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Abstract.— *We prove the monodromy conjecture for the topological zeta function for all non-degenerate surface singularities. Fundamental in our work is a detailed study of the formula for the zeta function of monodromy by Varchenko and the study of the candidate poles of the topological zeta function yielded by what we call 'B$_1$-facets'. In particular, new cases among the nondegenerate surface singularities for which the monodromy conjecture is proven now, are the non-isolated singularities, the singularities giving rise to a topological zeta function with multiple candidate pole and the ones for which the Newton polyhedron contains a B$_1$-facet.*

0. Introduction

In [We65], Weil introduced some zeta functions $\mathcal{Z}(K, f)$ that are integrals over a p-adic field K and that are associated to a polynomial $f(\underline{x}) \in K[\underline{x}]$. Igusa showed that these zeta functions are rational and he studied their poles (see [Ig75] and [Ig78]). The analogous integrals over $K = \mathbb{R}$ or $\mathbb{C}$ are meromorphic (see for example [At70] and [BeGe69]) and it is known that their poles are contained in the set of roots - and roots shifted by a negative integer - of the Bernstein polynomial b_f. By Malgrange ([Ma83]), if α is a root of b_f, then $e^{2\pi i \alpha}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0)$. So when $K = \mathbb{R}$ or $\mathbb{C}$, the poles of the zeta function induce eigenvalues of the local monodromy. The study of concrete examples made it natural to propose the following two conjectures at the p-adic side.

Monodromy conjecture. ([Ig88]) *Let $F \subset \mathbb{C}$ be a number field and $f \in F[\underline{x}]$. For almost all p-adic completions K of F, if s_0 is a pole of $\mathcal{Z}(K, f)$, then $e^{2\pi i Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of the hypersurface $f = 0$.*

The stronger conjecture at the level of the Bernstein-Sato polynomial: ([Ig88]) *Let $F \subset \mathbb{C}$ be a number field and $f \in F[\underline{x}]$. For almost all p-adic completions K of F, if s_0 is a pole of $\mathcal{Z}(K, f)$, then $Re(s_0)$ is a root of the (local) Bernstein-Sato polynomial $b_{f,0}(s)$ of f.*

One calls this zeta function $\mathcal{Z}(K, f)$ the Igusa zeta function. When Denef and Loeser introduced the topological zeta function in 1992 as a kind of limit of the Igusa zeta

function ([DL92]), an analogous version of the monodromy conjecture arose. The monodromy conjecture for the Igusa zeta function implies the monodromy conjecture for the topological zeta function.

The conjecture has been proven in several cases. Loeser verified the conjecture for the Igusa zeta function at the level of the Bernstein polynomial for plane curves (see [L88]). He also gave a proof for a class of polynomials in higher dimensions; the polynomial should be nondegenerate with respect to its Newton polyhedron and should satisfy some numerical conditions ([L90] and Section 2). Rodrigues gave an alternative proof for the monodromy conjecture for plane curves (see [Ro04]). Artal Bartolo, Cassou-Noguès, Luengo and Melle Hernández proved the monodromy conjecture for some surface singularities, such as for the topological zeta function for the superisolated ones (see [A-B,C-N,Lu,M-H02]), and for the Igusa zeta function for the quasi-ordinary polynomials in [A-B,C-N,Lu,M-H06]. In [LV09], Veys and the first author proved the monodromy conjecture for the topological zeta function for surface singularities that are general for a toric idealistic cluster.

In this paper we prove the monodromy conjecture for the topological zeta function for all nondegenerate surface singularities. In Section 1 we explain the objects that play a role in the monodromy conjecture, in particular the zeta function of monodromy and the local topological zeta function. When the singularity is nondegenerate with respect to its Newton polyhedron, then these zeta functions can be expressed in terms of their Newton polyhedron. We recall these formulas and some basic notions in the context of nondegenerate singularities in Section 2. We here also precise the conditions that the singularities have to satisfy to be covered by the proof of [L90] and which we can omit in this paper in dimension 3. In Section 3 we explain the strategy behind our proof and execute it. Fundamental is an analysis of the formula of Varchenko for the zeta function of monodromy ([Var76]). This allows us to show in dimension two and three that all facets induce eigenvalues of monodromy, except some special facets that we call B_1-facets. In Section 4 we will determine when these B_1-facets contribute a pole of the local topological zeta function. Mostly they do not. When they do contribute a pole, then we determine a vertex of the Newton polyhedron that gives rise to the corresponding eigenvalue of monodromy. A combination of these results leads to the monodromy conjecture for nondegenerate polynomials in dimension two and three. Although already completely known in dimension 2 (see [L88]), we still mention here our results in dimension 2 because we need them in dimension 3. In Section 5 we provide some examples.

At first glance it seems that the same strategy could lead to a proof in arbitrary dimension, but combinatorics are getting much more tricky and probably some new tools are needed.

1. The monodromy conjecture

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. Let F be the Milnor fibre of the Milnor fibration at the origin associated with f and write $h^* : H^i(F, \mathbb{C}) \to H^i(F, \mathbb{C})$, $i \geq 0$, for the monodromy transformation.

Definition 1. *The* zeta function of monodromy $Z(f, 0)$ *at the origin associated to the polynomial f is*

$$Z(f, 0)(t) := \prod_{i \geq 0} \left(det(id^* - th^*; H^i(F, \mathbb{C})) \right)^{(-1)^{(i+1)}}.$$

One calls α an *eigenvalue of monodromy of f at the origin* if α is an eigenvalue for some $h^* : H^i(F, \mathbb{C}) \to H^i(F, \mathbb{C})$.

In 1992 Denef and Loeser created a new zeta function which they called the topological zeta function because of the topological Euler–Poincaré characteristic $\chi(\cdot)$ turning up in it. Let $\pi : Z \to \mathbb{C}^n$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and we denote by N_j and by $\nu_j - 1$ the multiplicities of E_j in the divisor on Z of $f \circ \pi$ and $\pi^*(dx_1 \wedge \ldots \wedge dx_n)$, respectively. We denote also $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j \notin I} E_j)$, for $I \subset S$. Let the $E_j, j \in J := \{1, \cdots, r\} \subset S$, be the exceptional irreducible components of $\pi^{-1}(\{0\})$.

Definition 2. *The* local topological zeta function at the origin associated to f *is the rational function in one complex variable*

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

Denef and Loeser proved in [DL92] that every embedded resolution gives rise to the same function, so the topological zeta function is a well-defined singularity invariant. They also formulated the analogous version of the monodromy conjecture for the local topological zeta function.

Monodromy conjecture for the topological zeta function. *If s_0 is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of f at some point of the germ at 0 of the hypersurface $f = 0$.*

2. NONDEGENERATE SINGULARITIES

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial satisfying $f(0) = 0$. We write $f = \sum_{k \in \mathbb{Z}_{\geq 0}^n} a_k x^k$, where $k = (k_1, \ldots, k_n)$ and $x^k = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$. The *support of f* is $\text{supp} f = \{k \in \mathbb{Z}_{\geq 0}^n \,|\, a_k \neq 0\}$. The *Newton polyhedron Γ_0 of f at the origin* is the convex hull in $\mathbb{R}_{\geq 0}^n$ of

$$\bigcup_{k \in \text{supp} f} k + \mathbb{R}_{\geq 0}^n.$$

A *face* of the Newton polyhedron is the intersection of Γ_0 with a supporting hyperplane. A *facet* is a face of dimension $n - 1$. A polynomial $f(x_1, \ldots, x_n)$ is called *nondegenerate with respect to its Newton polyhedron Γ_0* if for every compact face τ of Γ_0 the polynomials $f_\tau := \sum_{k \in \tau} a_k x^k$ and $\partial f_\tau / \partial x_i, 1 \leq i \leq n$, have no common zeroes in $(\mathbb{C} \setminus \{0\})^n$. For a fixed Newton polyhedron Γ, almost all polynomials having Γ as its Newton polyhedron are nondegenerate with respect to Γ (see [AVG86, p.157]).

For nondegenerate polynomials a lot of invariants can be deduced from the Newton polyhedron. For instance it is possible to express the local topological zeta function and the monodromy zeta function in terms of the Newton polyhedron. Before giving these formulas, we recall the needed notions. If $a_1, \ldots, a_r \in \mathbb{R}^n \backslash \{0\}$, we call $\operatorname{cone}(a_1, \ldots, a_r) = \{\sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$ the *cone strictly positively spanned by the vectors* $a_1, \ldots, a_r$. Fix a cone Δ. If there exist vectors $a_1, \ldots, a_r \in \mathbb{R}^n \backslash \{0\}$ that are linearly independent over $\mathbb{R}$ such that $\Delta = \operatorname{cone}(a_1, \ldots, a_r)$, then Δ is called a *simplicial cone*. If moreover $a_1, \ldots, a_r$ can be chosen in $\mathbb{Z}^n$, we say Δ is a *rational simplicial cone*. For a rational simplicial cone Δ spanned by the primitive and linearly independent vectors $a_1, \ldots, a_r \in \mathbb{Z}^n$ we define the *multiplicity* of Δ, denoted by $\operatorname{mult}(\Delta)$, as the index of the lattice $\mathbb{Z}a_1 + \ldots + \mathbb{Z}a_r$ in the group of the points with integral coordinates of the vector space generated by $a_1, \ldots, a_r$. It is easy to verify that the multiplicity of Δ equals the number of elements in the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^r \lambda_i a_i \mid 0 \le \lambda_i < 1 \text{ for } i = 1, \ldots, r \right\}.$$

Moreover, $\operatorname{mult}(\Delta)$ is equal to the greatest common divisor of the determinants of the $(r \times r)$-matrices obtained by omitting columns from the matrix A with rows $a_1, \ldots, a_r$.

Let Γ_0 be a Newton polyhedron in $\mathbb{R}^n$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\ge 0}$ we put $N(a) := \inf_{x \in \Gamma_0} a \cdot x, \nu(a) := \sum_{i=1}^n a_i$ and $F(a) := \{x \in \Gamma_0 \mid a \cdot x = N(a)\}$. All $F(a), a \ne 0$, are faces of Γ_0. To a face τ of Γ_0 one associates a dual cone $\tau^\circ \subset \mathbb{R}^n$, defined as the closure in $\mathbb{R}^n$ of $\{a \in \mathbb{R}^n_{\ge 0} \mid F(a) = \tau\}$. This is a rational convex cone with vertex the origin, of dimension $n - \dim \tau$. Note that the map $\tau \mapsto \tau^\circ$ is inclusion-reversing. In particular if $\dim \tau = n - 1$ then τ° is a ray, say $\tau^\circ = a\mathbb{R}_{\ge 0}$ for some $a \in \mathbb{Z}^n_{\ge 0}$, and then the equation of the hyperplane through τ is $a \cdot x = N(a)$. If we demand that $\gcd(a_1, \ldots, a_n) = 1$, this a is uniquely defined. For a facet τ we also use the notation $N(\tau)$ and $\nu(\tau)$, meaning respectively $N(a)$ and $\nu(a)$ for this associated $a \in \mathbb{Z}^n_{\ge 0}$.

Suppose $\Delta = \mathbb{R}_{\ge 0}a_1 + \cdots + \mathbb{R}_{\ge 0}a_r$, with $a_1, \ldots, a_r \in \mathbb{Z}^n_{\ge 0}$ linearly independent and primitive (i.e. with relatively prime components). Then we put

$$J_\Delta(s) := \frac{\operatorname{mult}(\Delta)}{\prod_{i=1}^r (N(a_i)s + \nu(a_i))}.$$

The dual cone τ° is not necessarily a rational simplicial cone. However, there always exists a decomposition $\tau^\circ = \cup_{i=1}^k \Delta_i$ of τ° in rational simplicial cones Δ_i of dimension $\ell = \dim \tau^\circ$ satisfying $\dim(\Delta_i \cap \Delta_j) < \ell$ if $i \ne j$. Such a decomposition is even possible without introducing new rays (this is well-known, for a proof see [D95, Lemme 2]). To an arbitrary face τ of Γ_0 we now associate the rational function

$$J_\tau(s) := \sum_{i=1}^k J_{\Delta_i}(s),$$

where $\tau^\circ = \cup_{i=1}^k \Delta_i$ is a decomposition as mentioned above. Denef and Loeser showed in [DL92, Lemme 5.1.1] that this function does not depend on the chosen decomposition of τ°.

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Theorem 1. [DL92, Théorème 5.3] *If f is nondegenerate with respect to Γ_0, then the local topological zeta function at the origin is equal to*

$$Z_{top,f}(s) = \sum_{\tau \text{ vertex of } \Gamma_0} J_\tau(s) + \frac{s}{s+1} \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_0, \\ \dim \tau \geq 1}} (-1)^{\dim \tau}(\dim \tau)! \text{Vol}(\tau) J_\tau(s).$$

For a face τ of dimension 0, we put $\text{Vol}(\tau) = 1$. For every other compact face $\text{Vol}(\tau)$ is defined as the *volume* of τ for the volume form ω_τ. This is a volume form on $\text{Aff}(\tau)$, the affine space spanned by τ, such that the parallelepiped spanned by a lattice-basis of $\mathbb{Z}^n \cap \text{Aff}(\tau)$ has volume 1. The product $(\dim \tau)!\text{Vol}(\tau)$ is also called the normalized volume of the face τ. If τ is a simplicial facet, this normalized volume is equal to the multiplicity of the cone spanned by the vertices divided by the 'distance' of the facet to the origin. This distance is the number N in the equation $\sum_{i=1}^n a_i x_i = N$ of $\text{Aff}(\tau)$ (where $\gcd(a_1, \ldots, a_n) = 1$).

In this last theorem we clearly see that a set of candidate poles (containing all poles) of the local topological zeta function is given by -1 together with the rational numbers $-\nu(a)/N(a)$ for a orthogonal to a facet of the Newton polyhedron in the origin. We say that such a facet *contributes* the candidate pole.

Remark 1. There is also a graphical way to determine the candidate pole contributed by a facet τ. If $(r, \ldots, r)$ is the intersection point of the diagonal of the first quadrant with the affine hyperplane containing τ, then the candidate pole $-\nu(\tau)/N(\tau)$ is equal to $-1/r$.

Example 1. Note that $f(x, y) = x^2 y^7 + x^8 y + x^7 y^6 + x^{10} + y^{12}$ is nondegenerate with respect to its Newton polyhedron in the origin, so we can use Theorem 1 to compute its local topological zeta function. The equations and volumes of the compact facets τ_2, τ_3
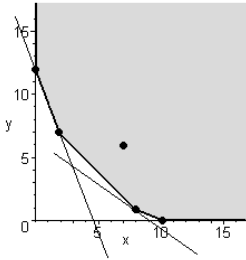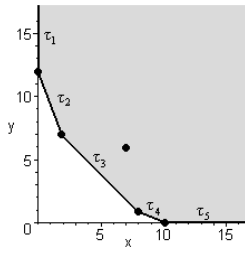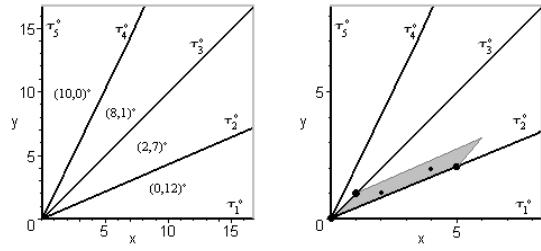

Figure 1   Figure 2   Figure 3

and τ_4 (see Figure 1) are respectively $5x + 2y = 24, \text{Vol}(\tau_2) = 1, x + y = 9, \text{Vol}(\tau_3) = 6$ and $x + 2y = 10, \text{Vol}(\tau_4) = 1$. The cones dual to the four vertices are all simplicial (see Figure 2). We compute that $\text{mult}(2, 7)^\circ = 3$ (see Figure 3), $\text{mult}(0, 12)^\circ = 2, \text{mult}(8, 1)^\circ = 1$ and $\text{mult}(10, 0)^\circ = 1$. The local topological zeta function is equal to

$$
\begin{aligned}
Z_{top,f}(s) &= \frac{2}{24s + 7} + \frac{3}{(9s + 2)(24s + 7)} + \frac{1}{(9s + 2)(10s + 3)} + \frac{1}{10s + 3} \\
&\quad - \frac{s}{s + 1}\left(\frac{1}{24s + 7} + \frac{6}{9s + 2} + \frac{1}{10s + 3}\right) \\
&= -\frac{135s^2 - 5s - 14}{(s + 1)(9s + 2)(24s + 7)}.
\end{aligned}
$$

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We now recall the formula for the zeta function of monodromy in the origin in terms of the Newton polyhedron. This was done by Varchenko (see [Var76]) and uses the formula of A'Campo (see [A'C75]). Again, let $f(x_1, \ldots, x_n)$ be a polynomial satisfying $f(0) = 0$ and suppose f is nondegenerate with respect to its Newton polyhedron Γ_0. Varchenko defines functions $\zeta_\tau(t)$ for the compact faces τ of Γ_0 for which there exists a subset $I \subset \{1, \ldots, n\}$ with $\#I = \dim(\tau) + 1$, such that $\tau \subset L_I := \{x \in \mathbb{R}^n \mid \forall i \notin I : x_i = 0\}$. Let $\sum_{i \in I} a_i x_i = m$ be the equation of $\mathrm{Aff}(\tau)$ in its corresponding L_I, where m and all a_i (for $i \in I$) are positive integers and the greatest common divisor of the $a_i, i \in I$, is equal to 1. We put

$$\zeta_\tau(t) := (1 - t^m)^{\dim(\tau)! \mathrm{Vol}(\tau)}.$$

Theorem 2. *[Var76] The zeta function of monodromy of f in the origin is equal to*

$$Z(f, 0)(t) = \prod \zeta_\tau(t)^{(-1)^{\dim(\tau)}},$$

where the product runs over all faces τ of Γ_0 for which ζ_τ is defined above.

Example 1 continued. We compute $\zeta_\tau(t)$ for τ equal to the vertices $(10, 0)$ and $(0, 12)$ and for τ equal to the compact facets. We get

$$\zeta_{\tau_2}(t) = (1 - t^{24}), \quad \zeta_{\tau_3}(t) = (1 - t^9)^6, \quad \zeta_{\tau_4}(t) = (1 - t^{10}).$$

For τ equal to $(10, 0)$ or $(0, 12)$, the corresponding equations are respectively $x = 10$ and $y = 12$ and the volumes are both 1. This results in

$$\zeta_{(10,0)}(t) = (1 - t^{10}), \quad \zeta_{(0,12)}(t) = (1 - t^{12}).$$

The zeta function of monodromy of f in the origin is equal to

$$Z(f, 0)(t) = \frac{1 - t^{12}}{(1 - t^{24})(1 - t^9)^6}.$$

$\square$

Notice that in the previous example all poles of $Z_{top,f}$ induce eigenvalues of monodromy. Indeed, $-3/10$ is a false candidate pole. This candidate pole is contributed by what we will call a B_1-*facet*. Also other kinds of facets can yield false candidate poles. The idea of Loeser in [L90] was to show that almost all compact facets give rise to an eigenvalue of monodromy. For τ and σ two facets of a Newton polyhedron, he defines $\beta(\tau, \sigma)$ to be the greatest common divisor of the (2×2)-minors of the matrix with as rows the relatively prime coefficients of the dual rays τ° and σ°. Then he puts

$$\lambda(\tau, \sigma) = \nu(\sigma) - \frac{\nu(\tau)}{N(\tau)} N(\sigma) \quad \text{and} \quad \mu(\tau, \sigma) = \frac{\lambda(\tau, \sigma)}{\beta(\tau, \sigma)}.$$

Using this terminology, his theorem is the following:

Theorem 3. [L90] *Let p be a prime number and K a finite extension field of $\mathbb{Q}_p$. Let $f \in K[x_1, \ldots, x_n]$ be a polynomial satisfying $f(0) = 0$. Suppose f is nondegenerate with respect to its Newton polyhedron Γ_0 in the origin. If a compact facet τ of Γ_0 satisfies the conditions*

1. $\nu(\tau)/N(\tau)$ is no integer;

2. for every facet σ of Γ_0 that is different from τ and has non-empty intersection with τ, we have that $\mu(\tau, \sigma)$ is no integer;

then $e^{-2\pi i \nu(\tau)/N(\tau)}$ is an eigenvalue of monodromy of f at the origin.

Actually, this result is a corollary of the stronger statement he proved about zeroes of the Bernstein-Sato polynomial. Notice that Theorem 3 can not be applied to facet τ_4 of Example 1. In dimension three we now want to omit these conditions and include also the poles coming from non-compact facets (such Newton polyhedra typically represent non-isolated singularities).

3. Determination of a set of eigenvalues of monodromy

3.1. Strategy

To explain the objective of this section, we introduce the notion of B_1-facet. We say that a facet τ of an n-dimensional Newton polyhedron is *non-compact for the variable* x_j $(1 \le j \le n)$ if for every point $p \in \tau$ the point $p + (0, \dots, 0, 1, 0, \dots, 0) \in \tau$, where $(0, \dots, 0, 1, 0, \dots, 0)$ is an n-tuple with 1 at place j and 0 everywhere else. We define the maps $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, \widehat{x_j}, \dots, x_n)$ for $j = 1, \dots, n$.

Definition 3. *A simplicial facet of an n-dimensional Newton polyhedron $(n \ge 2)$ is a B_1-simplex with respect to the variable x_i if it is a simplex with $n - 1$ vertices in the coordinate hyperplane $x_i = 0$ and one vertex at distance one of this hyperplane.*

A non-compact facet τ of an n-dimensional Newton polyhedron $(n \ge 3)$ is a (non-compact) B_1-facet with respect to the variable x_i if τ is non-compact for exactly one variable x_j and if $\pi_j(\tau)$ is a B_1-simplex in $\mathbb{R}^{n-1}$ with respect to x_i.

A B_1-facet is a B_1-simplex or a non-compact B_1-facet with respect to some variable.

Throughout this section we are going to prove the following result about facets that induce eigenvalues of monodromy.

Theorem 4. *Let f be a complex polynomial in two or three variables. Suppose that f is nondegenerate with respect to its Newton polyhedron Γ_0 in the origin. Then for every facet τ of Γ_0 that is not a B_1-facet, the complex number $e^{-2\pi i \nu(\tau)/N(\tau)}$ is an eigenvalue of monodromy of f at a point in the neighbourhood of the origin.*

The idea of the proof is the following. Remember that the zeta function of monodromy associated to an isolated singularity in odd dimensions is a polynomial, in even dimensions it is $(1 - t)$ divided by a polynomial. Indeed, for isolated singularities, Milnor proved that the cohomology groups $H^i(F, \mathbb{C})$ are $\{0\}$ for $i \ne 0$ and $i \ne n - 1$, where n denotes the dimension (see [Mi68]).

We will group the factors ζ_τ as defined in Section 2. For a compact facet τ of a two-dimensional Newton polyhedron, we define F_τ as

$$F_\tau := (\zeta_\tau)^{-1} \prod \zeta_p,$$

where the product runs over the vertices $p \in \tau$ that lie on coordinate axes. Then we see that in dimension two the zeta function of monodromy in the origin is equal to

$$Z(f,0)(t) = \prod_{\tau \text{ a compact facet}} F_\tau. \tag{1}$$

In Subsection 3.2. we will show that such a factor F_τ is the inverse of a polynomial (except when τ contains two vertices on the coordinate axes). This implies that a pole of such an F_τ is also a pole of the monodromy zeta function at the origin.

Note that Equation (1) is not valid if the Newton polyhedron has only one vertex and if this vertex is moreover of the form $(a,0)$ or $(0,a)$ for some $a \in \mathbb{Z}_{\geq 0}$. In this case the zeta function of monodromy is equal to $1/(1-t^a)$.

If we work with a three-dimensional Newton polyhedron, we first partition every compact facet in simplices. For such a simplex τ we define the factor F_τ as follows:

$$F_\tau := \zeta_\tau \prod_\sigma (\zeta_\sigma)^{-1} \prod_p \zeta_p,$$

where the first product runs over the line segments σ in τ that lie in coordinate planes and the second product runs over vertices p of τ that are intersection points of two such line segments (and hence lie on a coordinate axis). Note that for a partition of a compact facet τ into simplices $\tau_1, \cdots, \tau_r$ we have that $\zeta_\tau = \prod_{i=1}^r \zeta_{\tau_i}$. Following the formula of Varchenko, the zeta function of monodromy in the origin is

$$Z(f,0)(t) = \prod_\tau F_\tau \prod_\sigma (\zeta_\sigma)^{-1} \prod_p \zeta_p, \tag{2}$$

where the first product runs over all simplices τ obtained after subdividing the compact facets and the other products run over line segments σ in coordinate hyperplanes and vertices p on coordinate axes for which ζ_σ, respectively ζ_p, was not used in any F_τ. We will show that every F_τ is a polynomial and that for every compact (simplicial) facet τ that is not a B_1-simplex, the complex number $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ. When the Newton polyhedron does not contain any line segment in a coordinate plane that is not included in a compact facet, then every compact facet that is no B_1-simplex induces an eigenvalue of monodromy at the origin. When the Newton polyhedron does not satisfy this condition, we might need to consider the zeta function of monodromy at a point in the neighbourhood of the origin. Also for a non-compact facet that is no B_1-facet we consider the zeta function of monodromy in a point different from the origin to conclude that it induces an eigenvalue of monodromy. Lemma 9 is the key lemma to handle these situations.

3.2. MONODROMY EIGENVALUES IN DIMENSION 2

In this subsection we explore the situation in dimension two. Note that then the monodromy conjecture is already proven by Loeser (see [L88]). However, the result about facets that induce eigenvalues of monodromy in dimension 2 will be needed later on when we work in dimension 3.

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Proposition 5. *Let τ be a compact one-dimensional simplex of some two-dimensional Newton polyhedron. If τ does not contain two vertices on coordinate axes, then F_τ is the inverse of a polynomial. If moreover τ is no B_1-simplex, then $e^{-2\pi i\nu(\tau)/N(\tau)}$ is a pole of F_τ. If τ contains two vertices on coordinate axes, if τ is no B_1-simplex and $-\nu(\tau)/N(\tau) \neq -1$, then $e^{-2\pi i\nu(\tau)/N(\tau)}$ is a pole of F_τ.*

Proof. First suppose that τ does not contain any vertex on a coordinate axis. Then $F_\tau = (\zeta_\tau)^{-1}$ and the number $e^{-2\pi i\nu(\tau)/N(\tau)}$ is obviously a pole.

Then suppose that τ contains exactly one vertex on a coordinate axis. We introduce the integers k, l, m as in Figure 4 and put $g := \gcd(k - m, l)$.
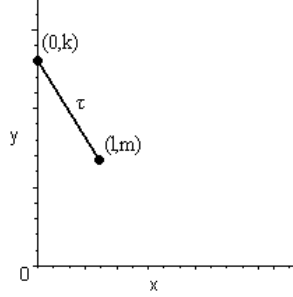

Figure 4

Then g is the (normalized) volume of the facet τ,

$$\text{Aff}(\tau) \leftrightarrow \frac{k - m}{g}x + \frac{l}{g}y = \frac{lk}{g} \quad \text{and} \quad F_\tau = \frac{(1 - t^k)}{(1 - t^{lk/g})^g}.$$

Since g divides l, F_τ is obviously the inverse of a polynomial. We need to verify if under the condition $l > 1$, the rational number $s_0 = -\frac{(l+k-m)/g}{lk/g}$ induces a pole $e^{2\pi is_0}$ of F_τ. If $g > 1$, this is obvious. If $g = 1$ it is only false if $e^{2\pi is_0}$ is a kth root of unity. This is equivalent with l being a divisor of $k - m$, which can only be true if $l = 1$.

Now suppose τ contains two vertices on coordinate axes. Note that then the zeta function of monodromy is equal to F_τ. Suppose the vertices are $(l, 0)$ and $(0, k)$ and suppose $g = \gcd(l, k)$. Then the supporting hyperplane of τ has equation $(k/g)x + (l/g)y = lk/g$ and

$$F_\tau = \frac{(1 - t^l)(1 - t^k)}{(1 - t^{lk/g})^g}.$$

If $g > 2$ it is obvious that $s_0 = -\frac{(l+k)/g}{lk/g}$ induces a pole of F_τ. If $g = 2$ this would only be false if $e^{2\pi is_0}$ is a lth and a kth root of unity. This means respectively that $k/2$ and $l/2$ are divisors of $(l + k)/2$, so $l = k = 2$ and $s_0 = -1$. This case was excluded. If $g = 1$, the complex number $e^{-2\pi i(l+k)/lk}$ is a pole unless l or k divides $l + k$, but since $g = 1$, this implies that l or k is 1 and τ is a B_1-simplex. ∎

The following theorem is Theorem 4 for a polynomial in two variables.

Theorem 6. *Let $f(x, y)$ be a complex polynomial that is nondegenerate with respect to its Newton polyhedron Γ_0. If $s_0 = -\nu(\tau)/N(\tau)$ for a facet τ that is not a B_1-simplex*

and not contained in a coordinate axis, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of f at a point in the neighbourhood of the origin.

Proof. If the facet τ is compact, this follows now immediately from Proposition 5 (taking into account that 1 is always an eigenvalue of monodromy).

Suppose τ is not compact, then $\text{Aff}(\tau) \leftrightarrow x = a$ for $a \in \mathbb{Z}_{>0}$ and $s_0 = -1/a$. For almost all $k \in \mathbb{C}\backslash\{0\}$, the Newton polyhedron of $f(x, y)$ at $(0, k)$ is equal to the Newton polyhedron of the polynomial $f(x, y - k)$ at the origin. We choose such a $k \in \mathbb{C}\backslash\{0\}$ and show that $e^{-2\pi i/a}$ is an eigenvalue of monodromy at the point $(0, k)$. Notice that this polyhedron only has one vertex, the point $(a, 0)$, and thus the polynomial $f(x, y - k)$ is nondegenerate with respect to its Newton polyhedron in the origin. So we can use the formula of Varchenko to compute the zeta function of monodromy of f at $(0, k)$. This zeta function is equal to $(1 - t^a)$. We conclude that $e^{-2\pi i/a}$ is an eigenvalue of monodromy of f at the point $(0, k)$. ∎

3.3. Monodromy eigenvalues in dimension 3

To prepare the proof of Theorem 4 in dimension 3, we first show Proposition 8 and Lemma 9. Lemma 9 is our key tool to handle non-isolated singularities. We will use the following property on greatest common divisors.

Lemma 7. *Let a, b, c and d be positive integer numbers. Then*

$$\gcd(a, b)\gcd(c, d) = \gcd(ac, ad, bc, bd).$$

Proposition 8. *Let τ be a compact simplex of some three-dimensional Newton polyhedron. Then F_τ is a polynomial. Moreover, if τ is not a B_1-simplex, then $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ.*

Proof. CASE 1: τ DOES NOT CONTAIN A SEGMENT IN A COORDINATE PLANE.

If the 2-simplex τ in Γ_0 does not contain a segment that is lying in a coordinate plane, then $F_\tau = \zeta_\tau$. For such a facet τ we clearly have that $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ.

CASE 2: τ CONTAINS EXACTLY ONE SEGMENT IN A COORDINATE PLANE.

<u>Part A.</u> As illustrated in Figure 5, let τ be the facet with vertices with coordinates $(a, b, c), (0, k, l)$ and $(0, m, n)$ and let σ be the segment with vertices $(0, k, l)$ and $(0, m, n)$. The equation of $\text{Aff}(\tau)$ is given by

$$x\left(k(c - n) + b(n - l) + m(l - c)\right) + y\left(a(l - n)\right) + z\left(a(m - k)\right) = a(ml - kn).$$

(The integers k, n, b and c might be zero, but not in a combination that introduces a new line segment in a coordinate plane.) Let g_τ be the greatest common divisor of the coefficients:

$$\begin{aligned} g_\tau &:= \gcd\left(k(c - n) + b(n - l) + m(l - c), a(l - n), a(m - k)\right), \\ &= \gcd\left((l - n)(m - b) + (m - k)(n - c), a(l - n), a(m - k)\right). \end{aligned}$$
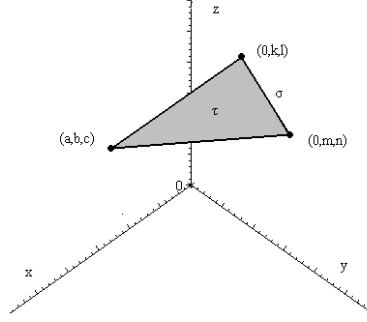
Figure 5

The normalized volume of a facet can be computed by putting its vertices in a matrix and dividing the determinant of this matrix by the distance $N(\tau)$. Hence we find $(\dim\tau)!\mathrm{Vol}(\tau) = g_\tau$ and $\zeta_\tau(t) = (1 - t^{\frac{a(ml-kn)}{g_\tau}})^{g_\tau}$.

The segment σ has as equation in the coordinate plane $x = 0$

$$y\,(a(l-n)) + z\,(a(m-k)) = a(ml-kn).$$

Let $g_\sigma := \gcd(\frac{a(l-n)}{g_\tau}, \frac{a(m-k)}{g_\tau})$. The normalized volume $(\dim\sigma)!\mathrm{Vol}(\sigma)$ of σ is equal to $\frac{g_\tau g_\sigma}{a} = \gcd(l-n, m-k)$ and thus $\zeta_\sigma(t) = (1 - t^{\frac{a(ml-kn)}{g_\tau g_\sigma}})^{\frac{g_\tau g_\sigma}{a}}$. We get

$$F_\tau = \frac{(1 - t^{\frac{a(ml-kn)}{g_\tau}})^{g_\tau}}{(1 - t^{\frac{a(ml-kn)}{g_\tau g_\sigma}})^{\frac{g_\tau g_\sigma}{a}}}.$$

As $\frac{g_\tau g_\sigma}{a}$ divides $l - n$ and $m - k$, it also divides g_τ. This shows that F_τ is a polynomial.

<u>Part B.</u> We will now show that $e^{-2\pi i\nu(\tau)/N(\tau)}$ is a zero of F_τ. The candidate pole $-\nu(\tau)/N(\tau)$ is equal to

$$-\frac{(k(c-n) + b(n-l) + m(l-c) + a(l-n) + a(m-k))/g_\tau}{a(ml-kn)/g_\tau}.$$

This rational number would not induce a zero of F_τ if

$$g_\sigma = a \text{ and } g_\tau g_\sigma \mid k(c-n) + b(n-l) + m(l-c) + a(l-n) + a(m-k).$$

Since $g_\tau g_\sigma$ is a divisor of the last two terms, it follows that g_σ must be equal to 1 to avoid a contradiction with the definition of g_τ. So also $a = 1$ and τ is a B_1-simplex.

We conclude that when τ is not a B_1-simplex, then $e^{-2\pi i\nu(\tau)/N(\tau)}$ is a zero of F_τ. When τ is a B_1-simplex (so $a = 1$), it can easily be seen that $g_\tau = \gcd(l - n, m - k)$. This implies that $g_\sigma = 1$ and $F_\tau = 1$.

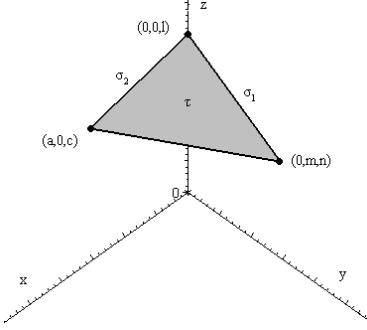CASE 3: τ CONTAINS EXACTLY TWO SEGMENTS IN COORDINATE PLANES.

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Figure 6

<u>Part A.</u> Let τ be the facet with vertices $(a,0,c),(0,0,l)$ and $(0,m,n)$. Let σ_1 be the segment with vertices $(0,0,l)$ and $(0,m,n)$ and let σ_2 be the segment with vertices $(0,0,l)$ and $(a,0,c)$, as illustrated in Figure 6. We first show that F_τ is a polynomial. The equation of $\mathrm{Aff}(\tau)$ is given by

$$x\left(m(l-c)\right) + y\left(a(l-n)\right) + zam = aml.$$

(One of the integers n or c might be zero, but not both.) Let g_τ be the greatest common divisor of the coefficients: $g_\tau := \gcd\left(m(l-c), a(l-n), am\right)$. We find $(\dim\tau)!\mathrm{Vol}(\tau) = g_\tau$ and hence $\zeta_\tau(t) = (1 - t^{\frac{aml}{g_\tau}})^{g_\tau}$.

The segment σ_1 has as equation in the coordinate plane $x = 0$

$$y\left(a(l-n)\right) + zam = aml.$$

Let $g_{\sigma_1} := \gcd(\frac{a(l-n)}{g_\tau}, \frac{am}{g_\tau})$. The normalized volume of σ is equal to $\frac{g_\tau g_{\sigma_1}}{a} = \gcd(l-n,m)$ and thus $\zeta_{\sigma_1}(t) = (1 - t^{\frac{aml}{g_\tau g_{\sigma_1}}})^{\frac{g_\tau g_{\sigma_1}}{a}}$. Analogously for the segment σ_2 we find that $\zeta_{\sigma_2}(t) = (1 - t^{\frac{aml}{g_\tau g_{\sigma_2}}})^{\frac{g_\tau g_{\sigma_2}}{m}}$, with $\frac{g_\tau g_{\sigma_2}}{m} = \gcd(l-c,a)$.

Let p be the intersection point of the segments σ_1 and σ_2. Then $\zeta_p(t) = (1 - t^l)$. We then get that

$$F_\tau = \frac{(1-t^l)(1-t^{\frac{aml}{g_\tau}})^{g_\tau}}{(1-t^{\frac{aml}{g_\tau g_{\sigma_1}}})^{\frac{g_\tau g_{\sigma_1}}{a}}(1-t^{\frac{aml}{g_\tau g_{\sigma_2}}})^{\frac{g_\tau g_{\sigma_2}}{m}}}.$$

By using Lemma 7 we can conclude that always

$$\frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m} \leq \gcd(l-n,m)\cdot\gcd(l-c,a) + 1$$
$$\leq g_\tau + 1$$

and that

$$\frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m} \leq g_\tau$$

whenever $\frac{g_\tau g_{\sigma_1}}{a} \neq 1$ and $\frac{g_\tau g_{\sigma_2}}{m} \neq 1$. If $g_\tau \geq \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m}$, then clearly F_τ is a polynomial. Suppose now that $\frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m} = g_\tau + 1$ and that $\frac{g_\tau g_{\sigma_1}}{a} = \gcd(l-n,m) = 1$ (which also implies that $g_{\sigma_2} = m$). To show that F_τ is a polynomial, we prove that a root of both factors in the denominator is also a root of $(1-t^l)$. We show that $\gcd\left(\frac{aml}{g_\tau g_{\sigma_1}}, \frac{aml}{g_\tau g_{\sigma_2}}\right) = l$.

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Now $\gcd\left(\frac{am}{g_\tau g_{\sigma_1}}, \frac{am}{g_\tau g_{\sigma_2}}\right) = \gcd(m, a/g_\tau) = \gcd(g_{\sigma_2}, g_{\sigma_1})$ and this last greatest common divisor is equal to 1 by definition of g_τ.

<u>Part B.</u> Now we investigate when $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ. When τ is a B_1-simplex, we may suppose $a = 1$. This implies $g_\tau = \gcd(l - n, m)$, $g_{\sigma_1} = 1$ and $\frac{g_\tau g_{\sigma_2}}{m} = 1$, which leads to $F_\tau = 1$. So from now on, we suppose $a, m > 1$. The candidate pole $s_0 = -\nu(\tau)/N(\tau)$ is equal to

$$-\frac{(m(l - c) + a(l - n) + am)/g_\tau}{aml/g_\tau}.$$

The number $e^{-2\pi i \nu(\tau)/N(\tau)}$ is certainly a zero of F_τ if $g_\tau > \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m}$. Let us see what happens when $g_\tau \leq \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{m}$. One can check from the definitions of g_τ and g_{σ_1} that $e^{2\pi i s_0}$ is a zero of the first factor of the denominator if and only if $g_{\sigma_1} = 1$. We also have the analogous statement for the second factor. Since $a, m > 1$ the number $e^{2\pi i s_0}$ is a zero of F_τ if this number is a zero of at most one of the two factors of the denominator. If $g_{\sigma_1} = g_{\sigma_2} = 1$ we are in the situation $g_\tau \leq \frac{g_\tau}{a} + \frac{g_\tau}{m}$. So $a = m = 2$. First note that then $g_\tau/2 = \gcd(l - n, 2)$ and $g_\tau/2 = \gcd(l - c, 2)$. We conclude that $\gcd(l - n, 2) = \gcd(l - c, 2)$ and thus $(l - n) + (l - c)$ is even. The candidate pole s_0 is equal to

$$-\frac{2(l - c) + 2(l - n) + 4}{4l}.$$

Since the numerator is divisible by 4, the number $e^{2\pi i s_0}$ is a zero of $(1 - t^l)$, which proves our result.

We conclude that when τ is not a B_1-simplex, then $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ. When τ is a B_1-simplex, then $F_\tau = 1$.

CASE 4: τ CONTAINS THREE SEGMENTS IN COORDINATE PLANES.

<u>Part A.</u> Let τ be the facet with vertices $(a, 0, 0), (0, 0, l)$ and $(0, m, 0)$. Let σ_1 be the segment with vertices $(0, 0, l)$ and $(0, m, 0)$, let σ_2 be the segment with vertices $(0, 0, l)$ and $(a, 0, 0)$ and let σ_3 be the segment with vertices $(0, m, 0)$ and $(a, 0, 0)$. This notation is illustrated in Figure 7.
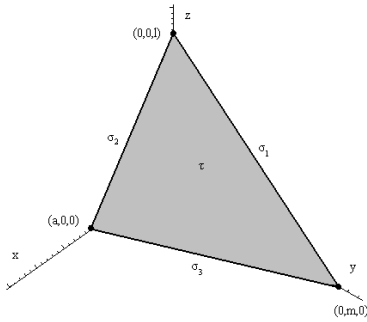

Figure 7

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The equation of τ is given by

$$x(ml) + y(al) + z(am) = aml.$$

We define $g_\tau, g_{\sigma_1}, g_{\sigma_2}$ and g_{σ_3} such that

$$g_\tau = \gcd(ml, al, am), \quad \frac{g_\tau g_{\sigma_1}}{a} = \gcd(l, m), \quad \frac{g_\tau g_{\sigma_2}}{l} = \gcd(a, m), \quad \frac{g_\tau g_{\sigma_3}}{m} = \gcd(a, l). \quad (3)$$

We then get that

$$F_\tau = \frac{(1 - t^l)(1 - t^m)(1 - t^a)(1 - t^{\frac{aml}{g_\tau}})^{g_\tau}}{(1 - t^{\frac{aml}{g_\tau g_{\sigma_1}}})^{\frac{g_\tau g_{\sigma_1}}{a}}(1 - t^{\frac{aml}{g_\tau g_{\sigma_2}}})^{\frac{g_\tau g_{\sigma_2}}{l}}(1 - t^{\frac{aml}{g_\tau g_{\sigma_3}}})^{\frac{g_\tau g_{\sigma_3}}{m}}}.$$

This is certainly a polynomial, since it is equal to the zeta function of monodromy of an isolated singularity.

<u>Part B.</u> If a, m or l equals 1, the facet is a B_1-simplex. Suppose $a = 1$. Then $g_\tau = \gcd(l, m)$, which implies $g_{\sigma_1} = 1$. Moreover we know $\frac{g_\tau g_{\sigma_2}}{l} = \frac{g_\tau g_{\sigma_3}}{m} = 1$. All this leads to $F_\tau = 1 - t$. Note that this could be known in advance, since in this case Γ_0 is the Newton polyhedron of a polynomial with a regular point in the origin. From now on, we suppose $a, l, m > 1$. We investigate whether $e^{-2\pi i\nu(\tau)/N(\tau)}$ is always a zero of F_τ. The candidate pole $s_0 = -\nu(\tau)/N(\tau)$ is equal to

$$-\frac{(lm + al + am)/g_\tau}{aml/g_\tau} = -\left(\frac{1}{a} + \frac{1}{m} + \frac{1}{l}\right).$$

When $g_\tau > \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{l} + \frac{g_\tau g_{\sigma_3}}{m}$, then $e^{2\pi i s_0}$ is certainly a zero of F_τ. First we claim for $j = 1, 2, 3$: the complex number $e^{2\pi i s_0}$ is a zero of the jth factor of the denominator if and only if $g_{\sigma_j} = 1$. This follows from the definitions of the greatest common divisors. Suppose now that $g_\tau = \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{l} + \frac{g_\tau g_{\sigma_3}}{m}$. It can only happen that $e^{2\pi i s_0}$ is not a zero of F_τ if $e^{2\pi i s_0}$ is a zero of the three factors in the denominator. This implies that $g_{\sigma_1} = g_{\sigma_2} = g_{\sigma_3} = 1$. On the other hand, we then have $\frac{1}{a} + \frac{1}{l} + \frac{1}{m} = 1$ and this means that $e^{2\pi i s_0} = 1$ is a zero of every factor in the numerator. In this way we find that $e^{2\pi i s_0}$ is a (multiple) zero of F_τ. So we are left with the case that $g_\tau < \frac{g_\tau g_{\sigma_1}}{a} + \frac{g_\tau g_{\sigma_2}}{l} + \frac{g_\tau g_{\sigma_3}}{m}$. If $e^{2\pi i s_0}$ is no zero of any factor of the denominator, then we are done. If only $g_{\sigma_1} = 1$, then we use $a > 1$ to conclude that $e^{2\pi i s_0}$ is a zero of F_τ. So we also suppose $g_{\sigma_2} = 1$.

- If $g_{\sigma_3} \neq 1$, the only situation where $e^{2\pi i s_0}$ might be canceled as a zero of F_τ, is when $g_\tau \leq g_\tau/a + g_\tau/l$. This implies $a = l = 2$ and $g_\tau = g_\tau/a + g_\tau/l$. In this case we find $s_0 = -\frac{m+1}{m}$ and moreover we see that $e^{2\pi i s_0}$ is a zero of the factor $(1 - t^m)$ of the numerator.

- If also $g_{\sigma_3} = 1$, we have $1 < \frac{1}{a} + \frac{1}{l} + \frac{1}{m}$. We may suppose $a = 2$. The equations (3) are now

$$g_\tau = 2 \gcd(l, m), \qquad g_\tau = l \gcd(2, m), \qquad g_\tau = m \gcd(2, l).$$

We can easily see that l and m can not be both odd. So suppose l is even. If also m is even, then $g_\tau = 2m = 2l$. We know that then the condition $1 < \frac{1}{2} + \frac{2}{m}$

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should be satisfied. Since m is even, we are in the situation $a = m = l = 2$. One calculates that then

$$F_\tau = \frac{(1-t^2)^3(1-t^2)^4}{(1-t^2)^2(1-t^2)^2(1-t^2)^2} = 1 - t^2,$$

which leads to the fact that $e^{2\pi i s_0} = e^{-3\pi i} = -1$ is a zero of F_τ.

If m is odd, we have $g_\tau = 2m = l$. In this case we can calculate

$$\begin{aligned} F_\tau &= \frac{(1-t^{2m})(1-t^m)(1-t^2)(1-t^{2m})^{2m}}{(1-t^{2m})^m(1-t^{2m})(1-t^{2m})^2} \\ &= (1-t^m)(1-t^2)(1-t^{2m})^{m-2}. \end{aligned}$$

Since $m \geq 3$, the candidate pole $s_0 = -\frac{m+3}{2m}$ induces a zero $e^{2\pi i s_0}$ of F_τ.

We conclude that $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of F_τ, except when τ is a B_1-simplex. In that case $F_\tau = (1-t)$. ∎

To prove Theorem 4 for f a polynomial in three variables, we need the following lemma about nondegeneracy.

Lemma 9. *If a complex polynomial $f(x, y, z)$ is nondegenerate with respect to its Newton polyhedron in the origin, then for almost all $k \in \mathbb{C}$ the polynomial $f(x, y, z - k)$ is nondegenerate with respect to its Newton polyhedron in the origin. (Analogously for the variables x and y.)*

Proof. We will denote the Newton polyhedron of $f(x, y, z - k)$, for $k \in \mathbb{C}$, by Γ_k. Notice that for almost all k, the polyhedron Γ_k is the projection of Γ_0 on the plane $z = 0$ times $\mathbb{R}_{\geq 0}$ and thus is the same for these k.

Suppose we have an infinite number of values of k for which the polynomial $f(x, y, z)$ is degenerate with respect to its Newton polyhedron in the point $(0, 0, k)$. As the Newton polyhedron only has a finite number of faces, there exists a compact line segment σ of Γ_k such that there are infinitely many $k \in \mathbb{C}$ for which g_σ has a singular point outside the coordinate axes. We now consider also k as a variable, so we look at $g_\sigma \in \mathbb{C}[x, y, k]$. This equation is quasi-homogeneous in x and y. If $\mathrm{Aff}(\sigma) \leftrightarrow bx + ay = c$, then we substitute x by u^a and y by v^b. Then $g_\sigma(u, v, k)$ is homogeneous in u and v. We dehomogenize by setting $v = 1$ and we decompose $g_\sigma(u, 1, k)$ into its irreducible components: $g_\sigma(u, 1, k) = \prod_{i=1}^p g_i^{m_i}(u, 1, k)$. As there are infinitely many values for k for which $g_\sigma(u, 1, k)$ has a singular point outside $u = 0$, it follows that there exists at least one component $g_i \in \{g_1, \dots, g_p\}$ that is not a monomial and for which the exponent m_i is greater than or equal to 2. Say that g_1 satisfies these conditions. We now homogenize again and we substitute u^a by x and v^b by y. If we substitute k in $g_\sigma(x, y, k)$ by the variable z, then for the non-compact facet τ' of Γ_0 that projects onto σ we have that $f_{\tau'}(x, y, z) = g_\sigma(x, y, z) = \prod_{i=1}^p g_i^{m_i}(x, y, z)$. This notation is illustrated in Figure 8.

Let us now look at the factor $g_1^{m_1}(x, y, z)$ that is quasi-homogeneous in x and y. We write $g_1 = M_1 + \cdots + M_d$ with the $M_i, 1 \leq i \leq d$, monomials in $\mathbb{C}[x, y, z]$. Let $N_1, \dots, N_e$ be the monomials in $\mathbb{C}[x, y, z]$ such that $f_{\tau'} = g_1^{m_1}(N_1 + \cdots + N_e)$. We
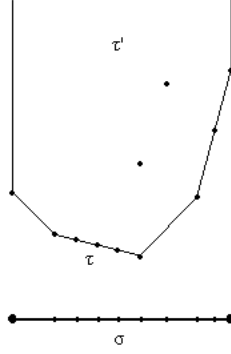
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Figure 8

can choose $T \subset \{1, \ldots, d\}$, $\#T \geq 2$, such that for every $i \in \{1, \ldots, e\}$, the support of $\{N_i M_t^{m_1}\}_{t \in T}$ is contained in a line ℓ_i and such that all support points of $\{N_i M_1^{m_1}, \ldots, N_i M_d^{m_1}\} \setminus \{N_i M_t^{m_1}\}_{t \in T}$ lie above ℓ_i. Note that $\ell_1, \ldots, \ell_e$ are parallel. Let ℓ be the lowest of the lines $\ell_1, \ldots, \ell_e$ and suppose $\ell_{i_1}, \ldots, \ell_{i_j} \subset \{\ell_1, \ldots, \ell_e\}$ coincide with ℓ. The support points of $f(x, y, z)$ on ℓ are exactly the support points of the polynomial $(N_{i_1} + \cdots + N_{i_j})(\sum_{t \in T} M_t)^{m_1}$. The line segment containing these support points is a compact face τ of Γ_0. The polynomial $f_\tau = (N_{i_1} + \cdots + N_{i_j})(\sum_{t \in T} M_t)^{m_1}$ has a singular point outside the coordinate planes. This implies that f is degenerate with respect to Γ_0 and leads to a contradiction. ∎

We can now prove Theorem 4 for a polynomial in three variables.

Theorem 10. *Let f be a complex polynomial in three variables. Suppose that f is nondegenerate with respect to its Newton polyhedron Γ_0 in the origin. Then for every facet τ of Γ_0 that is not a B_1-facet, the complex number $e^{-2\pi i \nu(\tau)/N(\tau)}$ is an eigenvalue of monodromy of f at a point in the neighbourhood of the origin.*

Proof. First suppose τ is compact. If every line segment of the Newton polyhedron that is contained in a coordinate plane is also contained in a compact facet, then we know from Equation (2) and Proposition 8 that the zeta function of monodromy at the origin is a product of polynomials.

If τ is a compact facet that is no union of B_1-simplices, then Proposition 8 says that $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a root of it. If τ is a compact facet which is a union of different B_1-simplices, then - up to permutation of the coordinates - the facet τ should have vertices $(a, 0, 0), (c, d, 0), (b, 1, 1)$ and $(e, 0, f)$ for $a, \ldots, f \in \mathbb{Z}_{\geq 0}$. One can verify that the candidate pole contributed by τ is equal to $-(1 - b + a)/a$. The factor $\zeta_{(a,0,0)} = (1 - t^a)$ was not used in any F_τ. From Equality (2) we deduce that $e^{-2\pi i(1-b+a)/a}$ is an eigenvalue of monodromy in the origin.

Let σ be a line segment in a coordinate plane, say in $z = 0$, but which is not contained in any compact facet. We suppose moreover that $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a pole of F_σ, where we use the notation F_σ as if σ was a facet of a two-dimensional Newton polyhedron in the plane $z = 0$. (Thus $F_\sigma = (\zeta_\sigma)^{-1}$, maybe multiplied with ζ_p if σ contains a point p on a coordinate axis.) If σ would be a B_1-simplex in the coordinate plane $z = 0$, then $F_\sigma = 1$, so this is also excluded. Now we choose a $k \in \mathbb{C}$ close to zero

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such that $f(x, y, z + k)$ is nondegenerate with respect to its Newton polyhedron in the origin (see Lemma 9). The support points of this new polynomial are the projections on the plane $z = 0$ of the old support points. So the monodromy zeta function of $f(x, y, z)$ at the point $(0, 0, -k)$ is equal to the monodromy zeta function of a polynomial in two variables with this projected polyhedron as its Newton polyhedron. From Proposition 5 we deduce that a pole of F_σ is also a pole of this monodromy zeta function.

Suppose now that τ is not compact, say for the variable z, and that it is no B_1-facet. So it has an equation of the form $ax + by = N$ for $a, b, N \in \mathbb{Z}$. Then we choose again a value $k \in \mathbb{C}$ for which $f(x, y, z + k)$ is nondegenerate with respect to its Newton polyhedron in the origin. Analogously as in the previous paragraph, we use the zeta function of monodromy of a polynomial in two variables. We know that there is a facet of the two-dimensional Newton polyhedron that has equation $ax + by = N$. Since this facet is no B_1-simplex, we conclude from Theorem 6 that $e^{-2\pi i \nu(\tau)/N(\tau)}$ is an eigenvalue of monodromy of f at a point in the neighbourhood of $(0, 0, -k)$. ∎

4. The monodromy conjecture for nondegenerate singularities in dimension 2 and 3

4.1. On false poles contributed by facets and the monodromy conjecture in dimension 2.

The poles of the local topological zeta function can be determined out of the resolution graph (see [Ve95, Theorem 4.3]). In [LVa09] we showed that, for nondegenerate plane curves, these poles are not so special in the sense that they are not selected. Indeed, all facets that are not B_1-simplices induce a pole of the local topological zeta function. Maybe this also offers some explanation why relations with for instance jumping numbers are not so surprising in dimension 2.

Theorem 11. *[LVa09, Theorem 3] Let f be a complex polynomial in two variables. Suppose that f is nondegenerate with respect to its Newton polyhedron Γ_0. Then for a candidate pole $s_0 \neq -1$ contributed by some facet it holds:*

$$s_0 \text{ is a pole of } Z_{top,f}$$
$$\Updownarrow$$
$$s_0 \text{ is contributed by a facet that is no } B_1\text{-simplex.}$$

Theorem 6 and Theorem 11 easily lead to a proof of the monodromy conjecture for nondegenerate polynomials in two variables.

Theorem 12. *Let $f(x, y)$ be a complex polynomial that is nondegenerate with respect to its Newton polyhedron in the origin. If s_0 is a pole of the local topological zeta function of f, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of f at some point of $\{f = 0\}$ close to 0.*

In Subsection 3.3 we proved that all facets, except the B_1-facets, induce eigenvalues of monodromy at a point in the neighbourhood of the origin. With the monodromy conjecture in mind, we will now investigate candidate poles contributed by B_1-facets. We will show that in almost all cases candidate poles of the local topological zeta function that are only contributed by B_1-simplices are no actual poles. Indeed, it will seem that there is only one exception in dimension 3, namely a candidate pole contributed by various B_1-facets only with respect to different variables having a line segment in common. We will come back to this later. Note that we will not mind to skip the candidate pole -1, since we know that $e^{-2\pi i} = 1$ is always an eigenvalue of monodromy.

Denef proved the following theorem (with unpublished proof) about these B_1-simplices:

Theorem 13. *Let f be a polynomial over $\mathbb{Z}_p$ in n variables. Suppose that f is nondegenerate over $\mathbb{Q}_p$ with respect to the compact faces of its Newton polyhedron. Suppose that $s_0 \neq -1$ is the real part of a candidate pole of Igusa's local zeta function such that s_0 is only contributed by one B_1-simplex. Then there is no pole of the p-adic Igusa zeta function with real part equal to s_0.*

We prove the following proposition about candidate poles contributed by B_1-facets of a 3-dimensional Newton polyhedron.

Proposition 14. *Let f be a complex polynomial in three variables that is nondegenerate with respect to its Newton polyhedron. Suppose $s_0 \neq -1$ is a candidate pole of the local topological zeta function only contributed by facets that are B_1-facets with respect to the same variable or by B_1-facets with respect to possibly different variables having an intersection of dimension at most zero. Then s_0 is no pole.*

Proof. We will consider seven separate cases. We investigate all possible configurations of B_1-facets (compact and non-compact) contributing the same candidate pole $s_0 \neq -1$.

CASE 1: s_0 IS CONTRIBUTED BY EXACTLY ONE B_1-SIMPLEX.

Suppose s_0 is only contributed by the B_1-simplex τ. We fix notation as in Figure 9. The vertices A and B of τ are contained in the coordinate plane $z = 0$. The vertex
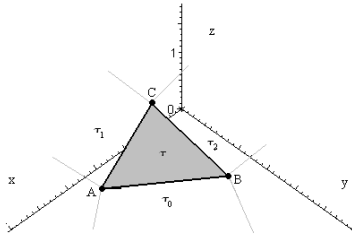

Figure 9

C lies at distance one. The other facets that contain the line segments AB, AC and

BC are respectively τ_0, τ_1 and τ_2. The facet τ_0 is not compact and is contained in the hyperplane $z = 0$. There might be other facets containing the vertices A, B or C. If for instance A is contained in more than three facets, the dual cone A° is not simplicial. However, we can take a subdivision of A° containing the cone Δ_A which we define as the cone spanned by the rays τ_0°, τ° and τ_1°. This is now the only simplicial cone in the subdivision of A° that contains τ°. The simplicial cones Δ_B and Δ_C are defined analogously. We introduce the following notation:

$$
\begin{aligned}
\mathrm{Aff}(\tau) &\leftrightarrow a_\tau^x x + a_\tau^y y + a_\tau^z z = N(\tau), \\
\mathrm{Aff}(\tau_1) &\leftrightarrow a_{\tau_1}^x x + a_{\tau_1}^y y + a_{\tau_1}^z z = N(\tau_1), \\
\mathrm{Aff}(\tau_2) &\leftrightarrow a_{\tau_2}^x x + a_{\tau_2}^y y + a_{\tau_2}^z z = N(\tau_2),
\end{aligned}
$$

where we suppose that the coefficients of the equations of the facets are relatively prime. Note that $\nu(\tau) = a_\tau^x + a_\tau^y + a_\tau^z, \nu(\tau_1) = a_{\tau_1}^x + a_{\tau_1}^y + a_{\tau_1}^z$ and $\nu(\tau_2) = a_{\tau_2}^x + a_{\tau_2}^y + a_{\tau_2}^z$. Following Theorem 1 we need to consider the following terms of the topological zeta function and prove that $-\nu(\tau)/N(\tau)$ is no pole of this sum.

$$
\begin{aligned}
&\frac{\mathrm{mult}(\Delta_A)}{(N(\tau)s + \nu(\tau))(N(\tau_1)s + \nu(\tau_1))} + \frac{\mathrm{mult}(\Delta_B)}{(N(\tau)s + \nu(\tau))(N(\tau_2)s + \nu(\tau_2))} \\
&+ \frac{\mathrm{mult}(\Delta_C)}{(N(\tau)s + \nu(\tau))(N(\tau_2)s + \nu(\tau_2))(N(\tau_1)s + \nu(\tau_1))} \\
&- \frac{s}{(s+1)}\frac{\mathrm{mult}(AC^\circ)\mathrm{Vol}(AC)}{(N(\tau)s + \nu(\tau))(N(\tau_1)s + \nu(\tau_1))} - \frac{s}{(s+1)}\frac{\mathrm{mult}(BC^\circ)\mathrm{Vol}(BC)}{(N(\tau)s + \nu(\tau))(N(\tau_2)s + \nu(\tau_2))} \\
&- \frac{s}{(s+1)}\frac{\mathrm{mult}(AB^\circ)\mathrm{Vol}(AB)}{(N(\tau)s + \nu(\tau))} + \frac{s}{(s+1)}\frac{\mathrm{mult}(\tau^\circ)\,2!\,\mathrm{Vol}(\tau)}{(N(\tau)s + \nu(\tau))}.
\end{aligned}
$$

To simplify this expression, we need a few easy results.

i) $\gcd(a_\tau^x, a_\tau^y) = 1, \gcd(a_{\tau_1}^x, a_{\tau_1}^y) = 1, \gcd(a_{\tau_2}^x, a_{\tau_2}^y) = 1$. The three facets τ, τ_1, τ_2 all contain two points of the form $(\cdot, \cdot, 0)$ and $(\cdot, \cdot, 1)$. Putting these in the equations, shows that there exist integers k, l, m, n such that

$$
a_\tau^z = ka_\tau^x + la_\tau^y, \quad a_{\tau_1}^z = ka_{\tau_1}^x + la_{\tau_1}^y, \quad a_\tau^z = ma_\tau^x + na_\tau^y, \quad a_{\tau_2}^z = ma_{\tau_2}^x + na_{\tau_2}^y.
$$

So a common divisor of the first two coefficients is also a divisor of the third one. Since the three coefficients are relatively prime, this leads to this first result.

ii) $\mathrm{mult}(\tau^\circ) = \mathrm{mult}(AB^\circ) = 1$. For the first multiplicity it is obvious that the multiplicity is equal to one. For the second, one can use the previous item.

iii) $2!\,\mathrm{Vol}(\tau) = \mathrm{Vol}(AB)$. The normalized volume of τ is equal to the quotient of the determinant of the matrix with the vertices and $N(\tau)$. The normalized volume of AB can be computed in the hyperplane $z = 0$. So we need the determinant of the matrix with the first two entries of A and B and divide this by the constant term of the equation of AB in $z = 0$. We use the first result again to see that this constant term is equal to $N(\tau)$. The two mentioned determinants are obviously equal.

iv) $\mathrm{Vol}(AC) = \mathrm{Vol}(BC) = 1$. This is immediate from the definition of volume.

v) $\mathrm{mult}(\Delta_A) = \mathrm{mult}(AC^\circ), \mathrm{mult}(\Delta_B) = \mathrm{mult}(BC^\circ)$. We can compute the multiplicity of Δ_A as

$$\mathrm{mult}(\Delta_A) = \begin{vmatrix} 0 & 0 & 1 \\ a_{\tau_1}^x & a_{\tau_1}^y & a_{\tau_1}^z \\ a_\tau^x & a_\tau^y & a_\tau^z \end{vmatrix}.$$

Using the equalities stated in i), this is equal to

$$\mathrm{mult}(AC^\circ) = \gcd\left(\begin{vmatrix} a_{\tau_1}^x & a_{\tau_1}^y \\ a_\tau^x & a_\tau^y \end{vmatrix}, \begin{vmatrix} a_{\tau_1}^x & a_{\tau_1}^z \\ a_\tau^x & a_\tau^z \end{vmatrix}, \begin{vmatrix} a_{\tau_1}^y & a_{\tau_1}^z \\ a_\tau^y & a_\tau^z \end{vmatrix} \right).$$

The second equality is analogous.

We use all this to simplify the part of the local topological zeta function written down above and find:

$$\frac{\mathrm{mult}(\Delta_A)}{(s+1)(N(\tau)s + \nu(\tau))(N(\tau_1)s + \nu(\tau_1))} + \frac{\mathrm{mult}(\Delta_B)}{(s+1)(N(\tau)s + \nu(\tau))(N(\tau_2)s + \nu(\tau_2))}$$
$$+ \frac{\mathrm{mult}(\Delta_C)}{(N(\tau)s + \nu(\tau))(N(\tau_2)s + \nu(\tau_2))(N(\tau_1)s + \nu(\tau_1))}.$$

To end the proof of this first case, one has to show that $N(\tau)s + \nu(\tau)$ is a divisor of

$$\mathrm{mult}(\Delta_A)(N(\tau_2)s + \nu(\tau_2)) + \mathrm{mult}(\Delta_B)(N(\tau_1)s + \nu(\tau_1)) + \mathrm{mult}(\Delta_C)(s+1).$$

This can be done by explicit computations using a computer algebra package.

CASE 2: s_0 IS CONTRIBUTED BY EXACTLY ONE B_1-FACET AND THIS B_1-FACET IS NOT COMPACT.

Suppose s_0 is only contributed by the non-compact B_1-facet τ. Say that τ is non-compact for the variable z and that it is a B_1-facet with respect to the variable y. We fix notation as in Figure 10.
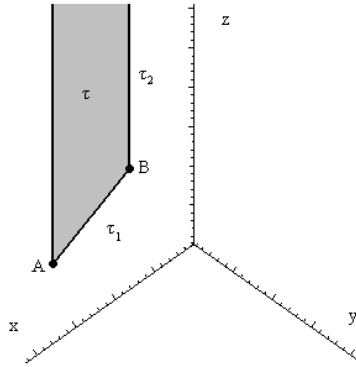

Figure 10

The vertex A is contained in the coordinate plane $y = 0$ and the vertex B is lying at a distance one from the hyperplane $y = 0$. We define the facet τ_1 as the facet that has the segment AB in common with τ. The non-compact facet that contains the vertex B and that has a half-line in common with τ will be called τ_2.

There might be other facets that contain the vertices A or B. However, we can investigate whether $-\nu(\tau)/N(\tau)$ is a pole of $Z_{top,f}$ by taking only the terms of $Z_{top,f}$ into account which are provided by the cone AB°, by the cone Δ_A spanned by the rays $\tau^\circ, (0,1,0), \tau_1^\circ$ and by the cone Δ_B spanned by the rays τ°, τ_1° and τ_2°:

$$-\frac{s}{(s+1)}\frac{\mathrm{mult}(AB^\circ)\mathrm{Vol}(AB)}{(N(\tau)s+\nu(\tau))(N(\tau_1)s+\nu(\tau_1))} + \frac{\mathrm{mult}(\Delta_A)}{(N(\tau)s+\nu(\tau))(N(\tau_1)s+\nu(\tau_1))}$$
$$+\frac{\mathrm{mult}(\Delta_B)}{(N(\tau)s+\nu(\tau))(N(\tau_2)s+\nu(\tau_2))(N(\tau_1)s+\nu(\tau_1))}.$$

As in Case 1 one finds that $\mathrm{mult}(\Delta_A) = \mathrm{mult}(AB^\circ)$ and $\mathrm{Vol}(AB) = 1$. If we set $A(k,0,l)$ and $B(m,1,n)$ and if we write

$$\begin{aligned}
\mathrm{Aff}(\tau) &\leftrightarrow x + (k-m)y = k, \\
\mathrm{Aff}(\tau_1) &\leftrightarrow a_{\tau_1}^x x + a_{\tau_1}^y y + a_{\tau_1}^z z = a_{\tau_1}^x k + a_{\tau_1}^z l = a_{\tau_1}^x m + a_{\tau_1}^y + a_{\tau_1}^z n, \\
\mathrm{Aff}(\tau_2) &\leftrightarrow ax + by = am + b,
\end{aligned}$$

with $\gcd(a_{\tau_1}^x, a_{\tau_1}^y, a_{\tau_1}^z) = 1 = \gcd(a,b)$, then one also easily computes that $\mathrm{mult}(\Delta_A) = a_{\tau_1}^z$ and $\mathrm{mult}(\Delta_B) = a_{\tau_1}^z(a(k-m)-b)$. The contribution to the topological zeta function written above then becomes

$$\frac{a_{\tau_1}^z(N(\tau_2)s+\nu(\tau_2)) + a_{\tau_1}^z(a(k-m)-b)(s+1)}{(s+1)(N(\tau)s+\nu(\tau))(N(\tau_1)s+\nu(\tau_1))(N(\tau_2)s+\nu(\tau_2))}$$

This can be simplified to

$$\frac{aa_{\tau_1}^z}{(s+1)(N(\tau_1)s+\nu(\tau_1))(N(\tau_2)s+\nu(\tau_2))},$$

which implies that $-\nu(\tau)/N(\tau)$ is no pole.

CASE 3: s_0 IS CONTRIBUTED BY EXACTLY TWO B_1-SIMPLICES WITH RESPECT TO THE SAME VARIABLE, HAVING A LINE SEGMENT IN COMMON.

Suppose s_0 is only contributed by the B_1-simplices τ and σ. We fix notation as in Figure 11. The vertices A, B and C are contained in the coordinate plane $z = 0$. The vertex D lies at distance one. The other facets that contain the line segments AD and CD are respectively τ_1 and τ_2. The facet τ_0 is not compact and is contained in the hyperplane $z = 0$. For A° and C° we use an analogous subdivision as in Case 1. So we define Δ_A as the cone spanned by the rays τ_1°, τ° and τ_0° and Δ_C as the cone spanned by the rays $\tau_2^\circ, \sigma^\circ$ and τ_0°. The cone B° is simplicial, so $\Delta_B := B^\circ$. The vertex D is contained in at least four facets. We define Δ_1 as the cone spanned by τ_1°, τ° and σ° and Δ_2 as the cone spanned by $\sigma^\circ, \tau_1^\circ$ and τ_2°. This can be seen in Figure 12. Now the proof goes further in the same way as in Case 1. The terms in the local topological zeta function originating
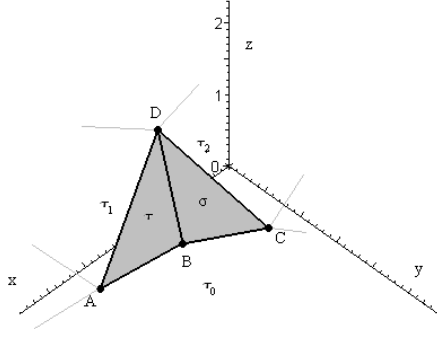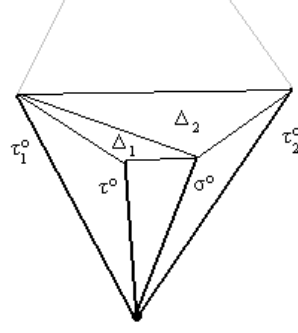
Figure 11   Figure 12

from AB and τ cancel. So do the terms coming from BC and σ. We find again that $\text{mult}(\Delta_A) = \text{mult}(AD^\circ), \text{mult}(\Delta_B) = \text{mult}(BD^\circ)$ and $\text{mult}(\Delta_C) = \text{mult}(CD^\circ)$. The relevant contribution to the topological zeta function is thus given by

$$\frac{\text{mult}(\Delta_A)}{(s+1)(N(\tau)s+\nu(\tau))(N(\tau_1)s+\nu(\tau_1))} + \frac{\text{mult}(\Delta_B)}{(s+1)(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))}$$
$$+ \frac{\text{mult}(\Delta_C)}{(s+1)(N(\sigma)s+\nu(\sigma))(N(\tau_2)s+\nu(\tau_2))} + \frac{\text{mult}(\Delta_1)}{(N(\tau)s+\nu(\tau))(N(\tau_1)s+\nu(\tau_1))(N(\sigma)s+\nu(\sigma))}$$
$$+ \frac{\text{mult}(\Delta_2)}{(N(\tau_2)s+\nu(\tau_2))(N(\tau_1)s+\nu(\tau_1))(N(\sigma)s+\nu(\sigma))}.$$

To conclude, one can calculate (using a computer algebra package) that the factors $(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))$ disappear when putting everything on the same denominator.

CASE 4: s_0 IS CONTRIBUTED BY EXACTLY TWO NON-COMPACT B_1-FACETS WITH RESPECT TO THE SAME VARIABLE, HAVING A LINE SEGMENT IN COMMON.

Suppose s_0 is only contributed by the B_1-facets τ and σ. Since σ and τ are B_1-facets with respect to the same variable, it is impossible that they have a non-compact half-line in common. We suppose τ to be non-compact for z and to be a B_1-facet for y. Let the segment AB be the intersection of τ and σ.

- If the B_1-facet σ is non-compact for y, then to be in the situation of Case 4 we have to ask that τ and σ both are B_1-facets for the variable x. The situation is pictured in Figure 13. Then $A(1,0,k)$ and $B(0,1,l)$ for $k,l \in \mathbb{Z}_{\geq 0}$. The equations of the planes through τ and σ are

$$\text{Aff}(\tau) \leftrightarrow x + y = 1, \qquad \text{Aff}(\sigma) \leftrightarrow (l-k)x + z = l.$$

Moreover, as τ and σ contribute the same candidate pole, one has that $l = 1$ and $k = 0$. This implies that the origin is not a singular point of $\{f = 0\}$ and thus $-\nu(\tau)/N(\tau) = -2$ is not a pole of $Z_{top,f}$.

- If the B_1-facet σ is non-compact for x, then the situation is as in Figure 14. We introduce $k,l,m,n \in \mathbb{Z}_{\geq 0}$ such that $A(k,0,l)$ and $B(m,1,n)$. Let τ_1 be the
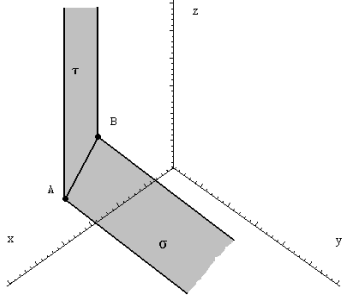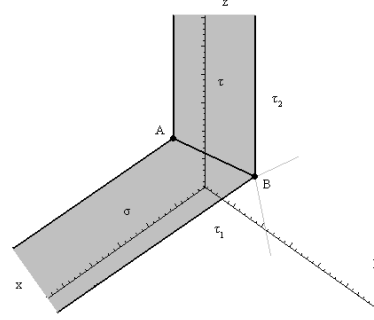
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Figure 13   Figure 14

non-compact facet containing the vertex B and sharing a half-line with σ and let τ_2 be the non-compact facet containing the vertex B and sharing a half-line with τ. Let Δ_A be the cone generated by the rays $(0,1,0), \tau^\circ, \sigma^\circ$, let Δ_{B1} be the cone spanned by the rays $\tau^\circ, \sigma^\circ, \tau_2^\circ$ and Δ_{B2} be the cone spanned by the rays $\tau_1^\circ, \sigma^\circ, \tau_2^\circ$. The relevant contribution to the topological zeta function is given by

$$\frac{-s\,\mathrm{mult}(AB^\circ)\mathrm{Vol}(AB)}{(s+1)(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))} + \frac{\mathrm{mult}(\Delta_A)}{(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))}$$
$$+\frac{\mathrm{mult}(\Delta_{B1})}{(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))(N(\tau_2)s+\nu(\tau_2))}$$
$$+\frac{\mathrm{mult}(\Delta_{B2})}{(N(\tau_2)s+\nu(\tau_2))(N(\tau_1)s+\nu(\tau_1))(N(\sigma)s+\nu(\sigma))}.$$

As in Case 1 we deduce that $\mathrm{mult}(AB^\circ) = \mathrm{mult}(\Delta_A) = 1$ and $\mathrm{Vol}(AB) = 1$. Moreover, τ and σ give rise to the same candidate pole, hence $k(n-1) = l(m-1)$. If $m = 1$, then also $n = 1$ (since we have $k \geq m$) and then $\nu(\tau)/N(\tau) = 1$. So suppose now that $m \neq 1$. Replacing l by $k(n-1)/(m-1)$, we find that in the above expression $(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))$ disappears when putting all terms on the same denominator.

CASE 5: s_0 IS CONTRIBUTED BY EXACTLY TWO B_1-FACETS - ONE OF THEM IS COMPACT AND THE OTHER ONE IS NOT COMPACT - WITH RESPECT TO THE SAME VARIABLE, HAVING A LINE SEGMENT IN COMMON.

Suppose that the facet τ is not compact for z and is a B_1-facet with respect to the variable y. Suppose that σ is a B_1-simplex that intersects τ in the segment AB and that induces the same candidate pole as τ. The facet σ should be a B_1-simplex with respect to x or to y to be in Case 5.

- Suppose that σ is a B_1-simplex for x. Then the situation is as in Figure 15, with $A(1,0,k), B(0,1,l)$ and $C(0,r,t)$. As τ and σ induce the same candidate pole, we find $k + l = 1$ and thus the origin is not a singular point of $\{f = 0\}$ and the only pole of the local topological zeta function is -1.
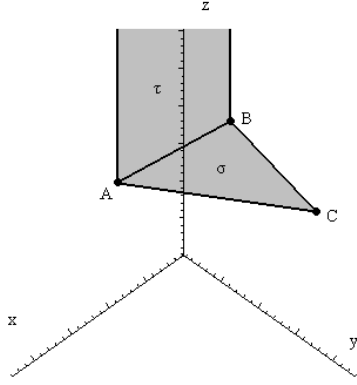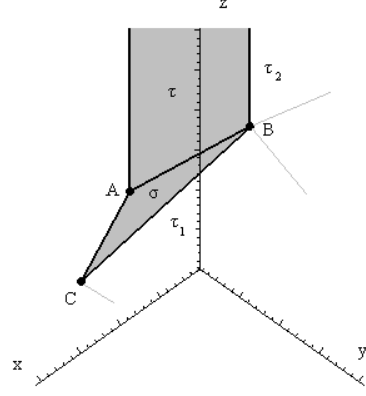
23

Figure 15　　　　　　Figure 16

- Suppose that σ is a B_1-simplex for y. This situation is represented in Figure 16.

  We take $k, l, m, n, r, t \in \mathbb{Z}_{\geq 0}$ such that $A(k, 0, l), B(m, 1, n)$ and $C(r, 0, t)$. Let τ_2 be the facet containing the vertex B and having a half-line in common with τ. Let τ_1 be a facet that intersects σ in the line segment BC. We write

$$
\begin{aligned}
\mathrm{Aff}(\tau) &\leftrightarrow x + (k - m)y = k, \\
\mathrm{Aff}(\sigma) &\leftrightarrow a_\sigma^x x + a_\sigma^y y + a_\sigma^z z = a_\sigma^x k + a_\sigma^z l \\
&= a_\sigma^x m + a_\sigma^y + a_\sigma^z n \\
&= a_\sigma^x r + a_\sigma^z t, \\
\mathrm{Aff}(\tau_1) &\leftrightarrow cx + dy + ez = cm + d + en = cr + et, \\
\mathrm{Aff}(\tau_2) &\leftrightarrow ax + by = am + b,
\end{aligned}
$$

with $\gcd(a_\sigma^x, a_\sigma^y, a_\sigma^z) = \gcd(c, d, e) = \gcd(a, b) = 1$.

Let Δ_A be the cone generated by the rays $(0, 1, 0), \tau^\circ, \sigma^\circ$, let Δ_{B1} be the cone spanned by the rays $\tau^\circ, \sigma^\circ, \tau_2^\circ$, let Δ_{B2} be the cone spanned by the rays $\tau_1^\circ, \sigma^\circ, \tau_2^\circ$ and let Δ_C be the cone spanned by $\tau_1^\circ, \sigma^\circ, (0, 1, 0)$.

The relevant contribution to the topological zeta function is

$$
\begin{aligned}
&\frac{2\mathrm{Vol}(\sigma)s}{(s+1)(N(\sigma)s + \nu(\sigma))} - \frac{s\,\mathrm{mult}(AB^\circ)\mathrm{Vol}(AB)}{(s+1)(N(\tau)s + \nu(\tau))(N(\sigma)s + \nu(\sigma))} \\
&- \frac{s\,\mathrm{mult}(BC^\circ)\mathrm{Vol}(BC)}{(s+1)(N(\tau_1)s + \nu(\tau_1))(N(\sigma)s + \nu(\sigma))} - \frac{s\,\mathrm{mult}(AC^\circ)\mathrm{Vol}(AC)}{(s+1)(N(\sigma)s + \nu(\sigma))} \\
&+ \frac{\mathrm{mult}(\Delta_A)}{(N(\tau)s + \nu(\tau))(N(\sigma)s + \nu(\sigma))} + \frac{\mathrm{mult}(\Delta_C)}{(N(\tau_1)s + \nu(\tau_1))(N(\sigma)s + \nu(\sigma))} \\
&+ \frac{\mathrm{mult}(\Delta_{B1})}{(N(\tau)s + \nu(\tau))(N(\sigma)s + \nu(\sigma))(N(\tau_2)s + \nu(\tau_2))} \\
&+ \frac{\mathrm{mult}(\Delta_{B2})}{(N(\tau_2)s + \nu(\tau_2))(N(\tau_1)s + \nu(\tau_1))(N(\sigma)s + \nu(\sigma))}.
\end{aligned}
$$

Analogously to Case 1, we find that

$$
\mathrm{mult}(AB^\circ) = \mathrm{mult}(\Delta_A) = a_\sigma^z, \ \mathrm{mult}(AC^\circ) = \gcd(a_\sigma^x, a_\sigma^z) = 1 = \mathrm{Vol}(AB) = \mathrm{Vol}(BC),
$$
$$
\mathrm{Vol}(AC) = 2\mathrm{Vol}(\sigma), \quad \mathrm{mult}(BC^\circ) = \mathrm{mult}(\Delta_C) = ea_\sigma^x - ca_\sigma^z.
$$

24

As τ and σ give rise to the same candidate pole (and thus the line through A and B intersects the diagonal of the first quadrant), we also have that $l(m-1) = k(n-1)$. If $m=1$, then also $n=1$ (because $k \geq m$ in this situation) and then $\nu(\tau)/N(\tau) = 1$. So suppose now that $m \neq 1$. Then we also use the following relations:

$$l = \frac{k(n-1)}{(m-1)}, \quad d = c(r-m) + e(t-n), \quad a_\sigma^y = a_\sigma^x(k-m) + a_\sigma^z(l-n), \quad a_\sigma^x = \frac{a_\sigma^z(t-l)}{(k-r)}$$

and we find that $(N(\tau)s+\nu(\tau))(N(\sigma)s+\nu(\sigma))$ disappears when putting everything on the same denominator.

CASE 6: s_0 IS CONTRIBUTED BY AT LEAST THREE B_1-FACETS WITH RESPECT TO THE SAME VARIABLE, HAVING LINE SEGMENTS IN COMMON.

The situation we study is as in Figure 17, where σ_1 and σ_2 might be non-compact.
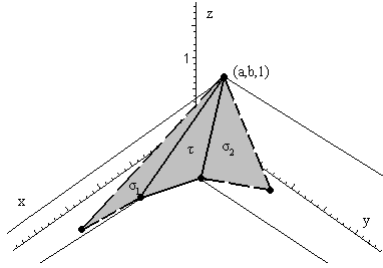

Figure 17

As the facets on the picture all contribute the same candidate pole, the diagonal of the first quadrant must contain the point $(a, b, 1)$. We conclude that this candidate pole is equal to -1.

CASE 7: s_0 IS CONTRIBUTED BY VARIOUS GROUPS OF B_1-FACETS AS IN THE PREVIOUS CASES, HAVING AT MOST A POINT IN COMMON.

In the previous cases we subdivided cones dual to the vertices in such a way that we can split the topological zeta function into parts corresponding to exactly one case. So Case 7 is an immediate consequence of the previous cases. ∎

4.3. THE MONODROMY CONJECTURE FOR NONDEGENERATE SURFACE SINGULARITIES.

In Theorem 10 we showed that every candidate pole contributed by a facet which is no B_1-facet induces an eigenvalue of monodromy and in Proposition 14 we showed that almost all candidate poles that are only contributed by B_1-facets are no actual poles. We can now finish our proof of the monodromy conjecture for nondegenerate singularities.

Theorem 15. *Let f be a polynomial in three variables which is nondegenerate with respect to its Newton polyhedron Γ_0. If s_0 is a pole of its local topological zeta function, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy at a point in the neighbourhood of the origin.*

Proof. It remains to study candidate poles that are contributed by two facets which are B_1-facets with respect to different variables and which have a 1-dimensional intersection. We may suppose that the two facets are no B_1-facets with respect to the same variable.

If the two B_1-facets have a half-line in common, say in the direction of the z-axis, then this half-line has equation $x = 1, y = 1$. The intersection point of this half-line with the diagonal of the first quadrant is the point $(1, 1, 1)$ and thus the candidate pole is -1.

If the common line segment is compact, one can check that the only possibility for this configuration is that this segment has vertices $(a, 0, 0)$ and $(b, 1, 1)$ for $a, b \in \mathbb{Z}_{\geq 0}$. The candidate pole contributed by these facets can be computed by searching the intersection point of the line through $(a, 0, 0)$ and $(b, 1, 1)$ with the diagonal of the first quadrant. This leads to the value $-\frac{1-b+a}{a}$. The complex number $e^{-2\pi i(1-b+a)/a}$ is an eigenvalue of monodromy, since the factor $\zeta_{(a,0,0)}$ does not show up in any F_τ and hence $(1 - t^a)$ is a factor of the zeta function of monodromy. (See Equation (2).) There might be a cancelation if $e^{-2\pi i(1-b+a)/a}$ is also a zero of some ζ_σ with σ a compact line segment which is not contained in any compact facet. However, in this case we can use the argument of the proof of Theorem 10 again to show that $e^{-2\pi i(1-b+a)/a}$ is an eigenvalue of monodromy at a point in the neighbourhood of the origin. This observation together with Theorem 10 and Proposition 14 lead to a proof of the monodromy conjecture for nondegenerate polynomials in three variables. ∎

5. Examples

Example 2. We study the polynomial $f(x, y, z) = x^{12} + y^{13} + z^{14} + xy^7 + y^2z^4 + x^2y^2z^3$. The picture of the Newton polyhedron in the origin of f is given in Figure 18. The polynomial is nondegenerate and its topological zeta function is

$$Z_{top,f}(s) = \frac{23310s^3 + 53211s^2 + 37410s + 8041}{(84s + 43)(s + 1)(42s + 17)(27s + 11)}.$$

We want to illustrate which facets induce eigenvalues of monodromy.

$\text{Aff}(\tau_1) \leftrightarrow x + 2y + 2z = 12$. This facet does not contain a line segment in a coordinate hyperplane. Hence $F_{\tau_1} = \zeta_{\tau_1} = (1 - t^{12})^2$. The contributed candidate pole $-5/12$ of the topological zeta function is not an actual pole. However, the facet is no B_1-simplex and $e^{-10\pi i/12}$ is an eigenvalue of monodromy.

$\text{Aff}(\tau_2) \leftrightarrow 7x + 30y + 6z = 84$. This facet contains two line segments in coordinate hyperplane. So we include these line segments and there intersection point in F_{τ_2} and find

$$F_{\tau_2} = \frac{(1 - t^{84})^4(1 - t^{14})}{(1 - t^{84})^2(1 - t^{14})^2} = \frac{(1 - t^{84})^2}{(1 - t^{14})}.$$

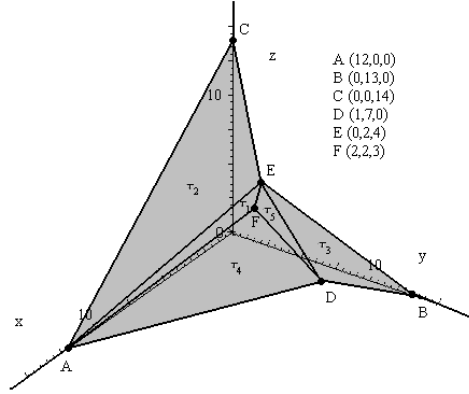This is a polynomial having $e^{-86\pi i/84}$ as a root.

Figure 18

$\mathrm{Aff}(\tau_3) \leftrightarrow 24x + 4y + 11z = 52$. Again, the polynomial F_{τ_3} is built with factors arising from the facet, two line segments and a vertex.

$$F_{\tau_3} = \frac{(1 - t^{52})(1 - t^{13})}{(1 - t^{13})(1 - t^{52})} = 1.$$

We see that the candidate pole contributed by τ_3 does not induce a zero of this polynomial. Notice that τ_3 a B_1-facet is. Indeed, its vertices are $(0, 13, 0), (0, 2, 4)$ and $(1, 7, 0)$.

$\mathrm{Aff}(\tau_4) \leftrightarrow 7x + 11y + 16z = 84$. The facet τ_4 contains one line segment in a coordinate hyperplane. Its contribution to the zeta function of monodromy is

$$F_{\tau_4} = \frac{(1 - t^{84})^3}{(1 - t^{84})} = (1 - t^{84})^2.$$

The candidate pole $-17/42$ (which is a pole) induces a zero of this polynomial.

$\mathrm{Aff}(\tau_5) \leftrightarrow 5x + 7y + 10z = 54$. The corresponding polynomial of τ_5 is equal to ζ_{τ_5}. So

$$F_{\tau_5} = 1 - t^{54}.$$

The pole $-11/27$ is contributed by τ_5 and $e^{-22\pi i/27}$ is a zero of F_{τ_5}.

The only vertex on a coordinate axis that belongs to more than one facet is the point $(12, 0, 0)$. To write down the monodromy zeta function of f in the origin, we take the product of $(1 - t^{12})$ with the factors F_{τ_i}. We conclude that

$$Z(f, 0)(t) = \frac{(1 - t^{12})^3(1 - t^{84})^4(1 - t^{54})}{1 - t^{14}}.$$

One can verify that the conditions for Theorem 3 are not satisfied for $\tau = \tau_4$. Indeed, take $\sigma = \tau_2$. Then $\beta(\tau, \sigma)$ is the greatest common divisor of the (2×2)-minors of the matrix

$$\begin{pmatrix} 7 & 11 & 16 \\ 7 & 30 & 6 \end{pmatrix},$$

which is equal to 1, and $\lambda(\tau, \sigma) = 43 - \frac{34}{84}84 = 9$. Hence, $\mu(\tau, \sigma)$ is an integer. $\square$

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The next example shows the situation when the Newton polyhedron contains a line segment in a coordinate plane which is not contained in a compact facet.

Example 3. We study the polynomial $f(x, y, z) = x^9y + x^2y^3 + xy^6z^2 + y^5z^5$. Its Newton polyhedron at the origin is pictured in Figure 19. It has 4 vertices: $A(9, 1, 0), B(2, 3, 0),$
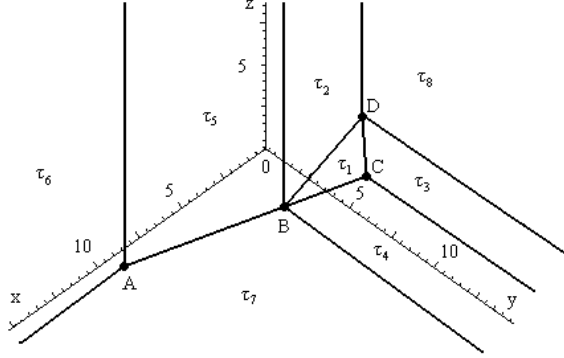

Figure 19

$C(1, 6, 2)$ and $D(0, 5, 5)$. The 8 facets have the following equations:

$$\begin{aligned}
\text{Aff}(\tau_1) &\leftrightarrow 11x + y + 4z = 25, & \text{Aff}(\tau_2) &\leftrightarrow x + y = 5, \\
\text{Aff}(\tau_3) &\leftrightarrow 3x + z = 5, & \text{Aff}(\tau_4) &\leftrightarrow 2x + z = 4, & \text{Aff}(\tau_5) &\leftrightarrow 2x + 7y = 25, \\
\text{Aff}(\tau_6) &\leftrightarrow y = 1, & \text{Aff}(\tau_7) &\leftrightarrow z = 0, & \text{Aff}(\tau_8) &\leftrightarrow x = 0.
\end{aligned}$$

Its local topological zeta function is

$$Z_{top,f}(s) = \frac{2685s^3 + 6026s^2 + 4110s + 864}{(5s + 2)(4s + 3)(25s + 9)(s + 1)(25s + 16)}.$$

The zeta function of monodromy at the origin is equal to $F_{\tau_1}(= \zeta_{\tau_1})$ divided by ζ_σ, where σ is the compact line segment AB. Both factors are equal to $(1 - t^{25})$, so the zeta function of monodromy at the origin is equal to 1. The facet τ_1 is no B_1-facet. It induces the pole $-16/25$ of the local topological zeta function. As in the proof of Theorem 10 we need to consider a complex number k such that $f(x, y, z - k)$ is nondegenerate with respect to its Newton polyhedron in the origin. This Newton
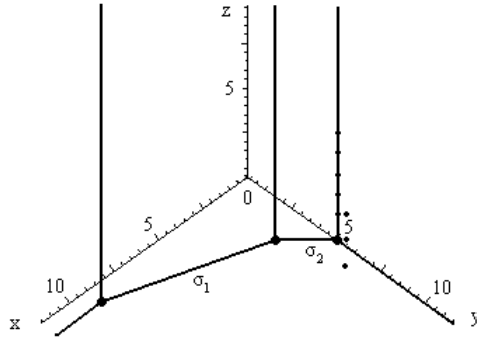

Figure 20

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polyhedron Γ_k is as in Figure 20. We have vertices $(9, 1, 0), (2, 3, 0)$ and $(0, 5, 0)$. The zeta function of monodromy of f in the point $(0, 0, k)$ is equal to

$$(\zeta_{\sigma_1})^{-1}(\zeta_{\sigma_2})^{-1}\zeta_{(0,5,0)} = \frac{(1 - t^5)}{(1 - t^5)^2(1 - t^{25})}.$$

We conclude that $e^{-32\pi i/25}$ is an eigenvalue of monodromy in a point in the neighbourhood of the origin. Moreover, note that the candidate pole $-4/5$, contributed by τ_3, is no pole. Indeed, this could be predicted since the facet τ_3 is a B_1-facet with respect to the variable x. □

References

[A'C75] N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helv. **50** (1975), 233-248.

[AVG86] V. Arnold, A. Varchenko and S. Goussein-Zadé, *Singularités des applications différentiables II,* Editions Mir, Moscou (1986).

[A-B,C-N,Lu,M-H02] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, A. Melle Hernández, *Monodromy conjecture for some surface singularities*, Ann. Scient. Ec. Norm. Sup. **35** (2002), 605-640.

[A-B,C-N,Lu,M-H06] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, A. Melle Hernández, *Quasi-ordinary power series and their zeta functions*, Memoirs of the A.M.S. **178**, no. 841 (2005).

[At70] M.F. Atiyah, *Resolution of singularities and division of distributions*, Comm. pure Appl. Math. **23** (1970), 145-150.

[BeGe69] I.N. Bernstein and S.I. Gel'fand, *Meromorphic property of the function P^λ*, Funct. Annal. Appl. **3** (1969), 68-69.

[D95] J. Denef, *Poles of p-adic Complex Powers and Newton Polyhedra,* Nieuw Archief voor Wiskunde (4), **13** (1995), 289-295.

[DL92] J. Denef and F. Loeser, *Caractéristique d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5, 4** (1992), 705-720.

[Ig75] J. Igusa, *Complex powers and asymptotic expansions I*, J. reine angew. Math. **268/269** (1974), 110-130; II, ibid **278/279** (1975), 307-321.

[Ig78] J. Igusa, *Lectures on forms of higher degree*, Tata Inst. Fund. Research, Bombay (1978).

[Ig88] J. Igusa, *B-functions and p-adic integrals*, Algebraic Analysis, Academic Press (1988), 231-241.

[LVa09] A. Lemahieu and L. Van Proeyen, *Poles of the topological zeta function for plane curves and Newton polyhedra*, C. R. Ac. Sci. **347** (2009), 637-642.

[LV09] A. Lemahieu and W. Veys, *Zeta functions and monodromy for surfaces that are general for a toric idealistic cluster*, International Mathematics Research Notices, No. 1 (2009) 1162.

[L88] F. Loeser, *Fonctions d'Igusa p-adiques et polynômes de Bernstein.* Amer. J. Math. **110** (1988), 1-22.

[L90] F. Loeser, *Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton*, J. reine angew. Math. **412** (1990), 75-96.

[Ma83] B. Malgrange, *Polynômes de Bernstein-Sato et cohomologie évanescente*, Astérisque **101/102** (1983), 243-267.

[Mi68] J. Milnor, *Singular points of complex hypersurfaces,* Ann. of Math. Studies Vol. **61**, Princeton University Press (1968).

[Ro04] B. Rodrigues, *On the monodromy conjecture for curves on normal surfaces*, Math. Proc. of the Cambridge Philosophical Society **136** (2004), 313-324.

[Var76] A.N. Varchenko, *Zeta-Function of Monodromy and Newton's Diagram,* Inventiones math. **37** (1976), 253-262.

[Ve95] W. Veys, *Determination of the poles of the topological zeta function for curves*, Manuscr. Math. **87** (1995), 435-448.

[We65] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1-87.