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**Abstract:** Piecewise affine (PWA) systems are often used to model gene regulatory networks. In this paper we elaborate on previous work about control problems for this class of models, using also some recent results guaranteeing the existence and uniqueness of limit cycles, based solely on a discrete abstraction of the system and its interaction structure. Our aim is to control the transition graph of the PWA system to obtain an oscillatory behaviour, which is indeed of primary functional importance in numerous biological networks; we show how it is possible to control the appearance or disappearance of a unique stable limit cycle by qualitative action on the degradation rates of the PWA system, both by static and dynamic feedback, i.e. the adequate coupling of a controlling subnetwork. This is illustrated on two classical gene network modules, having the structure of mixed feedback loops.

Keywords: Gene Networks, Feedback Control, Piecewise Affine, Periodic Solutions.

## 1. INTRODUCTION

Gene regulatory networks often display both robustness and steep, almost switch-like, response to transcriptional control. This motivates the use of an approximation of these response laws by a special class of diagonal piecewise affine differential (PWA) equations, to model genetic networks. Following (Glass [1975]), various authors have studied the mathematical properties these systems (Casey et al. [2006], Edwards [2000], Farcot [2006], Glass [1975], Gouzé et al. [2003]), as well as used them to model concrete biological systems (de Jong et al. [2004]).

The definition of PWA systems induces a partition of the state space in rectangular regions domains (or boxes), where the dynamics is affine. This partition leads to a qualitative description of the behaviour of PWA systems by a transition graph, describing the possible transitions between boxes.

Nowadays, the extraordinary development of biomolecular experimental techniques makes it possible to design and implement control laws in the cell system. The authors have recently developed a mathematical framework for controlling gene networks with hybrid feedback controls defined on each box (Farcot et al. [2008]). It is easy to see that this amounts to change the transition graph to obtain the desired one.

From another point of view, more oriented towards dynamical systems, it is also possible to obtain results concerning the limit cycles in PWA systems (see (Glass et al. [1978]) and the recent generalisation in (Farcot et al. [2009a])). The main results used in this paper are presented in Annex B.

Our aim in this paper is to control PWA systems to make a single stable limit cycle appear or disappear. To fulfil that goal, after some recalls concerning the PWA systems, the results on the control of the transition graph in the space of boxes are given (section 3), to obtain our main results illustrated by an example (section 4). More examples can be found in an extended version of the present paper (Farcot et al. [2009c]).

Related works on control aspects concern the affine or multi-affine hybrid systems (Habets et al. [2004], Belta et al. [2006]), or Poincaré-Bendixson theorem for hybrid systems (Matveev et al. [2000]). Other related works study the existence of limit cycles in the state space (Glass et al. [1978], Lu et al. [2009]). We are not aware of works linking control theory and limit cycle for this class of systems, and in $n$ dimensions.

## 2. PIECEWISE AFFINE MODELS

### *2.1 General formulation*

This section contains basic definitions and notations for piecewise affine models, of general form:

$$\frac{dx}{dt} = \kappa(x) - \Gamma(x)x \tag{1}$$

The variables $(x_1 \dots x_n)$ represent levels of expression of $n$ interacting genes, meaning in general concentrations of the mRNA or protein they code for. We will simply call *genes* the $n$ network elements in the following. Since gene transcriptional regulation is often considered to follow a

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steep sigmoid law, an approximation by a step function has been proposed to model the response of a gene (i.e. its rate of transcription) to the activity of its regulators (Glass [1975]). We use the notation:

$$\begin{cases} \mathsf{s}^+(x_i,\theta_i) = 0 & \text{if } x_i < \theta_i, \\ \mathsf{s}^+(x_i,\theta_i) = 1 & \text{if } x_i > \theta_i, \end{cases}$$

representing activation, whereas $\mathsf{s}^-(x,\theta) = 1 - \mathsf{s}^+(x,\theta)$ represents inhibition. Unless further precision is given, we leave this function undefined at its threshold value $\theta$.

The maps $\kappa : \mathbb{R}_+^n \to \mathbb{R}_+^n$ and $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^{n\times n}$ in (1) are usually multivariate polynomials (in general multi-affine), applied to step functions of the form $\mathsf{s}^\pm(x_i,\theta_i)$. For each $i \in \{1,\cdots,n\}$ the threshold values belong to a finite set: let $q_i$ be some positive integer, and denote

$$\Theta_i = \{\theta_i^0, \ldots, \theta_i^{q_i}\}. \tag{2}$$

The thresholds are assumed to be ordered (i.e. $\theta_i^j < \theta_i^{j+1}$), and the extreme values $\theta_i^0 = 0$ and $\theta_i^{q_i}$ represent the range of values taken by $x_i$ rather than thresholds.

$\Gamma$ is a diagonal matrix whose entries $\Gamma_{ii} = \gamma_i$ are degradation rates. Obviously, $\Gamma$ and the production rate $\kappa$ are piecewise-constant, taking fixed values in the rectangular domains obtained as Cartesian products of intervals bounded by threshold values in (2). These *boxes*, or *regular domains* (Plahte et al. [1998], de Jong et al. [2004]), are well characterised by integer vectors of the form $a = (a_1 \ldots a_n)$, with each $a_i \in \{0,\cdots,q_i-1\}$. Actually, such a vector $a$ identifies to a box $\mathcal{D}_a = \prod_{i=1}^n (\theta_i^{a_i}, \theta_i^{a_i+1})$. The set of boxes is then isomorphic to

$$\mathcal{A} = \prod_{i=1}^n \{0,\cdots,q_i-1\}, \tag{3}$$

Also, the following pairs of functions will be convenient notations: $\theta_i^\pm : \mathcal{A} \to \Theta_i$, $\theta_i^-(a) = \theta_i^{a_i-1}$ and $\theta_i^+(a) = \theta_i^{a_i}$.

Let us call *singular domains* the intersections of closure of boxes with threshold hyperplanes, where some $x_i \in \Theta_i \setminus \{\theta_i^0, \theta_i^{q_i}\}$. On these domains, the right-hand side of (1) is undefined in general. Although the notion of Filippov solution provides a generic solution to this problem (Gouzé et al. [2003]), in the case where the normal of the vector field has the same sign on both side of these singular hyperplanes, it is more simply possible to extend the flow by continuity. In the remaining of this paper, we will only consider trajectories which do not stay in any singular domain, a fact holding necessarily in absence of auto-regulation, i.e. when no $\kappa_i$ depends on $x_i$. This leads to the following hypothesis:

$\forall i \in \{1,\cdots,n\}$, $\kappa_i$ and $\gamma_i$ do not depend on $x_i$. **(H1)**

On any regular domain of index $a \in \mathcal{A}$, the rates $\kappa = \kappa(a)$ and $\Gamma = \Gamma(a)$ are constant, and thus equation (1) is affine. Its solution is explicitly known, for each coordinate $i$ :

$$\varphi_i(x,t) = x_i(t) = \phi_i(a) + e^{-\gamma_i t}\left(x_i(0) - \phi_i(a)\right), \tag{4}$$

where $t \in \mathbb{R}_+$ is such that $x(t) \in \mathcal{D}_a$, and

$$\phi(a) = (\phi_1(a)\cdots\phi_n(a)) = \left(\frac{\kappa_1(a)}{\gamma_1(a)}\cdots\frac{\kappa_n(a)}{\gamma_n(a)}\right).$$

It is clearly an attractive equilibrium of the flow (4). It will be called *focal point* in the following for reasons we explain now. Let us first make the generic assumption that no focal point lies on a singular domain:

$$\forall a \in \mathcal{A}, \quad \phi(a) \in \bigcup_{a'\in\mathcal{A}} \mathcal{D}_{a'}. \qquad \textbf{(H2)}$$

Then, if $\phi(a) \in \mathcal{D}_a$, it is an asymptotically stable steady state of system (1). Otherwise, the flow will reach the boundary $\partial\mathcal{D}_a$ in finite time. At this point, the value of $\kappa$ (and thus, of $\phi$) changes, and the flow changes its direction, evolving towards a new focal point. The same process carries on repeatedly. It follows that the continuous trajectories are entirely characterised by their successive intersections with the boundaries of regular domains.

This sequence depends essentially on the position of focal points with respect to thresholds. Actually, $\{x \,|\, x_i = \theta_i^-(a)\}$ (resp. $\{x \,|\, x_i = \theta_i^+(a)\}$) can be crossed if and only if $\phi_i(a) < \theta_i^-(a)$ (resp. $\phi_i(a) > \theta_i^+(a)$). Then, let us denote $I_{out}^+(a) = \{i \in \{1,\cdots,n\} | \, \phi_i > \theta_i^+(a)\}$, and similarly $I_{out}^-(a) = \{i \in \{1,\cdots,n\} | \, \phi_i < \theta_i^-(a)\}$. Then, $I_{out}(a) = I_{out}^+(a) \cup I_{out}^-(a)$ is the set of escaping directions of $\mathcal{D}_a$. Also, we call *walls* the intersections of threshold hyperplanes with the boundary of a regular domain.

When it is unambiguous, we will omit the dependence on $a$ in the sequel. Now, in each direction $i \in I_{out}$ the time at which $x(t)$ encounters the corresponding hyperplane, for $x \in \mathcal{D}_a$, is readily calculated:

$$\tau_i(x) = \frac{-1}{\gamma_i}\ln\left(\frac{\phi_i - \theta_i^\pm}{\phi_i - x_i}\right), \qquad i \in I_{out}^\pm. \tag{5}$$

Then, $\tau(x) = \min_{i\in I_{out}} \tau_i(x)$, is the *exit time* of $\mathcal{D}_a$ for the trajectory with initial condition $x$. We define a *transition map* $T^a : \partial\mathcal{D}_a \to \partial\mathcal{D}_a$:

$$\begin{aligned} T^a x &= \varphi\left(x, \tau(x)\right) \\ &= \phi + \alpha(x)(x-\phi). \end{aligned} \tag{6}$$

where $\alpha(x) = \exp(-\tau(x)\Gamma)$.

The map above is defined locally, on a domain $\mathcal{D}_a$. However, under our assumption **(H1)**, there is a well defined global transition map on the union of walls, denoted $T$.

To conclude this section let us define the *state transition graph* $\mathsf{TG}$ associated to a system of the form (1) as the pair $(\mathcal{A},\mathcal{E})$ of nodes and oriented edges, where $\mathcal{A}$ is defined in (3) and $(a,b) \in \mathcal{E} \subset \mathcal{A}^2$ if and only if $\partial\mathcal{D}_a \cap \partial\mathcal{D}_b \neq \varnothing$, and there exists a positive Lebesgue measure set of trajectories going from $\mathcal{D}_a$ to $\mathcal{D}_b$. It is not difficult to see that this is equivalent to $b$ being of the form $a \pm \mathrm{e}_i$, with $i \in I_{out}^\pm(a)$ and $\mathrm{e}_i$ a standard basis vector.

From now on, it will always be assumed that **(H1)** and **(H2)** hold.

## 3. PIECEWISE CONTROL

The recent advent of so called *synthetic biology* (Andrianantoandro et al. [2006], Kobayashi et al. [2004]) has led to a situation where gene regulatory processes are not only studied, but designed to perform certain functions. This motivates the use of tools from feedback control theory (Iglesias et al. [2009], Sontag [2005]).

In (Farcot et al. [2008]), we have presented an extension of systems of the form (1), where both production and decay terms have an input $u \in \mathbb{R}^p$, of which they were affine functions. In this context, we defined a class of qualitative control problems, and showed that were equivalent to some linear programming problems. As in this previous work, we consider here qualitative feedback laws, which depend only on the box containing the state vector, rather than its exact value. New results presented in this paper concern the control of periodic orbits, relying on recent results

recalled in Annex B.
This choice is motivated by robustness purposes, and recent experimental techniques allowing for the reversible induction of specific genes at a chosen instant, for instance using promoters inducible by ethanol (Deveaux et al. [2003]), or light (Shimizu-Sato et al. [2002], to name only two. Also, degradation rates may be modified, either directly by introducing a drug (Wyke et al. [2006]), or via a designed genetic circuit (Grilly et al. [2007]).

To simplify the presentation, we focus in this paper on the particular case where decay rates can be linearly controlled by a scalar and bounded input $u$. For each $i \in \{1, \cdots, n\}$, let us denote this as:

$$\frac{dx_i}{dt} = \kappa_i(x) - (\gamma_i^1(x)u + \gamma_i^0(x))x, \quad u \in [0, U] \subset \mathbb{R}_+, \quad (7)$$

where $\gamma_i^0$ and $\gamma_i^1$ are piecewise constant functions assumed to satisfy $\gamma_i^0 > 0$ and $\gamma_i^1 > -\frac{\gamma_i^0}{U}$, in any box. This ensures that decay rates are positive, but yet can be decreasing functions of $u$ (for $\gamma_i^1 < 0$).
Now, a feedback law depending only on the qualitative state of the system is simply expressed as the composite of a map $\bigcup_a \mathcal{D}_a \to \mathcal{A}$ indicating the box of the current state, with a function $u : \mathcal{A} \to [0, U]$ which represents the control law itself. In other words, in each box a constant input value is chosen. For a fixed law of this form, it is clear that the dynamics of (7) is entirely determined, and in particular we denote its transition graph by $\mathsf{TG}(u)$.
Let us now recall our definition of control problem.

**Global Control Problem**: Let $\mathsf{TG}^\star = (\mathcal{A}, \mathcal{E}^\star)$ be a transition graph. Find a feedback law $u : \mathcal{A} \to [0, U]$ such that $\mathsf{TG}(u) = \mathsf{TG}^\star$.

Clearly, $\mathcal{E}^\star$ cannot be arbitrary in $\mathcal{A}^2$, and must in particular contain only arrows of the form $(a, a \pm \mathrm{e}_i)$. Now in the present, restricted, context the equivalent linear programming problem described in (Farcot et al. [2008]) is very simple. For each $a \in \mathcal{A}$, the control problem above requires that the focal point $\phi(a, u(a))$ belongs to a certain union of boxes, i.e. its coordinates must satisfy inequalities of the form $\theta_i^{j^-(a)} < \kappa_i(a)/(\gamma_i^1(a)u(a) + \gamma_i^0(a)) < \theta_i^{j^+(a)}$, or equivalently

$$\frac{\kappa_i(a) - \gamma_i^0(a)\theta_i^{j^+(a)}}{\gamma_i^1(a)\theta_i^{j^+(a)}} < u(a) < \frac{\kappa_i(a) - \gamma_i^0(a)\theta_i^{j^-(a)}}{\gamma_i^1(a)\theta_i^{j^-(a)}} \quad (8)$$

if $\gamma_i^1(a) > 0$, and in reverse order otherwise. Hence, the solution set of the control problem is just the Cartesian product of all intervals of the form (8), when $a$ varies in $\mathcal{A}$. It is thus identical to a rectangle in $\mathbb{R}^{\#\mathcal{A}}$ (where $\#$ denotes cardinality), which is of full dimension if and only if the problem admits a solution.
Thanks to the explicit description (8), this set can be computed with a complexity which is linear in $\#\mathcal{A}$. The latter grows exponentially with the dimension of the system, but in practice, one will face problems where $\mathcal{E}$ and $\mathcal{E}^\star$ only differ on a subset of initial vertices, say $\mathcal{A}^\star$, and then the actual complexity will be of order $\#\mathcal{A}^\star$.
In addition to this type of control, we introduce in this note some first hints toward dynamic feedback control, where instead of a direct feedback $u$, one uses some additional variable (here a single one), evolving in time according to a system of the form (1), and coupled to the initial system.

## 4. CONTROL OF PERIODIC SOLUTIONS

We now illustrate by two modes of control how to preclude a limit cycle.

### *4.1 Static feedback control.*

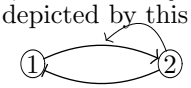
Consider the following two dimensional system:

$$\begin{cases} \dot{x}_1(t) = K_1\mathsf{s}^-(x_2) - (\gamma_1^1 u + \gamma_1^0)x_1 \\ \dot{x}_2(t) = K_2[\mathsf{s}^+(x_1, \theta_1^1)\mathsf{s}^+(x_2) + \mathsf{s}^+(x_1, \theta_1^2)\mathsf{s}^-(x_2)] - \gamma_2^0 x_2 \end{cases} \quad (9)$$

where $x_2$ has a single threshold, and $\mathsf{s}^\pm(x_2) = \mathsf{s}^\pm(x_2, \theta_2^1)$. We assume moreover that the following inequalities stand:

$$\gamma_1^1 > 0, \quad K_1 > \gamma_1^0\theta_1^2, \quad K_2 > \gamma_2^0\theta_2^1, \quad (10)$$

so that the first decay rate increases with $u$, and the interactions are *functional*: an activation of a variable leads to the corresponding focal point coordinate being above a variable's threshold (chosen as the highest one for $x_1$, since otherwise $\theta_1^2$ cannot be crossed from below). Remark that in this system, $x_2$ violates **(H1)**. However, it will appear soon that this autoregulation is only effective at a single wall, which is unstable, and thus can be ignored safely.
This system corresponds to a negative feedback loop, where $x_2$ is moreover able to modulate its activation by $x_1$: when $x_2$ is above its threshold, the interaction is more efficient, since it is active at a lower threshold $\theta_1^1 < \theta_1^2$. Biologically, this may happen if the proteins coded by $x_1$ and $x_2$ form a dimer, which activates $x_2$ more efficiently than $x_1$ protein alone. This is reminiscent of the *mixed feedback loop*, a very widespread module able to display various behaviours (Milo et al. [2002], François et al. [2005]). It might be depicted by this graph

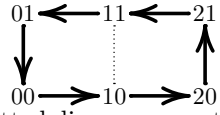


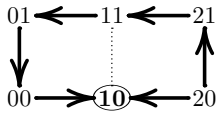

As seen in the equations, the scalar input is assumed to affect the first decay rate, but not the second (i.e. $\gamma_2^1 = 0$). Now, one readily computes the focal points of all boxes:

| 00 | 01 | 10 | 11 | 20 | 21 |
|---|---|---|---|---|---|
| $\phi_1$ | 0 | $\phi_1$ | 0 | $\phi_1$ | 0 |
| 0 | 0 | 0 | $\phi_2$ | $\phi_2$ | $\phi_2$ |

(11)

where $\phi_1$ is an abbreviation for $K_1/(\gamma_1^1 u + \gamma_1^0)$, and $\phi_2$ for $K_2/\gamma_2^0$. Under the constraints (10), this leads to the transition graph without input (i.e. $u = 0$ in all boxes):

$\mathsf{TG}(0) =$ 


The dotted line represents an unstable wall, for which Filippov theory would be required for full rigour. However, this wall is not reachable, and we ignore it afterwards.
Now, since this graph has a cycle, the two thresholds $\theta_1^1$, $\theta_1^2$ are crossed, and (11) is easily seen to imply condition (B.1), conclusion $B$) of Theorem 3 applies : there is a unique stable limit cycle.
Now, in accordance with the section's title, let us look for a $u$ leading to: $\mathsf{TG}^\star =$ 

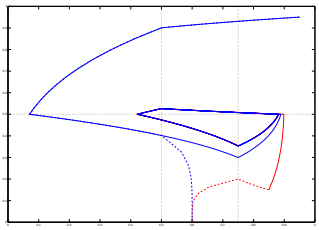


Fig. 1. Dashed lines: with feedback control. Plain lines: without. Two initial conditions, $(0.95, 0.95)$ in box 21 (blue curves) and $(0.85, 0.15)$ in 20 (red curves). The controlled and autonomous trajectories only diverge in box 10, 20 where the feedback is active. See parameters in Appendix A.1

Clearly in $\mathsf{TG}^\star$, the box $\mathcal{D}_{10}$ attracts trajectories from all other boxes, and contains its own focal point, which is thus a globally asymptotically stable equilibrium. The only states whose successors differ in $\mathsf{TG}(0)$, and $\mathsf{TG}^\star$ are 10 and 20, hence we assume $u(a) = 0$ for all other $a \in \mathcal{A}$, or $\mathcal{A}^\star = \{10, 20\}$ to recall the notations of previous section. Then, we have:

*Theorem 1.* The Control Problem of section 3 can be solved for system (9) under constraints (10).

**Proof.** Eq. (8) with $\theta_1^{j^-(a)} = \theta_1^1$ and $\theta_1^{j^+(a)} = \theta_1^2$ writes here

$$\frac{K_1 - \gamma_1^0\theta_1^2}{\gamma_1^1\theta_1^2} < u(a) < \frac{K_1 - \gamma_1^0\theta_1^1}{\gamma_1^1\theta_1^1} \tag{12}$$

for both $a \in \mathcal{A}^\star$. This always defines a nonempty interval by $\theta_1^1 < \theta_1^2$ □

An illustration on a numerical example is shown Figure 1.

### *4.2 Dynamic feedback control.*

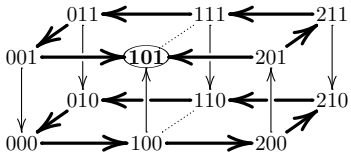
Now, let us focus on the question of realising an extended network which solves the same problem, by adding a variable to system (9). In other words, one now seeks to impose the dynamics described by $\mathsf{TG}^\star$ using dynamic feedback. Biologically, this amounts to designing a genetic construct whose promoter depends transcriptionaly on $x_1$ and $x_2$, and increases the degradation rate of $x_1$. Let us denote by $y$ the expression level of this additional gene. The most obvious version of such an extended system arises by increasing $y$ production rate exactly at boxes in $\mathcal{A}^\star$:

$$\begin{cases} \dot{x}_1(t) = K_1\mathsf{s}^-(x_2) - (\gamma_1^1\upsilon\,\mathsf{s}^+(y) + \gamma_1^0)x_1 \\ \dot{x}_2(t) = K_2[\mathsf{s}^+(x_1, \theta_1^1)\mathsf{s}^+(x_2) + \mathsf{s}^+(x_1, \theta_1^2)\mathsf{s}^-(x_2)] - \gamma_2^0 x_2 \\ \dot{y}(t) = \mathsf{s}^+(x_1, \theta_1^1)\mathsf{s}^-(x_2) - \gamma_y y \end{cases} \tag{13}$$

$\upsilon$ a constant in the interval (12), so that forcing $s^+(y) = 1$ would lead us back to a static feedback solution.
We consider without loss of generality that $y \in [0, 1/\gamma_y]$, since higher values of $y$ tend to $1/\gamma_y$ or 0. Also, $\mathsf{s}^+(y)$ is defined with respect to a threshold $\theta_y \in (0, 1)$. We also assume $\theta_y\gamma_y < 1$, ensuring that $y$ may cross its threshold when activated.

Now, (13) defines an autonomous systems of the form (1), whose transition graph has indeed a fixed point 101:

(14)




This fixed point corresponds the fixed point 10 of $\mathsf{TG}^\star$: in fact, the upper part of the graph above, where $\mathsf{s}^+(y) = 1$ is exactly $\mathsf{TG}^\star$. However, it is not invariant, and some trajectories can escape to $\mathsf{s}^+(y) = 0$, where we see $\mathsf{TG}(0)$, and thus the possibility of periodic solutions. Besides, there are other cycles in this graph.
Unlike static feedback control – and more realistically – the effect of $y$ on $\gamma_1$ takes some positive time, explaining why the situation is not a direct translation of previous case. We will now show that under additional constraints of the parameters governing $y$'s dynamics, it is possible to guarantee that $\mathcal{D}_{101}$ contains a globally asymptotically stable equilibrium. To achieve this, let us rephrase a lemma, proved as Lemma 1 in (Farcot [2006]):

*Lemma 1.* For any box, there is at most one pair of parallel walls successively crossed by solution trajectories of a system of the form (1).

In other words, there is at most one direction $i$ such that opposite walls, of the form $x_i = \theta_i^-$ and $x_i = \theta_i^+$, are crossed. Moreover, such an $i$ is characterised, see (Farcot [2006]), by the condition

$$\forall j \neq i, \quad \tau_i(\theta_i^-) < \tau_j(\theta_j^-), \tag{15}$$

under the assumption $I_{out} = I_{out}^+$ (which simplifies the description without loss of generality), i.e. all exiting walls occur at higher threshold values, of the form $\theta_i^+$, which is thus the threshold involved in the definition of $\tau_i$, Eq. (5). This allows us to prove the following result:

*Proposition 1.* Assume the following

$$(1 - \gamma_y\theta_y)^{\frac{1}{\gamma_y}} > \left(\frac{K_1 - \gamma_1^0\theta_1^2}{K_1 - \gamma_1^0\theta_1^1}\right)^{\frac{1}{\gamma_1^0}}$$

Then the steady state in box $\mathcal{D}_{101}$ attracts the whole state space of system (13).

**Proof.** Let us sketch only the proof, see (Farcot et al. [2009c]) for details. Any trajectory which does not enter $\mathcal{D}_{101}$ must cross the pair $W_1^\pm$ in succession. Now, from Lemma 1, among the two pairs of walls $W_1^\pm$ , $W_y^\pm$, only one can be crossed in succession by trajectories, and the inequality in the statement is derived from (15), ensuring that $W_y^\pm$ is the crossed pair of walls, and thus $\phi(101)$ is the only attractor. □

Some elementary calculus shows that the left-hand side in the inequality of proposition 1 is a increasing function of $\gamma_y$ when $\gamma_y \in (0, 1/\theta_y)$, as assumed previously. Thus, this inequality is equivalent to requiring a lower bound to $\gamma_y$, even though this bound does not have a simple explicit form. This fact can be given an intuitive explanation: $\gamma_y$ is inversely proportional to the characteristic time of the variable $y$, in each box. Hence, proposition 1 means that the dynamics of $y$ must be fast enough in order to retrieve the behaviour of the static feedback control, which

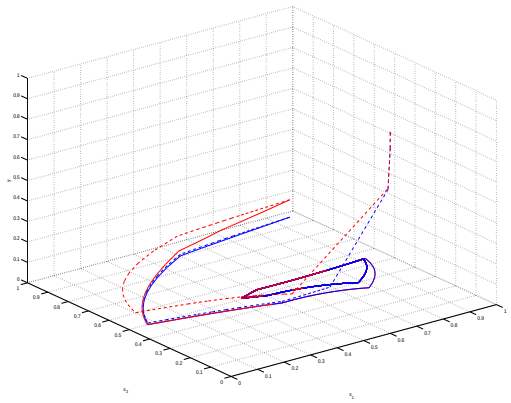


Fig. 2. Dashed line: inequality of proposition 1 satisfied. Plain line: inequality violated. Two common initial conditions, $(x_1, x_2, y) = (0.95; 0.95, 0.1)$ (blue) and $(0.95, 0.95, 0.95)$ (red). The value of $y$ has been divided by 10 to keep all variables in $[0, 1]$. In both cases a limit cycle is controlled into an equilibrium point. See parameters in Appendix A.2

corresponds to the limit of an instantaneous feedback. See Figure 2 for a numerical example.

The results of this section can be summarised as

*Theorem 2.* Consider a system of the form (9), with structural constraints (10). It has a unique, stable and globally attractive limit cycle in absence of input, i.e. $u = 0$.
Moreover, there exists a control law ensuring a unique, stable and globally attractive equilibrium point. This control can be achieved in two ways:

•) Using a scalar piecewise constant feedback $u$, such that $u(a)$ satisfies (12) for $a \in \{10, 20\}$.

•) Using dynamic feedback with a single additional variable $y$, as in (13), whose decay rate satisfies the condition in proposition 1, and with $v$ a solution of (12).

## 5. CONCLUSION

We have given, and illustrated by an example, a control methodology to make unique stable limit cycles appear or disappear in hybrid PWA systems. The obtained feedback laws are termed qualitative control because they depend only on a qualitative abstraction of the original system: its transition graph.
Future work suggested by this study are mostly related to the question of dynamic feedback. Actually, the example shows the effective possibility of using an additional variable to control a system, i.e. to design a controller system to be coupled to the original one. Moreover, the design of this dynamic feedback relied in a simple way on the static feedback problem. This technique should be formalised in more general terms, and applied to other examples in the future.

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## Appendix A. PARAMETER VALUES

### *A.1 Parameters for Figure 1*

| $K_1$ | $K_2$ | $\gamma_1^0$ | $\gamma_1^1$ | $\gamma_2^0$ | $\theta_1^1$ | $\theta_1^2$ | $\theta_2^1$ |
|---|---|---|---|---|---|---|---|
| 0.9 | 0.2 | 1 | 1 | 0.3 | 0.5 | 0.75 | 0.5 |

Moreover the value $u(a)$ is computed as the middle-point of the interval defined by (12).

### *A.2 Parameters for Figure 2*

| $K_1$ | $K_2$ | $\gamma_1^0$ | $\gamma_1^1$ | $\gamma_2^0$ | $\theta_1^1$ | $\theta_1^2$ | $\theta_2^1$ | $\theta_y$ |
|---|---|---|---|---|---|---|---|---|
| 0.9 | 0.2 | 1 | 1 | 0.3 | 0.5 | 0.75 | 0.5 | 0.5 |

To check the inequality in proposition 1, we need to compute $\left(\frac{\phi_1-\theta_1^2}{\phi_1-\theta_1^1}\right)^{\frac{1}{\gamma_1^0}} = 0.375$. Then, the two values of $\gamma_y$ we have tested are 0.1 and 1.7, for which $(1-\gamma_y\theta_y)^{\frac{1}{\gamma_y}}$ is respectively close to 0.599 (inequality satisfied) and 0.328 (inequality violated).

## Appendix B. STABILITY AND LIMIT CYCLES

This section summarize previous results about periodic solutions of systems of the form (1), see (Farcot et al. [2009a,b]) for more details.
Previous works include (Glass et al. [1978], Snoussi [1989], Mestl et al. [1995], Edwards [2000], Lu et al. [2009]). With the notable exception of (Snoussi [1989]), all these studies focused on the special case where $\Gamma$ is a scalar matrix, which greatly simplifies the analysis, since trajectories in each box are then straight lines towards the focal point.
In the rest of this section we consider a piecewise-affine system such that there exists a sequence $\mathcal{C} = \{a^0 \dots a^{\ell-1}\}$ of regular domains which is a cycle in the transition graph, and study periodic solutions in this sequence. We abbreviate the focal points of these boxes as $\phi^i = \phi(a^i)$. Let us now define a property of these focal points: we say that the points $\phi^i$ are *aligned* if

$$\forall i \in \{0, \cdots, \ell-1\},\ \exists! j \in \{1, \cdots, n\}, \quad \phi_j^{i+1} - \phi_j^i \neq 0, \tag{B.1}$$

where $\phi^\ell$ and $\phi^0$ are identified.
Since $\mathcal{C}$ is supposed to be a cycle in $\mathsf{TG}$, for each pair $(a^i, a^{i+1})$ of successive boxes there must be at least one coordinate at which their focal points differ, namely the only $s_i \in I_{out}(a^i)$ such that $a^{i+1} = a^i \pm \mathrm{e}_{s_i}$. We keep on denoting $s_i$ this *switching* coordinate in the following. Hence condition (B.1) means that $s_i$ is the only coordinate in which $\phi^i$ and $\phi^{i+1}$ differ. This implies in particular that $a^{i+1}$ is the only successor of $a^i$, i.e. there is no edge in $\mathsf{TG}$ from $\mathcal{C}$ to $\mathcal{A} \setminus \mathcal{C}$. It might seem intuitive in this case that all orbits in $\mathcal{C}$ converge either to a unique limit cycle, or to a point at the intersection of all crossed thresholds. However, this fact has only been proved for uniform decay rates (i.e. $\Gamma$ scalar), (Glass et al. [1978]), and its validity with distinct decay remains an open question.
If $\{s_i\}_{0\leq i<\ell} = \{1, \cdots, n\}$, i.e. all variables are switching along $\mathcal{C}$, then the intersection of all walls between boxes in $\mathcal{C}$ is either a single point, which we denote $\theta^{\mathcal{C}}$, or it is empty. The latter holds when two distinct thresholds are crossed in at least one direction. When defined, $\theta^{\mathcal{C}}$ is a fixed point for any continuous extension of the flow in $\mathcal{C}$, see (Farcot et al. [2009b]).
Let us now rephrase the main result from (Farcot et al. [2009b]).

*Theorem 3.* Let $\mathcal{C} = \{a^0, a^1 \cdots a^{\ell-1}\}$ denote a sequence of regular domains which is periodically visited by the flow, and whose focal points satisfy condition (B.1). Suppose also that all variables are switching at least once.
Let $W$ denote the wall $\partial\mathcal{D}_{a^0} \cap \partial\mathcal{D}_{a^1}$, and consider the first return map $\mathbf{T} : W \to W$ defined as the composite of local transition maps along $\mathcal{C}$.
*A*) If a single threshold is crossed in each direction, let $\lambda = \rho(D\mathbf{T}(\theta^{\mathcal{C}}))$, the spectral radius of the differential $D\mathbf{T}(\theta^{\mathcal{C}})$. Then, the following alternative holds:
**i)** if $\lambda \leqslant 1$, then $\forall x \in W$, $\mathbf{T}^n x \to \theta^{\mathcal{C}}$ when $n \to \infty$.
**ii)** if $\lambda > 1$ then there exists a unique fixed point different from $\theta^{\mathcal{C}}$, say $q = \mathbf{T}q$. Moreover, for every $x \in W \setminus \{\theta^{\mathcal{C}}\}$, $\mathbf{T}^n x \to q$ as $n \to \infty$.
*B*) If there are two distinct crossed thresholds in at least one direction, then the conclusion of **ii)** holds.

In (Farcot et al. [2009a]) we have resolved the alternative above for a particular class of systems: negative feedback loop systems of the form

$$\dot{x}_i = \kappa_i^0 + \mathsf{s}^{\varepsilon_i}(x_{i-1}, \theta_{i-1}) - \gamma_i x_i, \quad \varepsilon_i \in \{-,+\},\ 1 \leq i \leq n,$$

with subscripts understood modulo $n$, and an odd number of negative $\varepsilon_i$. It can be shown that there exists a cycle $\mathcal{C}$ in $\mathsf{TG}$ whose focal points satisfy (B.1). Then, in Theorem 3, it can be shown that A.**i)** holds in dimension $n = 2$, and A.**ii)** holds for all $n \geqslant 3$. We get

*Theorem 4.* Negative feedback loop systems admit a unique, global attractor.

- For $n = 2$, it is a stable fixed point, at the intersection $(\theta_1, \theta_2)$ of threshold lines.
- For $n \geq 3$, it is a stable limit cycle.