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Analysis of Output Frequencies of Nonlinear Systems

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***Abstract*—In this paper, an algorithm is derived for the determination of the output frequency ranges of nonlinear systems, which extends previous results on the output frequencies of nonlinear systems to a more general situation. The new results are significant for the analysis of the output frequency response of a wide class of nonlinear systems.**

***Index Terms*—Frequency response, nonlinear systems, output frequency range.**

I. Introduction

The frequency domain analysis of linear systems has been well established and widely studied in engineering systems. The study of nonlinear systems in the frequency domain was initiated in the late 1950s when the concept of generalized frequency response functions (GFRFs) was introduced [1]. The frequency domain approach for nonlinear systems is based on the Volterra series theory of nonlinear systems, and the GFRFs were defined as the multidimensional Fourier transformation of the Volterra kernels. Based on the GFRF concept, many results for the analysis of nonlinear systems in the frequency domain have been achieved [2]–[12]. Lang and Billings derived algorithms for computing the output frequencies/frequency ranges of nonlinear systems for both multiple and general inputs [6]—[8]. These results extend the well-known linear results where the output frequencies are the same as the frequencies of the input, to the nonlinear case, and indicate that the possible output frequencies of nonlinear systems are much richer than the frequencies of the input.

In order to extend these results to a more general case, this paper addresses the issue of the determination of the output frequency range of nonlinear systems when the system is subject to an input, the frequency components of which are located in a finite number of separate frequency intervals of different widths. Both an algorithm to compute the output frequency range and an explicit expression for the frequency ranges are derived.

The paper begins with an introduction of the output frequency response of nonlinear systems. This is followed by an overview of previous algorithms which provide the basis of this study. Then the new algorithm and an explicit expression for the output frequency range of nonlinear systems subject to inputs with frequency components located in a finite number of separate frequency intervals are derived. A simulation study is included to verify the effectiveness of the new results. Finally, some conclusions are drawn about the results achieved in the paper.

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II. Output Frequency Response of Nonlinear Systems

A. Output Spectrum

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series [13]

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \quad (1)$$

where $y(t)$ and $u(t)$ represent the system output and input respectively, $h_n(\tau_1,\ldots,\tau_n)$ is the nth-order Volterra kernel, and N denotes the maximum order of the system nonlinearities. In Lang and Billings [6], an expression for the output frequency response of the nonlinear systems was derived in a manner that reveals how the underlying nonlinear mechanisms operate on the input spectra to produce the system output frequency response, when the system is excited by the general input

$$\begin{aligned}u(t)&=\frac{1}{2\pi}\int_{0}^{\infty}2|U(j\omega)|\cos[\omega t+\angle U(j\omega)]d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)e^{j\omega t}d\omega.\end{aligned} \quad (2)$$

The result is given by

$$\begin{cases}\mathbf{Y}(j\omega)=\sum\limits_{n=1}^{N}\mathbf{Y}_n(j\omega)\ \forall\ \omega\\ \mathbf{Y}_n(j\omega)=\frac{\frac{1}{\sqrt{n}}}{(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}Y_n(j\omega_1,\ldots,j\omega_n)d\sigma_\omega\\ Y_n(j\omega_1,\ldots,j\omega_n)=H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)\end{cases} \quad (3)$$

where $\mathbf{Y}(j\omega)$ and $U(j\omega)$ represent the Fourier transforms of the system output and input, $\mathbf{Y}_n(j\omega)$ represents the system nth-order output frequency response, and

$$H_n(j\omega_1,\ldots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\ldots\tau_n)\times e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \quad (4)$$

is known as the nth-order GFRF, N is the maximum order of the system nonlinearity

$$\int_{\omega_1+\cdots+\omega_n=\omega}Y_n(j\omega_1,\ldots,j\omega_n)d\sigma_{\omega n} \quad (5)$$

denotes the integration of $Y_n(j\omega_1,\ldots,j\omega_n)$ over the n-dimensional hyperplane $\omega=\omega_1+\cdots+\omega_n$, and reveals the way in which the input spectrum makes a contribution, of degree n, to the output frequency component ω.

Equation (3) is a natural extension of the well-known linear relationship

$$Y(j\omega) = H(j\omega)U(j\omega) \tag{6}$$

to the nonlinear case, and compared with other results [11], [12], provides additional insight into the composition of the output frequency response of nonlinear systems. It is obvious that the nonlinear system output frequency response is much more complicated than in the linear case.

B. Output Frequencies

It is known from (3) that the possible output frequency range of a nonlinear system is the union of the frequency ranges produced by each order of the system nonlinearities

$$f_Y = \bigcup_{n=1}^{N} f_{Y_n} \tag{7}$$

where f_Y denotes the nonnegative frequency range of the system output and f_{Y_n} represents the nonnegative frequency range produced by the nth-order system nonlinearity.

For an input with spectrum described by

$$U(j\omega) = \begin{cases} U(j\omega), & \text{when } |\omega| \in [a,b] \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where

$$0 \le a < b < \infty \tag{9}$$

an algorithm was developed in [6] to compute the nonnegative parts of the output frequency range contributed by the nth order system nonlinearity such that (10), shown at the bottom of the page, where A' represents the transpose of matrix A and $\{\text{vector}\}$ denotes a set composed of the elements of the vector, $X(i,:)$ represents the ith row of matrix X, the two functions $f_1(\cdot)$ and $f_2(\cdot)$ are defined as

$$f_1(x,y) = \begin{cases} x, & \text{if } y \ge x \ge 0 \\ |y|, & \text{if } 0 \ge y \ge x \\ 0, & \text{if } y \ge 0 \ge x \end{cases} \tag{11}$$

and

$$f_2(x,y) = \begin{cases} y, & \text{if } y \ge x \ge 0 \\ |x|, & \text{if } 0 \ge y \ge x \\ \max(y,|x|), & \text{if } y \ge 0 \ge x \end{cases} \tag{12}$$

and

$$I_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{13}$$

Equations (7) and (10) can be used to numerically calculate the nonnegative output frequency range. A more transparent analytical relationship between f_Y and the input frequency range $[a,b]$ was derived in [7]. The result is (14), shown at the bottom of the page, where $p*$ can be taken as $1, 2, \ldots, \lfloor N/2 \rfloor$, the specific value of which depends on the system nonlinearities. If the system GFRFs $H_{N-(2i-1)}(\cdot) = 0$, for $i = 1, \ldots, q-1$, and $H_{N-(2q-1)}(\cdot) \ne 0$, then $p* = q$.

Results similar to the above for output frequencies of nonlinear systems were also reported in [15], where nonlinear filtering problems with communications systems were addressed.

$$\begin{cases} f_{y_n} = \bigcup_{i=1}^{2^{n-1}} \left[x_n^i, y_n^i\right] & n \ge 2 \\ \begin{bmatrix} x_n^i \\ y_n^i \end{bmatrix} = \begin{bmatrix} f_1\left(\min\{B_n A_n'(i,:)\}, \max\{B_n A_n'(i,:)\}\right) \\ f_2\left(\min\{B_n A_n'(i,:)\}, \max\{B_n A_n'(i,:)\}\right) \end{bmatrix} & \\ A_n = \begin{bmatrix} I_1 A_{n-1}(1,:) & I_2 \\ \vdots & \vdots \\ I_1 A_{n-1}(2^{n-2},:) & I_2 \end{bmatrix}, & A_1 = [1] \\ B_n = \begin{bmatrix} I_1 B_{n-1}(1,:) & B_1 \\ \vdots & \vdots \\ I_1 B_{n-1}(2^{n-1},:) & B_1 \end{bmatrix}, & B_1 = \begin{bmatrix} a \\ b \end{bmatrix} \end{cases} \tag{10}$$

$$\begin{cases} f_Y = f_{Y_N} \bigcup fY_{N-(2p^*-1)} & \\ fY_N = \begin{cases} \bigcup_{k=0}^{i*-1} I_k & \text{when } \frac{nb}{(a+b)} - \left\lfloor \frac{na}{(a+b)} \right\rfloor < 1 \\ \bigcup_{k=0}^{i*} I_k & \text{when } \frac{nb}{(a+b)} - \left\lfloor \frac{na}{(a+b)} \right\rfloor \ge 1 \end{cases} & \\ i^* = \left\lfloor \frac{na}{(a+b)} \right\rfloor + 1 & \lfloor\cdot\rfloor \text{ is an operand to take the integer part} \\ I_k = (na - k(a+b), nb - k(a+b)), & k = 0, \ldots, i^*-1 \\ I_{i^*} = (0, nb - i^*(a+b)) & \end{cases} \tag{14}$$

III. Analysis of Output Frequencies of Nonlinear Systems

The objective of this paper is to extend the results given by (7)–(14) to a more complicated case where system (1) is subject to a general input, the frequency components of which are located in a finite number of separate frequency intervals of different widths.

A. Computation of Non-Negative Output Frequency Ranges

Consider the case where system (1) is subject to an input with spectrum

$$U(j\omega) = \begin{cases} U(j\omega), & \text{when } |\omega| \in \bigcup_{i=1}^{m}[a_i, b_i] \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where

$$0 \le a_i < b_i < \infty, \quad i = 1, \dots, m \tag{16}$$

and

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m. \tag{17}$$

It is known from (3) that the frequency range of the nth-order nonlinear output should be determined by

$$\omega = \omega_1 + \cdots + \omega_n \tag{18}$$

with

$$\omega_l \in [-b_i, -a_i] \text{ or } [a_i, b_i] \quad i = 1, \dots, m, \;\; l = 1, \dots, n. \tag{19}$$

For the simplest case of $n = 1$, it is easy to show that the nonnegative output frequency range is $\bigcup_{i=1}^{m}[a_i, b_i]$, which follows exactly the linear system property. For the case of $n = 2$, considering $m = 2$, the frequency range can be written as

$$\omega = \omega_1 + \omega_2 \tag{20}$$

with

$$\omega_l \in [-b_i, -a_i] \text{ or } [a_i, b_i] \quad i = 1, 2, \;\; l = 1, 2. \tag{21}$$

From (20) and (21), the nonnegative output frequency range produced by second-order nonlinearity when the input spectrum is given by (15) can readily be obtained as

$$\begin{aligned} f_{Y_2} = & [2a_1, 2b_1] \cup [2a_2, 2b_2] \cup [0, b_1 - a_1] \\ & \cup [0, b_2 - a_2] \cup [a_1 + a_2, b_1 + b_2] \\ & \cup [a_2 - b_1, b_2 - a_1]. \end{aligned} \tag{22}$$

Based on the principle of deriving (22) from (20) and (21), the following general result can be obtained.

Proposition 1: The algorithm for evaluation of the nonnegative frequency range of the nth-order nonlinear output for general input (15) can be described as (23), shown at the bottom of the page, where $B_n(:, n(j-1)+1 : nj)$ denotes the rows of B_n, a $(2^n \times nm^n)$ matrix, with column number starting from $n(j-1)+1$ to nj, and

$$B_n = \begin{bmatrix} B'_n(:, 1 : nm) \\ B'_n(:, nm+1 : 2nm) \\ \vdots \\ B'_n(:, nm^n - nm + 1 : nm^n) \end{bmatrix}' \tag{24}$$

where the matrix block $B_n(:, (k_t \cdot nm - nm + 1) : k_t \cdot nm)$, $k_t = 1, 2, \dots, m^{n-1}$, can be written as

$$B_n(k_t) = [\, B^{(1)}(k_t) \quad B^{(2)}(k_t) \quad \cdots \quad B^{(m)}(k_t) \,] \tag{25}$$

with each subblock $B^{(i)}(k_t)$ being (26), shown at the bottom of the next page, where $i = 1, 2, \dots m$.

Equations (23)–(26) give the new algorithm to calculate the nonnegative output frequencies of system (1) under general inputs (15). The implementation of the algorithm which consists of (23)–(26) is straightforward using a matrix-oriented programming language such as MATLAB. The proof of Proposition 1 can be achieved by using the mathematical deduction approach. Although the basic idea for the proof is straightforward, the specific procedure involves much more complicated matrix manipulations. The details are, therefore, omitted due to space limitations.

$$\begin{cases} f_{Y_n} = \bigcup_{j=1}^{m^n} \bigcup_{i=1}^{2^{n-1}} \left[x_n^{i,j}, y_n^{i,j}\right] \\ \begin{bmatrix} x_n^{i,j} \\ y_n^{i,j} \end{bmatrix} = \begin{bmatrix} f_1\left(\min\left\{S_n^{i,j}\right\}, \max\left\{S_n^{i,j}\right\}\right) \\ f_2\left(\min\left\{S_n^{i,j}\right\}, \max\left\{S_n^{i,j}\right\}\right) \end{bmatrix} \\ S_n^{i,j} = B_n(:, n(j-1)+1 : nj) A'_n(i,:) \begin{cases} i = 1, \dots, 2^{n-1} \\ j = 1, \dots, m^n \end{cases} \\ A_n = \begin{bmatrix} I_1 A_{n-1}(1,:) & I_2 \\ \vdots & \vdots \\ I_1 A_{n-1}(2^{n-2},:) & I_2 \end{bmatrix}, & n \ge 2 \\ A_1 = [1], & B_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{bmatrix} \end{cases} \tag{23}$$

Remark 1: When $m = 1$ in Proposition 1, equation (23) is reduced to equation (10) indicating that the Proposition 1 is an extension of the previous result.

Remark 2: Consider the situation that $b_i = a_i + \varepsilon, i = 1, 2 \ldots, m$, B_1 can be written as

$$B_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ a_1+\varepsilon & a_2+\varepsilon & \cdots & a_m+\varepsilon \end{bmatrix}. \tag{27}$$

In this case, by taking $\varepsilon \to 0$, the results obtained by the algorithm can be viewed as the solution of the frequency range of the nth-order nonlinear output under a multitone input [6], [7].

For an illustration of the use of Proposition 1, consider an example where $a_1 = 0.3$, $b_1 = 0.5$, $a_2 = 1.0$, and $b_2 = 1.2$, which represents two separate frequency intervals of input frequencies.

From Proposition 1, $m = 2$ and the computation involves determination of

$$f_{Y_2} = \bigcup_{j=1}^{4}\bigcup_{i=1}^{2}\left[x_2^{i,j}, y_2^{i,j}\right]. \tag{28}$$

In this case, see equation (29) at the bottom of the page.

Therefore, for $j = 1, i = 1$

$$S_2^{1,1} = B_2(:,1:2)A_2'(1,:) = [0.6 \quad 0.8 \quad 0.8 \quad 1]$$

$$\begin{bmatrix} x_2^{1,1} \\ y_2^{1,1} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{1,1}\right\}, \max\left\{S_2^{1,1}\right\}\right] \\ f_2\left[\min\left\{S_2^{1,1}\right\}, \max\left\{S_2^{1,1}\right\}\right] \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}.$$

For $j = 1, i = 2$

$$S_2^{2,1} = B_2(:,1:2)A_2'(2,:) = [0 \quad -0.2 \quad 0.2 \quad 0]$$

$$\begin{bmatrix} x_2^{2,1} \\ y_2^{2,1} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{2,1}\right\}, \max\left\{S_2^{2,1}\right\}\right] \\ f_2\left[\min\left\{S_2^{2,1}\right\}, \max\left\{S_2^{2,1}\right\}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}.$$

For $j = 2, i = 1$

$$S_2^{1,2} = B_2(:,3:4)A_2'(1,:) = [1.3 \quad 1.5 \quad 1.5 \quad 1.7]$$

$$\begin{bmatrix} x_2^{1,2} \\ y_2^{1,2} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{1,2}\right\}, \max\left\{S_2^{1,2}\right\}\right] \\ f_2\left[\min\left\{S_2^{1,2}\right\}, \max\left\{S_2^{1,2}\right\}\right] \end{bmatrix} = \begin{bmatrix} 1.3 \\ 1.7 \end{bmatrix}.$$

For $j = 2, i = 2$,

$$S_2^{2,2} = B_2(:,3:4)A_2'(2,:) = [-0.7 \quad -0.9 \quad -0.5 \quad -0.7]$$

$$\begin{bmatrix} x_2^{2,2} \\ y_2^{2,2} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{2,2}\right\}, \max\left\{S_2^{2,2}\right\}\right] \\ f_2\left[\min\left\{S_2^{2,2}\right\}, \max\left\{S_2^{2,2}\right\}\right] \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix}.$$

For $j = 3, i = 1$

$$S_2^{1,3} = B_2(:,5:6)A_2'(1,:) = [1.3 \quad 1.5 \quad 1.5 \quad 1.7]$$

$$\begin{bmatrix} x_2^{1,3} \\ y_2^{1,3} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{1,3}\right\}, \max\left\{S_2^{1,3}\right\}\right] \\ f_2\left[\min\left\{S_2^{1,3}\right\}, \max\left\{S_2^{1,3}\right\}\right] \end{bmatrix} = \begin{bmatrix} 1.3 \\ 1.7 \end{bmatrix}.$$

For $j = 3, i = 2$

$$S_2^{2,3} = B_2(:,5:6)A_2'(2,:) = [0.7 \quad 0.5 \quad 0.9 \quad 0.7]$$

$$\begin{bmatrix} x_2^{2,3} \\ y_2^{1,3} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{2,3}\right\}, \max\left\{S_2^{2,3}\right\}\right] \\ f_2\left[\min\left\{S_2^{2,3}\right\}, \max\left\{S_2^{2,3}\right\}\right] \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix}.$$

For $j = 4, i = 1$

$$S_2^{1,4} = B_2(:,7:8)A_2'(1,:) = [2 \quad 2.2 \quad 2.2 \quad 2.4]$$

$$\begin{bmatrix} x_2^{1,4} \\ y_2^{1,4} \end{bmatrix} = \begin{bmatrix} f_1\left[\min\left\{S_2^{1,4}\right\}, \max\left\{S_2^{1,4}\right\}\right] \\ f_2\left[\min\left\{S_2^{1,4}\right\}, \max\left\{S_2^{1,4}\right\}\right] \end{bmatrix} = \begin{bmatrix} 2 \\ 2.4 \end{bmatrix}.$$

$$\begin{bmatrix} I_1 B_{n-1}(1, k_t(n-1)-(n-2):k_t(n-1)) & B_1(:,i) \\ I_1 B_{n-1}(2, k_t(n-1)-(n-2):k_t(n-1)) & B_1(:,i) \\ \vdots & \vdots \\ I_1 B_{n-1}(2^{n-1}, k_t(n-1)-(n-2):k_t(n-1)) & B_1(:,i) \end{bmatrix} \tag{26}$$

$$\left.\begin{aligned} B_1 &= \begin{bmatrix} 0.3 & 1.0 \\ 0.5 & 1.2 \end{bmatrix} \\ B_2 &= \begin{bmatrix} I_1B_1(1,1) & B_1(:,1) & I_1B_1(1,1) & B_1(:,2) & I_1B_1(1,2) & B_1(:,1) & I_1B_1(1,2) & B_1(:,2) \\ I_1B_1(2,1) & B_1(:,1) & I_1B_1(2,1) & B_1(:,2) & I_1B_1(2,2) & B_1(:,1) & I_1B_1(2,2) & B_1(:,2) \end{bmatrix} \\ &= \begin{bmatrix} 0.3 & 0.3 & 0.3 & 1.0 & 1.0 & 0.3 & 1.0 & 1.0 \\ 0.3 & 0.5 & 0.3 & 1.2 & 1.0 & 0.5 & 1.0 & 1.2 \\ 0.5 & 0.3 & 0.5 & 1.0 & 1.2 & 0.3 & 1.2 & 1.0 \\ 0.5 & 0.5 & 0.5 & 1.2 & 1.2 & 0.5 & 1.2 & 1.2 \end{bmatrix} \\ A_2 &= [I_1A_1(1,:) \quad I_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}\right\} \tag{29}$$

For $j = 4, i = 2$

$$S_2^{2,4} = B_2(:,7:8)A_2'(2,:) = [0 \quad -0.2 \quad 0.2 \quad 0]$$
$$\begin{bmatrix} x_2^{2,4} \\ y_2^{1,4} \end{bmatrix} = \begin{bmatrix} f_1 \left[\min\left\{S_2^{2,4}\right\}, \max\left\{S_2^{2,4}\right\}\right] \\ f_2 \left[\min\left\{S_2^{2,4}\right\}, \max\left\{S_2^{2,4}\right\}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}.$$

Consequently

$$f_{Y_2} = [0.6,1] \cup [0,0.2] \cup [1.3,1.7] \cup [0.5,0.9] \cup [1.3,1.7] \cup [0.5,0.9] \cup [2.0,2.4] \cup [0,0.2]. \quad (30)$$

Consider another example, where $a_1 = 0.3$, $b_1 = 0.5$, $a_2 = 1.0$, $b_2 = 1.1$, $a_3 = 2.1$ and $b_3 = 2.5$ represent three separate frequency intervals of different widths. In this case

$$B_1 = \begin{bmatrix} 0.3 & 1.0 & 2.1 \\ 0.5 & 1.1 & 2.5 \end{bmatrix}. \quad (31)$$

Following the same procedure, the output frequency range contributed by the second order nonlinear output can be obtained. The result is

$$\begin{aligned} f_{Y_2} = &[0.6,1] \cup [0,0.2] \cup [1.3,1.6] \cup [0.5,0.8] \cup [2.4,3] \\ &\cup [1.6,2.2] \cup [1.3,1.6] \cup [0.5,0.8] \cup [2,2.2] \cup [0,0.1] \\ &\cup [3.1,3.6] \cup [1,1.5] \cup [2.4,3] \cup [1.6,2.2] \cup [3.1,3.6] \\ &\cup [1,1.5] \cup [4.2,5] \cup [0,0.4]. \end{aligned} \quad (32)$$

B. Analytical Expression of the Output Frequency Ranges

In this section, an analytical relationship of the m separate frequency intervals of input frequencies $\bigcup_{i=1}^{m}[a_i, b_i]$ with the output frequency range f_{Y_n} is investigated.

Consider the case where the input frequencies are located in two separate frequency intervals of $[a_1, b_1]$ and $[a_2, b_2]$. The corresponding nth-order output frequency range $(f_{Y_n}) \cup (-f_{Y_n})$ is evaluated using Proposition 1 for $n = 1, 2, 3$ to yield

$$(f_{Y_1}) \cup (-f_{Y_1}) = [-b_2, -a_2] \cup [-b_1, -a_1] \cup [a_1, b_1] \cup [a_2, b_2] \quad (33)$$

$$\begin{aligned} (f_{Y_2}) \cup (-f_{Y_2}) = &[a_1 + a_2, b_1 + b_2] \cup [a_2 - b_1, b_2 - a_1] \\ &\cup [a_2 - b_2, b_2 - a_2] \cup [-2b_1, -2a_1] \\ &\cup [a_1 - b_1, b_1 - a_1] \cup [-2b_2, -2a_2] \\ &\cup [a_1 - b_2, b_1 - a_2] \cup [2a_1, 2b_1] \\ &\cup [-b_1 - b_2, -a_1 - a_2] \cup [2a_2, 2b_2] \end{aligned} \quad (34)$$

$$\begin{aligned} (f_{Y_3}) \cup (-f_{Y_3}) = &[a_1 + a_2 - b_1, b_1 + b_2 - a_1] \\ &\cup [a_1 + a_2 - b_2, b_1 + b_2 - a_2] \\ &\cup [a_2 - b_1 - b_2, b_2 - a_1 - a_2] \\ &\cup [a_1 - b_1 - b_2, b_1 - a_1 - a_2] \\ &\cup [2a_1 - b_2, 2b_1 - a_2] \cup [-3b_2, -3a_2] \\ &\cup [2a_1 - b_1, 2b_1 - a_1] \cup [-3b_1, -3a_1] \\ &\cup [2a_2 - b_2, 2b_2 - a_2] \cup [3a_2, 3b_2] \\ &\cup [2a_2 - b_1, 2b_2 - a_1] \cup [3a_1, 3b_1] \\ &\cup [2a_2 + a_1, 2b_2 + b_1] \\ &\cup [2a_1 + a_2, 2b_1 + b_2] \\ &\cup [a_1 - 2b_1, b_1 - 2a_1] \\ &\cup [a_1 - 2b_2, b_1 - 2a_2] \\ &\cup [a_2 - 2b_2, b_2 - 2a_2] \\ &\cup [a_2 - 2b_1, b_2 - 2a_1] \\ &\cup [-b_1 - 2b_2, -a_1 - 2a_2] \\ &\cup [-b_2 - 2b_1, -a_2 - 2a_1]. \end{aligned} \quad (35)$$

An observation of the composition of (33)–(35) indicates that the nth-order output frequency range $(f_{Y_n}) \cup (-f_{Y_n})$ can be described in Proposition 2 in the following.

Proposition 2: The nth-order output frequency range $(f_{Y_n}) \cup (-f_{Y_n})$ of system (1) subject to an input with spectrum (15) is given by

$$\bigcup_{\substack{\sum_{i=1}^{m}(k_{1i}+k_{2i})=n, \\ k_{1i},k_{2i}=0,1,\ldots,n \\ i=1,\ldots,m}} \left[\sum_{i=1}^{m} k_{1i}a_i - \sum_{i=1}^{m} k_{2i}b_i, \sum_{i=1}^{m} k_{1i}b_i - \sum_{i=1}^{m} k_{2i}a_i\right] \quad (36)$$

Proof of Proposition 2: From the analysis in Section III-A, It is known that the nth-order output frequency ω is related to the input frequencies $\omega_1, \omega_2, \ldots, \omega_n$ based on (18), and the values of the input frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are in $[-b_i, -a_i]$ or $[a_i, b_i]$ $(i = 1, 2, \ldots, m)$ as given by (19). Consequently if out of the n input frequencies, k_{1i} are taken in $[a_i, b_i]$ and k_{2i} are taken in $[-b_i, -a_i]$ $(i = 1, 2, \ldots, m)$, the minimum value of $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ thus obtained can be determined as $\sum_{i=1}^{m}(k_{1i}a_i - k_{2i}b_i)$, and the maximum value of $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ thus obtained can be determined as $\sum_{i=1}^{m}(k_{1i}b_i - k_{2i}a_i)$. Therefore, the range of $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ obtained for this particular choice of k_{1i} and k_{2i} $(i = 1, 2, \ldots, m)$ is

$$\left[\sum_{i=1}^{m}(k_{1i}a_i - k_{2i}b_i), \sum_{i=1}^{m}(k_{1i}b_i - k_{2i}a_i)\right] \quad (37)$$

where clearly k_{1i} and k_{2i} $(i = 1, 2, \ldots, m)$ are subject to the constraint

$$\sum_{i=1}^{m}(k_{1i} + k_{2i}) = n \quad k_{1i}, k_{2i} = 0, 1, \ldots, n \;\; i = 1, \ldots, m. \quad (38)$$

As a result, the nth-order output frequency range is the union of (37) with respect to those k_{1i} and k_{2i} $(i = 1, 2, \ldots, m)$ which are subject to the constraint (38). Thus, the result of Proposition 2 is proved.

Remark 3: When $m = 1$ in Proposition 2, (36) is reduced to the proposition in [7], i.e., the nth-order output frequency range $(f_{Y_n}) \cup (-f_{Y_n})$ is composed of the union of the $n+1$ intervals

$$[na - k(a+b), nb - k(a+b)], \quad k = 0, 1, \ldots n \quad (39)$$

when the system is subject to an input with its spectrum described by (8).

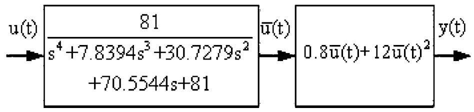

Fig. 1. Continuous time nonlinear Wiener model.

Proposition 2 is a rigorous theoretical result regarding the nth-order output frequency range of nonlinear systems when the systems are subject to an input the frequency components of which are located in separate frequency intervals. The proposal shows an explicit relationship between the nth-order nonlinear output frequency ranges and the ranges of input frequencies, therefore, it is significant for the theoretical analysis of nonlinear systems in the frequency domain. One can still use (36) to work out the frequency range $f_{Y_n} \cup (-f_{Y_n})$ via an exhaustive search for those k_{1i} and k_{2i} $(i = 1, 2, \ldots, m)$ which satisfy constraint (38) from $(n+1)^{2m}$ possible results. But given the straightforward implementation of Proposition 1 using a matrix oriented programming language and the wide application of MATLAB in many engineering fields, we suggest using Proposition 1 to numerically evaluate the output frequency range of nonlinear systems and using Proposition 2 for theoretical studies of the output frequencies of nonlinear systems. In addition, because Proposition 2 is based on the observation of the results obtained from Proposition 1, the computation procedure provided by Proposition 1 helps with understanding how the more compact and theoretically more significant Proposition 2 is reached.

Although the derivation for Propositions 1 and 2 above were made for continuous time systems, it is obvious that similar results also hold for discrete time nonlinear systems. It should be noted that Propositions 1 and 2 are valid for nonlinear systems which are asymptotically stable in the neighbourhood of the zero equilibrium point. This class of nonlinear systems is known as weakly nonlinear systems [13].

IV. Simulation Examples

To verify the output frequency ranges of nonlinear systems derived in the last section, two simulation examples are given in the following.

A. *Continuous Time Nonlinear Wiener Model*

Consider the continuous time nonlinear *Wiener model* described by

$$y(t) = 0.8\bar{u}(t) + 12\bar{u}^2(t) \tag{40}$$

as shown in Fig. 1 where

$$u(t) = \frac{1}{2\pi}\frac{[(\sin(0.5t) - \sin(0.3t)) + (\sin(1.2t) - \sin(1.0t))]}{t} \qquad -500 \le t \le 500 \tag{41}$$

s in the first block denotes Laplace operator, and all initial conditions for $t \le -500s$ are taken as zero.

The input and output of the system in the time and frequency domains are shown in Figs. 2 and 3, respectively.

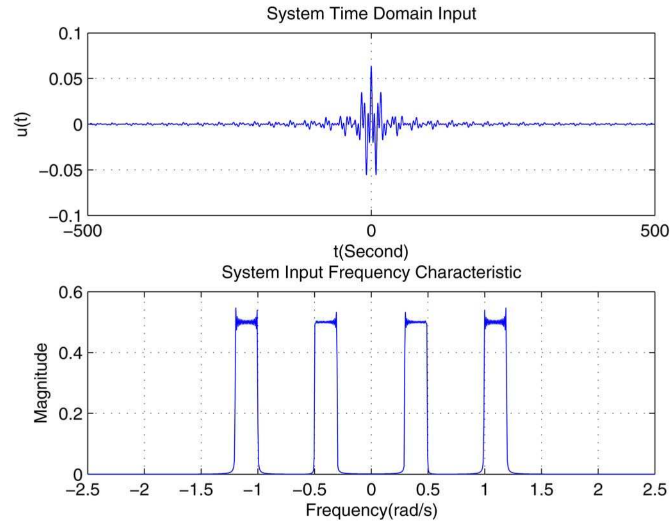

Fig. 2. Input signal in the time and frequency domain of the Wiener model.

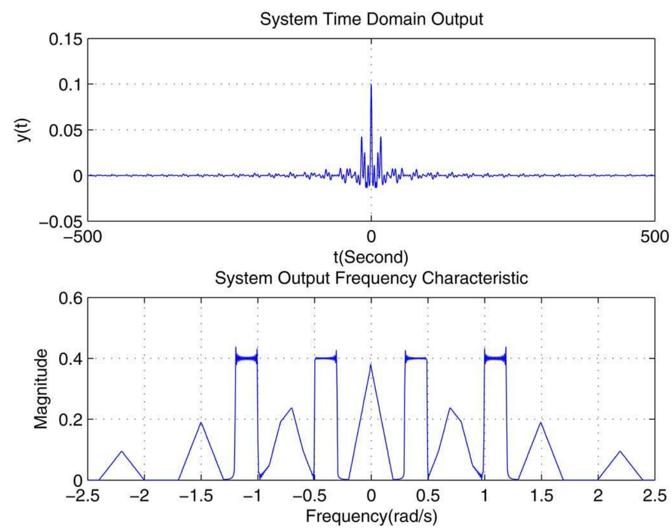

Fig. 3. Output signal in the time and frequency domain of the Wiener model.

Fig. 2 indicates that the system input frequency range is $[a_1, b_1] = [0.3, 0.5]$ and $[a_2, b_2] = [1.0, 1.2]$. The real output frequency range f_{Y_R} of the system is as shown in Fig. 3

$$f_{Y_R} = [0, 0.2] \cup [0.3, 0.5] \cup [0.5, 1] \cup [1.0, 1.2] \cup [1.3, 1.7] \cup [2.0, 2.4]. \tag{42}$$

Because the system includes nonlinearities up to second order and the input frequency range is $[0.3, 0.5]$ and $[1.0, 1.2]$, following the results in Section III, it can be shown that

$$\begin{aligned} f_Y = f_{Y_1} \cup f_{Y_2} &= [0.3, 0.5] \cup [1.0, 1.1] \\ &\cup [0.6, 1] \cup [0, 0.2] \cup [1.3, 1.7] \\ &\cup [0.5, 0.9] \cup [1.3, 1.7] \cup [0.5, 0.9] \\ &\cup [2.0, 2.4] \cup [0, 0.2] \\ &= [0.3, 0.5] \cup [1.0, 1.2] \\ &\cup [0, 0.2] \cup [0.5, 1] \cup [1.3, 1.7] \cup [2.0, 2.4] \end{aligned} \tag{43}$$

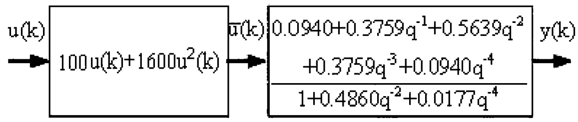

Fig. 4. Discrete-time nonlinear Hammerstein model.

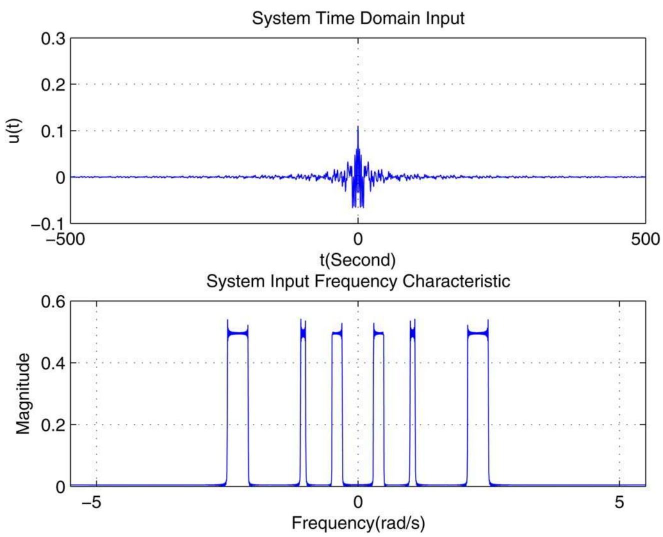

Fig. 5. Input signal in the time and frequency domain of the Hammerstein model.

which is exactly the same as what can be observed from Fig. 3. Therefore, the simulation study verifies the effectiveness of the new results developed in Section III.

B. Discrete Time Nonlinear Hammerstein Model

Consider the discrete time nonlinear system shown in Fig. 4 described by

$$\bar{u}(k) = 100u(k) + 1600u^2(k) \tag{44}$$

under the input given by equation

$$u(k) = \frac{1}{2\pi}\left[\frac{\sin(0.5k)-\sin(0.3k)}{k} + \frac{\sin(1.1k)-\sin(1.0k)}{k} + \frac{\sin(2.5k)-\sin(2.1k)}{k}\right]$$
$$-500 \le k \le 500 \tag{45}$$

where q^{-1} denotes the backward shift operator.

The input of the system in the time and frequency domains are shown in Fig. 5. By simulation analysis, the output of the system in the time and frequency domains were obtained, the results are shown in Fig. 6.

Fig. 5 indicates that the system input frequency range is $[a_1, b_1] = [0.3, 0.5]$, $[a_2, b_2] = [1.0, 1.1]$ and $[a_3, b_3] = [2.1, 2.5]$. The real system output frequency range f_{Y_R} can be observed in Fig. 6 as

$$f_{Y_R} = [0, 3] \cup [3.1, 3.6] \cup [4.2, 5]. \tag{46}$$

Since the system includes nonlinearities up to the second order and the input frequency range is $[0.3, 0.5]$, $[1.0, 1.1]$, and

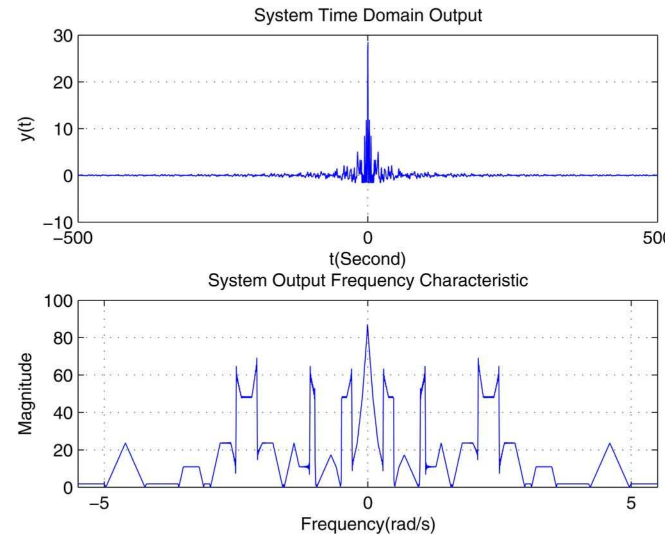

Fig. 6. Output signal in the time and frequency domain of the Hammerstein model.

$[2.1, 2.5]$, following the results in Section III, it can be shown that

$$\begin{aligned} f_Y = f_{Y_1} \cup f_{Y_3} &= [0.3, 0.5] \cup [1.0, 1.1] \cup [2.1, 2.5] \\ &\cup [0.6, 1] \cup [0, 0.2] \cup [1.3, 1.6] \\ &\cup [0.5, 0.8] \cup [2.4, 3] \cup [1.6, 2.2] \\ &\cup [1.3, 1.6] \cup [0.5, 0.8] \cup [2, 2.2] \\ &\cup [0, 0.1] \cup [3.1, 3.6] \cup [1, 1.5] \\ &\cup [2.4, 3] \cup [1.6, 2.2] \cup [3.1, 3.6] \\ &\cup [1, 1.5] \cup [4.2, 5] \cup [0, 0.4] \\ &= [0, 3] \cup [3.1, 3.6] \cup [4.2, 5]. \end{aligned} \tag{47}$$

Therefore, $f_Y = f_{Y_R}$, the computation result based on the analysis in Section III is again perfectly consistent with the simulation result.

V. Conclusion

For linear systems, the possible output frequencies are exactly the same as the frequency components in the input. For nonlinear systems, however, the situations are much more complicated. Normally the output frequency components are often much richer than that in the input. For nonlinear systems which can be described by a Volterra series model, both the algorithm for evaluating the output frequencies and an explicit expression for the relationship between the input and output frequency ranges have been derived in the authors' previous studies [6]–[8]. These extend the well known linear relationship between the input and output frequencies to the nonlinear case. In this paper, the results established in previous work have been further extended to a more general case where systems under study are subject to an input the frequency components of which are located in a finite number of separate frequency intervals. The new results have been proved theoretically, verified by simulation studies, and can be used to perform more sophisticated nonlinear system analysis and design in the frequency domain.

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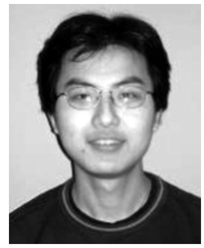

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