposition tree. More specifically we solve the case in which a node \( v \) on is labelled as \( p \)-component and the \( p \)-connected component \( H \) of \( G(v) \) is a graph with less than \( q \) vertices. In the remaining non-trivial cases, we use some results in [9] and [10] to calculate \( \Gamma(G(v)) \) and \( \chi_b(G(v)) \), respectively.

### 3 Greedy Coloring of \((q,q-4)\)-graphs

As first shown in [8], if \( G \) is the disjoint union of two graphs \( G_1 \) and \( G_2 \), then \( \Gamma(G) = \max\{ \Gamma(G_1), \Gamma(G_2) \} \). On the other hand, if \( G \) is the join of two graphs \( G_1 \) and \( G_2 \), then \( \Gamma(G) = \Gamma(G_1) + \Gamma(G_2) \). In [9] is shown how to determine the Grundy number for spiders.

**Lemma 2** ([9]). Let \( G \) be a spider with partition \((S,K,R)\) and \( n \) vertices. If \( G \) is a spider and \( \Gamma(R) \) is given, then \( \Gamma(G) \) can be determined in linear time.

Let \( G = (V,E) \) be a graph. A subset \( M \) of \( V \) with \( 1 \leq |M| \leq |V| \) is called a module if each vertex in \( V - M \) is either adjacent to all vertices of \( M \) or to none of them. A module \( M \) is called a homogeneous set if \( 1 < |M| < |V| \). The graph obtained from \( G \) by shrinking every maximal homogeneous set to one single vertex is called the characteristic graph of \( G \).

A graph is called split graph if its vertex set has a partition \((K,S)\) such that \( K \) induces a clique and \( S \) induces an stable set.

**Lemma 3** ([13]). A \( p \)-connected graph \( H \) is separable if and only if its characteristic graph is a split graph.
Lemma 5. Note that, if $M_1$ and $M_2$ are two modules of a graph $G$ such that $M_1 \cap M_2 = \emptyset$, then either the edges from \( \{v, w\} : v \in M_1, w \in M_2 \) belong to $G$ or $G$ has none of such edges.

Recall Lemma 3. Clearly, if the characteristic graph of a separable p-component $H$ with separation $(V_1, V_2)$ is the split graph $\langle K, S \rangle$, then every maximal homogeneous set $M^1_i \subseteq V_1$ shrinks to a vertex $v^1_i$ in the clique $K$, and every maximal homogeneous set $M^2_j \subseteq V_2$ shrinks to a vertex $v^2_j$ in the stable set $S$. We say that $H[M^1_i] = H^1_i$.

Let $H$ be a separable p-component with separation $(V_1, V_2)$. Observe that $H_1 = H[V_1]$ is the join of $H^1_1, \ldots, H^1_1$, since, between the graphs induced by two modules in the same graph, or there exist all the edges or none between them, and $H^1_1, \ldots, H^1_1$ are the graphs induced by the strong maximal modules of $H_1$. So, $\Gamma(H_1)$ is the Grundy number of the join of the graphs $H^1_1, \ldots, H^1_1$, which is $\sum_i \Gamma(H^1_i)$. Similarly, the Grundy number of some $H^2_2$ in $H_2 = H[V_2]$ with its neighborhood in $H_1$ is the Grundy number of the join of these graphs.

In [9], a relation between the Grundy number of a graph and the Grundy number of its modules is shown.

Proposition 4. Let $G, H_1, \ldots, H_n$ be disjoint graphs such that $n = |V(G)|$ and let $V(G) = \{v_1, \ldots, v_n\}$. Let $G'$ be the graph obtained by replacing $v_i \in V(G)$ by $H_i$, in such a way that there exist all the edges between the vertices of $H_i$ and $H_j$, $i \neq j$, if and only if $v_i v_j \in E(G)$. Then for every greedy coloring of $G'$ at most $\Gamma(H_i)$ colors contain vertices of the induced subgraph $G'[V(H_i)] \subseteq G'$, for all $i \in \{1, \ldots, n\}$.

According to Proposition 4, a greedy coloring of a graph $G$ restricted to its modules is a greedy coloring to them. The following result is a simple generalization of a result in [9]:

Lemma 5. Let $G$ be a graph and let $M$ be a module of $G$ such that $G[M] = H$ and in a greedy coloring that generates $\Gamma(G)$ there are $k$ colors in $H$. Let $G'$ be the graph obtained from $G$ by replacing $H$ by a complete graph $K_k$. Then, $\Gamma(G) = \Gamma(G')$.

Proof. Let $c$ be the coloring that generates $\Gamma(G)$. Let $A = \{\alpha_1, \ldots, \alpha_k\}$ be the set of colors of $c$ appearing on $H$. Let the vertices of the complete graph that replaces $H$ on $G'$ be $w_1, \ldots, w_k$ and $c'$ be the coloring of $G'$ defined by $c'(w_i) = \alpha_i$ for $i \in \{1, \ldots, k\}$ and $c'(v) = c(v)$ for each vertex $v \in V(G) - M$. It is a simple matter to check that $c'$ is a greedy coloring of $G'$. Hence $\Gamma(G') \geq \Gamma(G)$. Now let $\{S_1, \ldots, S_k\}$ be a greedy $k$-coloring of $H$ and $c'$ be a greedy $\Gamma(G')$-coloring of $G'$. It is important to see that there is a greedy $k$-coloring of $H$, by Proposition 4. Let $B = \{\beta_1, \ldots, \beta_k\}$ be the set of colors appearing on $K_k$ with $\beta_1 < \ldots < \beta_k$. Let $c$ be the coloring of $G$ which, for every $1 \leq i \leq k$, assigns the color $\beta_i$ to the vertices from $S_i$. Clearly, $c$ is a greedy coloring of $G$. So $\Gamma(G) \geq \Gamma(G')$. \(\square\)

We denote by $\theta_H$ an order that produces a coloring with $\Gamma(H)$ colors for $H$. In particular, we denote by $\theta'_j$ an order that produces a coloring with $\Gamma(H'_j)$ colors for $H'_j$. Theorem 6 is the main result of this section.

Theorem 6. Let $G$ be a $(q, q - 4)$-graph containing a separable p-component $H$ with separation $(V_1, V_2)$ and at most $q$ vertices, such that every vertex in $R = G - H$ is adjacent to all vertices in $H_1 = H[V_1]$ and to no vertex in $H_2 = H[V_2]$. Let $H^1_1, \ldots, H^1_1$ be the graphs induced by the maximal homogeneous sets of $H_1$ and $H^2_1, \ldots, H^2_1$ the graphs induced by the maximal homogeneous sets of $H_2$. Given $\chi(R)$ and $\Gamma(R)$, let $G'$ be the graph obtained from $G$ by replacing $R$ by a complete graph $K_{\chi(R)}$. Then:

(a) If $\Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H^2_i)$, then $\Gamma(G) = \Gamma(R) + \sum_{i=1}^m \Gamma(H^1_i)$;

(b) If $\Gamma(R) < \max_{1 \leq i \leq m} \Gamma(H^2_i)$, then $\Gamma(G) = \Gamma(G')$. 

Consider the following cases:

(i) There is a vertex \( v \in R \) colored \( c_{\text{max}} \):

Let \( c' = c(R \cup H_1) \). All colors in \( c \) should appear in \( c' \), since \( v \), to be colored \( c_{\text{max}} \), has to be adjacent to vertices colored with all colors different from \( c_{\text{max}} \), and a vertex in \( R \) has neighbors only in \( R \cup H_1 \). So, \( c' \) has more than \( \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^l) \) colors. Note that \( c' \) is not a greedy coloring to \( R \cup H_1 \), because a greedy coloring to \( R \cup H_1 \) has at most \( \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^l) \) colors, since \( R \cup H_1 \) is the join of \( R, H_1^1, \ldots, H_1^l \). Thus, there is a vertex \( u \in R \cup H_1 \) colored \( t \) that has no neighbor colored \( f \) in \( R \cup H_1 \), for some \( f < t \). Such vertex should be in \( H_1 \), since all neighbors of vertices in \( R \) are in \( R \cup H_1 \). Then, \( u \in H_1^l \) has a neighbor \( w \in H_2^l \) colored \( f \). Note that there exist all edges between \( H_1^l \) and \( H_2^l \). Some vertex \( z \in R \cup H_1 \) is also colored \( f \). It is easy to see that \( z \notin R \), otherwise \( u \) would have a neighbor in \( R \cup H_1 \) colored \( f \), since every vertex from \( R \) is adjacent all vertex in \( H_1 \).Lemma 3 shows that there is all possible edges between two modules of \( H_1 \). So, \( z \notin H_1^l \), for \( s \neq i \), because in this case also \( u \) would have a neighbor in \( R \cup H_1 \) already colored \( f \). Therefore \( z \in H_1^l \) and consequently \( z \) is adjacent to \( w \), since there must exist all possible edges between \( H_1^l \) and \( H_2^l \). But both are colored \( f \), and this coloring would be improper.

(ii) There is a vertex \( v \in H_2 \) colored \( c_{\text{max}} \):

For some \( s \in \{1, \ldots, m\} \), let \( v \in H_2 \) and \( c' = c(H_2^s \cup N(H_2^s)) \), \( N(H_2^s) \) being the graphs induced by the maximal homogeneous sets of \( H_1 \) such that the vertices are adjacent to the vertices of \( H_2^s \). All colors in \( c \) should appear in \( c' \), since \( v \) has to be adjacent to vertices colored with all colors different from \( c_{\text{max}} \) and a vertex in \( H_2^s \) has neighbors only in \( (H_2^s) \cup N(H_2^s) \). So, \( c' \) has more than \( \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^l) \) colors. Note that \( \Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H_i^l) \) implies \( \Gamma(R) \geq \Gamma(H_2^s) \). Therefore, \( \Gamma(H_2^s) + \sum_{i \in N(H_2^s)} \Gamma(H_i) \leq \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^l) \). Then \( c' \) is not a greedy coloring to \( (H_2^s) \cup N(H_2^s) \), because a greedy coloring to it has at most \( \Gamma(H_2^s) + \sum_{i \in N(H_2^s)} \Gamma(H_i) \) colors, since \( H_2^s \cup N(H_2^s) \) is the join of \( H_2^s, H_i^l, \forall i \in N(H_2^s) \). Thus, there is a vertex \( u \in H_2^s \cup N(H_2^s) \), colored \( t \), that has no neighbor colored \( f \) in \( H_2^s \cup N(H_2^s) \), for some \( f < t \). Such vertex should be in \( H_1 \), because all neighbors of vertices in \( H_2^s \) are in \( H_2^s \cup N(H_2^s) \). So, \( v \in H_1^l \), where \( H_1^l \in N(H_2^s) \), has a neighbor \( w \in R \cup H_1 \) colored \( f \). Observe that some vertex \( z \in H_2^s \cup N(H_2^s) \) is also colored \( f \). It is easy to see that \( z \notin H_2^s \). Otherwise, \( u \) would have a neighbor in \( H_2^s \cup N(H_2^s) \) colored \( f \) since every vertex in \( H_2^s \) is adjacent to every vertex in \( N(H_2^s) \). For the same reason, \( z \notin H_1^l \), for \( j \neq i \) and \( j \in N(H_2^s) \). Therefore \( z \in H_1^l \), but there is all possible edges between \( H_1^l \) and \( R \cup H_1 \) colored \( f \), what makes \( w \) and \( z \) neighbors. But both \( w \) and \( z \) are colored \( f \), and this coloring would be improper.

(iii) There is a vertex \( v \in H_1 \) colored \( c_{\text{max}} \):

To receive a color bigger than \( \Gamma(R) + \sum_{i=1}^l \Gamma(H_i) \), \( v \) must have at least \( \Gamma(R) + \sum_{i=1}^l \Gamma(H_i) \) neighbors of different colors. From its neighborhood in \( R \), \( v \) has at most \( \Gamma(R) \) neighbors with different colors, by Proposition 4. From the neighborhood of \( v \) in \( H_1^l \), for \( i \in \{1, \ldots, l\} \), \( v \) has at most \( \sum_{i=1}^l \Gamma(H_i^l) - 1 \) its own color, also by Proposition 4. So, it must appear another color \( c_n \) in a vertex \( w \in H_2^l \), where \( V(H_2^l) \in N(v) \). Since the vertices in \( R \) have no neighborhood with \( H_2, c_n \) must be bigger than all colors in \( R \) and \( w \) must be neighbor of vertices colored with all colors in \( R \). All these colors must appear in \( H_2^l \), because the neighbors from \( w \) outside \( H_2^l \) are
vertices in $H_1$, all neighbors from all vertices in $R$ and, therefore, with different colors of $R$.

We know that in $H^2_p$ appears at most $\Gamma(H^2_p)$ colors, what makes $w$ to have at most $\Gamma(H^2_p) - 1$ neighbors colored differently in $H^2_p$. But we know $\Gamma(H^2_p) \leq \Gamma(R)$ implies $\Gamma(H^2_p) - 1 < \Gamma(R)$.

So, all colors of $R$ cannot appear on the neighborhood of $w$, and such vertex cannot receive a different color.

(b) Since $\Gamma(R) < \max_{1 \leq j \leq m} \Gamma(H^2_p)$, in a greedy $\Gamma(G)$-coloring of $G$, by Proposition 4, there are $p < q$ colors on $R$. We do not know the exact value of $p$, but we know that $p$ goes from $\chi(R)$ to $\Gamma(R)$.

By Lemma 5, we can replace $R$ by a complete graph on $p$ vertices and we can obtain all possible ordinations of $V(G)$, which are $(q + p)!$ in total. So, we can calculate all greedy colorings for $G$ in $\sum_{p=\chi(R)} (p + q)! \leq q(2q)! = O(1)$ steps, for a fixed $q$. \qed

4 \hspace{0.5em} b\text{-coloring of } (q, q - 4)\text{-graphs}

In [10], Bonomo et al. presented a dynamic programming polynomial-time algorithm to compute the $b$-chromatic number of a $P_4$-sparse graph. For this, they introduced the dominance vector of a graph.

**Definition 7.** Let $G$ be a graph. Given a coloring of $G$, a vertex $v$ is said to be dominant if $v$ is adjacent to at least one vertex colored within each of the colors not assigned to $v$. The dominance vector $dom_G$ of $G$ is such that $dom_G[i]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of $G$ with $t$ colors, where $\chi(G) \leq t \leq |V(G)|$.

Note that a graph $G$ admits a $b$-coloring with $t$ colors if and only if $dom_G[i] = t$. So, the $b$-chromatic number $\chi_b(G)$ is the maximum number $t$ such that $dom_G[i] = t$. Thus, once calculated the dominance vector of a graph, we have its $b$-chromatic number. Bonomo et al. [10] proved that calculating the dominance vector is polynomial-time solvable for cographs and $P_4$-sparse graphs.

Lemmas 8 and 9 below from [10] show how to obtain the dominance vector for disjoint unions, joins and spiders. The calculation of $\chi(G)$ is from [14] and [15].

**Lemma 8** (Dominance vector for union and join operations [10]). Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \emptyset$ and let $t \geq \chi(G)$. If $G = G_1 \cup G_2$, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ and

$$dom_G[i] = \min\{t, dom_{G_1}[i] + dom_{G_2}[i]\}.$$

If $G = G_1 \vee G_2$, let $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$. Then, $\chi(G) = \chi(G_1) + \chi(G_2)$ and

$$dom_G[i] = \max_{a \leq j \leq b} \{dom_{G_1}[i] + dom_{G_2}[t - j]\}.$$

**Lemma 9** (Dominance vector for spiders [10]). Let $G$ be a spider with partition $(S, K, R)$, where $k = |S| = |K| \geq 2$. If $R$ is empty, consider $\chi(G[R]) = 0$ and $dom_{G[R]}[0] = 0$. Thus, $\chi(G) = k + \chi(G[R])$ and

(a) If $G$ is a thin spider, then

$$dom_G[i] = \begin{cases} k + dom_{G[R]}[i-k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ k, & \text{if } i = k + |R| + 1, \\ 0, & \text{if } i > k + |R| + 1. \end{cases}$$
(b) If $G$ is a thick spider, then

$$\text{dom}_G[i] = \begin{cases} 
  k + \text{dom}_{G[R]}[i-k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\
  \min\{k, 4k - 2i + 2|R|\}, & \text{if } k + |R| + 1 \leq i \leq 2k + |R|, \\
  0, & \text{if } i > 2k + |R|
\end{cases}$$

Using these lemmas, Bonomo et al. proved the theorem below.

**Theorem 10** (Bonomo et al. [10]). The dominance vector and the $b$-chromatic number of a cograph or $P_4$-sparse graph can be computed in $O(n^3)$ time.

Let $G = (V,E)$ be a graph and $M$ be a module of $G$. Let $G_M = G[M]$ and let $N(M)$ be the neighborhood of a vertex in $M$. Let $H, H_1$ and $H_2$ be the subgraphs of $G$ induced by $V \setminus M, N(M)$ and $V(H) \setminus N(M)$, respectively. If $H$ has less than $q$ vertices, $G$ is obtained by applying $p$-component($q$) operation over $(G_M, H = (H_1,H_2))$.

To calculate $\text{dom}_G[i]$, auxiliary lemma below shows us that there exists a good coloring such that: (a) all colors appears in $M$ or $H_1$ or (b) vertices of $M$ have distinct colors. Given a coloring $c$ of $G$ and a subgraph $G'$ of $G$, let $n(C)$ be the number of colors used in $C$ and let $(C, G')$ be the restriction of the coloring $C$ to $G'$.

**Lemma 11.** If $\chi(G) \leq t \leq |V(G)|$, then there is a proper coloring $C$ of $G$ with $t$ colors that maximizes the number of color classes with dominant vertices such that $n(C) = n(C,H_1) + n(C,G_M)$ or $n(C, G_M) = |V(G_M)|$.

**Proof.** Let $C$ be a coloring of $G$ with $t$ colors that maximizes the number of color classes with dominant vertices and then maximizes $n(C, G_M)$. Since $M$ is a module, each vertex in $G_M$ is adjacent to all vertices in $H_1$. Thus, $n(C) \geq n(C,H_1) + n(C,G_M)$. Suppose that $C$ does not satisfy the lemma. Since $n(C) > n(C,H_1) + n(C,G_M)$, then there is a color $c$ that appears only in vertices of $H_2$ and thus no vertex of $G_M$ is dominant in $C$. Since $n(C,G_M) < |V(G_M)|$, then there are two vertices $v$ and $v'$ of $G_M$ that have the same color in $C$. Consider the coloring $C'$ obtained from $C$ by coloring $v$ with color $c$. Note that any dominant vertex in $C$ is also a dominant vertex in $C'$ and thus $C'$ also has a maximum number of color classes with dominant vertices among colorings with $t$ colors. Note that $n(C', G_M) > n(C, G_M)$. Suppose again that $C'$ does not satisfy the lemma. So, we can repeat this argument until we obtain a coloring $C^*$ such that all vertices of $G_M$ are colored with distinct colors. Thus, $n(C^*, G_M) = |V(G_M)|$ as desired. \hfill \Box

Applying this lemma, we have four possible cases:

- (a) all colors appears in $M$ or $H_1$
  - (a.1) There is no dominant vertex in $H_2$
  - (a.2) There is a dominant vertex in $H_2$

- (b) Vertices of $M$ have distinct colors
  - (b.1) There are colors in $M$ that are not in $H$
  - (b.2) Every color in $M$ appears in $H$

Case (b.2) is easy to handle because it implies that $|M| \leq |V(H)|$. Since we will force that $|V(H)| \leq q$, we can obtain all colorings of $G$ with $t$ colors in constant time. To deal with cases (a.1), (a.2) and (b.1), we have to define some parameters.
Let $C(t)$ be the set of all colorings of $H$ with $t$ colors and let $C(t, t')$ be subset of $C(t)$ with colorings of $H$ such that $H_1$ uses $t'$ colors. Let $C \subseteq C(t, t')$. For $H' \subseteq H$, let $c(C, H')$ denote the set of colors used in $H'$. We say that a vertex $v$ in $H_1$ is partially dominant if $v$ is adjacent to at least one vertex receiving each color in $c(C, H_1)$. Let $d_1(C)$ be the number of colors classes of $C$ with partially dominant vertices in $H_1$. Let $d_2(C)$ be the number of color classes of $c(C, H_2) \setminus c(C, H_1)$ with a dominant vertex. Let $d_3(C)$ be the number of color classes in $c(C, H_1)$ with either a dominant vertex in $H_2$ or a partially dominant vertex in $H_1$. Let $J \subseteq c(C, H_2) \setminus c(C, H_1)$. We say that a vertex $v$ in $H_1$ is $J$-dominant if $v$ is adjacent to at least one vertex receiving each color in $c(C, H) \setminus J$. Let $d_4(C, J)$ be the number of color classes of $C$ with either a dominant vertex in $H_2$ or a $J$-dominant vertex in $H_1$ and $d_5(C, j) = \sup\{d_4(C, J), J \subseteq c(C, H_2) \setminus c(C, H_1), |J| = j\}$.

Let $\chi(G) \leq t \leq |V|$, let $t_1 = \max\{t - |V(G_M)|, 0\}$, let $t_2 = \min\{|V(H_1)|, t - \chi(G_M)\}$, let $t_3 = \min\{|V(H)|, t\}$, let $t_4 = \min\{t - |V(G_M)|, |V(H)|\}$ and let

$$
\tau_1(t) = \sup_{t_1 \leq t' \leq t_2} \{\text{dom}_{G_M}[t - t'] + d_1(C) \mid C \subseteq C(i, t')\}
$$

$$
\tau_2(t) = \sup_{t_1 \leq t' \leq t_2} \{\min\{t - t', d_2(C) + \text{dom}_{G_M}[t - t']\} + d_3(C) \mid C \subseteq C(t, t')\}
$$

$$
\tau_3(t) = \sup_{t_1 \leq t' \leq t_2, 0 \leq t' \leq t} \{d_5(C, \hat{i} + |V(G_M)| - t) \mid C \subseteq C(i, t')\}
$$

Excluding case (b.2) by forcing that $|V(G)| > 2|V(H)|$, we have the important lemma below.

**Lemma 12.** If $\chi(G) \leq t \leq |V(G)|$ and $|V(G)| > 2|V(H)|$, then

$$
\text{dom}_{G}[t] = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}
$$

**Proof.** Let $C$ be a coloring of $G$ with $t$ colors that maximizes the number of color classes with dominant vertices. According to Lemma 11, suppose that either $n(C, H_1) + n(C, G_M) = t$ or $n(C, H_1) + n(C, G_M) < t$ and $n(C, G_M) = |V(G_M)|$. Let $i = n(C, H)$.

The first case considered is (a) when $n(C, H_1) + n(C, G_M) = t$. Note that if $v$ is a dominant vertex in $C$, then $v$ is dominant in $(C, G_M)$ if $v \in V(G_M)$ and $v$ is partially dominant in $(C, H)$ if $v \in V(H_1)$. Let $t = n(C, H_1)$. Since $\chi(G_M) \leq n(C, G_M) = |V(G_M)|$, then $t - |V(G_M)| \leq |V(G)| - |V(H_1)| \leq t - \chi(G_M)$. We also get that $t' \leq |V(H_1)|$ and, thus, $t_1 \leq t' \leq t_2$.

Now, consider (a.1) that there is no dominant vertex in $H_2$. In this case, $t' \leq \hat{i} \leq \min\{|V(H)|, t\}$. We also have that the number of color classes of $C$ with dominant vertices of colors that appear in $H_1$ is precisely $d_1(C, H)$ and the with dominant vertices of colors that appear in $G_M$ is at most $\text{dom}_{G_M}[t - t']$. Thus, if $n(C, H_1) + n(C, G_M) = t$ and there is no dominant vertex of $C$ in $H_2$, then $\text{dom}_{G}[t] \leq \tau_1(t)$.

Now, consider (a.2) that there is at least one dominant vertex $u$ in $H_2$. Since $u$ is adjacent to every other color of $C$ and every neighbour of $u$ is in $H$, then $i = t$. Note that the number of color classes of $C$ with dominant vertices of colors that appear in $H_1$ is precisely $d_3(C, H)$. The number of color classes of $C$ with dominant vertices of colors that appear in $G_M$ is at most $\min\{t - t', d_2(C) + \text{dom}_{G}[t - t']\}$. Thus, if $n(C, H_1) + n(C, G_M) = t$ and there is at least one dominant vertex of $C$ in $H_2$, then $\text{dom}_{G}[t] \leq \tau_2(t)$.

The second case considered is (b) when $n(C, H_1) + n(C, G_M) \leq t$ and $n(C, G_M) = |V(G_M)|$. If (b.2) $c(C, G_M) \leq c(C, H)$, then $n(C, G_M) = |V(G_M)|$ implies that $|M| \leq |V(H)|$ and $|V(G)| \leq |V(H)|$, a contradiction. Thus, (b.1) there is a color unique to vertices in $G_M$. Note also that $n(C, H_1) + n(C, G_M) < t$ implies that there is a color unique to vertices in $H_2$. Thus, all dominant vertices of $C$ are in $H_1$. Note that $i \leq |V(H)|$ and $i \leq t$ and, thus, $i \leq t_3$. We also have that $t \leq \hat{i} + |V(G_M)|$ which implies that $\hat{i} \geq t_1$. Let $J = c(C, G_M) \cap c(C, H)$. Note that $t = \hat{i} + |V(G_M)| - |J|$ which implies that
$|J| = \hat{t} + |V(G_M)| - t$. Since $J$ is a subset of $c(C, H) \setminus c(C, H_1)$, then $|J| = \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$ which implies that $t' \leq t - |V(G_M)|$. Since $H_1$ has at most $V(H_1)$ colors, then $t' \leq t_4$. Now, note that every dominant vertex of $C$ is a $J$-dominant vertex of $H_1$ in $(C, H)$. Thus, the number of color classes with dominant vertices in $C$ is $d_4((C, H), \hat{t})$, which is at most $d_5((C, H), \hat{t} + |V(G_M)| - t)$. Thus, if $n(C, H_1) + n(C, G_M) < t$ and $|V| > 2|V(H)|$, then $\text{dom}_G[t] \leq \tau_3(t)$.

We can conclude from the previous paragraphs that if $|V| > 2|V(H)|$, then $\text{dom}_G[t] \leq \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}$. To do so, let $C_M$ be a coloring of $G_M$ with $t - t'$ colors and $\text{dom}_{G_M}[t - t']$ color classes with dominant vertices and $C_M'$ be a coloring of $G_M$ with $|V(G_M)|$ colors.

Suppose that $t_1 < t' \leq t_2$. If $\hat{t} \leq t$, then rename the colors in $c(C_H, H_2) \setminus c(C_H, H_1)$ to colors in the set $c(C_M)$ and let $C$ be the coloring of $G$ obtained by piecing together this coloring with $C_M$. Note that $C$ has precisely $t$ colors and there are $\text{dom}_{G_M}[t - t']$ color classes with dominant vertices in colors of $C(C, G_M)$ and $d_1(C_H)$ color classes with dominant vertices in colors of $c(C, H_1)$. Since $c(C, G_M) \cap c(C, H_1) = \emptyset$, then $C$ has at least $\text{dom}_{G_M}[t - t'] + d_1(C)$ color classes with dominant vertices. This implies that $\text{dom}_G[t] \geq \tau_1(t)$.

Now, suppose that $\hat{t} = t$. Let $C_M = \{c_1, \ldots, c_{t'-t}\}$ and suppose that the color classes with indices in $\{1, \ldots, \text{dom}_{G_M}[t - t']\}$ have dominant vertices in $C_M$. Now let $C_H$ be obtained from $C_H$ by renaming the colors in $c(C_H, H_2) \setminus c(C_H, H_1)$ to colors in $c(C_M)$ in such a way that the color classes with the highest indices have dominant vertices. Note that this is possible as $c(C_H, H_2) \setminus c(C_H, H_1)$ has size precisely $t - t'$. Let $C$ be obtained by piecing together the colorings $C_M$ and $C_H$. Note that $C$ has precisely $t$ colors, $d_3(C_H)$ color classes in $c(C, H_1)$ with dominant vertices and $\min\{t - t', d_2(C) + \text{dom}_G[t - t']\}$ color classes in $c(C, G_M)$ with dominant vertices. This implies that $\text{dom}_G[t] \geq \tau_2(t)$.

Now, suppose that $0 \leq t' \leq t_1$ and $t_1 \leq \hat{t} \leq t_3$. Note that $0 \leq \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$, as $\hat{t} \geq t_1 = t - |V(G_M)|$ and $t' \leq t_4 \leq t - |V(G_M)|$. Thus, let $J$ be a subset of $c(C_H, H_2) \setminus c(C_H, H_1)$ such that $d_4(C_H, J) + \text{dom}_G[t - t'] + |V(G_M)| - t$. Let $C_H$ be obtained by renaming the colors of $C_H$ in the set $J$ to colors in $C_M$ so that $|c(C_H) \cap c(C_M)| = |J| = \hat{t} + |V(G_M)| - t$. Let $C$ be obtained by piecing together the colorings $C_H$ and $C_M$. Note that $n(C) = n(C, H) + n(C, G_M) - |J| = \hat{t} + |V(G_M)| - |J| = t$. This implies that $\text{dom}_G[t] \geq \tau_3(t)$.

**Lemma 13.** Let $q > 0$ be a fixed integer, let $H$ be a graph with less than $q$ vertices and let $H_1$ and $H_2$ be induced subgraphs of $H$ such that $V(H_1)$ and $V(H_2)$ are a vertex partition of $H$. Given a graph $G_M$ with $n$ vertices, let $G$ be the graph obtained by applying $p$-component operation over $(G_M, H = (H_1, H_2))$ (just join all edges between $G_M$ and $H_1$). Then, given the chromatic number $\chi(G_M)$ and the dominance vector $\text{dom}_M$ of $G_M$, we can calculate the chromatic number $\chi(G)$ in time $\Theta(n)$ and the dominance vector $\text{dom}_G$ of $G$ in time $\Theta(n^2)$.

**Proof.** Since $|V(H)| \leq q$, where $q$ is an integer fixed, we have that parameters $\tau_1(t), \tau_2(t)$ and $\tau_3(t)$ can be obtained in linear time (once fixed $t'$ and $\hat{t}$, the value in sup can be obtained in constant time that depends only on $q$). If $|V(G)| \leq 2|V(H)| \leq 2q$, then we can calculate $\text{dom}_G[t]$ in constant time. If $|V(G)| > 2|V(H)|$, then, applying Lemma 12, we have $\text{dom}_G[t]$ in linear time. So we can obtain the dominance vector $\text{dom}_G$ of $G$ in time $\Theta(n^2)$ for all possible values of $t$.

**References**


