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QUASIHOMOGENEOUS THREE-DIMENSIONAL REAL ANALYTIC LORENTZ METRICS DO NOT EXIST

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ABSTRACT. We show that a germ of a real analytic Lorentz metric on $\mathbb{R}^3$ which is *locally homogeneous* on an open set containing the origin in its closure is necessarily locally homogeneous in the neighborhood of the origin. We classifiy Lie algebras that can act *quasihomogeneously*—meaning they act transitively on an open set admitting the origin in its closure, but not at the origin—and isometrically for such a metric. In the case that the isotropy at the origin of a quasihomogeneous action is semi-simple, we provide a complete set of normal forms of the metric and the action.

1. INTRODUCTION

A Riemannian or pseudo-Riemannian metric is called *locally homogeneous* if any two points can be connected by flowing along a finite sequence of local Killing fields. The study of such metrics is a traditional field in differential and Riemannian geometry. In dimension two, they are exactly the semi-Riemannian metrics of constant sectional curvature. Locally homogeneous Riemannian metrics of dimension three are the subject of Thurston's 3-dimensional geometrization program [Thu97]. The classification of compact locally homogeneous Lorentz 3-manifolds was given in [DZ10].

The most symmetric geometric structures after the locally homogeneous ones are those which are *quasihomogeneous*, meaning they are locally homogeneous on an open set containing the origin in its closure, but not locally homogeneous in the neighborhood of the origin. In particular, all the scalar invariants of a quasihomogeneous geometric structure are constant. Recall that, for Riemannian metrics, constant scalar invariants implies local homogeneity (see [PTV96] for an effective result).

In a recent joint work with A. Guillot, the first author obtained the classification of germs of quasihomogeneous, real analytic, torsion free, affine connections on surfaces [DG13]. The article [DG13] also classifies the quasihomogeneous germs of real analytic, torsion free, affine connections which extend to *compact* surfaces. In particular, such germs of quasihomogeneous connections do exist.

The first author proved in [Dum08] that *a real analytic Lorentz metric on a compact* 3*-manifold which is locally homogeneous on a nontrivial open set is locally homogeneous on all of the manifold*. In other words, quasihomogeneous real analytic Lorentz metrics do not extend to *compact* threefolds. The same is known to be true, by work of the second author, for real analytic Lorentz metrics on compact manifolds of higher dimension, under the assumptions that the Killing algebra is semi-simple, the metric is geodesically complete, and the universal cover is acyclic [Mel09]. In the smooth category, A. Zeghib proved in [Zeg96] that compact Lorentz 3-manifolds which admit essential Killing fields are necessarily locally homogeneous.

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Here we simplify arguments of [Dum08] and introduce new ideas in order to dispense with the compactness assumption and prove the following local result:

Theorem 1. *Let g be a real-analytic Lorentz metric in a connected open neighborhood U of the origin in $\mathbf{R}^3$. If g is locally homogeneous on a nontrivial open subset in U, then g is locally homogeneous on all of U.*

We also present a new, alternative approach to the problem, relying on the Cartan connection associated to a Lorentzian metric. This approach yields a nice alternate proof of our results.

Our work is motivated by Gromov's *open-dense orbit theorem* [DG91, Gro88] (see also [Ben97, Fer02]). Gromov's result asserts that, if the pseudogroup of local automorphisms of a *rigid geometric structure*—such as a Lorentz metric or a connection—acts with a dense orbit, then this orbit is open. In this case, the rigid geometric structure is locally homogeneous on an open dense set. Gromov's theorem says little about this maximal open and dense set of local homogeneity, which appears to by mysterious (see [DG91, 7.3.C]). In many interesting geometric situations, it may be all of the connected manifold. This was proved, for instance, for Anosov flows preserving a pseudo-Riemannian metric arising from differentiable stable and unstable foliations and a transverse contact structure [BFL92]. In [BF05], the authors deal with this question, and their results indicate ways in which some rigid geometric structures cannot degenerate off the open dense set.

The composition of this article is the following. In Section 2 we use the geometry of Killing fields and geometric invariant theory to prove that the Killing Lie algebra of a three-dimensional quasihomogeneous Lorentz metric g is a three-dimensional, solvable, nonunimodular Lie algebra. We also show that g is locally homogeneous away from a totally geodesic surface S, on which the isotropy is a one parameter semi-simple group or a one parameter unipotent group. In the case of semi-simple isotropy, Theorem 1 is proved in Section 3. In the case of unipotent isotropy, Theorem 1 is proved in Section 4. Section 5 provides an alternative proof of Theorem 1 using the formalism of Cartan connections.

2. Killing Lie Algebra. Invariant Theory

Let g be a real analytic Lorentz metric defined in a connected open neighborhood U of the origin in $\mathbb{R}^3$, which we assume is also simply connected. In this section we recall the definition and several properties of the Killing algebra of (U,g). These were proved in [Dum08] without use of the compactness assumption. For completeness, we briefly explain their derivation again here.

Classically, (see, for instance [Gro88, DG91]) one considers the k-jet of g by taking at each point $u \in U$ the expression of g up to order k in exponential coordinates. In these coordinates, the 0-jet of g is the standard flat Lorentz metric $dx^2+dy^2-dz^2$. At each point $u \in U$, the space of exponential coordinates is acted on simply transitively by $O(2,1)$, the identity component of which is isomorphic to $PSL(2,\mathbb{R})$. The space of all exponential coordinates in U compatible with a fixed orientation and time orientation is a principal $PSL(2,\mathbb{R})$-bundle over U, which we will call the orthonormal frame bundle and denote by $R(U)$.

Geometrically, the k-jets of g form an analytic $PSL(2,\mathbb{R})$-equivariant map $g^{(k)} : R(U) \to V^{(k)}$, where $V^{(k)}$ is the finite dimensional vector space of k-jets at 0 of Lorentz metrics on $\mathbb{R}^3$ with fixed 0-jet $dx^2+dy^2-dz^2$. The group $O^0(2,1) \simeq PSL(2,\mathbb{R})$ acts linearly on this space, in which the origin corresponds to the k-jet of the flat metric. One can find the details of this classical construction in [DG91].

Recall also that a local vector field is a *Killing field* for a Lorentz metric g if its flow preserves g wherever it is defined. Note that local Killing fields preserve orientation and time orientation, so they act on $R(U)$. The collection of all germs of local Killing fields at a point u has the structure of a finite dimensional Lie algebra $\mathfrak{g}$ called the *local Killing algebra* of g at u. At a given point $u \in U$, the subalgebra $\mathfrak{i}$ of the local Killing algebra consisting of the local Killing fields X with $X(u)=0$ is called the *isotropy* algebra at u.

The proof of theorem 1 will use analyticity in an essential way. We will make use of an extendability result for local Killing fields proved first by Nomizu in the Riemannian setting [Nom60] and generalized then for any rigid geometric structure by Amores and Gromov [Amo79, Gro88, DG91]. This phenomenon states that a local Killing field of g can be extended by monodromy along any curve γ in U, and the resulting Killing field only depends on the homotopy type of γ. Because U is assumed connected and simply connected, *local Killing fields extend to all of U*. Therefore, the local Killing algebra at any $u \in U$ equals the algebra of globally defined Kiling fields on U, which we will denote by $\mathfrak{g}$.

Definition 2. The Lorentz metric g is *locally homogeneous on an open subset $W \subset U$*, if for any $w \in W$ and any tangent vector $V \in T_w W$, there exists $X \in \mathfrak{g}$ such that $X(w) = V$. In this case, we will say that the Killing algebra $\mathfrak{g}$ is *transitive on W*.

Notice that Nomizu's extension phenomenon does not imply that the extension of a family of pointwise linearly independent Killing fields stays linearly independent. The assumption of theorem 1 is that $\mathfrak{g}$ is transitive on a nonempty open subset $W \subset U$. Choose three elements $X, Y, Z \in \mathfrak{g}$ that are linearly independent at a point $u_0 \in W$. The function $\mathrm{vol}_g(X(u), Y(u), Z(u))$ is analytic on U and nonzero in a neighborhood of u_0. The vanishing set of this function is a closed analytic proper subset S' of U containing the points where $\mathfrak{g}$ is not transitive. Its complement is an open dense set of U on which $\mathfrak{g}$ is transitive. Therefore, we can assume henceforth that *g is a quasihomogeneous Lorentz metric in the neighborhood U of the origin in $\mathbb{R}^3$*, with Killing algebra $\mathfrak{g}$.

We will next derive some basic properties of $\mathfrak{g}$ that follow from quasihomogeneity. Let S be the complement of the maximal open subset of U on which $\mathfrak{g}$ acts transitively–that is, of a maximal locally homogeneous subset of U. It is an intersection of closed, analytic sets, so *S is closed and analytic.*

Lemma 3 ([Dum08] lemme 3.2(i))**.** *The Killing algebra $\mathfrak{g}$ cannot be both 3-dimensional and unimodular.*

Proof. Let K_1, K_2 and K_3 be a basis of the Killing algebra. Again consider the analytic function $v = vol_g(K_1, K_2, K_3)$. Since $\mathfrak{g}$ is unimodular and preserves the volume form of g, the function v is nonzero and constant on each open set where $\mathfrak{g}$ is transitive. On the other hand, v vanishes on S: a contradiction. □

Lemma 4 ([Dum08] lemme 2.1, proposition 3.1, lemme 3.2(i))**.**

(i) The dimension of the isotropy at a point $u \in U$ is $\neq 2$.
(ii) The Killing algebra $\mathfrak{g}$ is of dimension 3.
(iii) The Killing algebra $\mathfrak{g}$ is solvable.

Proof. (i) Assume by contradiction that the isotropy algebra $\mathfrak{i}$ at a point $u \in U$ has dimension two. Elements of $\mathfrak{i}$ act linearly in exponential coordinates at u. Since elements of $\mathfrak{i}$ preserve g, they preserve, in particular, the k-jet of g at u, for all $k \in \mathbb{N}$. This gives an embedding of $\mathfrak{i}$ in the Lie algebra of $PSL(2,\mathbb{R})$ such that the corresponding two dimensional connected subgroup of $PSL(2,\mathbb{R})$ preserves the k-jet of g at u, for all $k \in \mathbb{N}$. But *stabilizers in a finite dimensional linear algebraic $PSL(2,\mathbb{R})$-action are of dimension $\neq 2$*. Indeed, it suffices to check this statement for irreducible linear representations of $PSL(2,\mathbb{R})$, for which it is well-known that the stabilizer in $PSL(2,\mathbb{R})$ of a nonzero element is one dimensional [Kir74].

It follows that the stabilizer in $PSL(2,\mathbb{R})$ of the k-jet of g at u is of dimension three and hence equals $PSL(2,\mathbb{R})$. Consequently, in exponential coordinates at u, each element of $\mathfrak{sl}(2,\mathbb{R})$ gives rise to a local linear vector field which preserves g, because it preserves all k-jets of the analytic metric g at u. The isotropy algebra $\mathfrak{i}$ thus contains a copy of the Lie algebra of $PSL(2,\mathbb{R})$: a contradiction, since $\mathfrak{i}$ is of dimension two.

(ii) Since g is quasihomogeneous, the Killing algebra is of dimension at least 3. For a three-dimensional Lorentz metric, the maximal dimension of the Killing algebra is 6. This characterizes Lorentz metrics of constant sectional curvature. Indeed, in this case, the isotropy is, at each point, of dimension three (see, for instance, [Wol67]). These Lorentz metrics are locally homogeneous.

Assume, by contradiction, that the Killing algebra of g is of dimension 5. Then, on any open set of local homogeneity the isotropy is two-dimensional. This is in contradiction with point (i).

Assume, by contradiction, that the Killing algebra of g is of dimension 4. Then, at a point $s \in S$, the isotropy has dimension ≥ 2. Hence, point (i) implies that the isotropy at s has dimension three and thus is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Moreover, the standard linear action of the isotropy on T_sU preserves the image of the evaluation morphism $ev(s) : \mathfrak{g} \to T_sU$, which is a line. But the standard 3-dimensional $PSL(2,\mathbb{R})$-representation does not admit invariant lines: a contradiction. Therefore, the Killing algebra is three-dimensional.

(iii) A Lie algebra of dimension three is semi-simple or solvable [Kir74]. Since semi-simple Lie algebras are unimodular, Lemma 3 implies that $\mathfrak{g}$ is solvable. □

Let us recall Singer's result [Sin60, DG91, Gro88] which asserts that *g is locally homogeneous if and only if the image of $g^{(k)}$ is exactly one $PSL(2,\mathbb{R})$-orbit in $V^{(k)}$, for a certain k (big enough).* This theorem is the key ingredient in the proof of the following fact.

Proposition 5 ([Dum08] lemme 2.2)**.** *If g is quasihomogeneous, then the Killing algebra $\mathfrak{g}$ does not preserve any nontrivial vector field of constant norm ≤ 0.*

Proof. Let $k \in \mathbb{N}$ be given by Singer's theorem. First suppose, for a contradiction, that there exists an isotropic vector field X in U preserved by $\mathfrak{g}$. Then the $\mathfrak{g}$-action on $R(U)$, lifted from the action on U, preserves the subbundle $R'(U)$, where $R'(U)$ is a reduction of the structural group $PSL(2,\mathbb{R}) \cong O^o(2,1)$ to the stabilizer of an isotropic vector in the standard linear representation on $\mathbb{R}^3$:

$$H = \left\{ \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \in PSL(2,\mathbb{R}) \ : \ T \in \mathbb{R} \right\}$$

Restricting to exponential coordinates with respect to frames preserving X gives an H-equivariant map $g^{(k)} : R'(U) \to V^{(k)}$. On each open set W on which g is locally homogeneous, the image $g^{(k)}(R'(W))$ is exactly one H-orbit $\mathscr{O} \subset V^{(k)}$. Let $s \in S$ be a point in the closure of W. Then the image under $g^{(k)}$ of the fiber $R'(W)_s$ lies in the closure of $\mathscr{O}$. But H is unipotent, and a classical result due to Kostant and Rosenlicht [Ros61] asserts that *for algebraic representations of unipotent groups, the orbits are closed.* This implies that the image $g^{(k)}(R'(W)_s)$ is also $\mathscr{O}$. Since $\mathfrak{g}$ acts transitively on S, this holds for all $s \in S$.

Any open set of local homogeneity in U admits points of S in its closure. It follows that the image of $R'(U)$ under $g^{(k)}$ is exactly the orbit $\mathscr{O}$. Singer's theorem implies that g is locally homogeneous, a contradiction to quasihomogeneity.

If, for a contradiction, there exists a $\mathfrak{g}$-invariant vector field X in U of constant strictly negative g-norm, then the $\mathfrak{g}$-action on $R(U)$ preserves a subbundle $R'(U)$ with structural group H', where H' is the stabilizer of a strictly negative vector in the standard linear representation of $PSL(2,\mathbb{R})$ on $\mathbb{R}^3$. In this case, H' is a compact one parameter group in $PSL(2,\mathbb{R})$. The previous argument works, replacing the Kostant-Rosenlicht theorem by the obvious fact that orbits of smooth compact group actions are closed. □

Lemma 6 (compare [Dum08], proposition 3.3)**.**

(i) S is a connected, real analytic submanifold of codimension one.
(ii) The isotropy at a point of S is unipotent or semisimple.
(iii) The restriction of g to S is degenerate.

Proof. (i) The fact that S is a real analytic set was already established above: it coincides with the vanishing of the analytic function $v = vol_g(K_1, K_2, K_3)$, where K_1, K_2 and K_3 form a basis of the Killing algebra. If needed, one can shrink the open set U in order to get S connected. By point (i) in Lemma 4, the isotropy algebra at points in S has dimension one or three. We prove that this dimension must be equal to one.

Assume, for a contradiction, that there exists $s \in S$ such that the isotropy at s has dimension three. Then, the isotropy algebra at s is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. On the other hand, since both are 3-dimensional, the isotropy algebra at s is isomorphic to $\mathfrak{g}$. Hence, $\mathfrak{g}$ is semi-simple, which contradicts Lemma 4 (iii).

It follows that the isotropy algebra at each point $s \in S$ is of dimension one. Equivalently, the evaluation morphism $ev(s) : \mathfrak{g} \to T_sU$ has rank two. Since the $\mathfrak{g}$-action preserves S, this implies that S is a smooth submanifold of codimension one in U and T_sS coincides with the image of $ev(s)$. Moreover, $\mathfrak{g}$ acts transitively on S.

(ii) Let $\mathfrak{i}$ be the isotropy Lie algebra at $s \in S$. It corresponds to a 1-parameter subgroup of $PSL(2,\mathbb{R})$, which is elliptic, semi-simple, or unipotent. In any case, there is a tangent vector $V \in T_sU$ annihilated by $\mathfrak{i}$. Then $\mathfrak{i}$ also vanishes along the curve $\exp_s(tV)$, where defined. Because points of $U \setminus S$ have trivial isotropy, this curve must be contained in S. Thus the fixed vector V of the flow of $\mathfrak{i}$ is tangent to S.

If $\mathfrak{i}$ is elliptic, it preserves a tangent direction at s transverse to the invariant subspace $T_sS \subset T_sU$. Within T_sS, there must also be an invariant line independent from V. But now an elliptic flow with three invariant lines must be trivial. We conclude that $\mathfrak{i}$ is semi-simple or unipotent.

(iii) If the isotropy is unipotent, the vector V annihilated by $\mathfrak{i}$ must be isotropic, and the invariant subspace T_sS must equal $V^{\perp}$. So S is degenerate in this case.

If $\mathfrak{i}$ is semi-simple, then V is spacelike. The other two eigenvectors of $\mathfrak{i}$ have nontrivial eigenvalues and must be isotropic. On the other hand, $\mathfrak{i}$ preserves the plane T_sS, so it preserves a line of T_sU transverse to S and a line independent from V in T_sS. These lines must be the eigenspaces of $\mathfrak{i}$. If the plane $T_sS \subset T_sU$ contains an isotropic line and is transverse to an isotropic line, then it is degenerate. □

According to Lemma 6 we have two different geometric situations, which will be treated separately in Sections 3 and 4.

3. No quasihomogeneous Lorentz metrics with semi-simple isotropy

If the isotropy at $s \in S$ is semisimple, then it fixes a vector $V \in T_sS$ of positive g-norm. Using the transitive $\mathfrak{g}$-action on S, we can extend V to a $\mathfrak{g}$-invariant vector field X on S with constant positive g-norm. In this section *we assume that the isotropy is semi-simple*. We can suppose that X is of constant norm equal to 1.

Recall that the affine group of the real line Aff is the group of transformations of $\mathbb{R}$ given by $x \mapsto ax + b$, with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$. If Y is the infinitesimal generator of the one parameter group of homotheties and H the infinitesimal generator of the one parameter group of translations, then $[Y,H] = H$.

Lemma 7 (compare [Dum08], proposition 3.6)**.**

(i) The Killing algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R} \oplus \mathfrak{aff}$. The isotropy corresponds to the one parameter group of homotheties in Aff.
(ii) The vector field X is the restriction to S of a central element X' in $\mathfrak{g}$.
(iii) The restriction of the Killing algebra to S has, in adapted analytic coordinates (x,h), a basis $(-h\frac{\partial}{\partial h}, \frac{\partial}{\partial h}, \frac{\partial}{\partial x})$.
(iv) In the above coordinates, the restriction of g to S is dx^2.

Proof. (i) We show first that the derived Lie algebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is 1-dimensional. It is a general fact that the derived algebra of a solvable Lie algebra is nilpotent [Kir74]. Remark first that $[\mathfrak{g},\mathfrak{g}] \neq 0$. Indeed, otherwise

$\mathfrak{g}$ is abelian and the action of the isotropy $\mathfrak{i} \subset \mathfrak{g}$ at a point $s \in S$ is trivial on $\mathfrak{g}$ and hence on T_sS, which is identified with $\mathfrak{g}/\mathfrak{i}$. The isotropy action on the tangent space T_sS being trivial implies that the isotropy action is trivial on T_sU (An element of $O(2,1)$ which acts trivially on a plane in $\mathbb{R}^3$ is trivial). This implies that the isotropy is trivial at $s \in S$: a contradiction. As $\mathfrak{g}$ is 3-dimensional, $\mathfrak{g}'$ is a nilpotent Lie algebra of dimension 1 or 2, hence $\mathfrak{g}' \simeq \mathbb{R}$, or $\mathfrak{g}' \simeq \mathbb{R}^2$.

Assume, for a contradiction, that $\mathfrak{g}' \simeq \mathbb{R}^2$. We first prove that the isotropy $\mathfrak{i}$ lies in $[\mathfrak{g},\mathfrak{g}]$. Suppose this is not the case. Then $[\mathfrak{g},\mathfrak{g}] \simeq \mathbb{R}^2$ acts freely and transitively on S, preserving the vector field X. Then X is the restriction to S of a Killing vector field $X' \in [\mathfrak{g},\mathfrak{g}]$.

Let Y be a generator of the isotropy at $s \in S$. Since X is fixed by the isotropy, one gets, in restriction to S, the following Lie bracket relation: $[Y,X'] = [Y,X] = aY$, for some $a \in \mathbb{R}$. On the other hand, by our assumption, $Y \notin [\mathfrak{g},\mathfrak{g}]$, meaning that $a = 0$. This implies that X' is a central element in $\mathfrak{g}$. In particular, $\mathfrak{g}'$ is one-dimensional: a contradiction. Hence $\mathfrak{i} \subset [\mathfrak{g},\mathfrak{g}]$.

Now let Y be a generator of $\mathfrak{i}$, $\{Y,X'\}$ be generators of $[\mathfrak{g},\mathfrak{g}]$, and $\{Y,X',Z\}$ be a basis of $\mathfrak{g}$. The tangent space of S at a point $s \in S$ is identified with $\mathfrak{g}/\mathfrak{i}$. Denote $\bar{X}',\bar{Z}$ the projections of X' and Z to this quotient. The infinitesimal action of Y on this tangent space is given in the basis $\{\bar{X}',\bar{Z}\}$ by the matrix

$$ad(Y) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

because $\mathfrak{g}' \simeq \mathbb{R}^2$ and $ad(Y)(\mathfrak{g}) \subset \mathfrak{g}'$. Moreover, $ad(Y) \neq 0$, since the restriction of the isotropy action to T_sS is injective. From this form of $ad(Y)$, we see that the isotropy is unipotent with fixed direction $\mathbb{R}X'$: a contradiction.

We have proved that $[\mathfrak{g},\mathfrak{g}]$ is 1-dimensional. Notice that $\mathfrak{i} \neq [\mathfrak{g},\mathfrak{g}]$. Indeed, if they are equal, then the action of the isotropy on the tangent space T_sU at $s \in S$ is trivial: a contradiction.

Let H be a generator of $[\mathfrak{g},\mathfrak{g}]$, and Y the generator of $\mathfrak{i}$. Then $[Y,H] = aH$, with $a \in \mathbb{R}$. If $a = 0$, then the image of $ad(Y)$, which lies in $[\mathfrak{g},\mathfrak{g}]$, belongs to the kernel of $ad(Y)$, which contradicts semisimplicity of the isotropy. Therefore $a \neq 0$ and we can assume, by changing the generator Y of the isotropy, that $a = 1$, so $[Y,H] = H$.

Let $X' \in \mathfrak{g}$ be such that $\{X',H\}$ span the kernel of $ad(H)$. Then $\{Y,X',H\}$ is a basis of $\mathfrak{g}$. There is $b \in \mathbb{R}$ such that $[X',Y] = bH$. After replacing X' by $X' + bH$, we can assume $[X',Y] = 0$. It follows that $\mathfrak{g}$ is the Lie algebra $\mathbb{R} \oplus \mathfrak{aff}(\mathbf{R})$. The Killing field X' spans the center, the isotropy Y spans the one parameter group of homotheties, and H spans the one parameter group of translations.

(ii) This comes from the fact that X is the unique vector field tangent to S invariant by the isotropy.

(iii) The commuting Killing vector fields X' and H are nonsingular on S. This implies that, in adapted coordinates (x,h) on S, $H = \frac{\partial}{\partial h}$ and $X = \frac{\partial}{\partial x}$. Because $[Y,X] = 0$, the restriction of Y to S has the expression $f(h)\frac{\partial}{\partial h}$, with f an analytic function vanishing at the origin. The Lie bracket relation $[Y,H] = H$ reads

$$\left[f(h)\frac{\partial}{\partial h}, \frac{\partial}{\partial h}\right] = \frac{\partial}{\partial h},$$

and leads to $f(h) = -h$.

(iv) Since $H = \frac{\partial}{\partial h}$ and $X = \frac{\partial}{\partial x}$ are Killing fields, the restriction of g to S admits constant coefficients with respect to the coordinates (x,h). Since H is expanded by the isotropy, it follows that H is of constant g-norm equal to 0. On the other hand, X is of constant g-norm equal to one. It follows that the expression of g on S is dx^2. □

Lemma 8. *Assume $\mathfrak{g}$ as in Lemma 7 acts quasihomogeneously on (U,g). In adapted analytic coordinates (x,h,z) on U,*

$$g = dx^2 + dhdz + Cz^2dh^2 + Dzdxdh \qquad \text{for some } C,D \in \mathbb{R}$$

Moreover, in these coordinates, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial h}$, and $-h\frac{\partial}{\partial h} + z\frac{\partial}{\partial z}$ are Killing fields.

Proof. Consider the commuting Killing vector fields X' and H constructed in Lemma 7. Their restrictions to S have the expressions $H = \partial/\partial h$ and $X = \partial/\partial x$. Recall that H is of constant g-norm equal to 0 and X is of constant g-norm equal to one. Point (iv) in Lemma 7 also shows that $g(X,H) = 0$ on S. Moreover, being central, X' is of constant g-norm equal to one on all of U.

Define a geodesic vector field Z as follows. At each point in S, there exists a unique vector Z, transverse to S, such that $g(Z,Z) = 0$, $g(X,Z) = 0$, and $g(H,Z) = 1$. Now Z uniquely extends in a neighborhood of the origin to a geodesic vector field. The image of S through the flow along Z defines a foliation by surfaces. Each leaf is given by $exp_S(zZ)$, for some z small enough. The leaf S corresponds to $z = 0$.

Since X' and H are Killing, they commute with Z. Let (x,h,z) be analytic coordinates in the neighborhood of the origin such that $X' = \partial/\partial x, H = \partial/\partial h, Z = \partial/\partial z$. The scalar product $g(Z,X')$ is constant along the orbits of Z. This comes from the following classical computation :

$$Z \cdot g(X',Z) = g(\nabla_Z X',Z) + g(X',\nabla_Z Z) = 0$$

since $\nabla_Z Z = 0$ and $\nabla . X'$ is skew-symmetric with respect to g. The same is true for $g(Z,H)$. Moreover, the invariance of the metric by the commutative Killing algebra generated by X' and H implies that $dxdz = 0$ and $dhdz = 1$ over all of U. The coefficients of dh^2 and $dxdh$ depend only on z. Then

$$g = dx^2 + dhdz + c(z)dh^2 + d(z)dxdh$$

with c and d analytic functions which both vanish at $z = 0$.

Next we use the invariance of g by the isotropy $\mathbb{R}Y$. Recall that $[Y,X'] = 0$ and $[Y,H] = H$. Since the isotropy preserves X, it must preserve also the two isotropic directions of $X^{\perp}$. Moreover, since $g(H,Z) = 1$, the isotropy must expand H and contract Z at the same rate. This implies the Lie bracket relation $ad(Y) \cdot Z = [Y,Z] = -Z$. Now, since Y and X' commute, the general expression for Y is

$$Y = u(h,z)\frac{\partial}{\partial h} + v(h,z)\frac{\partial}{\partial z} + t(h,z)\frac{\partial}{\partial x}$$

with u, v, and t analytic functions, where $u(h,0) = -h$, and v and t vanish on $\{z = 0\}$.

The other Lie bracket relations read

$$[u(h,z)\frac{\partial}{\partial h} + v(h,z)\frac{\partial}{\partial z} + t(h,z)\frac{\partial}{\partial x}, \frac{\partial}{\partial h}] = \frac{\partial}{\partial h}$$

and

$$[u(h,z)\frac{\partial}{\partial h} + v(h,z)\frac{\partial}{\partial z} + t(h,z)\frac{\partial}{\partial x}, \frac{\partial}{\partial z}] = -\frac{\partial}{\partial z}$$

The first relation gives

$$\frac{\partial u}{\partial h} = -1 \qquad \frac{\partial v}{\partial h} = 0 \qquad \frac{\partial t}{\partial h} = 0$$

The second one leads to

$$\frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 1 \qquad \frac{\partial t}{\partial z} = 0$$

We get

$$u(h,z) = -h \qquad v(h,z) = z \qquad t(h,z) = 0$$

Hence, in our coordinates, $Y = -h\partial/\partial h + z\partial/\partial z$. The invariance of g under the action of this linear vector field implies $c(e^{-t}z)e^{2t} = c(z)$ and $d(e^{-t}z)e^{t} = d(z)$, for all $t \in \mathbb{R}$. This implies then that $c(z) = Cz^2$ and $d(z) = Dz$, with C, D real constants. □

3.1. **Computation of the Killing algebra.** We need to understand now whether the metrics

$$g_{C,D} = dx^2 + dhdz + Cz^2dh^2 + Dzdxdh$$

constructed in Lemma 8 really are quasihomogeneous. In other words, do the metrics in this family admit other Killing fields than $\partial/\partial x$, $\partial/\partial h$ and $-h\partial/\partial h + z\partial/\partial z$? In this section we compute the full Killing algebra of $g_{C,D}$. In particular, we obtain that the metrics $g_{C,D} = dx^2 + dhdz + Cz^2dh^2 + Dzdxdh$ always admit additional Killing fields and, by lemma 4 (ii) are locally homogeneous.

The formula for the Lie derivative of g (see, eg, [KN96]) gives

$$(L_T g_{C,D})\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = T \cdot g_{C,D}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) + g_{C,D}\left(\left[\frac{\partial}{\partial x_i}, T\right], \frac{\partial}{\partial x_j}\right) + g_{C,D}\left(\frac{\partial}{\partial x_i}, \left[\frac{\partial}{\partial x_j}, T\right]\right)$$

Let $T = \alpha\partial/\partial x + \beta\partial/\partial h + \gamma\partial/\partial z$. The pairs

$$\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (1)\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)\ (2)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)\ (3)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\ (4)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial h}\right)\ (5)\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial z}\right)\ (6)\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial h}\right)$$

give the following system of PDEs on α, β and γ in order for T to be a Killing field:

$$0 = \beta_z, \tag{1}$$
$$0 = \alpha_x + Dz\beta_x, \tag{2}$$
$$0 = \beta_x + Dz\beta_z + \alpha_z, \tag{3}$$
$$0 = \gamma D + Dz\alpha_x + Cz^2\beta_x + \gamma_x + \alpha_h + Dz\beta_h, \tag{4}$$
$$0 = \beta_h + Cz^2\beta_z + Dz\alpha_z + \gamma_z, \tag{5}$$
$$0 = zC\gamma + Cz^2\beta_h + Dz\alpha_h + \gamma_h. \tag{6}$$

The following proposition finishes the proof of Theorem 1 in the case of semi-simple isotropy on S:

Proposition 9. *The Lorentz metrics $g_{C,D}$ are locally homogeneous for all $C, D \in \mathbb{R}$.*

Proof. It is straightforward to verify that

$$T = Dh\frac{\partial}{\partial x} + \frac{1}{2}(D^2 - C)h^2\frac{\partial}{\partial h} + ((C - D^2)zh - 1)\frac{\partial}{\partial z}$$

satisfies equations (1)–(6). Note that $T(0) = -\partial/\partial z$, so $T \notin \mathfrak{g}$, and (U, g) is locally homogeneous. □

We explain now our method to find the extra Killing field T in Proposition 9, and we compute the full Killing algebra, which we will denote $\mathfrak{l}$, of $g_{C,D}$. Recall the n-dimensional Lorentzian manifolds $\mathrm{AdS}^n, \mathrm{Min}^n$, and dS^n, of constant sectional curvature $-1, 0$, and 1, respectively (see, eg, [Wol67]). Recall also that AdS^3 is isometric to $SL(2,\mathbb{R})$ with the bi-invariant Cartan-Killing metric.

Proposition 10.

(i) If $D \neq 0$ and $C \neq 0, D^2$, then $(U, g_{C,D})$ is locally isometric to a left-invariant metric on $SL(2,\mathbb{R})$ with $\mathfrak{l} \cong \mathbb{R} \oplus sl(2,\mathbb{R})$. The isotropy is diagonally embedded in the direct sum.

(ii) *If* $D \neq 0$ *and* $C = D^2$*, then* $(U, g_{C,D})$ *is locally isometric to a left-invariant metric on the Heisenberg group with* $\mathfrak{l} \cong \mathbb{R} \ltimes \mathfrak{heis}$*. The isotropy is the* $\mathbb{R}$ *factor, which acts by a semi-simple automorphism of* $\mathfrak{heis}$*.*

(iii) *If* $C = 0$ *and* $D \neq 0$*, then* $(U, g_{C,D})$ *is locally isometric to* AdS^3*, so* $\mathfrak{l} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$*.*

(iv) *If* $C \neq 0$ *and* $D = 0$*, then* $(U, g_{C,D})$ *is locally isometric to* $\mathbb{R} \times dS^2$*, for which* $\mathfrak{l} \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$*. The isotropy is generated by a semi-simple element of* $\mathfrak{sl}(2,\mathbb{R})$*.*

(v) *If* $C = 0$ *and* $D = 0$*, then* $(U, g_{C,D})$ *is locally isometric to* Min^3*, so* $\mathfrak{l} \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$*.*

Proof. Recall that all Lorentz metrics $g_{C,D}$ admit the Killing fields X', Y and H, for which the Lie bracket relations are $[Y, X'] = [H, X'] = 0$ and $[Y, H] = H$. Moreover, Proposition 9 shows that all Lorentz metrics $g_{C,D}$ are locally homogeneous and their Killing algebra $\mathfrak{l}$ is of dimension at least 4. Assuming $g_{C,D}$ is not of constant sectional curvature, then Lemma 4 (i) implies $\dim \mathfrak{l} = 4$. We first derive some information on the algebraic structure of $\mathfrak{l}$ in this case.

If $\dim \mathfrak{l} = 4$, then it is generated by X', Y, H, and an additonal Killing field T. Since the isotropy $\mathbb{R}Y$ at the origin fixes the spacelike vector $X(0)$ and expands H, we can choose a fourth generator T of $\mathfrak{g}$ evaluating at the origin to a generator of the second isotropic direction of the Lorentz plane $X(0)^{\perp}$. Then $[Y, T] = -T + aY$ for some constant $a \in \mathbb{R}$, and we can replace T with $T - aY$ in order that $[Y, T] = -T$. Now T is an eigenvector of $ad(Y)$. Since X' and Y commute, T is also an eigenvector of $ad(X')$, so $[T, X'] = cT$, for some $c \in \mathbb{R}$.

The Jacobi relation

$$[Y, [T, H]] = [[Y, T], H] + [T, [Y, H]] = [-T, H] + [T, H] = 0$$

says that $[T, H]$ commutes with Y. The centralizer of Y in $\mathfrak{l}$ is $\mathbb{R}Y \oplus \mathbb{R}X'$. We conclude $[H, T] = aX' - bY$, for some $a, b \in \mathbb{R}$.

(i) Assume $D \neq 0$ and $C \neq 0, D^2$. A straightforward computation shows that $g_{C,D}$ is not of constant sectional curvature. We will construct a Killing field $T = \alpha\partial/\partial x + \beta\partial/\partial h + \gamma\partial/\partial z$, meaning the functions α, β and γ solve the PDE system (1)–(6), with $c = 0$ and $a = 1$.

First we use the Lie bracket relations derived above for T and $\mathfrak{g}$. Remark that, since T and X' commute, the coefficients α, β and γ of T do not depend on the coordinate x; in particular, equation (2) is satisfied. The relation $[H, T] = aX' - bY$ reads, when $a = 1$,

$$\left[\frac{\partial}{\partial h}, T\right] = \frac{\partial}{\partial x} + b\left(h\frac{\partial}{\partial h} - z\frac{\partial}{\partial z}\right)$$

This leads to $\alpha_h = 1, \beta_h = bh$, and $\gamma_h = -bz$. Using equation (1), we obtain $\beta = \frac{1}{2}bh^2$. Now equation (4) gives $\gamma = -bzh - 1/D$.

Equation (6) now reads

$$0 = zC(-\frac{1}{D} - zbh) + Cz^2bh + Dz - bz = -\frac{Cz}{D} + Dz - bz$$

which yields $b = D - C/D$. Now γ can be written $-1/D - zh(D - C/D)$.

Equation (3) says $\alpha_z = 0$, so we conclude $\alpha = h$. The resulting vector field is

$$T = h\frac{\partial}{\partial x} + \frac{1}{2}(D - \frac{C}{D})h^2\frac{\partial}{\partial h} + \left(zh(\frac{C}{D} - D) - \frac{1}{D}\right)\frac{\partial}{\partial z} \tag{7}$$

Note that the coefficients of T also satisfy equation (5), so T is indeed a Killing field.

We obtained this solution setting $c = 0$, so the Lie algebra $\mathfrak{l}$ generated by $\{T, X', Y, H\}$ contains X' as a central element. We also set $a = 1$, and found $b = D - C/D$, so $[H, T] = X' + (C/D - D)Y$. It is straightforward to verify that for T as above, $[Y, T] = -T$. Under the hypothesis $C \neq D^2$, the Lie subalgebra generated by

$\{X' - (C/D - D)Y, H, T\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and acts transitively on U. Consequently, $g_{C,D}$ is locally isomorphic to a left invariant Lorentz metric on $SL(2,\mathbb{R})$. The full Killing algebra is $\mathfrak{l} \cong \mathbb{R} \oplus sl(2,\mathbb{R})$, and the isotropy $\mathbb{R}Y$ is diagonally embedded in the direct sum. This terminates the proof of point (i).

(ii) When $D \neq 0$ and $C = D^2$, then (7) still solves the Killing equations. The bracket relations are the same, but now $[H,T] = X'$. Then $\mathfrak{l} \cong \mathbb{R} \ltimes \mathfrak{heis}$, where the $\mathfrak{heis}$ factor is generated by $\{H, T, X'\}$, which acts transitively, and $\mathbb{R}$ factor is generated by the isotropy Y, which acts by a semi-simple automorphism on $\mathfrak{heis}$. Up to homothety, there is a unique left-invariant Lorentz metric on Heis in which X' is spacelike, by Proposition 1.1 of [DZ10], where it is called the *Lorentz-Heisenberg geometry*.

(iii) When $C = 0$ and $D \neq 0$, then (7) again solves the Killing equations. It now simplifies to

$$T = h\frac{\partial}{\partial x} + \frac{1}{2}Dh^2\frac{\partial}{\partial h} + \left(-zhD - \frac{1}{D}\right)\frac{\partial}{\partial z}$$

The bracket $[H,T] = X' - DY$, and $\mathfrak{l}$ still contains a copy of $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$, with center generated by X' and $\mathfrak{sl}(2,\mathbb{R})$ generated by $\{X' - DY, H, T'\}$. The $\mathfrak{sl}(2,\mathbb{R})$ factor still acts simply transitively. On the other hand, one directly checks that $\alpha = \beta = 0$ and $\gamma = e^{-Dx}$ is a solution of the PDE system, meaning that $e^{-Dx}\partial/\partial z$ is also a Killing field. From

$$\left[X', e^{-Dx}\frac{\partial}{\partial z}\right] = -De^{-Dx}\frac{\partial}{\partial z} \neq 0$$

it is clear that this additional Killing field does not belong to the subalgebra generated by $\{T, X', Y, H\}$, in which X' is central. It follows that the Killing algebra is of dimension at least five, which implies that $g_{0,D}$ is of constant sectional curvature. Since $g_{0,D}$ is locally isomorphic to a left invariant Lorentz metric on $SL(2,\mathbb{R})$, this implies that the curvature is negative. Up to normalization, $g_{0,D}$ is locally isometric to AdS^3.

(iv) The Killing field T in (7) multiplied by D gives

$$T_D = Dh\frac{\partial}{\partial x} + \frac{1}{2}(D^2 - C)h^2\frac{\partial}{\partial h} + \left(zh(C - D^2) - 1\right)\frac{\partial}{\partial z}$$

Setting $C \neq 0$ and $D = 0$ gives

$$T_0 = -\frac{Ch^2}{2}\frac{\partial}{\partial h} + (zhC - 1)\frac{\partial}{\partial z}$$

which is indeed a Killing field of $g_{C,0}$. The brackets are $[X', T_0] = 0, [H, T_0] = CY$, and $[Y, T_0] = -T_0$. As in case (i), the Killing Lie algebra contains a copy of $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$, with center generated by X', and $\mathfrak{sl}(2,\mathbb{R})$ generated by (Y, H, T_0). Here the isotropy generator Y lies in the $\mathfrak{sl}(2,\mathbb{R})$-factor, which acts with two-dimensional orbits. This local $\mathfrak{sl}(2,\mathbb{R})$-action defines a two-dimensional foliation tangent to $X'^{\perp}$. Recall that X' is of constant g-norm equal to one, so $X'^{\perp}$ has Lorentzian signature. The metric is, up to homotheties on the two factors, locally isomorphic to the product $\mathbb{R} \times \mathrm{dS}^2$.

(v) If $C = D = 0$, then $g_{C,D}$ is flat and $\mathfrak{l} \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$. □

As a by-product of the proof of Theorem 1 in the case of semi-simple isotropy, we have obtained the following more technical result:

Proposition 11. *Let g be a real-analytic Lorentz metric in a neighborhood of the origin in $\mathbf{R}^3$. Suppose that there exists a three-dimensional subalgebra $\mathfrak{g}$ of the Killing Lie algebra acting transitively on an open set admitting the origin in its closure, but not in the neighborhood of the origin. If the isotropy at the origin is a one parameter semi-simple subgroup in $O(2,1)$, then*

(i) *There exist local analytic coordinates* (x,h,z) *in the neighborhood of the origin and real constants* C,D *such that*

$$g = g_{C,D} = dx^2 + dhdz + Cz^2dh^2 + Dzdxdh.$$

(ii) *The algebra* $\mathfrak{g}$ *is solvable, and equals, in these coordinates,*

$$\mathfrak{g} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial h}, -h\frac{\partial}{\partial h} + z\frac{\partial}{\partial z} \rangle.$$

In particular, $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{aff}(\mathbf{R})$*, where* $\mathfrak{aff}(\mathbf{R})$ *is the Lie algebra of the affine group of the real line.*

(iii) *All the metrics* $g_{C,D}$ *are locally homogeneous. They admit a Killing field* $T \notin \mathfrak{g}$ *of the form*

$$T = Dh\frac{\partial}{\partial x} + \frac{1}{2}(D^2 - C)h^2\frac{\partial}{\partial h} + ((C - D^2)zh - 1)\frac{\partial}{\partial z}$$

The possible geometries on $(U, g_{C,D})$ *are given by (i) - (v) of Proposition 10.*

4. No quasihomogeneous Lorentz metrics with unipotent isotropy

We next treat the unipotent case of Lemma 6. The following results can be found in [Dum08] Propositions 3.4 and 3.5 in Section 3.1, where they are proved without making use of compactness. See also [Zeg96, Proposition 9.2] for point (iii).

Proposition 12.

(i) *The surface* S *is totally geodesic.*

(ii) *The Levi-Civita connection* ∇ *restricted to* S *is either flat, or locally isomorphic to the canonical bi-invariant connection on the affine group of the real line* Aff*.*

(iii) *The restriction of the Killing algebra* $\mathfrak{g}$ *to* S *is isomorphic either to to the Lie algebra of the Heisenberg group in the flat case, or otherwise to a solvable subalgebra* $\mathfrak{sol}(1,a)$ *of* $Aff \times Aff$*, spanned by the elements* $(t,0),(0,t)$ *and* (w,aw)*, where* t *is the infinitesimal generator of the one parameter group of translations,* w *the infinitesimal generator of the one parameter group of homotheties, and* $a \in \mathbb{R}$*.*

Recall that, as S has codimension one, the restriction to S of the Killing Lie algebra $\mathfrak{g}$ of g is an isomorphism. The Heisenberg group is unimodular, so by Lemma 4, $\mathfrak{g}$ is isomorphic to $\mathfrak{sol}(1,a)$, with $a \neq -1$, and *S is non flat*.

Recall that in dimension three, the curvature is completely determined by its Ricci tensor *Ricci* which is a quadratic form. The Ricci curvature is determined by the Ricci operator, which is a field of g-symmetric endomorphisms $A : TU \to TU$ such that $Ricci(u,v) = g(Au,v)$, for any tangent vectors u,v.

Proposition 13.

(i) *The three eigenvalues of the Ricci operator are equal to* 0 *everywhere on* U*.*

(ii) *The metric* g *is curvature homogeneous; more precisely, in an adapted framing on* U*, the Ricci operator reads*

$$A = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha \in \mathbb{R}^*$$

Proof. (i) Pick a point s in S. The Ricci operator $A(s)$ must be invariant by the unipotent isotropy. The action of the isotropy on T_sU fixes an isotropic vector $e_1 = X(s)$ tangent to S and so preserves the degenerate plane $e_1^{\perp} = T_sS$. In order to define an adapted basis, consider two vectors $e_2, e_3 \in T_sU$ such that

$$g(e_1,e_2) = 0 \qquad g(e_2,e_2) = 1 \qquad g(e_3,e_3) = 0 \qquad g(e_2,e_3) = 0 \qquad g(e_3,e_1) = 1$$

The action on T_sU of the one parameter group of isotropy is given in the basis (e_1,e_2,e_3) by the matrix

$$L_t = \begin{pmatrix} 1 & t & -\frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R}$$

First we show that $A(s) : T_sU \to T_sU$ has, in our adapted basis, the following form:

$$\begin{pmatrix} \lambda & \beta & \alpha \\ 0 & \lambda & -\beta \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

Since $A(s)$ is invariant by the isotropy, it commutes with L_t for all t. Each eigenspace of $A(s)$ is preserved by L_t and conversely. As L_t does not preserve any non trivial splitting of T_sU, it follows that all eigenvalues of B are equal to some $\lambda \in \mathbb{R}$. Moreover, the unique line and plane invariant by L_t must also be invariant by $A(s)$, so $A(s)$ is upper-triangular in the basis (e_1,e_2,e_3). A straightforward calculation of the top corner entry of $A(s)L_t = L_tA(s)$ leads to the relation on the β entries and thus our claimed form for $A(s)$.

Now the g-symmetry of $A(s)$ means $g(A(s)e_2,e_3) = g(e_2,A(s)e_3)$, which gives $\beta = 0$. Since the symmetric functions of the eigenvalues of A are scalar invariants, they must be constant on all of U. This implies that the three eigenvalues of A are equal to λ on all of U. It remains only to prove that $\lambda = 0$. Consider an open set in U on which the Killing algebra $\mathfrak{sol}(1,a)$ is transitive. On this open set g is locally isomorphic to a left invariant Lorentz metric on the Lie group $SOL(1,a)$.

The sectional and Ricci curvatures and Ricci operator of a left-invariant Lorentz metric on a given Lie group can be calculated, starting from the Koszul formula, in terms of the brackets between left-invariant vector fields forming an adapted framing of the metric. In [CK09] Calvaruso and Kowalski calculated Ricci operators for left invariant Lorentz metrics on three dimensional Lie groups, assuming they are not symmetric (see also [Nom79], [CP97], [Cal07]). (Under our assumptions the isotropy at points of $U \backslash S$ is trivial, so we need consider only non-symmetric left-invariant metrics). A consequence of their Theorems 3.5, 3.6, and 3.7 is that the Ricci operator of a left-invariant, non-symmetric Lorentz metric on a *nonunimodular* three-dimensional Lie group admits a triple eigenvalue λ if and only if $\lambda = 0$, and the Ricci operator is nilpotent of order two. We conclude $\lambda = 0$, so $A(s)$ has the form claimed. Moreover, A is nilpotent of order two on $U \setminus S$.

(ii) Because $\mathfrak{g}$ acts transitively on S, there is an adapted framing along S in which $A \equiv A(s)$. The parameter α in $A(s)$ cannot vanish; otherwise the curvature of g vanishes on S and (S,∇) is flat, which was proved to be impossible in Proposition 12. Now the Ricci operator on S is nontrivial and lies in the closure of the $PSL(2,\mathbb{R})$-orbit $\mathscr{O}$ of the Ricci operator on $U \setminus S$. But we know from (i) that on $U \backslash S$, the Ricci operator is g-symmetric and nilpotent of order 2, so it has the same form as $A(s)$, meaning it also belongs to the $PSL(2,\mathbb{R})$-orbit of

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Now $Ricci(u,u)$ is a quadratic form of rank one and $Ricci(u,u) = g(W,u)^2$, for some non-vanishing isotropic vector field W on U, which coincides with X on S. Invariance of $Ricci$ by $\mathfrak{g}$ implies invariance of W, and then Proposition 5 implies that g is locally homogeneous.

5. Alternate proofs using the Cartan connection

The aim of this section is to give a second proof of Theorem 1 using the Cartan connection associated to a Lorentz metric. We still consider g a Lorentz metric defined in a connected open neighborhood U of the origin in $\mathbb{R}^3$.

5.1. **Introduction to the Cartan connection.** Let $\mathfrak{h} = \mathfrak{o}(1,2) \ltimes \mathbb{R}^{1,2}$. Let $P \cong \mathrm{O}(1,2)$ so $\mathfrak{p} \subset \mathfrak{h}$. Let $\pi : B \to U$ be the principal P-bundle of normalized frames on U, in which the Lorentz metric g has the matrix form

$$\mathbb{I} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

(Note that B is nearly the same as the bundle $R(U)$ from Section 2, though it has been enlarged to allow all possible orientations and time orientations.)

The *Cartan connection* associated to (U,g) is the 1-form $\omega \in \Omega^1(B,\mathfrak{h})$ formed by the sum of the Levi-Civita connection of the metric $\nu \in \Omega^1(B,\mathfrak{p})$ and the tautological 1-forms $\theta \in \Omega^1(B,\mathbb{R}^{1,2})$. The form ω satisfies the following axioms for a Cartan connection:

1. It gives a parallelization of B—that is, for all $b \in B$, the restriction $\omega_b : T_bB \to \mathfrak{h}$ is an isomorphism.
2. It is P-equivariant: for all $p \in P$, the pullback $R_p^*\omega = \mathrm{Ad}\ p^{-1} \circ \omega$.
3. It recognizes fundamental vertical vector fields: for all $X \in \mathfrak{p}$, if $X^\ddagger$ is the vertical vector field on B generated by X, then $\omega(X^\ddagger) \equiv X$.

The *Cartan curvature* of ω is

$$K(X,Y) = \mathrm{d}\omega(X,Y) + [\omega(X),\omega(Y)]$$

This 2-form is always semibasic, meaning $K_b(X,Y)$ only depends on the projections of X and Y to $\pi(b) \in U$; in particular, K vanishes when either input is a vertical vector. We will therefore express the inputs to K_b as tangent vectors at $\pi(b)$. Torsion-freeness of the Levi-Civita connection implies that K has values in $\mathfrak{p}$. Thus K is related to the usual Riemannian curvature tensor $R \in \Omega^2(U) \otimes \mathrm{End}(TM)$ by

$$\theta_b \circ R_{\pi(b)}(u,v) \circ \theta_b^{-1} = K_b(u,v)$$

The benefit here of working with the Cartan curvature is that, when applied to Killing vector fields, it gives a precise relation between the brackets on the manifold U and the brackets in the Killing algebra $\mathfrak{g}$.

The P-equivariance of ω leads to P-equivariance of K: $R_p^*K(X,Y) = \mathrm{Ad}\ p^{-1} \circ K(X,Y)$. The infinitesimal version of this statement is, for $A \in \mathfrak{p}$,

$$K([A^\ddagger,X],Y) + K(X,[A^\ddagger,Y]) = [K(X,Y),A]$$

A Killing field Y on U lifts to a vector field on B, which we will also denote Y, with $L_Y\omega = 0$. Note that also $L_YK = 0$ in this case. Thus if X and Y are Killing fields, then

$$X.\omega(Y) = \omega[X,Y] \qquad \text{and} \qquad Y.\omega(X) = \omega[Y,X]$$

In this case,

$$\begin{aligned} K(X,Y) &= X.\omega(Y) - Y.\omega(X) - \omega[X,Y] + [\omega(X),\omega(Y)] \\ &= \omega[X,Y] - \omega[Y,X] - \omega[X,Y] + [\omega(X),\omega(Y)] \\ &= \omega[X,Y] + [\omega(X),\omega(Y)] \end{aligned}$$

so, when X and Y are Killing, then

$$\omega[X,Y] = [\omega(Y),\omega(X)] + K(X,Y) \tag{8}$$

Via the parallelization given by ω, the semibasic, $\mathfrak{p}$-valued 2-form K corresponds to a P-equivariant, automorphism-invariant function

$$\kappa : B \to \wedge^2\mathbb{R}^{1,2*} \otimes \mathfrak{p}$$

The reader can find more details about the geometry of Cartan connections in the book [Sha97].

5.2. **Curvature representation.** Denote $\{e,h,f\}$ a basis of $\mathbb{R}^{1,2}$ in which the inner product is given by $\mathbb{I}$. Let E,H,F be generators of $\mathfrak{p}$ with matrix expression in the basis $\{e,h,f\}$

$$E = \begin{pmatrix} 0 & -1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ & 1 & 0 \end{pmatrix}$$

Therefore this representation of $\mathfrak{p}$ is equivalent to ad $\mathfrak{p}$ via the isomorphism sending $\{e,h,f\}$ to $\{E,H,F\}$.

Denote by $\{e^*,h^*,f^*\}$ the dual basis of $\{e,h,f\}$, determined by the property that the row vectors for e^*,h^*, and f^* in terms of the basis $\{e,h,f\}$ form the identity matrix. Thus $e^*(x) = \langle f,x\rangle$, $h^*(x) = \langle h,x\rangle$, and $f^*(x) = \langle e,x\rangle$. Denote by * the corresponding isomorphism $\mathbb{R}^{2,1} \to \mathbb{R}^{2,1*}$, and also for the inverse isomorphism. For $p \in \mathrm{SO}(2,1)$ and $x \in \mathbb{R}^{2,1}$, we have $(p.x)^* = p^*.x^*$ for the *dual represention* $p^*.x^* = x^* \circ p^{-1}$.

Now $E \mapsto e$, $H \mapsto h$, $F \mapsto f$ defines an $\mathfrak{o}(2,1)$-equivariant isomorphism from $\mathfrak{o}(2,1)$ to $\mathbb{R}^{2,1}$—that is, for each $X \in \mathfrak{o}(2,1)$, the matrix of ad X in the basis E,H,F equals the matrix of X in the basis e,h,f.

Next consider the volume form $vol = e^* \wedge h^* \wedge f^*$, which is invariant by the dual action of $\mathrm{SO}(2,1)$. Then define an isomorphism $\tau : \wedge^2\mathbb{R}^{2,1*} \to \mathbb{R}^{2,1*}$ by

$$vol = \mu \wedge \langle \tau(\mu)^*, \ \rangle$$

It is straightforward to see that this map is an equivariant linear isomorphism: $\tau(p^*\mu) = p^*\tau(\mu)$ for all $p \in \mathrm{SO}(2,1)$. In terms of a basis,

$$\begin{aligned} \tau \quad : \quad & e^* \wedge h^* \mapsto e^* \\ & f^* \wedge e^* \mapsto h^* \\ & h^* \wedge f^* \mapsto f^* \end{aligned}$$

The tensor product of these isomorphisms $\wedge^2\mathbb{R}^{2,1*} \to \mathbb{R}^{2,1*}$ and $\mathfrak{o}(2,1) \to \mathbb{R}^{2,1}$ gives an $\mathrm{SO}(2,1)$-equivariant isomorphism $\wedge^2\mathbb{R}^{2,1*} \otimes \mathfrak{o}(2,1) \to \mathbb{R}^{3\times 3}$, where the representation on 3×3 matrices is by conjugation.

Now one can easily identify the three irreducible components of this representation. The first, denoted E_0, is the 1-dimensional trivial representation. It corresponds to the scalar matrices. We will denote by m_d a vector spanning E_0. The next irreducible component E_1 corresponds to the matrices in $\mathfrak{o}(2,1)$. Now consider the polar decomposition of $\mathfrak{gl}(3)$ with respect to the Lorentzian inner product. The subspace E_1 comprises matrices X satisfying $X\mathbb{I} = -\mathbb{I}X^t$. An $\mathrm{SO}(2,1)$-invariant complementary subspace consists of those satisfying $X\mathbb{I} = \mathbb{I}X^t$. This subspace splits into E_0 and the last irreducible component, E_2, which is 5-dimensional.

The component E_0 corresponds to scalar curvature, while E_2 corresponds to the tracefree Ricci curvature; more precisely, our realization of $E_0 \oplus E_2$ as a representation contained in $\mathfrak{gl}(3)$ gives Ricci endomorphisms. In dimension 3, the curvature tensor is determined by the Ricci curvature, so we will focus below on the components of the curvature in $E_0 \oplus E_2$. A basis of this subspace is described in the following table, which lists the expressions for each basis vector as elements of $\mathbb{R}^{3\times 3}$ and as elements of $\wedge^2\mathbb{R}^{2,1*} \otimes \mathfrak{o}(2,1)$:

	$\mathbb{R}^{3\times 3}$	$\wedge^2\mathbb{R}^{2,1*}\otimes\mathfrak{o}(2,1)$
m_d	$e^*\otimes e+h^*\otimes h+f^*\otimes f$	$e^*\wedge h^*\otimes E+f^*\wedge e^*\otimes H+h^*\wedge f^*\otimes F$
m_{e^2}	$f^*\otimes e$	$h^*\wedge f^*\otimes E$
m_{eh}	$h^*\otimes e+f^*\otimes h$	$f^*\wedge e^*\otimes E+h^*\wedge f^*\otimes H$
m_{2h^2-ef}	$2h^*\otimes h-e^*\otimes e-f^*\otimes f$	$2f^*\wedge e^*\otimes H+h^*\wedge e^*\otimes E+f^*\wedge h^*\otimes F$
m_{hf}	$e^*\otimes h+h^*\otimes f$	$e^*\wedge h^*\otimes H+f^*\wedge e^*\otimes F$
m_{f^2}	$e^*\otimes f$	$e^*\wedge h^*\otimes F$

Assume now that g is quasihomogeneous. Recall that, by the results in Section 2, the Killing algebra $\mathfrak{g}$ is three-dimensional. It acts transitively on U, away from a two-dimensional, degenerate submanifold S passing through the origin. Moreover, $\mathfrak{g}$ acts transitively on S and the isotropy at points of S is conjugated to a one parameter semi-simple group or to a one parameter unipotent group in $PSL(2,\mathbb{R})$. We will study the interaction of $\mathfrak{g}$, $\omega(\mathfrak{g})$, and κ, both on and off S.

5.3. **Semisimple isotropy.** Let b_0 be a point of B lying over the origin and assume that the isotropy action of $\mathfrak{g}$ at 0 is semisimple, as in Section 3. A semisimple element of $\mathfrak{p}$ is conjugate in P to the element H, so up to changing the choice of $b_0\in\pi^{-1}(0)$, we may assume that $\omega_{b_0}(\mathfrak{g})\cap\mathfrak{p}$ is spanned by H.

Proposition 14. *(compare Lemma 7 (i)) If the isotropy of* $\mathfrak{g}$ *at the origin is semisimple, then* $\mathfrak{g}\cong\mathbb{R}\oplus\mathfrak{aff}(\mathbb{R})$.

Proof. Let $Y\in\mathfrak{g}$ have $\omega_{b_0}(Y)=H$, so the corresponding Killing field vanishes at the origin. The projection $\overline{\omega_{b_0}(\mathfrak{g})}$ of $\omega_{b_0}(\mathfrak{g})$ to $\mathbb{R}^{1,2}$ is 2-dimensional, degenerate, and H-invariant. Again, by changing the point b_0 in the fiber above the origin, we may conjugate by an element normalizing $\mathbb{R}H$ so that this projection is $\text{span}\{e,h\}$. Therefore, there is a basis $\{X,Y,Z\}$ of $\mathfrak{g}$ such that

$$\omega_{b_0}(X)=h+\alpha E+\beta F\qquad\text{and}\qquad\omega_{b_0}(Z)=e+\gamma E+\delta F$$

for some $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. Because $K_{b_0}(Y,\cdot)=0$, equation (8) gives

$$\omega_{b_0}[Y,X]=[h+\alpha E+\beta F,H]=-\alpha E+\beta F\in\omega_{b_0}(\mathfrak{g})$$

so $\alpha=\beta=0$ and $[Y,X]=0$. A similar computation gives

$$\omega_{b_0}[Y,Z]=[e+\gamma E+\delta F,H]=-e-\gamma E+\delta F$$

so $\delta=0$, and $[Y,Z]=-Z$.

Infinitesimal invarance of K by Y gives

$$K_{b_0}([Y,X],Z)+K_{b_0}(X,[Y,Z])=Y.(K(X,Z))_{b_0}=H^{\ddagger}.(K(X,Z))_{b_0}=[-H,K_{b_0}(X,Z)]$$

which reduces to $K_{b_0}(X,Z)=[H,K_{b_0}(X,Z)]$. Since K takes values in $\mathfrak{p}$ we get

$$K_{b_0}(X,Z)=\kappa_{b_0}(h,e)=rE\qquad\text{for some } r\in\mathbb{R}$$

Now equation (8) gives for X and Z,

$$\begin{aligned}\omega_{b_0}[X,Z]&=[e+\gamma E,h]+rE\\&=-\gamma e+rE\end{aligned}$$

so $r=-\gamma^2$ and $[X,Z]=-\gamma Z$.

The structure of the algebra $\mathfrak{g}$ in the basis $\{Y,X,Z\}$ is

$$\mathrm{ad}\, Y = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad \mathrm{ad}\, X = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\gamma \end{pmatrix} \qquad \mathrm{ad}\, Z = \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & \gamma & 0 \end{pmatrix}$$

This $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$, with center generated by $\gamma Y - X$. □

Let $W = X - \gamma Y$. Note that $W(0)$ has norm 1 because $\overline{\omega_{b_0}(W)} = h$. As in Section 3, where the central element of $\mathfrak{g}$ is called X', the norm of W is constant 1 on U because it is $\mathfrak{g}$-invariant and equals 1 at a point of S. Existence of a Killing field of constant norm 1 has the following consequences for the geometry of U:

Proposition 15.

(i) The local $\mathfrak{g}$-action on U preserves a splitting of TU into three line bundles, $L^- \oplus \mathbb{R}W \oplus L^+$, with L^- and L^+ isotropic.

(ii) The distributions $L^- \oplus \mathbb{R}W$ and $L^+ \oplus \mathbb{R}W$ are each tangent to $\mathfrak{g}$-invariant, degenerate, totally geodesic foliations $\mathscr{P}^-$ and $\mathscr{P}^+$, respectively; moreover, the surface S is a leaf of one of these foliations, which we may assume is $\mathscr{P}^+$.

Proof. (i) Because $\mathfrak{g}$ preserves W, it preserves $W^\perp$, which is a 2-dimensional Lorentz distribution. A 2-dimensional Lorentz vector space splits into two isotropic lines preserved by all linear isometries. Therefore $W^\perp = L^- \oplus L^+$, with both line bundles isotropic and $\mathfrak{g}$-invariant.

(ii) Because the flow along W preserves L^- and L^+, the distributions $L^- \oplus \mathbb{R}W$ and $L^+ \oplus \mathbb{R}W$ are involutive, and thus they each integrate to foliations $\mathscr{P}^-$ and $\mathscr{P}^+$ by degenerate surfaces.

Let $x \in U$. Let $V^+ \in \Gamma(L^-)$ and $V^+ \in \Gamma(L^+)$ be vector fields with $V^\pm(x) \neq 0$ and $[W,V^\pm](x) = 0$. It is well known that a Killing field of constant norm is geodesic: $\nabla_W W = 0$. Moreover, because $g(V^\pm,V^\pm)$ is constant zero, $W.g(V^\pm,V^\pm) = V^\pm.g(V^\pm,V^\pm) = 0$, from which

$$g_x(\nabla_W V^\pm, V^\pm) = g_x(\nabla_{V^\pm} W, V^\pm) = g_x(\nabla_{V^\pm} V^\pm, V^\pm) = 0$$

The tangent distributions $T\mathscr{P}^\pm$ equal $(V^\pm)^\perp$, and it is now straightforward to verify from the axioms for ∇ that $\mathscr{P}^-$ and $\mathscr{P}^+$ are totally geodesic through x.

The Killing field W is tangent to the surface S. Because S is degenerate, $TS^\perp$ is an isotropic line of $W^\perp$ and therefore coincides with L^+ or L^-. We can assume it is L^+, so S is a leaf of $\mathscr{P}^+$; in particular, we have shown S is totally geodesic. □

Proposition 16.

(i) For $x \in U$ and $u,v \in T\mathscr{P}^\pm_x$, the curvature $R_x(u,v)$ annihilates $(\mathscr{P}^\pm_x)^\perp$.

(ii) The Ricci endomorphism at x preserves each of the line bundles $L^+, \mathbb{R}W$, and L^-.

Proof. (i) The argument is the same for $\mathscr{P}^+$ and $\mathscr{P}^-$, so we write it for $\mathscr{P}^-$. Let $x \in U \backslash S$. Because $\mathfrak{g}$ acts transitively on a neighborhood of x, there is a Killing field A^- evaluating at x to a nonzero element of $L^-(x)$. Note that $[A^-,W] = 0$. The orbit of x under A^- and W coincides near x with an open subset of $\mathscr{P}^-_x$. Because L^- is $\mathfrak{g}$-invariant, the values of A^- in this relatively open set belong to L^-.

Now $A^-.g(A^-,A^-) = 0$ implies $g(\nabla_{A^-}A^-,A^-) = 0$, and $A^-.g(A^-,W) = 0$ gives

$$0 = g_x(\nabla_{A^-}A^-,W) + g_x(A^-,\nabla_{A^-}W) = g_x(\nabla_{A^-}A^-,W)$$

using that $\mathscr{P}^-_x$ is totally geodesic. Therefore $(\nabla_{A^-}A^-)_x = aA^-$ for some $a \in \mathbb{R}$. The flows along A^- and W act locally transitively on $\mathscr{P}^-_x$ preserving the connection ∇ and commuting with A^-. Thus $\nabla_{A^-}A^- \equiv aA^-$ on a neighborhood of x in $\mathscr{P}^-_x$.

Next, $W.g(A^-,W)=0$ gives

$$0=g(\nabla_W A^-,W)+g(A^-,\nabla_W W)=g(\nabla_W A^-,W)$$

using that W is geodesic. Therefore $(\nabla_W A^-)_x = bA^-$ for some $b\in\mathbb{R}$. Again invariance of ∇, A^-, and W implies that $\nabla_W A^- \equiv bA^-$ on a neighborhood of x in $\mathscr{P}^-_x$. Now we compute

$$R_x(A^-,W)A^- = (\nabla_{A^-}\nabla_W - \nabla_W\nabla_{A^-} - \nabla_{[A^-,W]})A^- = \nabla_{A^-}(bA^-) - \nabla_W(aA^-) = abA^- - baA^- = 0$$

This property of the curvature we have proved on $U\backslash S$ remains true on S because it is a closed condition.

(ii) It suffices to show that the Ricci endomorphism preserves $L^-\oplus\mathbb{R}W = T\mathscr{P}^-$ and $L^+\oplus\mathbb{R}W = T\mathscr{P}^+$. Again, we just write the proof for $\mathscr{P}^-$. The Ricci endomorphism preserves $T\mathscr{P}^-$ if and only if $\mathrm{Ricci}_x(u,v) = \mathrm{Ricci}_x(v,u) = 0$ for any $u\in L^-_x$, $v\in T\mathscr{P}^-_x$. Assume $u\neq 0$ and complete it to a normalized basis $\{u,w,z\}$ of T_xU with $w=W(x)$, $z\in L^+_x$, and $g_x(u,z)=1$. Then

$$\mathrm{Ricci}_x(v,u) = g_x(R(v,u)u,z)+g_x(R(v,w)u,w)+g_x(R(v,z)u,u) = 0+0+0 = 0.$$

□

Let $\mathscr{R}$ be the $\mathfrak{g}$-invariant reduction of B to the subbundle comprising frames $(x,(v^-,W(x),v^+))$ with $v^-\in L^-_x$ and $v^+\in L^+_x$. Now $\mathscr{R}$ is a principal A-bundle, where $\mathbb{R}^*\cong A<P$ is the subgroup with matrix form

$$A=\left\{\begin{pmatrix}\lambda^2 & & \\ & 1 & \\ & & \lambda^{-2}\end{pmatrix} \ : \ \lambda\in\mathbb{R}^*\right\}$$

Proposition 16 translates to the following statement on $\mathscr{R}$.

Proposition 17. *For any $b\in\mathscr{R}$, the component $\bar\kappa_b$ in the representation $E_0\oplus E_2$, corresponding to the Ricci endomorphism, is diagonal, so has the form*

$$\bar\kappa_b = ym_d + zm_{2h^2-ef} \qquad y,z\in\mathbb{R}$$

Note that for the vertical vector field $H^\ddagger$, we have $H^\ddagger.\bar\kappa_b = -H\cdot\bar\kappa_b = 0$. Because this curvature function is also $\mathfrak{g}$-invariant, it is constant on $\mathscr{R}_{U\backslash S}$. By continuity, we conclude that on all $\mathscr{R}$,

$$\bar\kappa \equiv ym_d + zm_{2h^2-ef} \qquad y,z\in\mathbb{R}$$

Since $\mathfrak{g}$ acts transitively on $U\setminus S$ and preserves $\mathscr{R}$, for any $b\in\mathscr{R}$ there exists a sequence a_n in A such that $a_n\cdot\omega_b(\mathfrak{g}) = \mathrm{Ad}\, a_n\cdot\omega_b(\mathfrak{g}) \to \omega_{b_0}(\mathfrak{g})$. Let us consider such a sequence a_n corresponding to a point $b\in B$ lying above $U\setminus S$. Then we prove the following

Lemma 18. *Write*

$$a_n = \begin{pmatrix}\lambda_n^2 & & \\ & 1 & \\ & & \lambda_n^{-2}\end{pmatrix}, \ \lambda_n\in\mathbb{R}^*$$

Then $\lambda_n\to\infty$.

Proof. First note that λ_n cannot converge to a nonzero number, because in this case $\lim_n a_n\cdot\omega_b(\mathfrak{g}) = \omega_{b_0}(\mathfrak{g})$ would still project onto $\mathbb{R}^{1,2}$ modulo $\mathfrak{p}$, contradicting that the $\mathfrak{g}$-orbit of 0 is two-dimensional. This also shows that a_n cannot admit a convergent subsequence, meaning that a_n goes to the infinity in A.

The space $\omega_b(\mathfrak{g})$ can be written as $\mathrm{span}\{e+\rho(e), h+\rho(h), f+\rho(f)\}$ for $\rho:\mathbb{R}^{1,2}\to\mathfrak{p}$ a linear map. The space $a_n\cdot\omega_b(\mathfrak{g})$ contains $\lambda_n^{-2}f + a_n\rho(f)$, so it contains $f+\lambda_n^2 a_n\cdot\rho(f)$. If $\lambda_n\to 0$, then this last term converges to $f+\xi \in \omega_{b_0}(\mathfrak{g})$, for some $\xi \in \mathfrak{p}$ (because the adjoint action of a_n on $\mathfrak{p}$ is diagonal with eigenvalues λ_n^2, 1 and λ_n^{-2}). But $\overline{\omega_{b_0}(\mathfrak{g})}$ is spanned by e and h, so this is a contradiction. □

Differentiating the function $\bar{\kappa} : B \to \mathbb{V}^{(0)} = E_0 \oplus E_2$ gives, via the parallelization of B arising from ω, a P-equivariant, automorphism-invariant function $D^{(1)}\bar{\kappa} : B \to \mathbb{V}^{(1)} = \mathfrak{h}^* \otimes \mathbb{V}^0$, and similarly, by iteration, functions $D^{(i)}\bar{\kappa} : B \to \mathbb{V}^{(i)} = \mathfrak{h}^* \otimes \mathbb{V}^{(i-1)}$. For vertical directions $X \in \mathfrak{p}$, the derivative is determined by equivariance: $X^\ddagger.\bar{\kappa} = -X \cdot \bar{\kappa}$. Our goal, in order to show local homogeneity of U, is to show that $D^{(i)}\bar{\kappa}$ has values on B in a single P-orbit. Because $\bar{\kappa}$ determines κ for 3-dimensional metrics, it will follow that $D^{(i)}\kappa$ has values on B in a single P-orbit, which suffices by Singer's theorem to conclude local homogeneity (see Proposition 3.8 in [Mel11] for a version of Singer's theorem for real analytic Cartan connections and also [Pec14] for the smooth case). By P-equivariance of these functions, it suffices to show that the values on $\mathscr{R}$ lie in a single A-orbit. We will prove the following slightly stronger result:

Proposition 19. *The curvature $\bar{\kappa}$ and all of its derivatives $D^{(i)}\bar{\kappa}$ are constant on $\mathscr{R}$.*

Proof. Recall that

$$\bar{\kappa} \equiv y m_d + z m_{2h^2-ef}$$

on all of $\mathscr{R}$, for some fixed $y, z \in \mathbb{R}$. The proof proceeds by induction on i. Suppose that for $i \geq 0$, the derivative $D^{(i)}\bar{\kappa}$ is constant on $\mathscr{R}$, so that in particular, the value $D^{(i)}\bar{\kappa}$ is annihilated by H. As in the proof for $i = 0$ above, to show that $D^{(i+1)}\bar{\kappa}$ is constant on $\mathscr{R}$, it suffices to show that $H^\ddagger.D^{(i+1)}\bar{\kappa}_b = -H \cdot D^{(i+1)}\bar{\kappa}_b = 0$ at a single point $b \in \mathscr{R}|_{U\backslash S}$.

To complete the induction step, we will need the following information on $\omega_b(\mathfrak{g})$.

Lemma 20. *At $b \in \mathscr{R}$ lying over $x \in U\backslash S$, the Killing algebra evaluates to*

$$\omega_b(\mathfrak{g}) = span\{e + \gamma E + \beta H, h - \gamma H, f + \alpha H + \delta F\}, \qquad \gamma, \beta, \alpha, \delta \in \mathbb{R}$$

Proof. Write

$$\omega_b(\mathfrak{g}) = \mathrm{span}\{e + \rho(e), h + \rho(h), f + \rho(f)\}$$

From proposition 14, we know that

$$a_n \cdot \omega_b(\mathfrak{g}) \to \omega_{b_0}(\mathfrak{g}) = \mathrm{span}\{e + \gamma E, h, H\}$$

Now lemma 18 implies that $\rho(h)$ and $\rho(f)$ both have zero component on E. Indeed, since this component is dilated by λ_n^2, it must vanish in order that $E \notin \omega_{b_0}(\mathfrak{g})$.

At the point b, let A^- be a Killing field with $\pi_{*b}A^- \in L^-_{\pi(b)}$, so we can assume $\overline{\omega_b(A^-)} = e$. We have $\omega_b(A^-) = e + \rho(e)$ and $\omega_b(W) = h + \rho(h)$. Recall from proposition 14 that $\kappa_{b_0}(h,e) = rE$. The fact that $\bar{\kappa}_b = \bar{\kappa}_{b_0}$ implies that the full curvature $\kappa_b = \kappa_{b_0}$, so also

$$\kappa_b(e,h) = K_b(A^-, W) = rE$$

On the other hand, equation (8) gives

$$0 = \omega_b[A^-, W] = [h + \rho(h), e + \rho(e)] + rE$$

so

$$\rho(h)e = \rho(e)h \qquad \text{and} \qquad [\rho(h), \rho(e)] = -rE$$

Writing $\rho(e) = \gamma E + \beta H + \delta F$ and $\rho(h) = \beta' H + \delta' F$ gives $\beta' = -\gamma$ and $\delta = \delta' = 0$ from the first equation. Note that the second equation gives $\gamma^2 = -r$, which is consistent with proposition 14. □

We now use $\mathfrak{g}$-invariance of $D^{(i)}\bar{\kappa}$. For abitrary $X \in \mathfrak{h}$, write $X^{\ddagger}$ for the coresponding ω-constant vector field on B. Lemma 20 gives

(1) $(e+\gamma E+\beta H)^{\ddagger}(b).D^{(i)}\bar{\kappa} \equiv 0$
(2) $(h-\gamma H)^{\ddagger}(b).D^{(i)}\bar{\kappa} \equiv 0$
(3) $(f+\alpha H+\delta F)^{\ddagger}(b).D^{(i)}\bar{\kappa} \equiv 0$

From (1),

$$\begin{aligned} D^{(i+1)}\bar{\kappa}_b(e) &= -(\gamma E+\beta H)^{\ddagger}(b).D^{(i)}\bar{\kappa} \\ &= (\gamma E+\beta H)\cdot D^{(i)}\bar{\kappa}_b \\ &= \gamma E\cdot D^{(i)}\bar{\kappa}_b \end{aligned}$$

Then

$$\begin{aligned} (H\cdot D^{(i+1)}\bar{\kappa}_b)(e) &= H\cdot(D^{(i+1)}\bar{\kappa}_b(e)) - D^{(i+1)}\bar{\kappa}_b([H,e]) \\ &= H\cdot(\gamma E)\cdot D^{(i)}\bar{\kappa}_b - D^{(i+1)}\bar{\kappa}_b(e) \\ &= \gamma([H,E]+EH)\cdot D^{(i)}\bar{\kappa}_b - \gamma E\cdot(D^{(i)}\bar{\kappa}_b) \\ &= \gamma E\cdot D^{(i)}\bar{\kappa}_b - \gamma E\cdot D^{(i)}\bar{\kappa}_b = 0 \end{aligned}$$

Item (2) gives, by a similar calculation,

$$D^{(i+1)}\bar{\kappa}_b(h) = -\gamma H\cdot D^{(i)}\bar{\kappa}_b = 0$$

and

$$(H\cdot D^{(i+1)}\bar{\kappa}_b)(h) = 0$$

Finally, (3) gives

$$D^{(i+1)}\bar{\kappa}_b(f) = \delta F\cdot D^{(i)}\bar{\kappa}_b$$

and again

$$(H\cdot D^{(i+1)}\bar{\kappa}_b)(f) = 0$$

We have thus shown vanishing of $H\cdot D^{(i+1)}\bar{\kappa}_b$ on $\mathbb{R}^{1,2}$. The remainder of $\mathfrak{h}$ is obtained by taking linear combinations with $\mathfrak{p}$. The H-invariance of $D^{(i)}\bar{\kappa}$ and P-equivariance of $D^{(i+1)}\bar{\kappa}$ give, for $X \in \mathfrak{p}$,

$$\begin{aligned} (H\cdot D^{(i+1)}\bar{\kappa}_b)(X) &= H\cdot(D^{(i+1)}\bar{\kappa}_b(X)) - D^{(i+1)}\bar{\kappa}_b([H,X]) \\ &= -H\cdot X\cdot D^{(i)}\bar{\kappa}_b + [H,X]\cdot D^{(i)}\bar{\kappa}_b \\ &= -X\cdot H\cdot D^{(i)}\bar{\kappa}_b = 0 \end{aligned}$$

The conclusion is $H\cdot D^{(i+1)}\bar{\kappa}_b = 0$, as desired. □

Now if $\bar{\kappa}$ and all its derivatives are constant on $\mathscr{R}$, then U is curvature homogeneous to all orders, and therefore, U is locally homogeneous by Singer's theorem for Cartan connections [Mel11, Pec14].

Let us consider now the remaining case where the isotropy at the origin is unipotent.

5.4. **Unipotent isotropy.**

Proposition 21. *If the isotropy at $0 \in S$ is unipotent, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sol}(a,b)$, for $b \neq -a$.*

Proof. Let $Y \in \mathfrak{g}$ generate the isotropy at 0. There is $b_0 \in \pi^{-1}(0)$ for which $\omega_{b_0}(Y) = E$. The projection $\overline{\omega_{b_0}(\mathfrak{g})}$ of $\omega_{b_0}(\mathfrak{g})$ to $\mathbb{R}^{1,2}$ is 2-dimensional and E-invariant, so it must be span$\{e,h\}$. Therefore, there is a basis $\{X,Y,Z\}$ of $\mathfrak{g}$ such that

$$\omega_{b_0}(X) = e + \alpha H + \beta F \qquad \text{and} \qquad \omega_{b_0}(Z) = h + \gamma H + \delta F$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Because $K_{b_0}(Y, \cdot) = 0$, equation (8) gives

$$\omega_{b_0}[Y,X] = [e + \alpha H + \beta F, E] = \alpha E - \beta H \in \omega_{b_0}(\mathfrak{g})$$

so $\beta = 0$ and $[Y,X] = \alpha Y$. A similar computation gives

$$\omega_{b_0}[Y,Z] = [h + \gamma H + \delta F, E] = e + \gamma E - \delta H$$

so $\delta = -\alpha$, and $[Y,Z] = X + \gamma Y$.

Infinitesimal invariance of K by Y gives

$$K_{b_0}([Y,X],Z) + K_{b_0}(X,[Y,Z]) = [-E, K_{b_0}(X,Z)]$$

But the left side is 0 because $[Y,X](0) = 0$ and $[Y,Z](0) = X(0)$. Therefore E commutes with $K_{b_0}(X,Z) \in \mathfrak{p}$, which means

$$K_{b_0}(X,Z) = rE \qquad \text{for some } r \in \mathbb{R}$$

Now equation (8) gives for X and Z,

$$\begin{aligned} \omega_{b_0}[X,Z] &= [h + \gamma H - \alpha F, e + \alpha H] + rE \\ &= \gamma e + \alpha h - \alpha^2 F + rE \end{aligned}$$

In order that this element belongs to $\omega_{b_0}(\mathfrak{g})$, one must have $\alpha = 0$ or $\gamma = 0$. First consider $\gamma = 0$. The structure of the algebra $\mathfrak{g}$ in the basis $\{Y,X,Z\}$ is

$$\text{ad } Y = \begin{pmatrix} 0 & \alpha & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \qquad \text{ad } X = \begin{pmatrix} -\alpha & & r \\ & 0 & \\ & & \alpha \end{pmatrix} \qquad \text{ad } Z = \begin{pmatrix} 0 & -r & \\ -1 & 0 & \\ & -\alpha & 0 \end{pmatrix}$$

This algebra is unimodular so in this case does not arise, by Lemma 3.

Next consider $\alpha = 0$. Then the Lie algebra is

$$\text{ad } Y = \begin{pmatrix} 0 & & \gamma \\ & 0 & 1 \\ & & 0 \end{pmatrix} \qquad \text{ad } X = \begin{pmatrix} 0 & & r \\ & 0 & \gamma \\ & & 0 \end{pmatrix} \qquad \text{ad } Z = \begin{pmatrix} -\gamma & -r & \\ -1 & -\gamma & \\ & & 0 \end{pmatrix}$$

In order that $\mathfrak{g}$ not be unimodular, γ must be nonzero (notice also that for $\gamma = r = 0$, we would get a Heisenberg algebra). We obtain a solvable Lie algebra

$$\mathfrak{g} \cong \mathbb{R} \ltimes_{\varphi} \mathbb{R}^2, \qquad \text{where } \varphi = \begin{pmatrix} -\gamma & -r \\ -1 & -\gamma \end{pmatrix}$$

If $r > 0$, then

$$\mathfrak{g} \cong \mathfrak{sol}(a,b), \qquad \text{where } a = -\gamma + \sqrt{r},\ b = -\gamma - \sqrt{r}$$

Conversely, φ is $\mathbb{R}$-diagonalizable only if $r > 0$. □

Proposition 22. *(compare Lemma 13 (i))*

(i) At points of S, the three eigenvalues of the Ricci operator are equal.
(ii) This triple eigenvalue is positive if and only if the Killing algebra $\mathfrak{sol}(a,b)$ is $\mathbb{R}$-diagonalizable.

Proof. (i) The invariance of the Ricci endomorphism $\bar{\kappa}_{b_0}$ by E means (see the table in Subsection 5.2):

$$\bar{\kappa}_{b_0} \in \text{span}\{m_d, m_{e^2}\}.$$

The triple eigenvalue is the coefficient of m_d.

(ii) The full curvature $\kappa_{b_0} \in E_0 \oplus E_1 \oplus E_2$ is E-invariant, so it has components on m_d and m_{e^2}, as above, plus possibly a third component on

$$m_e = h^* \otimes e - f^* \otimes h = f^* \wedge e^* \otimes E - h^* \wedge f^* \otimes H \in E_1$$

Referring to the column labeled $\wedge^2 \mathbb{R}^{1,2*} \otimes \mathfrak{p}$ in the table reveals that m_d is the only possible component of κ_{b_0} assigning a nonzero value to the input pair (e,h). Therefore the parameter r in the proof of Proposition 21 coincides with the coefficient of m_d in κ_{b_0} and with the triple eigenvalue of the Ricci endomorphism at 0. □

But, by the point (iii) in Proposition 12, we know that the Killing algebra $\mathfrak{sol}(a,b)$ is $\mathbb{R}$-diagonalizable. This implies that $r > 0$.

On the other hand, recall that in [CK09] Calvaruso and Kowalski classified Ricci operators for left invariant Lorentz metrics g on three dimensional Lie groups. In particular, they proved (see their Theorems 3.5, 3.6 and 3.7) that a Ricci operator of a left invariant Lorentz metric on a *nonunimodular* three-dimensional Lie group admits a triple eigenvalue $r \neq 0$ if and only if g is of constant sectional curvature. Since on $U \setminus S$, our Lorentz metric g is locally isomorphic to a left invariant Lorentz metric on the nonunimodular Lie group $SOL(a,b)$ corresponding to the Killing algebra, this implies that g is of constant sectional curvature. In particular, g is locally homogeneous.

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