[For solving Cauchy singular integral equations]

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THÈSE

présentée pour obtenir le diplôme de

Doctorat Sciences Université Mohamed Khider Biskra, Algerie

Docteur de l'Université Jean Monnet, Saint Etienne, France

Discipline :
Mathématiques Appliquées et Applications des Mathématiques

*Spécialité :*Analyse Numérique

Sur la Résolution des Equations Intégrales Singulières à Noyau de Cauchy

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par

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soutenue le 27 avril 2011
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Remerciements

Je tiens à remercier M. le Dr Said Guedjiba, pour son soutien exemplaire durant la préparation de ce projet. Je lui suis également reconnaissant pour les qualités scientifiques et pédagogiques de son encadrement et pour la disponibilité sans faille dont il a fait preuve durant ma formation doctorale. Dès le début, sa confiance m'a donné l'énergie et l'inspiration me permettant de surmonter toutes les difficultés.

Je tiens également à remercier Monsieur le Professeur Mario Ahues, qui m'a accueilli au sein du LaMUSE. Je le remercie pour son aide précieuse durant la préparation des premières étapes de ce travail. Il a su me guider sur le chemin de la recherche.

Mes remerciements les plus respectueux vont à Monsieur Alain Largillier. Veuillez trouver ici l'expression de mes remerciements les plus sincères ainsi que la marque de mon profond respect.

Je remercie Pr. Lazhar Bougoffa et Pr. Amina Benbernou qui ont bien voulu être rapporteurs de cette thèse.

Ma sincère reconnaissance à tous les membres du jury pour l'honneur qu'ils me font en acceptant de le présider et d'examiner ce travail.

Je tiens à remercier Madame Laurence Grammont, pour avoir accepté d'être membre de jury de cette thèse.

J'adresse toute ma gratitude à Monsieur le Professeur Brahim Mezerdi, qu'il trouve ici l'expression de ma profonde reconnaissance pour avoir accepté d'examiner cette thèse.

Je remercie très sincèrement Monsieur le Professeur Khaled Melkemi, d'avoir accepté d'être président du jury.

J'adresse aussi mes sincères remerciements à M. Zouhir Mokhtari, Chef du département de Mathématiques à l'université Mohamed Khider, Biskra.

Je tiens à remercier tous les membres du département de Mathématiques de la Faculté des Sciences et Techniques de Saint-Étienne, en particulier M. Grigory Panasenko et M. Mahdi Boukrouche .

Je tiens à remercier mon ami Hamza et sa famille. Merci pour l'accueil chaleureux que vous m'avez prodigué lors de mon arrivée.

Je tiens à remercier tout particulièrement mes amis Nazih, Fayçal, Hicham, Ahmed, Lamine, Nachite. Qu'ils trouvent ici mes chaleureux remerciements.

Quant à ma femme et mon fils Abdeldjalil, je leur adresse mes plus chaleureux remerciements. Leur soutien moral, indispensable pour moi, m'a permis de tenir bon et de ne pas abandonner dans les moments difficiles. Ils ont su me soutenir et parfois me supporter. Qu'ils soient certains de toute ma reconnaissance et de tout mon amour.

Finalement, je tiens à remercier mes parents pour leur soutien exemplaire et leurs

sacrifices loyaux durant ces longues années de quête sur la voie du savoir.

Dédicace

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A Mon Fils Abdeldjalil

A Mes Parents...

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Introduction

In the last two decades, Cauchy integral equations have assumed an increasing relevance. This is due to the great variety of problems arising in sciences, engineering and technology, which can be described by such equations. Cauchy integral and Integro-differential equations involve important mathematical techniques, because they are encountered by mathematicians, and physical and social scientists, in their investigations. These equations were described in many available books, concerning theory and applications (cf. [2], [7], [9], [10], [16], [20], [34], [37], [39], [45], [70], [89], [92], [95], [108], [109], [117], [122], [125], [130], [139]-[141]).
Several books include the latest questions related to high technology on solving very important theoretical and practical problems on solid mechanics, fracture mechanics, structural analysis, elastodynamics, fluid mechanis and aerodynamics, by using singular integral equation methods (cf. [1], [13], [48], [71], [90], [96]-[98], [118], [119], [126]). In ([132]-[138]), the author presented many papers relating to boundedness of singular integral operators with Cauchy kernel in weighted spaces.
In ([18], [14], [72]-[78]), the authors have studied the Cauchy integral equations in weigthted spaces of continuous functions, using Jacobi weights; they introduced a certain number of polynomial approximation spaces. In ([50]-[69]), the authors presented analytical theories and numerical evaluation methods for solving Cauchy integral equations, in an accessible manner for a variety of applications to problems in the theory of three-dimensional elasticity. In the last decades several papers have been published on the convergence of the quadrature rules for evaluating Cauchy singular integrals, (see [21]-[31], [32], [102]-[107]). In ([80]-[88]), the author has investigated the algebra of Cauchy integral operators with piecewise continuous coefficients on reflexive Orlicz spaces, and he has presented the necessary conditions for Fredholmness of singular integral operators in reflexive weighted rearrangement-invariant spaces.
During the last 30 years, there has been a substantial increase in interest in the numerical solution of the Fredholm singular integral and integro-differential equations with Cauchy kernel (cf. [35, 40, 41]). These equations have important applications in mathematical physics, applied mathematics, and numerical analysis. The mathematical formulation of physical phenomena often involves Cauchy type, see, for example the excellent book by Muskhelishvili (cf. [110]), and the references therein. Various of Fredholm singular integral equations with Cauchy kernel have been solved numerically in recent times by several authors using approximate methods. Recently, Chakrabarti, and Martha, have developed a straightforward method, involving expansion of the unknown function of a Fredholm integral equation of the second kind, in terms of polynomials, and have used the method of least-squares (cf. [19]). Eshkuvatov et al have described a special approximate method for solving Fredholm

integral equation of the first kind, with Cauchy type (cf. [36]). Elliott, described a classical collocation method for singular integral equations having a Cauchy kernel, and showed that, under reasonable conditions, the approximate solutions converge to the solution of the original equation (cf. [35]). Golberg, analyzed and obtained convergence proofs of some numerical methods, for solving several classes of Cauchy singular integral equations (cf. [40, 41, 42]). In 2007 Mandal and Bera employed a simple method based on polynomial approximation of a function to obtain approximate solution of a class of singular integral equations of the second kind (cf. [100]). Also Many different methods have been developed to obtain an approximate solution of these equations (cf. [12, 91, 123, 131]). Other techniques for solving Fredholm singular integro-differential equations with Cauchy kernel have been presented in several works. Badr, presented a Galerkin approach for solving the linear integro-differential equation of the second kind with Cauchy kernel by using the orthogonal basis of Legendre polynomials (cf. [11]). In 2006 Maleknejad and Arzhang have presented a Taylor-series expansion method for a class of Fredholm singular integro-differential equation with Cauchy kernel, and used the truncated Taylor-series polynomial of the unknown function and transform the integro-differential equation into an nth order linear ordinary differential equation with variable coefficients (cf. [99]). In 2008 Subhra Bhattacharya, and Mandal have presented a method based on polynomial approximation using Bernstein polynomial basis, to obtain approximate numerical solutions of singular integro-differential equations with Cauchy kernel, and compared their numerical results with those obtained by various Galerkin methods (cf. [100]). So the polynomial approximation play an important role in the numerical computation of the integral and integro-differential equations (cf. [35, 40, 41, 120, 121]).
The purpose of this thesis, is to develop and illustrate various new methods for solving many classes of Cauchy singular integral and integro-differential equations. This work is organized as follows:

In the beginning, we briefly recall a few basic concepts from general theoretical framework, such as bounded and compact operators, Hilbert spaces and adjoint operators, spectral theory framework, convergence of operators, approximation based on projections, and some classification of integral equations.

In chapter one, we study the successive approximation method for solving a Cauchy singular integral equations of the first kind in the general case. We prove the convergence of the method in this general case. The proposed method has been tested for two kernels which are particularly important in practice.

In chapter two, we present two methods for solving Cauchy integral equation of the second kind: Firstly we present a collocation method based on trigonometric polynomials combined with a regularization procedure, for solving Cauchy integral equations of the second kind, in $L^2([0, 2\pi], \mathbb{C})$. A system of linear equations is involved. We prove the existence of the solution for a double projection scheme, and we perform the error analysis. Numerical examples illustrate the theoretical results. Secondly we solve directly Cauchy integral equation on the real line using Fourier expansion in Sobolev spaces.

The purpose of chapter three, is to approximate the solution of an operator equation involving a non compact bounded operator in Hilbert spaces, using projection methods. We prove the existence of the solution for the approximate equation, and we perform the error analysis. We apply the method for solving the Cauchy integral equations in $L^2(0,1)$ for two cases : Galerkin projections and Kulkarni projections respectively, using a sequence of orthogonal finite rank projections. Numerical examples illustrate the theoretical results.

In chapter four, we introduce a collocation method for Cauchy integro-differential equations, using airfoil polynomials of the first kind. According to this method, we obtain a system of linear equations. We give some sufficient conditions for the convergence of this method. In the end, we investigate the computational performance of our approach through some numerical examples. w In chapter five, we propose two methods for solving integro-differential equations with Cauchy kernel: First, we present a modified projection method based on Legendre polynomials. A system of linear equations is solved. Second, we present a Sloan projection method for solving integro-differential equations with Cauchy kernel, using Legendre polynomials. We give numerical examples.

The last chapter deals with regularization for Cauchy integral equation of the second kind. We apply three projection methods to the regularized equation. First we use Kantorovich projection, and perform the error analysis. After we study the Sloan projection and prove some results about the error analysis. Finally Galerkin projection is established and its error analysis is discussed.

Preliminaries

We begin by recalling briefly a few basic concepts from general theoretical framework, such as bounded and compact operators, Hilbert spaces and adjoint operators, spectral theory framework, convergence of operators, approximation based on projections, and some classification of integral equations.

Linear Operators

Let X be a Banach space, and let T be a linear operator defined on X. By the nullspace of T, $\mathcal{N}(T)$ we mean the space of vectors annihilated by T, so

$$\mathcal{N}(T) := \{\varphi \in X : T\varphi = 0\}.$$

The image (or range) of T is defined by

$$\mathcal{R}(T) := \{T\varphi : \varphi \in X\}.$$

A linear operator T from a normed space X into a normed space Y is called bounded if there exists a positive number M such that

$$\|Tx\| \leq M \|x\| \text{ for all } x \in X,$$

and

$$\|T\| := \sup_{\|x\| \leq 1} \|Tx\|,$$

is the norm of T. $BL(X, Y)$ will denote the space of bounded linear operators from X into Y, and $BL(X)$ those from X into itself.

T is called compact if it maps each bounded set in X into a relatively compact set in Y. That is T is compact if the set $\{Tx : \|x\| \leq 1\}$ has compact closure in Y.

We recall that a subset U of a normed space X is called compact if every open covering of U contains a finite subcovering.

A subset of a normed space is called relatively compact if its closure is compact.

Theorem 1. *A bounded subset of a finite-dimensional normed space is relatively compact.*

Proof. See [91]. □

We have also:

1. Compact linear operators are bounded.

2. A linear operator T from a finite-dimensional normed space X into a normed space Y is bounded.

3. The identity operator $I : X \to X$ is compact if and only if X has finite dimension.

4. A bounded operator T from a normed space X into a normed space Y, with finite-dimensional range $T(X)$ is compact. Such an operator is called a finite-rank one.

5. Linear combinations of compact operators are compact.

6. If (T_n) is a sequence of compact operators and $\|T_n - T\| \to 0$ as $n \to \infty$, then T is compact.

Theorem 2. *Let X be a Banach space, and let T be a bounded linear operator from X into X, with*

$$\|T\| < 1.$$

Then the $I - T$ has a bounded inverse on X that is given by the Neumann series

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j,$$

and satisfies

$$\left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}.$$

Proof. See [91]. □

A Banach space X is called a Hilbert space if the norm on X is induced by an inner product, that is, by a Hermitian positive definite sesquilinear form $\langle .,. \rangle$, as follows:

$$\|x\| := \langle x, x \rangle^{1/2} \text{ for } x \in X.$$

Let X be a Hilbert space and $T \in BL(X)$. The adjoint of T is the unique operator $T^* \in BL(X)$ such that

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^*\psi \rangle \text{ for all } \varphi, \psi \in X.$$

We say that T is normal if $T^*T = TT^*$, and that T is selfadjoint if $T^* = T$.

The folowing results hold in a Hilbert space X:

1. Schwarz Inequality: For all $x, y \in X$,

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| .$$

2. Let $T \in BL(X)$, then

$$\|T\| = \|T^*T\|^{1/2} .$$

3. Let $T \in BL(X)$, then

$$\mathcal{N}(T) = \mathcal{R}(T^*)^{\perp} \text{ and } \overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^{\perp}.$$

4. Suppose that $T \in BL(X)$ is compact. The linear equation

$$(T - zI)x = y,$$

has a unique solution $x \in X$ in the case where the corresponding homogenous equation

$$(T - zI)x = 0,$$

has only the trivial solution.

Spectral Theory Framework

Let $T \in BL(X)$, we recall the following definitions:

The resolvent of T is defined by

$$\mathrm{re}\,(T) := \{z \in \mathbb{C} : T - zI \text{ is invertible}\}.$$

For $z \in \mathrm{re}\,(T)$, the resolvent operator of T at z is defined by

$$\mathrm{R}(T, z) := (T - zI)^{-1}.$$

The spectrum of T is the set

$$\mathrm{sp}\,(T) := \{z \in \mathbb{C} : z \notin \mathrm{re}\,(T)\}.$$

If X is finite dimensional, then $\mathrm{sp}\,(T)$ consists of the eigenvalues of T.

The spectral radius of T is defined by

$$\rho(T) := \sup\{|\lambda| : \lambda \in \mathrm{sp}\,(T)\}.$$

We have:

1. If $z \in \mathrm{re}\,(T)$, then $\mathrm{R}(T, z) \in BL(X)$.

2. If $T^* = T$, then $\mathrm{sp}\,(T) \subseteq \mathbb{R}$.

3. If $T^* = -T$, then $\mathrm{sp}\,(T) \subseteq \mathrm{i}\mathbb{R}$.

4. If $T \in BL(X)$ is normal, then $\rho(T) = \|T\|$.

5. Let $T \in BL(X)$. If $z \in \mathbb{C}$ is given, then for every $y \in X$, the linear equation

$$(T - zI)x = y,$$

has a unique solution $x \in X$ determined by y if and only if $z \in \mathrm{re}\,(T)$.

Sequences of Operators

Let $(T_n)_{n\geq 1}$ be a sequence of bounded linear operators on a Banach space X. Let $T \in BL(X)$. Unless otherwise mentioned, the convergence is as $n \to \infty$.

Let us consider three well-known modes of convergence.

The pointwise convergence, denoted by $T_n \xrightarrow{p} T$:

$$\|T_n x - Tx\| \to 0 \text{ for every } x \in X.$$

The norm convergence, denoted by $T_n \xrightarrow{n} T$:

$$\|T_n - T\| \to 0.$$

The collectively compact convergence, denoted by $T_n \xrightarrow{cc} T$:

$T_n \xrightarrow{p} T$, and for some positive integer n_0, the set

$$\cup_{n\geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\},$$

is a relatively compact subset of X.

If $T_n \xrightarrow{n} T$ or $T_n \xrightarrow{cc} T$, then clearly $T_n \xrightarrow{p} T$. But the converse is not true.

In [4], the authors have studied a new mode of convergence called: the ν-convergence, denoted by $T_n \xrightarrow{\nu} T$:

$$(\|T_n\|) \text{ is bounded }, \quad \|(T_n - T)T\| \to 0, \quad \|(T_n - T)T_n\| \to 0.$$

Theorem 3. *1. If $T_n \xrightarrow{n} T$, then $T_n \xrightarrow{\nu} T$. Conversely, if $0 \notin \text{sp}(T)$ and $T_n \xrightarrow{\nu} T$, then $T_n \xrightarrow{n} T$.*

2. If $T_n \xrightarrow{cc} T$ and T is compact, then $T_n \xrightarrow{\nu} T$.

3. Let $T_n \xrightarrow{\nu} T$ and $U_n \xrightarrow{\nu} U$. Then $T_n + U_n \xrightarrow{\nu} T + U$ if and only if $(T_n - T)U \xrightarrow{n} 0$.

Proof. See [4] □

Approximation Based on Projections

Let (π_n) be a sequence of bounded projections defined on X, that is each π_n is bounded linear operator and $\pi_n^2 = \pi_n$, hence $\|\pi_n\| \geq 1$.

The following three conditions are equivalent one with each other if π_n is a bounded projection defined on Hilbert space X:

$$\pi_n^* = \pi_n, \quad \|\pi_n\| \leq 1, \quad \mathcal{N}(\pi_n) = \mathcal{R}(\pi_n)^{\perp}.$$

If one of these conditions is satisfied, then π_n is called an orthogonal projection.

Define

$$T_n^P := \pi_n T, \quad T_n^S := T\pi_n, \quad T_n^G := \pi_n T \pi_n, \quad T_n^K := \pi_n T + T\pi_n - \pi_n T \pi_n.$$

The bounded operators $T_n^P, T_n^S, T_n^G, T_n^K$, are known as the projection approximation of T, the Sloan approximation of T, the Galerkin approximation of T, and the Kulkarni approximation of T, respectively.

Theorem 4. *Let $T \in BL(X)$ and $\pi_n \xrightarrow{p} I$. Then*

1. $T_n^P \xrightarrow{p} T$, $T_n^S \xrightarrow{p} T$ *and* $T_n^G \xrightarrow{p} T$.
2. *If T is a compact operator, then* $T_n^P \xrightarrow{n} T$, $T_n^S \xrightarrow{\nu} T$, $T_n^G \xrightarrow{\nu} T$.
3. *If T is a compact operator and* $\pi_n^* \xrightarrow{p} I^*$, *then* $T_n^S \xrightarrow{n} T$, $T_n^G \xrightarrow{n} T$.

Proof. See [4] □

Classification of Integral Equations

An integral equation is an equation for an unknown fuction φ, where φ appears also under the integral sign. The classification of integral equations centres on many basic characteristics:

1. Limits of integration
   - Both fixed: Fredholm equation.
   - One variable: Volterra equation.
2. Placement of the unknown function
   - Only inside integral: First kind.
   - Both inside and outside integral: Second kind.

   The Fredholm integral equation of the first kind is represented by

   $$\int_a^b k(s,t)\varphi(t)dt = g(s), \quad a \leq s \leq b.$$

   The Volterra integral equation of the second kind is represented by

   $$\varphi(s) = \int_a^s k(s,t)\varphi(t)dt + g(s), \quad a \leq s \leq b.$$

3. Nature of the known function
   - Identically zero: Homogeneous.
   - Not identically zero: Inhomogeneous.

   For example, the equation

   $$\varphi(s) = \int_a^b k(s,t)\varphi(t)dt, \quad a \leq s \leq b,$$

   is referred to as the homogeneous Fredholm integral equation of the second kind.

4. Linearity: The equation is linear with respect to the unknown function or not.

- Linear integral equations.
- Nonlinear integral equations.

For example, the equation

$$\varphi(s) - \int_a^s k(s, t, \varphi(t))dt = g(s), \quad a \leq s \leq b,$$

is a nonlinear Volterra integral equation of the second kind, where k is not a linear function of its third variable.

5. Depending on the kind of the integral

- Regular integral equations.
- Singular integral equations.

A special and important example of a singular integral equation is the Cauchy or strongly singular integral equation. In this case the integral must be understood as the Cauchy principal value.

An example of a Cauchy singular kernel is

$$k(s, t) = \frac{1}{s - t}, \quad s \neq t.$$

Sometimes, integral equations occur with additional derivatives of the unknown function (under the integral or ouside). In this case, the problem is called an integro-differential equation.

For example, the equation

$$\varphi'(s) + \oint_a^b \frac{\varphi(t)}{t - s} dt = f(s), \quad a \leq s \leq b,$$

is an integro-differential equation with Cauchy kernel.

Chapter 1

Solving Cauchy Integral Equations of the First Kind by Iterations

1.1 Introduction

The successive approximation method is applied for the first time by N.I. Ioakimidis (cf. [49]), to solve practical cases of a Cauchy singular integral equation: the airfoil one (cf. [131]). In this chapter we study a more general case. We prove the convergence of the method in this general case. The proposed method has been tested for two kernels which are particularly important in practice.
Cauchy singular integral equations of the first kind are often encountered in contact and fracture problems in solid mechanics. Sokhottski, Harnack, Mushkelishvili (cf. [110]), Privalov, Magnaradze, Mikhlin, Khvedelidze, Vekua, Kupradze, Gakhov, Golberg, Elliott, Srivastav, Sesko, Erdogan, Junghannes, Linz, Ioakimidis and others have investigated such type of equation. The solutions of these problems may be obtained analytically using the theory developed by Mushkelishvili. Cauchy integral equations are usually difficult to solve analytically, and it is required to obtain approximate solutions. So many different methods have been developed to obtain an approximate solution of a Cauchy integral equation (cf. [12], [91], [123]). In 1988, Ioakimidis solved the airfoil equation with the successive approximation method for the first time. In this chapter this method is applied for solving a Cauchy singular integral equations of the first kind in the general case. The convergence of the method is investigated.

1.2 Development of the Method

We consider the Cauchy integral equation of the first kind

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\varphi(t)}{t-x}dt = g(x), \quad -1 < x < 1, \tag{1.1}$$

where v and g are known functions and φ is the unknown. We shall assume that g has derivatives of all orders on $[-1,1]$, and that

$$\begin{aligned}
\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)}{t-x}dt &= 1, \quad -1<x<1, && (1.2)\\
\frac{1}{\pi}\int_{-1}^{1}|v(t)|dt &\leq 1, \quad -1<x<1, && (1.3)\\
\frac{1}{\pi}\oint_{-1}^{1}\frac{1}{v(t)(t-x)}dt &= 1, \quad -1<x<1, && (1.4)\\
\frac{1}{\pi}\int_{-1}^{1}\frac{1}{|v(t)|}dt &\leq 1, \quad -1<x<1.
\end{aligned}$$

We substract the singularity of equation (1.1) at $t=x$ as follows:

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\varphi(x)}{t-x}dt+\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi(t)-\varphi(x)\right]}{t-x}dt=g(x), \quad -1<x<1. \tag{1.5}$$

Using (1.2) we rewrite equation (1.5) as:

$$\varphi(x)+\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi(t)-\varphi(x)\right]}{t-x}dt=g(x), \quad -1<x<1.$$

We obtain

$$\varphi(x)=g(x)-\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi(t)-\varphi(x)\right]}{t-x}dt, \quad -1<x<1.$$

Now, we apply to this equation the successive approximation method:

$$\varphi_{n+1}(x)=g(x)-\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi_n(t)-\varphi_n(x)\right]}{t-x}dt, \quad -1<x<1.$$

Using (1.2), we have:

$$\varphi_{n+1}(x)=\varphi_n(x)+g(x)-\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\varphi_n(t)}{t-x}dt, \quad -1<x<1, \tag{1.6}$$

where

$$\varphi_0(x)=0, \quad -1<x<1.$$

Let

$$R_n(x)=\varphi(x)-\varphi_n(x) \tag{1.7}$$

From (1.6) we obtain, for $-1<x<1$,

$$\begin{aligned}
\varphi_{n+1}(x)-\varphi(x) &= \varphi_n(x)-\varphi(x)+g(x)\\
&\quad -\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi_n(t)-\varphi(t)\right]}{t-x}dt\\
&\quad -\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\varphi(t)}{t-x}dt.
\end{aligned}$$

From (1.7) and (1.1),

$$-R_{n+1}(x) = -R_n(x) + \frac{1}{\pi}\oint_{-1}^{1} \frac{v(t)R_n(t)}{t-x}dt, \quad -1 < x < 1.$$

Then

$$R_{n+1}(x) = R_n(x) - \frac{1}{\pi}\oint_{-1}^{1} \frac{v(t)R_n(t)}{t-x}dt, \quad -1 < x < 1. \tag{1.8}$$

Now, using (1.2), equation (1.8) becomes:

$$R_{n+1}(x) = -\frac{1}{\pi}\oint_{-1}^{1} \frac{v(t)\left[R_n(t) - R_n(x)\right]}{t-x}dt, \quad -1 < x < 1,$$

and hence

$$R'_{n+1}(x) = -\frac{1}{\pi}\oint_{-1}^{1} v(t)\frac{R_n(t) - R_n(x) - (t-x)R'_n(x)}{(t-x)^2}dt, \quad -1 < x < 1,$$

But following Taylor's theorem with integral remainder,

$$R_n(t) - R_n(x) - (t-x)R'_n(x) = \int_0^1 (1-s)R''_n(x + s(t-x))(t-x)^2 ds.$$

Hence,

$$\|R'_{n+1}\| \leq \frac{1}{2}\|R''_n\|.$$

Recursively,

$$\|R^{(j)}_{n+1}\| \leq \frac{1}{j+1}\|R^{(j+1)}_n\|, \quad j \in \mathbb{N}.$$

Thus

$$\|R^{(j)}_{n-j+1}\| \leq \frac{1}{j+1}\|R^{(j+1)}_{n-j}\|, \quad j \in \mathbb{N}.$$

Multiplying memberwise for $j \in [\![0, n]\!]$, we get

$$\|R_{n+1}\| \leq \frac{1}{(n+1)!}\|R^{(n+1)}_0\|,$$

but from (1.7),

$$\|R^{(n+1)}_0\| = \|\varphi^{(n+1)}\|.$$

So

$$\|R_n\| \leq \frac{1}{n!}\|\varphi^{(n)}\|, \quad n \in \mathbb{N}.$$

From (1.1) and by the Sohngen inversion formula,

$$\varphi(x) = -\frac{1}{\pi}\oint_{-1}^{1} \frac{g(t)}{v(t)\,(t-x)}dt, \quad -1 < x < 1. \tag{1.9}$$

Equation (1.9) takes the form

$$\varphi(x) = -g(x) - \frac{1}{\pi}\oint_{-1}^{1} \frac{g(t)-g(x)}{v(t)\,(t-x)}dt, \quad -1 < x < 1.$$

By standard calculus,

$$\|\varphi^{(n)}\| \leq \|g^{(n)}\| + \frac{1}{n+1}\|g^{(n+1)}\|.$$

Using (1.4),

$$\|R_n\| \leq \frac{1}{n!}\|g^{(n)}\| + \frac{1}{(n+1)!}\|g^{(n+1)}\|, \quad n \in \mathbb{N}.$$

Hence, if

$$\lim_{n\to\infty}\left(\frac{1}{n!}\|g^{(n)}\| + \frac{1}{(n+1)!}\|g^{(n+1)}\|\right) = 0,$$

then the successive approximation method converges.

1.3 Numerical Experiments

The proposed method has been tested for the two following kernels which are particularly important in practice:

1.3.1 Case A:

Let

$$v(t) = \sqrt{\frac{1+t}{1-t}}.$$

All the hypotheses on v are satisfied. From (1.6) we obtain

$$\varphi_{n+1}(x) = \varphi_n(x) + g(x) - \frac{1}{\pi}\oint_{-1}^{1} \frac{\varphi_n(t)}{t-x}\sqrt{\frac{1+t}{1-t}}\,dt, \quad -1 < x < 1. \tag{1.10}$$

But (cf. [94]),

$$\frac{1}{\pi}\oint_{-1}^{1} \frac{\varphi(t)}{t-x_i}\sqrt{\frac{1+t}{1-t}}\,dt = \sum_{j=1}^{m} \frac{2\,(1+t_j)}{2m+1}\,\frac{\varphi(t_j)}{t_j-x_i},$$

where the points t_j and x_i are given by

$$\begin{aligned} t_j &= \cos\left(\frac{2j-1}{2m+1}\pi\right), \quad j \in [\![1, m]\!], \\ x_i &= \cos\left(\frac{2i\pi}{2m+1}\right), \quad i \in [\![1, m]\!]. \end{aligned}$$

From (1.10),

$$\begin{aligned}\varphi_{n+1}(x_i) &= \varphi_n(x_i) + g(x_i) - \sum_{j=1}^{m} \frac{2\,(1+t_j)}{2m+1}\frac{\varphi_n(t_j)}{t_j - x_i}, \quad i \in [\![1, m]\!],\\ \varphi_{n+1}(t_k) &= \varphi_n(t_k) + g(t_k) - \sum_{p=1}^{m} \frac{2\,(1+x_p)}{2m+1}\frac{\varphi_n(x_p)}{x_p - t_k}, \quad k \in [\![1, m]\!].\end{aligned}$$

1.3.2 Case B:

Let

$$v(t) = \sqrt{\frac{1-t}{1+t}}.$$

All the hypotheses on v are satisfied. This case has been investigated by Ioakimidis in 1988. From (1.6),

$$\varphi_{n+1}(x) = \varphi_n(x) - g(x) + \frac{1}{\pi}\oint_{-1}^{1} \frac{\varphi_n(t)}{t-x}\sqrt{\frac{1-t}{1+t}}\,dt, \quad -1 < x < 1. \tag{1.11}$$

But

$$\frac{1}{\pi}\oint_{-1}^{1} \frac{\varphi(t)}{t-x_i}\sqrt{\frac{1-t}{1+t}}\,dt = \sum_{j=1}^{n} \frac{2\,(1-t_j)}{2m+1}\frac{\varphi(t_j)}{t_j - x_i},$$

where the points t_j and x_i are given by

$$\begin{aligned} t_j &= \cos\left(\frac{2j}{2m+1}\pi\right), \quad j \in [\![1, m]\!],\\ x_i &= \cos\left(\frac{2i-1}{2m+1}\pi\right), \quad i \in [\![1, m]\!].\end{aligned}$$

From (1.11),

$$\begin{aligned}\varphi_{n+1}(x_i) &= \varphi_n(x_i) - g(x_i) + \sum_{j=1}^{m} \frac{2\,(1-t_j)}{2m+1}\frac{\varphi_n(t_j)}{t_j - x_i}, \quad i \in [\![1, m]\!],\\ \varphi_{n+1}(t_k) &= \varphi_n(t_k) - g(t_k) + \sum_{p=1}^{m} \frac{2\,(1-x_p)}{2m+1}\frac{\varphi_n(x_p)}{x_p - t_k}, \quad k \in [\![1, m]\!].\end{aligned}$$

Chapter 2

Fourier Expansion for Cauchy Integral Equations of the Second Kind

2.1 Introduction

In the first section of this chapter we present a collocation method based on trigonometric polynomials combined with a regularization procedure, for solving Cauchy integral equations of the second kind, in $L^2([0, 2\pi], \mathbb{C})$. A system of linear equations is involved. We prove the existence of the solution for a double projection scheme, and we perform the error analysis. Some numerical examples illustrate the theoretical results. In second section we present a direct method for solving Cauchy integral equation on the real line.

Cauchy integral equations appear in many applications in scientific fields such as unsteady aerodynamics and aero elastic phenomena, visco elasticity, fluid dynamics, electrodynamics. There is a theoretical study on some kind of Cauchy integral equations in [110]. Many Cauchy integral equations are difficult to solve analytically, and it is required to obtain approximate solutions. In ([128]), the author has studied a reduction of some class of singular integral equations to regular Fredholm integral equations in $L^p([-1, 1], \mathbb{C})$. The purpose of this chapter is firstly to approximate the solution of a Cauchy integral equation of the second kind in $L^2([0, 2\pi], \mathbb{C})$, using collocation, trigonometric polynomials and a regularization procedure, secondly to solve directly Cauchy integral equation on the real line using Fourier expansion in Sobolev spaces.

2.2 Collocation Method for Cauchy Integral Equations Using Trigonometric Polynomials in $L^2([0,2\pi],\mathbb{C})$

2.2.1 Description of The Method

For each nonzero real constant μ, and each real function f, consider the problem of finding a function φ, such that

$$\mu\varphi(s) - \oint_0^{2\pi} \frac{\varphi(t)}{t-s}dt = f(s), \quad 0 \le s \le 2\pi, \tag{2.1}$$

where the integral is understood to be the Cauchy principal value:

$$\oint_0^{2\pi} \frac{\varphi(t)}{t-s}dt = \lim_{\epsilon\to 0}\Big[\int_0^{s-\epsilon} \frac{\varphi(t)}{t-s}dt + \int_{s+\epsilon}^{2\pi} \frac{\varphi(t)}{t-s}dt\Big].$$

Equation (2.1) is a Cauchy integral equation of the second kind. Letting

$$T\varphi(s) := \oint_0^{2\pi} \frac{\varphi(t)}{t-s}dt, \quad 0 \le s \le 2\pi,$$

equation (2.1) reads as

$$\mu\varphi - T\varphi = f.$$

Theorem 5. *For each $f \in L^2([0,2\pi],\mathbb{C})$, equation (2.1) has a unique solution $\varphi \in L^2([0,2\pi],\mathbb{C})$, and the Cauchy integral operator T is bounded and skew-Hermitian from $L^2([0,2\pi],\mathbb{C})$ into itself.*

Proof. See [124]. □

Let $X := L^2([0,2\pi],\mathbb{C})$, and X_n denote the space spanned by the first $2n+1$ of trigonometric polynomials. Define σ_n to be the orthogonal projection from X onto X_n. Hence, for $\psi \in X$,

$$\lim_{n\to\infty} \|\sigma_n\psi - \psi\|_2 = 0.$$

Let be $0 \le s_{n,1} < s_{n,2} < \ldots < s_{n,2n+1} \le 2\pi$. For each $i \in [\![1, 2n+1]\!]$ consider the hat function $e_{n,i}$ in $C^0([0,2\pi],\mathbb{C})$, such that, for each $j \in [\![1, 2n+1]\!]$,

$$e_{n,i}(s_{n,j}) = \delta_{i,j}.$$

Let Y_n be the space spanned by these hat functions, which has dimension $2n+1$. Define the interpolatory projection operator π_n from $C^0([0,2\pi],\mathbb{C})$ onto Y_n:

$$\pi_n h(s) := \sum_{j=1}^{2n+1} h(s_{n,j})e_{n,j}(s), \quad h \in C^0([0,2\pi],\mathbb{C}).$$

We recall that (see [4, 8]):

$$\lim_{n\to\infty} \|\pi_n h - h\|_\infty = 0.$$

Define the regularized operator T_ϵ for $\epsilon > 0$:

$$T_\epsilon\varphi(s) := \oint_0^{2\pi} \frac{(t-s)\varphi(t)}{(t-s)^2+\epsilon^2}dt, \quad 0 \le s \le 2\pi,$$

which is compact and skew-Hermitian from $L^2([0,2\pi],\mathbb{C})$ into itself. Let φ_ϵ be the solution of the regularized integral equation

$$(\mu I - T_\epsilon)\varphi_\epsilon = f,$$

and consider the approximate operator

$$T_{\epsilon,n} := \pi_n T_\epsilon \sigma_n.$$

Theorem 6. *For n large enough, the operator $\mu I - T_{\epsilon,n}$ is invertible, the constant*

$$\beta_\epsilon := \sup_n \|(\mu I - T_{\epsilon,n})^{-1}\|$$

is finite, and the solution $\psi_{\epsilon,n}$ of the equation

$$(\mu I - T_{\epsilon,n})\psi_{\epsilon,n} = f,$$

converges to the solution φ of equation (2.1) if, first, $n \to \infty$ and then $\epsilon \to 0$.

Proof. Since T_ϵ is compact, the theory developped in [4, 8] shows that the inverse operator $(I - T_{\epsilon,n})^{-1}$ exists and is uniformly bounded for n large enough. Since

$$\begin{aligned}
\psi_{\epsilon,n} - \varphi_\epsilon &= [(\mu I - T_{\epsilon,n})^{-1} - (\mu I - T_\epsilon)^{-1}]f \\
&= (\mu I - T_{\epsilon,n})^{-1}[T_\epsilon - T_{\epsilon,n}](\mu I - T_\epsilon)^{-1}f \\
&= (\mu I - T_{\epsilon,n})^{-1}[T_\epsilon - T_{\epsilon,n}]\varphi_\epsilon,
\end{aligned}$$

we get

$$\|\psi_{\epsilon,n} - \varphi_\epsilon\|_2 \le \beta_\epsilon \|(T_\epsilon - T_{\epsilon,n})\varphi_\epsilon\|_2 \to 0 \text{ as } n \to \infty.$$

Since T_ϵ is skew-Hermitian,

$$\|(\mu I - T_\epsilon)^{-1}\| \le \frac{1}{|\mu|},$$

independently of ϵ. Hence, the constant

$$\gamma := \sup_\epsilon \|(\mu I - T_\epsilon)^{-1}\|$$

is finite and from

$$\varphi_\epsilon - \varphi = (\mu I - T_\epsilon)^{-1}[T - T_\epsilon]\varphi$$

we get

$$\|\varphi_\epsilon - \varphi\|_2 \le \gamma \|[T - T_\epsilon]\varphi\|_2 \to 0 \text{ as } \epsilon \to 0.$$

Hence

$$\|\psi_{\epsilon,n} - \varphi\|_2 \le \|\varphi_\epsilon - \varphi\|_2 + \|\psi_{\epsilon,n} - \varphi_\epsilon\|_2 \to 0,$$

if, first, $n \to \infty$, and then $\epsilon \to 0$. □

The collocation method leads to the following linear system

$$(\mu I - T_\epsilon\sigma_n)\psi_{\epsilon,n}(s_{n,i}) = f(s_{n,i}), \quad i \in [\![1, 2n+1]\!].$$

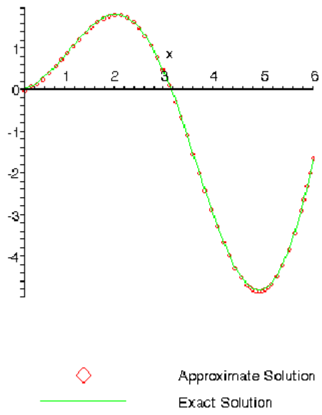

Figure 2.1: $n = 9$

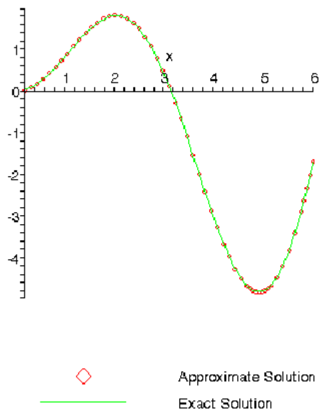

Figure 2.2: $n = 23$

2.2.2 Numerical Example

Let $\mu = -1$ and, for $s \in [0, 2\pi]$,

$$f(s) := -s[\sin s + \text{Si}\ (s) \cos s - \sin(s)\text{Ci}\ (s) - \text{Si}\ (s - 2\pi) \cos s + \text{Ci}\ (2\pi - s) \sin s],$$

where Si is the sine integral function, and Ci is the cosine integral function. The exact solution of equation (2.1) is then

$$\varphi(s) := s \sin s, \quad 0 \leq s \leq 2\pi.$$

For the regularization process, take $\epsilon = 10^{-4}$. For the numerical approximation take $n = 9$, $n = 23$ and $n = 62$. The results are exhibited in figures 2.1, 2.2 and 2.3 respectively.

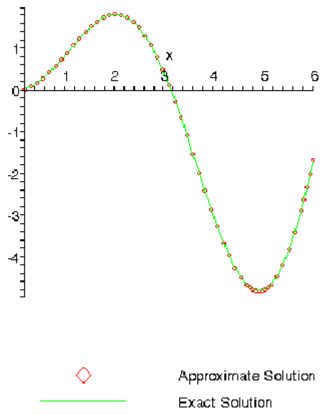

Figure 2.3: $n = 62$

2.3 Direct Solution of Cauchy Integral Equation on the Real Line

We consider the Cauchy integral equation of the second kind on the real line

$$\varphi(x) + \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt = g(x), \quad x \in \mathbb{R}. \tag{2.2}$$

We assume that g is 2π-periodic.
Let

$$\phi_m(t) = e^{\mathrm{i}mt}, \quad m \in \mathbb{Z}.$$

Theorem 7. *For $m \in \mathbb{Z}$, we have*

$$\oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t-x} dt = -\mathrm{i}\pi\phi_m(x), \quad x \in \mathbb{R}.$$

Proof. We have

$$\oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t-x} dt = m \oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{mt-mx} dt,$$

hence

$$\oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t-x} dt = \oint_{-\infty}^{+\infty} \frac{e^{\mathrm{i}y}}{y-s} dy.$$

It is well known (cf.[79]) that

$$\oint_{-\infty}^{+\infty} \frac{e^{\mathrm{i}y}}{y-s} dy = -\mathrm{i}\pi e^{\mathrm{i}s},$$

so that

$$\oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t-x} dt = -\mathrm{i}\pi e^{\mathrm{i}mx} = -\mathrm{i}\pi\phi_m(x).$$

□

Let

$$\varphi(x) = \sum_{-\infty}^{+\infty} a_m \phi_m(x).$$

Soient $p > 0$ et $H^p([0, 2\pi], \mathbb{C})$ l'espace de Sobolev classique, for $0 \le p < \infty$, the Sobolev space of all functions $\varphi \in L^2([0, 2\pi], \mathbb{C})$ such that

$$\sum_{-\infty}^{+\infty} (1 + m^2)^p |a_m|^2 < \infty.$$

We introduce the following norm in $H^p([0, 2\pi], \mathbb{C})$:

$$\|\varphi\|_p = \left\{ \sum_{-\infty}^{+\infty} (1 + m^2)^p |a_m|^2 \right\}^{\frac{1}{2}}.$$

Let

$$A\varphi(x) = \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{\varphi(t)}{t - x} dt.$$

Theorem 8. *If $p > q$ the operator A is bounded from $H^p([0, 2\pi], \mathbb{C})$ into $H^q([0, 2\pi], \mathbb{C})$.*

Proof. If

$$\varphi(x) = \sum_{-\infty}^{+\infty} a_m \phi_m(x),$$

we get

$$A\varphi(x) = \sum_{-\infty}^{+\infty} \frac{a_m}{\pi} \oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t - x} dt.$$

But

$$A\phi_m(x) = -\mathrm{i}\phi_m(x),$$

so that

$$\|A\varphi\|_q = \left\{ \sum_{-\infty}^{+\infty} (1 + m^2)^q |a_m|^2 \right\}^{\frac{1}{2}}.$$

Since

$$(1 + m^2)^q \le (1 + m^2)^p,$$

we obtain

$$\sum_{-\infty}^{+\infty} (1 + m^2)^q |a_m|^2 \le \sum_{-\infty}^{+\infty} (1 + m^2)^p |a_m|^2.$$

Thus

$$\|A\varphi\|_q \le \|\varphi\|_p.$$

□

Theorem 9. *If $p > q$ then $H^p([0, 2\pi], \mathbb{C})$ is dense in $H^q([0, 2\pi], \mathbb{C})$, with compact imbedding from $H^p([0, 2\pi], \mathbb{C})$ into $H^q([0, 2\pi], \mathbb{C})$.*

Proof. (see [91]). □

We rewrite equation (2.2) as:

$$\sum_{-\infty}^{+\infty} a_m \left\{ \phi_m(x) + \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{\phi_m(t)}{t - x} \right\} = g(x).$$

Hence

$$\sum_{-\infty}^{+\infty} a_m \left\{ \phi_m(x) - \mathrm{i}\phi_m(x) \right\} = g(x),$$

that is

$$\sum_{-\infty}^{+\infty} a_m \left(1 - \mathrm{i}\right) \phi_m(x) = g(x).$$

On the space $L^2([0, 2\pi], \mathbb{C})$, the inner product

$$\langle \varphi, \psi \rangle = \int_0^{2\pi} \varphi(t)\overline{\psi(t)} dt.$$

$\{\phi_m\}_{m \in \mathbb{Z}}$ is an orthogonal system, so

$$\sum_{-\infty}^{+\infty} a_m \left(1 - \mathrm{i}\right) \langle \phi_m, \phi_k \rangle = \langle g, \phi_k \rangle, \quad k \in \mathbb{Z}.$$

But

$$\langle \phi_m, \phi_k \rangle = \begin{cases} 2\pi & m = k \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$a_k \left(1 - \mathrm{i}\right) \langle \phi_k, \phi_k \rangle = \langle g, \phi_k \rangle.$$

Thus

$$a_k = \frac{1}{2\pi \left(1 - \mathrm{i}\right)} \langle g, \phi_k \rangle.$$

Chapter 3

Two Projection Methods for Skew-Hermitian Operator Equations

3.1 Introduction and Mathematical Background

In this chapter we present a projection method for solving an operator equations with bounded operator in Hilbert spaces. We prove the existence of the solution for the approximate equation, and we perform the error analysis. We apply the method for solving the Cauchy integral equations in $L^2([0,1],\mathbb{C})$ for two cases: Galerkin projections and Kulkarni projections respectivly, using a sequence of orthogonal finite rank projections. Numerical examples illustrate the theoretical results.

Since 1980, many papers have been dedicated to the numerical solution of operator equations in the compact case, using Galerkin and other projection methods. In [4], the authors have studied some finite rank approximations using bounded finite rank projections. In [3], the authors have used a projection approximation for solving weakly singular Fredholm integral equations of the second kind. In [93], the author has proposed a more accurate approximation for compact operator equations. The goal of this chapter is to apply two projection methods to an integral equation with singular kernel. The abstract framework is that of bounded but noncompact skew-Hermitian operators in a Hilbert space. The application will deal with a Cauchy integral equation in $L^2([0,1],\mathbb{C})$ with two discretizations: the classical Galerkin and the new Kulkarni approximations built upon a sequence of orthogonal finite rank sequence of projections.

Let H be a Hilbert space, and T a bounded operator from H into itself. For a given function $f \in H$, we consider the problem of finding a function $\varphi \in H$ such that

$$\varphi - T\varphi = f. \tag{3.1}$$

Let T^* be the adjoint of T. We assume that equation (3.1) has a unique solution $\varphi \in H$, and that T is skew-Hermitian: is $T^* = -T$. Let $(T_n)_{n\geq 1}$ be a sequence of skew-Hermitian operators from H into itself.

Theorem 10. *For all n, the operator $I - T_n$ is invertible, and*

$$\|(I - T_n)^{-1}\| \leq 1.$$

Proof. Since

$$(\mathrm{i}T_n)^* = -\mathrm{i}T_n^* = \mathrm{i}T_n,$$

the operator $\mathrm{i}T_n$ is self-adjoint, and hence $\mathrm{sp}\,(T_n) \subseteq \mathrm{i}\mathbb{R}$, where sp denotes the spectrum. This shows that $1 \notin \mathrm{sp}\,(T_n)$ and hence the operator $I - T_n$ is invertible. On the other hand, for all $x \in H$,

$$\mathrm{Re}\,\langle (I-T_n)x, x\rangle = \frac{1}{2}\left[\langle (I-T_n)x, x\rangle + \overline{\langle (I-T_n)x, x\rangle}\right] = \langle x, x\rangle\,,$$

hence

$$\|x\|^2 \leq |\mathrm{Re}\,\langle (I-T_n)x, x\rangle| \leq |\langle (I-T_n)x, x\rangle| \leq \|(I-T_n)x\|\,\|x\|\,,$$

so $\|(I-T_n)^{-1}\| \leq 1$. □

3.2 Cauchy Integral Equations of the Second Kind

Set $H := L^2([0,1],\mathbb{C})$, and consider the following Cauchy integral equation of the second kind

$$\varphi(s) - \oint_0^1 \frac{\varphi(t)}{t-s}dt = f(s), \quad 0 \leq s \leq 1, \tag{3.2}$$

where f is a known function. The above integral is understood to be the Cauchy principal value:

$$\oint_0^1 \frac{\varphi(t)}{t-s}dt = \lim_{\epsilon\to 0}\left[\int_0^{s-\epsilon} \frac{\varphi(t)}{t-s}dt + \int_{s+\epsilon}^1 \frac{\varphi(t)}{t-s}dt\right].$$

Letting

$$T\varphi(s) := \oint_0^1 \frac{\varphi(t)}{t-s}dt, \quad 0 \leq s \leq 1,$$

equation (3.2) is equivalent to the equation (3.1). We recall that for each $f \in H$, equation (3.2) has a unique solution $\varphi \in H$, and the Cauchy integral operator T is bounded from H into itself, further $T^* = -T$ (see [124]). Let $(s_{n,j})_{j=0}^n$ be a grid on $[0,1]$ such that

$$0 \leq s_{n,0} < s_{n,1} < \ldots < s_{n,n} \leq 1.$$

Set

$$h_{n,i} := s_{n,i} - s_{n,i-1}, \quad i \in [\![1, n]\!], \quad h_n := (h_{n,1}, h_{n,2}, \ldots, h_{n,n}).$$

Let us consider $(\Pi_n)_{n\geq 1}$, a sequence of bounded projections each one of finite rank, such that

$$\Pi_n x := \sum_{j=1}^n \langle x, e_{n,j}\rangle\, e_{n,j},$$

where

$$e_{n,j} := \frac{\phi_{n,j}}{\sqrt{h_{n,j}}}, \quad \phi_{n,j}(s) := \begin{cases} 1 \text{ for } s \in]s_{n,j-1}, s_{n,j}[, \\ 0 \text{ otherwise.}\end{cases}$$

Let

$$J_n := \{s_{n,j}, \quad j \in [\![0, n]\!]\}.$$

Define the modulus of continuity of the function $\psi \in H$ relative to h_n as follows:

$$\omega_2(\psi, J_n) := \sup_{0\leq\delta\leq h_n} \left(\int_0^1 |\psi(\tau+\delta) - \psi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}}.$$

All functions are extended by 0 outside $[0,1]$. We recall that

$$\omega_2(\psi, J_n) \to 0 \text{ as } n \to \infty \text{ for all } \psi \in H,$$

and that, for all $\psi \in H$ (cf. [5]),

$$\|(I - \Pi_n)\psi\|_2 \leq \omega_2(\psi, J_n). \tag{3.3}$$

3.2.1 Galerkin Approximation

In the past two decades, several results have been established for solving compact operator equations using the Galerkin method. In this section we use the Galerkin method for approximate the solution of our bounded equation. Since $\Pi_n^* = \Pi_n$ we get $T_n^* = -T_n$, with $T_n = T_n^G := \Pi_n T \Pi_n$, hence the following Galerkin approximate equation

$$\varphi_n^G - T_n \varphi_n^G = \Pi_n f, \tag{3.4}$$

has a unique solution φ_n^G, given by

$$\varphi_n^G = \sum_{j=1}^n x_{n,j} e_{n,j}$$

for some scalars $x_{n,j}$. Equation (3.4) reads as

$$\sum_{j=1}^n x_{n,j} \left[e_{n,j} - \Pi_n T e_{n,j} \right] = \Pi_n f,$$

so that

$$\sum_{j=1}^n x_{n,j} \left[e_{n,j} - \sum_{i=1}^n \langle T e_{n,j}, e_{n,i} \rangle e_{n,i} \right] = \sum_{i=1}^n \langle f, e_{n,i} \rangle e_{n,i},$$

that is to say, the coefficients $x_{n,j}$ are obtained by solving the following linear system

$$(I - A_n) x_n = b_n,$$

where

$$\begin{aligned} A_n(k,j) &:= \frac{1}{\sqrt{h_{n,j} h_{n,k}}} \int_{s_{n,k-1}}^{s_{n,k}} \oint_{s_{n,j-1}}^{n,s_j} \frac{dt}{t-s} ds, \\ b_n(k) &:= \frac{1}{\sqrt{h_{n,k}}} \int_{s_{k-1}}^{s_k} f(s) ds. \end{aligned}$$

Theorem 11. *The following estimate holds:*

$$\|\varphi_n^G - \varphi\|_2 \leq \omega_2(f, J_n) + \omega_2(T\varphi, J_n) + \pi\omega_2(\varphi, J_n).$$

Proof. In fact

$$\begin{aligned}
\varphi_n^G - \varphi &= (I - T_n^G)^{-1}\Pi_n f - (I - T)^{-1} f \\
&= (I - T_n^G)^{-1}\Pi_n f - (I - T_n^G)^{-1} f + (I - T_n^G)^{-1} f - (I - T)^{-1} f \\
&= (I - T_n^G)^{-1}(\Pi_n - I) f + (I - T_n^G)^{-1}\left[(I - T) - (I - T_n^G)\right](I - T)^{-1} f \\
&= (I - T_n^G)^{-1}\left[(\Pi_n - I) f + (T_n^G - T)\varphi\right].
\end{aligned}$$

It is proved in Theorem 1 that $\left\|(I - T_n^G)^{-1}\right\| \leq 1$. Since

$$(T_n^G - T)\varphi = (\Pi_n - I)T\varphi + \Pi_n T(\Pi_n - I)\varphi,$$

and since $\|\Pi_n\| = 1$ and $\|T\| \leq \pi$ (cf. [124]), then using (3.3), we get the desired result. □

3.2.2 Kulkarni Approximation

In [93] the author has proposed to approximate a linear operator T by the following finite rank operator

$$T_n^K := \Pi_n T + T\Pi_n - \Pi_n T\Pi_n.$$

Theory has been developped for the compact case. In this section, we propose to approximate our noncompact bounded operator T by this finite rank operator. Let φ_n^K be the approximate solution of the equation (3.2) using T_n^K. As in [93], let

$$u_n := \Pi_n \varphi_n^K.$$

Since $\Pi_n u_n = u_n$, there exist scalars $c_{n,j}$ such that

$$u_n = \sum_{j=1}^{n} c_{n,j} e_{n,j}.$$

Following [93],

$$u_n - \left[\Pi_n T\Pi_n + \Pi_n T(I - \Pi_n)T\Pi_n\right] u_n = \Pi_n f + \Pi_n T(I - \Pi_n) f,$$

so that

$$\sum_{j=1}^{n} c_{n,j}\left[e_{n,j} - (\Pi_n T e_{n,j} + \Pi_n T(I - \Pi_n)T e_{n,j})\right] = \Pi_n f + \Pi_n T(I - \Pi_n) f,$$

and hence

$$\sum_{j=1}^{n} c_{n,j}\left[e_{n,j} - \sum_{k=1}^{n}(\langle T e_{n,j}, e_{n,k}\rangle + \langle T(I - \Pi_n)T e_{n,j}, e_{n,k}\rangle) e_{n,k}\right] =$$

$$= \sum_{k=1}^{n} \langle f, e_{n,k} \rangle e_{n,k} + \sum_{k=1}^{n} \langle T(I-\Pi_n) f, e_{n,k} \rangle e_{n,k}.$$

Performing the inner product with $e_{n,i}$, we obtain the linear system

$$c_{n,i} - \sum_{j=1}^{n} c_{n,j} \left[\langle T e_{n,j}, e_{n,i} \rangle + \langle T(I-\Pi_n) T e_{n,j}, e_{n,i} \rangle \right] = \langle f, e_{n,i} \rangle + \langle T(I-\Pi_n) f, e_{n,i} \rangle, \quad i \in [\![1, n]\!],$$

which becomes

$$c_{n,i} - \sum_{j=1}^{n} \left[\langle T e_{n,j}, e_{n,i} \rangle + \left\langle T^2 e_{n,j}, e_{n,i} \right\rangle - \sum_{k=1}^{n} \langle T e_{n,j}, e_{n,k} \rangle \langle T e_{n,k}, e_{n,i} \rangle \right] c_{n,j}$$

$$= \langle f, e_{n,i} \rangle + \langle T f, e_{n,i} \rangle - \sum_{k=1}^{n} \langle f, e_{n,k} \rangle \langle T e_{n,k}, e_{n,i} \rangle, \quad i \in [\![1, n]\!]. \quad (3.5)$$

The following computations are needed:

$$\begin{aligned} \langle T e_{n,j}, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_{s_{j-1}}^{s_j} \frac{dt}{t-s} ds, \\ \left\langle T^2 e_{n,j}, e_{n,i} \right\rangle &= \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_{0}^{1} \frac{1}{t-s} \oint_{s_{j-1}}^{s_j} \frac{d\tau}{\tau - t} dt ds, \\ \langle T f, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_{0}^{1} \frac{f(t)}{t-s} dt ds, \\ \langle f, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} f(s) ds. \end{aligned}$$

Once the system (3.5) is solved, the solution φ_n^K is built through

$$\varphi_n^K = u_n + (I-\Pi_n) T u_n + (I-\Pi_n) f.$$

Hence

$$\varphi_n^K(s) = u_n(s) + \oint_0^1 \frac{u_n(t)}{t-s} dt - \sum_{k=1}^{n} \frac{\phi_{n,k}(s)}{h_{n,k}} \int_{s_{k-1}}^{s_k} \oint_0^1 \frac{u_n(\tau)}{\tau - t} d\tau dt + f(s) - \sum_{k=1}^{n} \frac{\phi_{n,k}(s)}{h_{n,k}} \int_{s_{k-1}}^{s_k} f(t) dt.$$

Theorem 12. *The following estimate holds:*

$$\|\varphi_n^K - \varphi\|_2 \leq \left[2\omega_2(\varphi, J_n) \left\| (I-\Pi_n) T^2 (I-\Pi_n) \varphi \right\|_2 \right]^{\frac{1}{2}}.$$

Proof. On one hand

$$\varphi_n^K - \varphi = (f + T_n^K \varphi_n^K) - (f + T\varphi) = T_n^K (\varphi_n^K - \varphi) + (T_n^K - T)\varphi.$$

Thus

$$(I - T_n^K)(\varphi_n^K - \varphi) = (T_n^K - T)\varphi,$$

so that

$$\varphi_n^K - \varphi = (I - T_n^K)^{-1} (T_n^K - T)\varphi,$$

which lends to

$$\left\|\varphi_n^K - \varphi\right\|_2 \leq \left\|(I - T_n^K)^{-1}\right\| \left\|(T_n^K - T)\varphi\right\|_2 .$$

On the other hand,

$$\left(T_n^K - T\right)\varphi = \left[\Pi_n T(I - \Pi_n) - T(I - \Pi_n)\right]\varphi = -\left(I - \Pi_n\right) T \left(I - \Pi_n\right)\varphi.$$

Since $\left\|(I - T_n^K)^{-1}\right\| \leq 1$,

$$\left\|\varphi_n^K - \varphi\right\|_2 \leq \left\|(I - \Pi_n) T (I - \Pi_n)\varphi\right\|_2 \leq \omega_2(T(I - \Pi_n)\varphi, J_n).$$

But

$$\begin{aligned}
\omega_2^4(T(I-\pi_n)\varphi, J_n) &= \sup_{0\leq\delta\leq h_n} \langle T(I-\pi_n)(\varphi(.+\delta)-\varphi), T(I-\pi_n)(\varphi(.+\delta)-\varphi)\rangle^2 \\
&= \sup_{0\leq\delta\leq h_n} \left\langle \varphi(.+\delta)-\varphi, -(I-\Pi_n)T^2(I-\Pi_n)[\varphi(.+\delta)-\varphi]\right\rangle^2 \\
&\leq \sup_{0\leq\delta\leq h_n} \int_0^1 |\varphi(\tau+\delta)-\varphi(\tau)|^2 d\tau \times \\
&\quad \int_0^1 \left|(I-\Pi_n)T^2(I-\Pi_n)[\varphi(\tau+\delta)-\varphi(\tau)]\right|^2 d\tau \\
&\leq 4\omega_2^2(\varphi, J_n)\left\|(I-\Pi_n)T^2(I-\Pi_n)\varphi\right\|_2^2,
\end{aligned}$$

and we get the desired result. □

3.3 Numerical Examples

Example 3.1

We consider the following Cauchy integral equation

$$\varphi(s) - \oint_0^1 \frac{\varphi(t)}{t-s}dt = s^2 - 2s + \frac{1}{2} + (s^2 - s)\ln\left(\frac{s}{1-s}\right), \quad 0 < s < 1.$$

The right hand side has been built so that the exact solution to this equation be

$$\varphi(s) = s^2 - s.$$

We present in table (3.1) the corresponding absolute errors for this example.

Example 3.2

We consider the following Cauchy integral equation

$$\varphi(s) - \oint_0^1 \frac{\varphi(t)}{t-s}dt = \frac{4 + \ln s + 2\ln 2 + \pi s - 4\ln(1-s)}{4(s^2+1)}, \quad 0 < s < 1.$$

The right hand side has been built so that the exact solution to this equation be

$$\varphi(s) = \frac{1}{s^2+1}.$$

We present in table (3.2) the corresponding absolute errors for this example.

n	$\left\|\varphi - \varphi_n^G\right\|_2$	$\left\|\varphi - \varphi_n^K\right\|_2$
3	6.69e-2	5.83e-3
5	3.82e-2	3.17e-3
7	2.64e-2	2.54e-3
15	1.17e-2	7.17e-4
25	6.88e-3	1.75e-4

Table 3.1: *Example 3.1*

n	$\left\|\varphi - \varphi_n^G\right\|_2$	$\left\|\varphi - \varphi_n^K\right\|_2$
4	4.04e-2	6.44e-3
6	2.63e-2	3.12e-3
10	1.55e-2	1.00e-3
12	1.29e-2	9.54e-4
20	7.70e-3	3.35e-4

Table 3.2: *Example 3.2*

3.4 Conclusions

This work extends the application of projection methods to singular integral equations of Cauchy type. As it has already established for sufficiently differentiable kernels (see [93]), the Kulkarni approximation gives more accurate results than the classical Galerkin approximation. In exchange, from a computational point of view, the complexity of Kulkarni approximation doubles Galerkin's one since one more evaluation of the integral operator is needed to build each coefficient of the matrix associated to the auxiliary linear system.

Chapter 4

Collocation Method for Solving Integro-Differential Equations with Cauchy Kernel

4.1 Introduction

This chapter investigates the numerical solution for a class of integro-differential equations with Cauchy kernel by using airfoil polynomials of the first kind. According to this method, we obtain a system of linear algebraic equations. We give some sufficient conditions for the convergence of this method. In the end, we investigate the computational performance of our approach through some numerical examples. The last two decades have been witnessing a strong interest among physicists, engineers and mathematicians for the theory and numerical modeling of integral and integro-differential equations. These equations are solved analytically see, for example the excellent book by Muskhelishvili (cf. [110]), and the references therein. But only special cases of these equations are solved analytically, so we should solve other classes of these equations by using numerical methods, several methods have been recently developed for the numerical solution of the integro-differential equations. Specifically, in [11], Badr presented a Galerkin approach for solving the integro-differential equation of the second kind with Cauchy kernel by using the orthogonal basis of Legendre polynomials. In [46], the authors have considered a method based on projector-splines for the numerical solution of integro-differential equation. In [99], the authors presented a Taylor-series expansion method for a class of Fredholm singular integro-differential equation with Cauchy kernel, and used the truncated Taylor-series polynomial of the unknown function, and transform the integro-differential equation into a linear ordinary differential equation of order n with variable coefficients. In [15], we find a method based on polynomial approximation using Bernstein polynomial basis, to obtain approximate numerical solution of a singular integro-differential equation with Cauchy kernel, and compared the numerical results obtained with those obtained by various Galerkin methods. A great deal of effort has been made in the development of numerical techniques for the approximate solution of Cauchy integral equations, (cf. [12], [19], [35], [36], [40], [41], [42], [43], [91], [94], [100], [123], [131]). Several methods use polynomial techniques, (cf. [8], [35], [40], [42], [44], [120], [121]). So different kinds of polynomials play an essential role in approximation theory, and have many interesting applications, particularly they may be applied to solve integro-differential equations.

In this chapter, we will propose to employ a method based on the airfoil polynomials of the first kind, for solving the Fredholm singular integro-differential equation with Cauchy kernel.

The chapter is organized as follows: In the next section we will discuss airfoil polynomials and their key properties. In section 2 we give the description and development of the method, and we discuss estimates for the rate of convergence of the method. In section 3 for showing efficiency of this method, we use numerical examples. Section 4 is devoted to the conclusion of this chapter.

We recall that the so-called airfoil polynomials are used as expansion functions to compute the pressure on an airfoil in steady or unsteady subsonic flow.

The airfoil polynomail t_n of the first kind is defined by

$$t_n(x) \quad = \quad \frac{\cos[(n+\frac{1}{2})\arccos x]}{\cos(\frac{1}{2}\arccos x)}.$$

The airfoil polynomail u_n of the second kind is defined by

$$u_n(x) \quad = \quad \frac{\sin[(n+\frac{1}{2})\arccos x]}{\sin(\frac{1}{2}\arccos x)}.$$

4.2 The Approximate Solution

Given a function f and a constant λ, consider the problem of finding a function φ such that

$$\varphi'(x) + \frac{\lambda}{\pi} \oint_{-1}^{1} \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1. \tag{4.1}$$

The above equation called Fredholm integro-differential equation with Cauchy kernel. We will propose an approximate solution for equation (4.1). For this purpose, we will introduce an approximation using the airfoil polynomials of the first kind t_n as

$$\varphi_n(x) = \omega(x) \sum_{i=0}^{n} a_i t_i(x),$$

where

$$\omega(x) = \sqrt{\frac{1+x}{1-x}}.$$

The formula (cf. [33])

$$(1+x)t_i'(x) = (i+\frac{1}{2})u_i(x) - \frac{1}{2}t_i(x),$$

gives

$$\varphi_n'(x) = \sum_{i=0}^{n} a_i \{\omega'(x)t_i(x) + \frac{\omega(x)}{1+x}[(i+\frac{1}{2})u_i(x) - \frac{1}{2}t_i(x)]\}. \tag{4.2}$$

On the other hand (cf. [33]),

$$\frac{1}{\pi}\oint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{t_i(t)}{t-x}dt = u_i(x). \tag{4.3}$$

Consider the set of $n+1$ collocation points x_j, which are the zeros of u_{n+1}:

$$x_j = -\cos\frac{2j-1}{2n+3}\pi, \quad j \in [\![0, n]\!].$$

Let us introduce the following notations

$$(A\varphi)(x) := \varphi'(x), \quad -1 < x < 1.$$

$$(T\varphi)(x) := \frac{\lambda}{\pi}\oint_{-1}^{1}\frac{\varphi(t)}{t-x}dt, \quad -1 < x < 1.$$

Equation (4.1) can be written in the following operator form

$$A\varphi + T\varphi = f.$$

Denote by $C^{0,\lambda}([-1,1],\mathbb{R})$ the space of all functions φ defined on $[-1,1]$ satisfying the following Hölder condition: $\exists M \geq 0$ such that

$$\forall x_1, x_2 \in [-1,1], \quad |\varphi(x_1) - \varphi(x_2)| \leq M\,|x_1 - x_2|^{\lambda},$$

where $0 < \lambda \leq 1$.

Let

$$H := \left\{\varphi \in L^2([-1,1],\mathbb{R}) \ : \ \varphi' \in L^2([-1,1],\mathbb{R}), \quad \varphi(-1) = 0\right\}.$$

The operator T is bounded from $L^2([-1,1],\mathbb{R})$ into itself and also from $C^{0,\lambda}([-1,1],\mathbb{R})$ into itself (cf.[110]). We recall that

$$(A^{-1}y)(s) = \int_{-1}^{s} y(t)dt,$$

and that $A^{-1} : L^2[-1,1] \to H$ is compact (cf.[17]).

Consider hat functions $e_0, e_1, e_2, \ldots, e_n$ in $C^0([-1,1],\mathbb{R})$ such that

$$e_j(x_k) = \delta_{j,k}.$$

Define the projection operators P_n from $C^0([-1,1],\mathbb{R})$ into the space of continuous functions by

$$P_n g(x) := \sum_{j=0}^{n} g(x_j)e_j(x).$$

Let us define the operators

$$D_n := A^{-1}P_nT, \quad D := A^{-1}T.$$

Consider the following approximate equation in the unknown φ_n:

$$\varphi_n + D_n\varphi_n = A^{-1}f.$$

Theorem 13. *Assume that $f \in C^0([-1,1],\mathbb{R})$. There exists a positive constant M, such that*

$$\|\varphi_n - \varphi\|_\infty \leq M\|D_n\varphi - D\varphi\|_\infty.$$

for n large enough.

Proof. It is well-known that $\|P_n g - g\|_\infty \to 0$, for all $g \in C^0([-1,1],\mathbb{R})$. Since A^{-1} is compact, it is clear that D is compact. In (cf. [8] and [91]) it is shown that the inverse operator $(I + D_n)^{-1}$ exists and is uniformly bounded for n large enough. On the other hand,

$$\varphi_n - \varphi = \left[A^{-1}f - D_n\varphi_n\right] - \left[A^{-1}f - D\varphi\right],$$

hence

$$\varphi_n - \varphi = \left[D\varphi - D_n\varphi_n\right].$$

This leads to

$$\varphi_n - \varphi = \left[(D - D_n)\varphi - D_n(\varphi_n - \varphi)\right].$$

Thus

$$(I + D_n)(\varphi_n - \varphi) = (D - D_n)\varphi.$$

Consequently

$$\begin{aligned} \varphi_n - \varphi &= (I + D_n)^{-1}\left[(D - D_n)\varphi\right], \\ \|\varphi_n - \varphi\|_\infty &\leq M\|(D - D_n)\varphi\|_\infty, \end{aligned}$$

where

$$M := \sup_{n \geq N} \left\|(I + D_n)^{-1}\right\|,$$

which is finite. □

Finally, the following system follows:

$$A\varphi_n(x_j) + T\varphi_n(x_j) = f(x_j), \quad j \in [\![0, n]\!].$$

By (4.2) and (4.3),

$$\sum_{i=0}^{n} a_i\{\omega'(x_j)t_i(x_j) + \frac{\omega(x_j)}{1 + x_j}[(i + \frac{1}{2})u_i(x_j) - \frac{1}{2}t_i(x_j)] + \lambda u_i(x_j)\} = f(x_j), \quad j \in [\![0, n]\!].$$

4.3 Numerical Results and Discussion

In order to illustrate the performance of our method, we report in this section, numerical results of some examples, selected integro-differential equations, solved by the method of this study. In these numerical computations each table shows the numerical error of our approximate solution.

x	$n=5$	$n=20$	$n=116$
-0.8	0.1640e-1	0.183e-2	0.16e-3
-0.6	0.1855e-1	0.185e-2	0.15e-3
-0.4	0.2204e-1	0.181e-2	0.2e-4
-0.2	0.2119e-1	0.184e-2	0.12e-3
0.0	0.189e-1	0.19e-2	0.2e-4
0.2	0.1856e-1	0.171e-2	0.3e-4
0.4	0.1992e-1	0.179e-2	0.8e-4
0.6	0.2044e-1	0.174e-2	0.8e-4
0.8	0.2148e-1	0.177e-2	0.9e-4

Table 4.1: *Example 4.1*

x	$n=5$	$n=49$	$n=135$
-0.8	0.737e-2	0.21001e-3	0.40012e-4
-0.6	0.927e-2	0.23e-3	0.80002e-4
-0.4	0.1439e-1	0.17e-3	0.60007e-4
-0.2	0.1171e-1	0.23001e-3	0.90003e-4
0.0	0.59827e-2	0.17379e-3	0.45706e-4
0.2	0.52100e-2	0.14001e-3	0.60001e-4
0.4	0.1147e-1	0.21004e-3	0.60024e-4
0.6	0.1616e-1	0.26002e-3	0.6e-4
0.8	0.282e-2	0.22001e-3	0.30065e-4

Table 4.2: *Example 4.2*

Example 4.1

Let us first consider the following integro-differential equation

$$\varphi'(x) + \frac{1}{\pi^5} \oint_{-1}^{1} \frac{\varphi(t)}{t-x} dt = \frac{1}{\pi^4} x^2 + 2(\frac{1}{\pi^5} + 1)x - \frac{1}{\pi^4}$$

The exact solution is

$$\varphi(x) = x^2 - 1.$$

Table (4.1) gives the numerical results for Example 4.1.

Example 4.2

In this example we consider the following integro-differential equation

$$\varphi'(x) + \oint_{-1}^{1} \frac{\varphi(t)}{t-x} dt = \pi x^3 - 5x^2 - \pi x + \frac{7}{3} + (x^3 - x)\ln(x+1) - (x^3 - x)\ln(x-1).$$

The exact solution for this equation is

$$\varphi(x) = -x^3 + x.$$

Table (4.2) shows the rate of convergence of the method. The results confirm the convergence properties proved above.

4.4 Concluding Remarks

Cauchy kernel are important in many fields of applied mathematics. The method can be developed and applied to other class of integral and integro-differential equations. The advantage of this method is that we can eliminate the singularity, and compute an approximate solution through a system of linear equations.

Chapter 5

Projection Methods for Integro-Differential Equations with Cauchy Kernel

5.1 Introduction and Mathematical Background

In this chapter we present two methods for solving Cauchy integro-differential equations. First, we present a projection method based on Legendre polynomials, for solving integro-differential equations with Cauchy kernel, in $L^2([-1,1],\mathbb{C})$. The proposed numerical procedure leads to solve a system of linear equations. We prove the existence of the solution for the approximate equation, and we perform the error analysis. Numerical examples illustrate the theoretical results. Next, we propose a Sloan projection method for the approximate solution of an integro-differential equations with Cauchy kernel in $L^2([-1,1]\times[-1,1],\mathbb{C})$ using Legendre polynomials. A system of linear equations is to be solved.
The theory of integro-differential equations with Cauchy kernel has important applications in the mathematical modelling of many scientific fields such as fluid dynamics, electrodynamics, elasticity. Many integro-differential equations need to be solved numerically. Several authors have been studied projection approximations for solving integral equations with different numerical procedures, the theory of projection approximations is developed in [4]. In [4], the authors have studied some finite rank approximations using bounded finite rank projections. Projection approximation methods play an essential role in approximation theory, and have many interesting applications, particularly to solve integral equations. In [3], the authors have used a projection approximation for solving weakly singular Fredholm integral equations of the second kind. Let $\mathcal{H} := L^2([-1,1],\mathbb{C})$, be the space of complex-valued Lebesgue square integrable (classes of) functions on $[-1,1]$. The purpose of this chapter is firstly to introduce a projection method based on the Legendre polynomials, for solving integro-differential equations with Cauchy kernel in $\mathcal{H}$. The purpose of the second method is to approximate the solution of integro-differential equations with Cauchy Kernel using Sloan projection in the first time.

Let the universe of our discours be the Hilbert space $\mathcal{H}$. Set

$$\mathcal{D} := \{\varphi \in \mathcal{H} \ : \ \varphi' \in \mathcal{H},\, \varphi(-1) = 0\},$$

and consider the integro-differential equation with Cauchy kernel

$$\varphi'(s) + \oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt = f(s), \quad -1 < s < 1, \tag{5.1}$$

where the integral is understood as the Cauchy principal value:

$$\oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt = \lim_{\epsilon \to 0} \left(\int_{-1}^{s-\epsilon} \frac{x(t)}{t-s} dt + \int_{s+\epsilon}^{1} \frac{x(t)}{t-s} dt \right).$$

Letting

$$\begin{aligned} T\varphi(s) &:= \oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt, \quad -1 < s < 1, \\ A\varphi(s) &:= \varphi'(s), \quad -1 < s < 1, \end{aligned}$$

the operator T is bounded from $\mathcal{H}$ into itself and

$$A^{-1}y(s) = \int_{-1}^{s} y(t) dt, \quad -1 < s < 1,$$

is compact. Equation (5.1) can be rewritten as

$$\varphi + A^{-1}T\varphi = A^{-1}f.$$

Let

$$K := A^{-1}T$$

which is compact. We assume that -1 is not an eigenvalue of K.
Let $(L_n)_{n\geq 0}$ be the sequence of Legendre polynomials which is an orthogonal basis for $\mathcal{H}$:

$$\langle L_j, L_k \rangle = \delta_{jk} \frac{2}{2j+1},$$

and

$$e_j := \sqrt{\frac{2j+1}{2}} L_j,$$

the corresponding normalized sequence.

Let $(\pi_n)_{n\geq 0}$ be the sequence of bounded finite rank orthogonal projections defined by

$$\pi_n x := \sum_{j=0}^{n-1} \langle x, e_j \rangle e_j.$$

Hence, for $\psi \in \mathcal{H}$,

$$\lim_{n\to\infty} \|\pi_n \psi - \psi\| = 0.$$

5.2 A Projection Method for Integro-Differential Equations with Cauchy Kernel

Let $\mathcal{H}_n$ denote the space spanned by the first n of Legendre polynomials. It is clear that $A^{-1}(\mathcal{H}_n) = \mathcal{H}_{n+1}$. The approximate problem is the following equation for φ_n:

$$\varphi_n + A^{-1}\pi_n T\varphi_n = A^{-1}\pi_n f.$$

Clearly $\varphi_n \in \mathcal{D} \cap \mathcal{H}_{n+1}$. We introduce the following notations:

$$K := A^{-1}T, \quad K_n := A^{-1}\pi_n T, \quad g := A^{-1}f, \quad g_n := A^{-1}\pi_n f,$$

and we assume that -1 is not an eigenvalue of K. Hence the equation

$$(I + K)\,\varphi = g,$$

is approximated by

$$(I + K_n)\,\varphi_n = g_n.$$

For all $x \in \mathcal{H}$,

$$\lim_{n\to\infty} \|K_n x - Kx\| = 0,$$

and since A^{-1} is compact,

$$\lim_{n\to\infty} \|\,(K_n - K)\,K\| = 0, \quad \lim_{n\to\infty} \|\,(K_n - K)\,K_n\| = 0.$$

Writing

$$\varphi_n = \sum_{j=0}^{n} x_{n,j} e_j,$$

the $n+1$ unknowns $x_n(j)$ solve

$$\begin{aligned}\sum_{j=0}^{n} x_n(j)\left[e'_j + \pi_n T e_j\right] &= \pi_n f,\\ \sum_{j=0}^{n} x_n(j) e_j(-1) &= 0.\end{aligned}$$

This leads to a linear system

$$A_n x_n = b_n,$$

where, for $i \in [\![0, n-1]\!]$ and $j \in [\![0, n]\!]$,

$$\begin{aligned}A_n(i,j) &:= \sqrt{\frac{2i+1}{2}}\sqrt{\frac{2j+1}{2}}\left[\int_{-1}^{1} L'_j(s)L_i(s)ds + \int_{-1}^{1}(\oint_{-1}^{1}\frac{L_j(\tau)}{\tau - s}d\tau)L_i(s)ds\right],\\ A_n(n,j) &:= e_j(-1),\\ b_n(i) &:= \sqrt{\frac{2i+1}{2}}\int_{-1}^{1} f(s)L_i(s)ds,\\ b_n(n) &:= 0.\end{aligned}$$

Since K is compact, the theory developped in [4] shows that for n large enough, the operator $I + K_n$ is invertible, and its inverse is uniformly bounded with respect to n.

Let $s > 0$ and $H^s([-1,1],\mathbb{C})$ be the classical Sobolev space, and let $\|.\|_s$ denote its norm. (For details, see [10].) Remark that

$$(I + A^{-1}T)(H^s([-1,1],\mathbb{C})) = H^s([-1,1],\mathbb{C}).$$

We recall that (cf. [10]) there exists $c > 0$ such that, for all $\psi \in H^s([-1,1],\mathbb{C})$,

$$\|(I - \pi_n)\psi\| \le cn^{-s}\|\psi\|_s. \tag{5.2}$$

Theorem 14. *Assume that $f \in H^s([-1,1],\mathbb{C})$ for some $s > 0$. Then, there exists $\alpha > 0$ such that*

$$\|\varphi_n - \varphi\| \le \alpha[n^{1-s}\|T\varphi\|_{s-1} + n^{-s}\|f\|_s].$$

Proof. We have

$$\begin{aligned} \varphi_n - \varphi &= \left[(I + K_n)^{-1} g_n - (I + K)^{-1} g\right] + (I + K_n)^{-1} g - (I + K_n)^{-1} g \\ &= (I + K_n)^{-1} \left[(K - K_n)\varphi + g_n - g\right], \end{aligned}$$

and hence

$$\|\varphi_n - \varphi\| \le C\left[\|(K - K_n)\varphi\| + \|A^{-1}\|\|(I - \pi_n) f\|\right].$$

On the other hand,

$$(K - K_n)\varphi = A^{-1}(I - \pi_n) T\varphi.$$

But $f \in H^s([-1,1],\mathbb{C})$, so $\varphi \in H^s([-1,1],\mathbb{C})$ and $T\varphi \in H^{s-1}([-1,1],\mathbb{C})$. Using (5.2), the desired result follows. □

5.3 Sloan Projection Method

Consider the approximate problem of finding $\varphi_n^S \in D$ such that

$$\varphi_n^S + K\pi_n\varphi_n^S = A^{-1}f. \tag{5.3}$$

Clearly, if such a function exists, it belongs to $\mathcal{D}$.

Applying the operator π_n to both sides of equation (5.3) we get

$$\pi_n\varphi_n^S + \pi_n K\pi_n\varphi_n^S = \pi_n A^{-1}f,$$

or, equivalently,

$$\sum_{j=0}^{n-1}\left\langle \varphi_n^S, e_j\right\rangle e_j + \sum_{j=0}^{n-1}\left\langle \varphi_n^S, e_j\right\rangle \pi_n K e_j = \sum_{j=0}^{n-1}\left\langle A^{-1}f, e_j\right\rangle e_j,$$

and performing the inner product with e_i we get the following system:

$$\left\langle \varphi_n^S, e_i\right\rangle + \sum_{j=0}^{n-1}\left\langle \varphi_n^S, e_j\right\rangle \langle \pi_n K e_j, e_i\rangle = \left\langle A^{-1}f, e_i\right\rangle, \quad i \in [\![0, n-1]\!].$$

Since $\pi_n^* = \pi_n$, and $\pi_n e_i = e_i$,

$$\left\langle \varphi_n^S, e_i \right\rangle + \sum_{j=0}^{n-1} \left\langle \varphi_n^S, e_j \right\rangle \langle K e_j, e_i \rangle = \left\langle A^{-1} f, e_i \right\rangle, \quad i \in [\![0, n-1]\!]. \tag{5.4}$$

Since K is compact, $(I_n + A_n)^{-1}$ exists for n large enough (see [4]). Once the system (5.4) is solved, φ_n^S is recovered as

$$\varphi_n^S(s) = \int_{-1}^{s} f(t)dt - \sum_{j=0}^{n-1} x_n(j) \sqrt{\frac{2j+1}{2}} \int_{-1}^{s} \oint_{-1}^{1} \frac{L_j(\tau)}{\tau - t} d\tau dt.$$

Let

$$M \quad := \quad \sup_{n \geq N} \left\| (I + K\pi_n)^{-1} \right\|,$$

which is finite.

Theorem 15. *Assume that $f \in H^s([-1,1], \mathbb{C})$ for some $s > 0$. Then, there exists $\beta > 0$ such that*

$$\|\varphi_n^S - \varphi\| \leq M\beta \|K\| \, n^{-s} \|\varphi\|_s.$$

Proof. We have

$$\begin{aligned} \varphi_n^S - \varphi &= \left(A^{-1} f - K\pi_n \varphi_n^S\right) - \left(A^{-1} f - K\varphi\right) \\ &= K\left(I - \pi_n\right)\varphi + K\pi_n\left(\varphi - \varphi_n^S\right), \end{aligned}$$

and hence

$$\left(\varphi_n^S - \varphi\right) = (I + K\pi_n)^{-1} K\left(I - \pi_n\right)\varphi.$$

But $f \in H^s([-1,1], \mathbb{C})$, so $\varphi \in H^s([-1,1], \mathbb{C})$. Using (5.2), the desired result follows. □

5.4 Numerical Example

In this section, we present a numerical example to illustrate the theoretical results obtained in the above sections. Tables 5.1 shows the absolute error as a function of n.

Example 5.1

Let f be defined so that the exact solution be

$$\varphi(s) = \frac{s+1}{s^2+1}.$$

n	$\|\varphi - \varphi_n\|_2$	$\|\varphi - \varphi_n^S\|_2$
4	2.40e-2	2.15e-2
5	9.88e-3	7.92e-3
6	4.08e-3	2.59e-3
7	1.68e-3	9.85e-4
8	6.99e-4	3.44e-4
9	2.89e-4	1.32e-4
10	1.20e-4	4.81e-5
11	4.96e-5	1.88e-5
12	2.05e-5	6.98e-6

Table 5.1: *Example 5.1*

Chapter 6

Regularization and Projection Approximations for Cauchy Integral Equations of the Second Kind

6.1 Introduction and Mathematical Background

In this chapter, we derived the regularization to the solution of Cauchy integral equation, and we apply the projection to the obtained equation. First we us Kantorovich projection, and we perform the error analysis. After we study the Sloan projection and we prove some results about the error analysis. In the end of this chapter Galerkin projection is estabished and its error analysis is discussed.

Let $C^{0,\alpha}([-1,1],\mathbb{C})$, $0<\alpha\leq 1$ be the space of all α-Hölder continuous functions. Let us denote by $H^*([-1,1],\mathbb{C})$, $0<\alpha\leq 1$ the space of all functions φ which satisfy the following conditions:

- φ is α-Hölder continuous on every closed subinterval of $(-1,1)$,
- $$\varphi(t)=\frac{\varphi^*(t)}{(t-c)^{\mu}},\quad 0\leq\mu<1,$$

  near $c=\pm 1$. φ^* is Hölder continuous function.

Consider the following Cauchy integral equation of the second kind

$$a\vartheta(x)+\frac{b}{\pi}\oint_{-1}^{1}\frac{\vartheta(t)}{t-x}dt-\mu\int_{-1}^{1}k(x,t)\vartheta(t)dt=g(x),\quad -1<x<1, \tag{6.1}$$

where a,b are constants, such that $a^2+b^2=1$. We assume that $g\in C^{0,\alpha}([-1,1],\mathbb{C})$. Following [110],

$$\vartheta(x)=\omega(x)\varphi(x),$$

where the function φ is a Hölder continuous function, and ω is a weight function defined as

$$\omega(x):=(1-x)^{\alpha}(1+x)^{\beta},$$

where α and β are given by

$$\alpha := \frac{1}{2\pi i}\log\frac{a-ib}{a+ib}+\nu, \quad \beta := -\frac{1}{2\pi i}\log\frac{a-ib}{a+ib}+\nu', \quad -1<\alpha,\beta<1.$$

ν and ν' are integers related to the following index

$$\kappa := -(\alpha+\beta) = -(\nu+\nu').$$

Hence

$$a\omega(x)\varphi(x)+\frac{b}{\pi}\oint_{-1}^{1}\frac{\omega(t)\varphi(t)}{t-x}dt-\mu\int_{-1}^{1}k(x,t)\omega(t)\varphi(t)dt = g(x), \quad -1<x<1.$$

Let K^0 be the operator defined by

$$K^0\phi(x) := a\omega(x)\phi(x)+\frac{b}{\pi}\oint_{-1}^{1}\frac{\omega(t)\phi(t)}{t-x}dt, \quad \phi\in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1,$$

and K^1 be the operator defined by

$$K^1\phi(x) := \int_{-1}^{1}k(x,t)\omega(t)\phi(t)dt, \quad \phi\in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1.$$

First, we will discuss the solution of the following Cauchy integral equation of the first kind

$$K^0\varphi(x) = h(x), \quad h\in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1.$$

The solution of this equation has the following boundary behavior in $[-1,1]$:

- If $-1<\alpha<0$ and $0<\beta<1:$ The solution is continuous in $(-1,1)$, bounded at $x=-1$ and may have a weak singularity at $x=1$.
- If $0<\alpha<1$ and $-1<\beta<0:$ The solution is continuous in $(-1,1)$, bounded at $x=1$ and may have a weak singularity at $x=-1$.
- If $0<\alpha<1$ and $0<\beta<1:$ The solution is continuous in $(-1,1)$, and bounded at both ends $x=-1$ and $x=1$.
- If $-1<\alpha<0$ and $-1<\beta<0:$ The solution is continuous in $(-1,1)$, and may have a weak singularities both at $x=-1$ and $x=-1$.

According to [47], the Cauchy integral equation of the first kind has the solution

$$\varphi(x) = a\frac{h(x)}{\omega(x)}-\frac{b}{\pi}\oint_{-1}^{1}\frac{h(t)}{\omega(t)}\frac{dt}{t-x}+\frac{b}{\pi}C,$$

where

- For $\kappa=1$, the solution is unbounded at both ends $x=-1$ and $x=1$, and not unique since C is an arbitrary constant.
- For $\kappa=0$, the solution is unique since $C=0$.

- For $\kappa = -1$, the equation has a solution if and only if

$$\int_{-1}^{1} \frac{h(t)}{\omega(t)} dt = 0,$$

  and in this case the solution is unique with $C = 0$.

Let K be the operator defined by

$$K\phi(x) := a\omega^*(x)\phi(x) - \frac{b}{\pi} \oint_{-1}^{1} \frac{\omega^*(t)\phi(t)}{t-x} dt, \quad \phi \in C^{0,\alpha}([-1,1], \mathbb{C}), \quad -1 < x < 1,$$

where

$$\omega^*(x) := \frac{1}{\omega(x)}.$$

If

$$h := g + \mu K^1 \phi,$$

then we get the following equivalent equation with a similar discussion as above

$$\varphi - \mu K K^1 \varphi = Kg + \frac{b}{\pi} C.$$

Let

$$f := \frac{-1}{\mu}(Kg + \frac{b}{\pi} C), \quad \lambda := \frac{1}{\mu}, \quad T := KK^1.$$

Then φ solves

$$(T - \lambda I)\varphi = f. \tag{6.2}$$

Using the Proposition 1.1.2 in [47], if we assume that $k \in C^1([-1,1]^2, \mathbb{C})$, $g \in C^1([-1,1], \mathbb{C})$, then the operator T is compact from $X := C^0([-1,1], \mathbb{C})$ into X.

6.2 Finite Rank Approximations and Regularization

In this section we introduce a grid $(x_{n,j})_{j=0}^{n}$ on $[-1,1]$ such that

$$-1 < x_{n,0} < x_{n,1} < \ldots < x_{n,n-1} < x_{n,n} < 1.$$

Consider hat functions $e_0, e_1, e_2, \ldots, e_n$ in $C^0([-1,1], \mathbb{C})$ such that

$$e_j(x_{n,k}) = \delta_{j,k}.$$

Define the projection π_n from $C^0([-1,1], \mathbb{C})$ into itself by

$$\pi_n g(x) := \sum_{j=0}^{n} g(x_{n,j}) e_j(x).$$

We recall that by [4, 8]

$$\lim_{n\to\infty} \|\pi_n g - g\|_\infty = 0.$$

The solution φ of (6.2) satisfies

$$\varphi = \frac{1}{\lambda}(T\varphi - f).$$

Let

$$\psi := T\varphi = \lambda\varphi + f.$$

We get

$$\varphi = \frac{1}{\lambda}(\psi - f),$$

so that

$$\psi = \frac{1}{\lambda}(T\psi - Tf). \tag{6.3}$$

Hence

$$\pi_n\psi = \frac{1}{\lambda}(\pi_n T\psi - \pi_n Tf). \tag{6.4}$$

Let us approximate the solution of (6.3) by ψ_n^P (P for Kantorovich) such that

$$\psi_n^P = \frac{1}{\lambda}(\pi_n T\psi_n^P - \pi_n Tf). \tag{6.5}$$

Theorem 16. *The inverse operator $(I - \frac{1}{\lambda}\pi_n T)^{-1}$ exists and it is uniformly bounded for n large enough, and*

$$\left\|\psi - \psi_n^P\right\| \leq M_0 \left\|\psi - \pi_n\psi\right\|.$$

Proof. In fact

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \psi_n^P) = \psi - \frac{1}{\lambda}\pi_n T\psi - \psi_n^P + \frac{1}{\lambda}\pi_n T\psi_n^P.$$

From (6.5),

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \psi_n^P) = \psi - \frac{1}{\lambda}\pi_n T\psi + \frac{1}{\lambda}\pi_n Tf.$$

Thus

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \psi_n^P) = \psi - \frac{1}{\lambda}(\pi_n T\psi - \pi_n Tf).$$

Using (6.4), we get

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \psi_n^P) = \psi - \pi_n\psi,$$

and the resulat follows, with

$$M_0 := \sup_n \left\|(I - \frac{1}{\lambda}\pi_n T)^{-1}\right\|.$$

Since T is compact, $\lim_{n\to\infty} \|\pi_n T - T\|_\infty = 0$, and hence $M_0 < \infty$. □

Let

$$\tilde{\varphi}_n^P := \frac{1}{\lambda}(\psi_n^P - f).$$

Theorem 17. *The iterated approximation* $\tilde{\varphi}_n^P$ *satisfies*

$$\left\|\tilde{\varphi}_n^P - \varphi\right\| \leq C_1 \left\|T - \pi_n T\right\|. \quad \left\|\tilde{\varphi}_n^P - \varphi\right\| \leq \frac{M_0}{|\lambda|} \left\|\psi - \pi_n \psi\right\|.$$

Proof. We have

$$(\pi_n T - \lambda I)\tilde{\varphi}_n^P = \frac{1}{\lambda}(\pi_n T - \lambda I)(\psi_n^P - f),$$

so

$$(\pi_n T - \lambda I)\tilde{\varphi}_n^P = \frac{1}{\lambda}(\pi_n T\psi_n^P - \lambda\psi_n^P - \pi_n Tf + \lambda f).$$

But (6.5) implies

$$\pi_n T\psi_n^P - \lambda\psi_n^P = \pi_n Tf.$$

Hence

$$(\pi_n T - \lambda I)\tilde{\varphi}_n^P = f.$$

Thus

$$\begin{aligned}
\tilde{\varphi}_n^P - \varphi &= (\pi_n T - \lambda I)^{-1} f - (T - \lambda I)^{-1} f \\
&= (\pi_n T - \lambda I)^{-1} (T - \lambda I)^{-1} \left[(T - \lambda I) - (\pi_n T - \lambda I)\right] f \\
&= (\pi_n T - \lambda I)^{-1} (T - \lambda I)^{-1} \left[T - \pi_n T\right] f.
\end{aligned}$$

Finally, we have

$$\left\|\tilde{\varphi}_n^P - \varphi\right\| \leq C_1 \left\|T - \pi_n T\right\|.$$

Where

$$C_0 := \sup_n \left\|(\pi_n T - \lambda I)^{-1}\right\|,$$

$$C_1 := \left\|(T - \lambda I)^{-1}\right\| \left\|f\right\| C_0.$$

Also

$$\begin{aligned}
\tilde{\varphi}_n^P - \varphi &= \frac{1}{\lambda}(\psi_n^P - f) - \frac{1}{\lambda}(\psi - f) \\
&= \frac{1}{\lambda}(\psi_n^P - \psi).
\end{aligned}$$

Using the above Theorem we get the desired bound. □

6.3 Sloan Projection

Using (6.3),

$$\psi_n^S = \frac{1}{\lambda}(T\pi_n\psi_n^S - Tf). \tag{6.6}$$

Theorem 18. *The inverse operator* $(T\pi_n - \lambda I)^{-1}$ *exists, it is uniformaly bounded for* n *large enough, and*

$$\left\|\psi - \psi_n^S\right\| \leq M_1 \left\|T\right\| \left\|\pi_n\psi - \psi\right\|,$$

where

$$M_1 := \sup_n \left\|(T\pi_n - \lambda I)^{-1}\right\|.$$

Proof. We remark that

$$(T\pi_n - \lambda I)(\psi - \psi_n^S) = T\pi_n\psi - \lambda\psi - T\pi_n\psi_n^S + \lambda\psi_n^S.$$

It follows from (6.6) that

$$(T\pi_n - \lambda I)(\psi - \psi_n^S) = T\pi_n\psi - \lambda\psi - Tf.$$

By (6.6),

$$(T\pi_n - \lambda I)(\psi - \psi_n^S) = T\pi_n\psi - T\psi,$$

and the result follows. Since T is compact, $\lim_{n\to\infty} \left\|\pi_n T - T\right\|_\infty = 0$, and hence M_1 is finite. □

6.4 Galerkin Projection

By using Galerkin projection from (6.3),

$$\psi_n^G = \frac{1}{\lambda}(\pi_n T\pi_n\psi_n^G - \pi_n Tf) \tag{6.7}$$

Theorem 19. *The inverse operator* $(I - \frac{1}{\lambda}\pi_n T)^{-1}$ *exists and it is uniformaly bounded for* n *large enough, and*

$$\left\|\psi - \psi_n^G\right\| \leq \gamma \left\|(\pi_n\psi - \psi\right\|,$$

where

$$\gamma := \sup_n \left\|(I - \frac{1}{\lambda}\pi_n T)^{-1}\right\|.$$

Proof. We have

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \pi_n\psi_n^G) = \psi - \frac{1}{\lambda}\pi_n T\psi - \pi_n\psi_n^G + \frac{1}{\lambda}\pi_n T\psi_n^G.$$

From (6.7),

$$\begin{aligned}(I - \frac{1}{\lambda}\pi_n T)(\psi - \pi_n\psi_n^G) &= \psi - \frac{1}{\lambda}\pi_n T\psi + \frac{1}{\lambda}\pi_n Tf \\ &= \psi - \frac{1}{\lambda}(\pi_n T\psi - \pi_n Tf).\end{aligned}$$

By (6.4),

$$(I - \frac{1}{\lambda}\pi_n T)(\psi - \pi_n\psi_n^G) = \psi - \pi_n\psi.$$

Hence,

$$\psi - \pi_n\psi_n^G = (I - \frac{1}{\lambda}\pi_n T)^{-1}(\psi - \pi_n\psi),$$

and we get the desired result. Since T is compact, $\lim_{n\to\infty} \left\|\pi_n T - T\right\|_\infty = 0$, and hence $\gamma < \infty$. □

Conclusions and perspectives

In this thesis, new numerical schemes based on the projection and collocation methods have been constructed and justified for approximate solutions of Cauchy integral and integro-differential equations. We have developed a projection method for solving an operator equation with bounded noncompact operator in Hilbert spaces.

This work may be extended to other type of Cauchy integral and integro-differential equations.

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RESUME

L'objectif de ce travail est la résolution des équations intégrales singulières à noyau Cauchy. On y traite les équations singulières de Cauchy de première espèce par la méthode des approximations successives. On s'intéresse aussi aux équations intégrales à noyau de Cauchy de seconde espèce, en utilisant les polynômes trigonométriques et les techniques de Fourier.

Dans la même perspective, on utilise les polynômes de Tchebychev de quatrième degré pour résoudre une équation intégro-différentielle à noyau de Cauchy.

En suite, on s'intéresse à une autre équation intégro-différentielle à noyau de Cauchy, en utilisant les polynômes de Legendre, ce qui a donné lieu à développer deux méthodes basées sur une suite de projections qui converge simplement vers l'identité.

En outre, on exploite les méthodes de projection pour les équations intégrales avec des operateurs intégraux bornés non compacts et on a appliqué ces méthodes à l'équation intégrale singulière à noyau de Cauchy de deuxième espèce.

Mots clés: Equations intégrales, approximations successives, méthodes de projection, méthodes de collocation, noyau de Cauchy.

Summary

The purpose of this thesis is to develop and illustrate various new methods for solving many classes of Cauchy singular integral and integro-differential equations.

We study the successive approximation method for solving Cauchy singular integral equations of the first kind in the general case, then we develop a collocation method based on trigonometric polynomials combined with a regularization procedure, for solving Cauchy integral equations of the second kind.

In the same perspective, we use a projection method for solving operator equation with bounded noncompact operators in Hilbert spaces.

We apply a collocation and projection methods for solving Cauchy integro-differential equations, using airfoil and Legendre polynomials.

Keywords: Integral equations, successive approximations, projection methods, collocation methods, Cauchy kernel.

Thèse de Doctorat Cotutelle Internationale

Présenté par : Abdelaziz MENNOUNI

Sur la Résolution des Equations Intégrales Singulières à Noyau de Cauchy

Résumé

À Biskra, le 27 avril 2011

Le domaine des équations intégrales, aussi vaste qu'il le soit, a connu ces dernières années une attention considérable. En effet, les équations intégrales interviennent dans le traitement de différents problèmes apparraissant dans les domaines des sciences physiques et de la technologie tels que: le transfert radiatif, la diffusion, l'élasticité, où les équations intégrales singulières à noyau Cauchy trouvent leur place.

Concernant notre thème: équations intégrales singulières à noyau Cauchy, on s'intéresse à la résolution de ce type d'équations par différentes méthodes telles que celles de projection et collocation.

On vise en premier lieu à généraliser une méthode des approximations successives pour résoudre une équation intégrale singulière à noyau de Cauchy de première espèce, appliquée à deux cas d'intérêt pratique.

Puis, en se servant des méthodes de projection, on traite des équations fonctionnelles avec des opérateurs bornés (non compacts); notamment la méthode de Galerkin et la méthode de Kulkarni.

D'autre part on traite les équations intégro-différentielles à noyau de Cauchy via la méthode de collocation et la méthode de projection.

Un autre objectif réalisé est le traitement des équations intégrales singulières à noyau de Cauchy de deuxième espèce en utilisant les polynômes trigonométriques et une procédure de régularisation.

Le présent travail est organisé comme suit: On commence par un rappel général de la théorie des opérateurs bornés, puis un rappel de théorie spectrale, enfin on donne brièvement une classification des équations intégrales et une introduction à notre thème.

Dans le premier chapitre on traite une équation intégrale singulière à noyau de Cauchy de première espèce, en utilisant la méthode des approximations successives.

Soit l'équation intégrale singulière à noyau de Cauchy de première espèce

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\varphi(t)}{t-x}dt = g(x), \quad -1 < x < 1,$$

où v et g sont deux fonctions connues, et φ est l'inconnue. On suppose que

$$\begin{aligned}
\frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)}{t-x}dt &= 1, \quad -1<x<1,\\
\frac{1}{\pi}\int_{-1}^{1}|v(t)|dt &\leq 1, \quad -1<x<1,\\
\frac{1}{\pi}\oint_{-1}^{1}\frac{1}{v(t)(t-x)}dt &= 1, \quad -1<x<1,\\
\frac{1}{\pi}\int_{-1}^{1}\frac{1}{|v(t)|}dt &\leq 1, \quad -1<x<1.
\end{aligned}$$

Par la méthode des approximations successives on obtient

$$\varphi_{n+1}(x) = g(x) - \frac{1}{\pi}\oint_{-1}^{1}\frac{v(t)\left[\varphi_n(t)-\varphi_n(x)\right]}{t-x}dt, \quad -1<x<1.$$

Considérons

$$R_n(x) = \varphi(x) - \varphi_n(x).$$

On a

$$\|R_n\| \leq \frac{1}{n!}\|g^{(n)}\| + \frac{1}{(n+1)!}\|g^{(n+1)}\|, \quad n\in\mathbb{N}.$$

Si

$$\lim_{n\to\infty}\left(\frac{1}{n!}\|g^{(n)}\| + \frac{1}{(n+1)!}\|g^{(n+1)}\|\right) = 0,$$

alors la méthode des approximations successives converge.

La méthode proposée a été testée pour les deux cas suivants:

$$v(t) = \sqrt{\frac{1+t}{1-t}},$$

et

$$v(t) = \sqrt{\frac{1-t}{1+t}},$$

respéctivement.

Le deuxième chapitre traite des equations intégrales singulières à noyau de Cauchy de deuxième éspèce en utilisant les polynômes trigonométriques. Ce chapitre contient deux sections:

Dans la première section, nous présentons une méthode de collocation basée sur les polynômes trigonométriques combinée à une procédure de régularisation, pour résoudre l'équation intégrale de Cauchy de seconde espèce

$$\mu\varphi(s) - \oint_{0}^{2\pi}\frac{\varphi(t)}{t-s}dt = f(s), \quad 0\leq s\leq 2\pi.$$

Cette équation s'écrit

$$\mu\varphi - T\varphi = f,$$

où

$$T\varphi(s) := \oint_0^{2\pi} \frac{\varphi(t)}{t-s} dt, \quad 0 \leq s \leq 2\pi.$$

Soient $X := L^2([0, 2\pi], \mathbb{C})$, et X_n l'espace engendré par les polynômes trigonométriques de degré $\leq 2n+1$. Soit σ_n la projection orthogonale de X sur X_n. Alors, pour tout $\psi \in X$,

$$\lim_{n\to\infty} \|\sigma_n\psi - \psi\|_2 = 0.$$

Soient $0 \leq s_{n,1} < s_{n,2} < \ldots < s_{n,2n+1} \leq 2\pi$. Pour tout $i \in [\![1, 2n+1]\!]$ considérons la fonction chapeau $e_{n,i}$ dans $C^0([0, 2\pi], \mathbb{C})$, telle que, pour chaque $j \in [\![1, 2n+1]\!]$,

$$e_{n,i}(s_{n,j}) = \delta_{i,j}.$$

Soit Y_n l'espace engendré par ces fonctions chapeau, qui est de dimension $2n+1$. Nous définissons la projection interpolation π_n de $C^0([0, 2\pi], \mathbb{C})$ sur Y_n:

$$\pi_n h(s) := \sum_{j=1}^{2n+1} h(s_{n,j}) e_{n,j}(s), \quad h \in C^0([0, 2\pi], \mathbb{C}).$$

Définissons l'opérateur de régularisation T_ϵ pour $\epsilon > 0$:

$$T_\epsilon\varphi(s) := \oint_0^{2\pi} \frac{(t-s)\varphi(t)}{(t-s)^2 + \epsilon^2} dt, \quad 0 \leq s \leq 2\pi,$$

qui est compact et sesqui-hermitien de X dans lui-même.

Soit φ_ϵ la solution de l'équation intégrale régularisée:

$$(\mu I - T_\epsilon)\varphi_\epsilon = f,$$

et considérons l'opérateur d'approximation

$$T_{\epsilon,n} := \pi_n T_\epsilon \sigma_n.$$

Théorème 0.1 *Pour n assez grand, l'operateur $\mu I - T_{\epsilon,n}$ est inversible, la constante*

$$\beta_\epsilon := \sup_n \|(\mu I - T_{\epsilon,n})^{-1}\|$$

est finie et la solution $\psi_{\epsilon,n}$ de l'equation

$$(\mu I - T_{\epsilon,n})\psi_{\epsilon,n} = f,$$

converge vers la solution φ si, en premier lieu, $n \to \infty$ puis $\epsilon \to 0$.

La méthode de collocation conduit au système linéaire

$$(\mu I - T_\epsilon\sigma_n)\psi_{\epsilon,n}(s_{n,i}) = f(s_{n,i}), \quad i \in [\![1, 2n+1]\!].$$

Dans la deuxième section, nous présentons une méthode directe basée sur les polynômes trigonométriques pour résoudre l'équation intégrale singulière de Cauchy de deuxième espèce

$$\varphi(x) + \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt = g(x), \quad x \in \mathbb{R}.$$

Supposons que g est 2π-periodique.
Soit

$$\phi_m(t) = e^{\mathrm{i}mt}, \quad m \in \mathbb{Z}.$$

Alors

$$\varphi(x) = \sum_{-\infty}^{+\infty} a_m \phi_m(x),$$

où

$$a_k = \frac{1}{2\pi\,(1-\mathrm{i})} \langle g, \phi_k \rangle .$$

Dans le troisième chapitre on applique les méthodes de projection à des opérateurs bornés non compacts.

Soit H un espace de Hilbert, et T un opérateur borné de H dans lui-même. Pour une fonction donnée $f \in H$, le problème est de trouver une fonction $\varphi \in H$ telle que

$$\varphi - T\varphi = f.$$

On suppose que cette équation admet une solution unique $\varphi \in H$, et que T est sesqui-hermitien: i.e $T^* = -T$. Soit $(T_n)_{n\geq 1}$ une suite d'opérateurs sesqui-hermitiens de H dans lui-même.

Théorème 0.2 *Pour tout n, l'opérateur $I - T_n$ est inversible et*

$$\|(I - T_n)^{-1}\| \leq 1.$$

Soit $H := L^2([0,1], \mathbb{C})$, considérons l'équation intégrale singulière de Cauchy

$$\varphi(s) - \oint_0^1 \frac{\varphi(t)}{t-s} dt = f(s), \quad 0 \leq s \leq 1,$$

où f est une fonction connue.

Cette équation s'écrit sous la forme:

$$(I - T)\varphi = f,$$

où

$$T\varphi(s) := \oint_0^1 \frac{\varphi(t)}{t-s} dt, \quad 0 \leq s \leq 1.$$

Rappelons que pour tout $f \in H$, cette équation admet une solution unique $\varphi \in H$, et l'opérateur intégral T est borné de H dans lui-même, de plus $T^* = -T$.

Soit $(s_{n,j})_{j=0}^{n}$ une grille sur $[0,1]$ telle que

$$0 \leq s_{n,0} < s_{n,1} < \ldots < s_{n,n} \leq 1.$$

Posons

$$h_{n,i} := s_{n,i} - s_{n,i-1}, \quad i \in [\![1,n]\!], \quad h_n := (h_{n,1}, h_{n,2}, \ldots, h_{n,n}).$$

Considérons la suite $(\Pi_n)_{n\geq 1}$, des projections bornées et de rang fini, telle que

$$\Pi_n x := \sum_{j=1}^{n} \langle x, e_{n,j} \rangle \, e_{n,j},$$

où

$$e_{n,j} := \frac{\phi_{n,j}}{\sqrt{h_{n,j}}}, \quad \phi_{n,j}(s) := \begin{cases} 1 \text{ si } s \in]s_{n,j-1}, s_{n,j}[, \\ 0 \text{ sinon.} \end{cases}$$

Soit

$$J_n := \{s_{n,j}, \quad j \in [\![0,n]\!]\}.$$

Nous définissons le module de L^2-intégrabilité de la fonction $\psi \in H$ par rapport à h_n comme suit:

$$\omega_2(\psi, J_n) := \sup_{0\leq\delta\leq\|h_n\|_\infty} \left(\int_0^1 |\psi(\tau+\delta) - \psi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}}.$$

Toutes les fonctions sont prolongées par 0 en dehors de $[0,1]$. Nous rappelons que

$$\lim_{n\to\infty} h_n = 0 \Longrightarrow \lim_{n\to\infty} \omega_2(\psi, J_n) = 0 \text{ pour tout } \psi \in H,$$

et que, pour tout $\psi \in H$,

$$\|(I - \Pi_n)\psi\|_2 \leq \omega_2(\psi, J_n).$$

Dans un premier temps considérons la projection de Galerkin

$$T_n = T_n^G := \Pi_n T \Pi_n,$$

alors l'equation d'approximation de Galerkin suivante:

$$\varphi_n^G - T_n \varphi_n^G = \Pi_n f,$$

admet une solution unique φ_n^G, donnée par

$$\varphi_n^G = \sum_{j=1}^{n} x_n(j) e_{n,j}.$$

En résolvant le système linéaire ci-dessous, on retrouve les coefficients $x_n(j)$:

$$(I - A_n)x_n = b_n,$$

où

$$\begin{aligned} A_n(k,j) &:= \frac{1}{\sqrt{h_{n,j} h_{n,k}}} \int_{s_{n,k-1}}^{s_{n,k}} \oint_{s_{n,j-1}}^{n,s_j} \frac{dt}{t-s} ds, \\ b_n(k) &:= \frac{1}{\sqrt{h_{n,k}}} \int_{s_{k-1}}^{s_k} f(s) ds. \end{aligned}$$

Théorème 0.3 *L'estimation suivante a lieu:*

$$\|\varphi_n^G - \varphi\|_2 \leq \omega_2(f, J_n) + \omega_2(T\varphi, J_n) + \pi\omega_2(\varphi, J_n).$$

En deuxième lieu considérons la projection de Kulkarni

$$T_n^K := \Pi_n T + T\Pi_n - \Pi_n T \Pi_n.$$

La théorie a été développée pour le cas compact. Ici, nous proposons d'approcher notre opérateur borné non compact T.

Soit φ_n^K solution approchée de Kulkarni. On pose

$$u_n := \Pi_n \varphi_n^K.$$

Alors

$$u_n = \sum_{j=1}^{n} c_{n,j} e_{n,j}.$$

En résolvant le système linéaire ci-dessous, on retrouve les coefficients $c_{n,j}$:

$$c_{n,i} - \sum_{j=1}^{n} \left[\langle Te_{n,j}, e_{n,i} \rangle + \left\langle T^2 e_{n,j}, e_{n,i} \right\rangle - \sum_{k=1}^{n} \langle Te_{n,j}, e_{n,k} \rangle \langle Te_{n,k}, e_{n,i} \rangle \right] c_{n,j}$$

$$= \langle f, e_{n,i} \rangle + \langle Tf, e_{n,i} \rangle - \sum_{k=1}^{n} \langle f, e_{n,k} \rangle \langle Te_{n,k}, e_{n,i} \rangle, \quad i \in [\![1, n]\!].$$

Rappelons que

$$\begin{aligned}
\langle Te_{n,j}, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_{s_{j-1}}^{s_j} \frac{dt}{t-s} ds, \\
\left\langle T^2 e_{n,j}, e_{n,i} \right\rangle &= \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_0^1 \frac{1}{t-s} \oint_{s_{j-1}}^{s_j} \frac{d\tau}{\tau - t} dt ds, \\
\langle Tf, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} \oint_0^1 \frac{f(t)}{t-s} dt ds, \\
\langle f, e_{n,i} \rangle &= \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} f(s) ds.
\end{aligned}$$

La solution φ_n^K est donnée par

$$\varphi_n^K = u_n + (I - \Pi_n) T u_n + (I - \Pi_n) f,$$

et donc

$$\varphi_n^K(s) = u_n(s) + \oint_0^1 \frac{u_n(t)}{t-s} dt - \sum_{k=1}^{n} \frac{\phi_{n,k}(s)}{h_{n,k}} \int_{s_{k-1}}^{s_k} \oint_0^1 \frac{u_n(\tau)}{\tau - t} d\tau dt + f(s) - \sum_{k=1}^{n} \frac{\phi_{n,k}(s)}{h_{n,k}} \int_{s_{k-1}}^{s_k} f(t) dt.$$

Théorème 0.4 *L'estimation suivante est vérifiée:*

$$\|\varphi_n^K - \varphi\|_2 \leq \left[2\omega_2(\varphi, J_n)\left\|(I-\Pi_n)T^2\left(I-\Pi_n\right)\varphi\right\|_2\right]^{\frac{1}{2}}.$$

Dans le quatrième chapitre, on présente la méthode de collocation en utilisant les polynomes de Tchebychev pour approcher la solution de l'équation intégro-différentielle

$$\varphi'(x) + \frac{\lambda}{\pi}\oint_{-1}^{1}\frac{\varphi(t)}{t-x}dt = f(x), \quad -1 < x < 1.$$

où f est une fonction connue et λ est une constante.

Cette équation s'écrit:

$$A\varphi + T\varphi = f,$$

où

$$(A\varphi)(x) := \varphi'(x), \quad -1 < x < 1.$$

$$(T\varphi)(x) := \frac{\lambda}{\pi}\oint_{-1}^{1}\frac{\varphi(t)}{t-x}dt, \quad -1 < x < 1.$$

Nous allons introduire une approximation en utilisant les polynômes de Tchebychev de troisième espèce t_n comme suit:

$$\varphi_n(x) = \omega(x)\sum_{i=0}^{n} a_i t_i(x),$$

où

$$\omega(x) := \sqrt{\frac{1+x}{1-x}}.$$

Nous rappelons que les polynômes de Tchebychev de troisième espèce t_n sont définis par:

$$t_n(x) \quad := \quad \frac{\cos[(n+\frac{1}{2})\arccos x]}{\cos(\frac{1}{2}\arccos x)},$$

et les polynômes de Tchebychev de quatrième espèce u_n sont définis par:

$$u_n(x) \quad := \quad \frac{\sin[(n+\frac{1}{2})\arccos x]}{\sin(\frac{1}{2}\arccos x)}.$$

Considérons l'ensemble des zeros x_j de u_{n+1}:

$$x_j := -\cos\frac{2j-1}{2n+3}\pi, \quad j \in [\![0, n]\!].$$

Soit $C^{0,\lambda}([-1,1],\mathbb{R})$ l'espace des fonctions φ sur $[-1,1]$ satisfaisant la condition de Hölder suivante: $\exists M \geq 0$ telle que:

$$\forall x_1, x_2 \in [-1,1]\,, \quad |\varphi(x_1) - \varphi(x_2)| \leq M\,|x_1 - x_2|^{\lambda}\,,$$

où $0 < \lambda \leq 1$.

On pose

$$H := \left\{\varphi \in L^2([-1,1],\mathbb{R}) \ : \ \varphi' \in L^2([-1,1],\mathbb{R}), \quad \varphi(-1) = 0\right\}.$$

L'opérateur T est borné de $L^2([-1,1],\mathbb{R})$ dans lui-même et aussi de $C^{0,\lambda}([-1,1],\mathbb{R})$ dans lui-même.

Rappelons que

$$(A^{-1}y)(s) = \int_{-1}^{s} y(t)dt,$$

et que $A^{-1} : L^2([-1,1],\mathbb{R}) \to H$ est compact.

Considérons les fonctions chapeau $e_0, e_1, e_2, \ldots, e_n$ dans $C^0([-1,1],\mathbb{R})$ telles que

$$e_j(x_k) = \delta_{j,k}.$$

Soit la projection P_n de $C^0([-1,1],\mathbb{R})$ sur l'espace des fonctions continues, définie par:

$$P_n g(x) := \sum_{j=0}^{n} g(x_j) e_j(x).$$

On pose

$$D_n := A^{-1} P_n T, \quad D := A^{-1} T.$$

Considérons l'équation de l'inconnue φ_n suivante:

$$\varphi_n + D_n \varphi_n = A^{-1} f.$$

Théorème 0.5 *Soit $f \in C^0([-1,1],\mathbb{R})$. Il existe une constante positive M, telle que*

$$\|\varphi_n - \varphi\|_\infty \quad \leq \quad M \|D_n \varphi - D\varphi\|_\infty.$$

pour n assez grand.

On trouve, le système:

$$A\varphi_n(x_j) + T\varphi_n(x_j) = f(x_j), \quad j \in [\![0, n]\!].$$

Et donc

$$\sum_{i=0}^{n} a_i \{\omega'(x_j) t_i(x_j) + \frac{\omega(x_j)}{1 + x_j} [(i + \frac{1}{2}) u_i(x_j) - \frac{1}{2} t_i(x_j)] + \lambda u_i(x_j)\} = f(x_j), \quad j \in [\![0, n]\!].$$

Dans le cinquième chapitre nous présentons deux méthodes pour résoudre l'équation intégro-différentielle à noyau de Cauchy

$$\varphi'(s) + \oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt = f(s), \quad -1 \leq s \leq 1.$$

Cette équation s'écrit

$$\varphi + A^{-1} T \varphi = A^{-1} f,$$

où

$$\begin{aligned} T\varphi(s) &:= \oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt, \quad -1 \leq s \leq 1, \\ A\varphi(s) &:= \varphi'(s), \quad -1 \leq s \leq 1. \end{aligned}$$

Soient

$$\mathcal{H} := L^2([-1,1], \mathbb{R}),$$

.

$$\mathcal{D} := \{\varphi \in \mathcal{H} \ : \ \varphi' \in \mathcal{H}, \ \varphi(-1) = 0\}.$$

Soit $(L_n)_{n \geq 0}$ la suite des polynômes de Legendre. On pose

$$e_j := \sqrt{\frac{2j+1}{2}} L_j,$$

Soit $(\pi_n)_{n \geq 0}$ la projection orthogonale définie par:

$$\pi_n x := \sum_{j=0}^{n-1} \langle x, e_j \rangle \, e_j.$$

En premier lieu, nous présentons le problème d'approximation

$$\varphi_n + A^{-1} \pi_n T \varphi_n = A^{-1} \pi_n f.$$

Soit $\mathcal{H}_n$ sous-espace engendré par les n premiers polynômes de Legendre.

Il est clair que $\varphi_n \in \mathcal{D} \cap \mathcal{H}_{n+1}$. Cela conduit au système linéaire suivant:

$$A_n x_n = b_n,$$

où , pour $i \in [\![0, n-1]\!]$ et $j \in [\![0, n]\!]$,

$$\begin{aligned} A_n(i,j) &:= \sqrt{\frac{2i+1}{2}} \sqrt{\frac{2j+1}{2}} \left[\int_{-1}^{1} L_j'(s) L_i(s) ds + \int_{-1}^{1} (\oint_{-1}^{1} \frac{L_j(\tau)}{\tau - s} d\tau) L_i(s) ds \right], \\ A_n(n,j) &:= e_j(-1), \\ b_n(i) &:= \sqrt{\frac{2i+1}{2}} \int_{-1}^{1} f(s) L_i(s) ds, \\ b_n(n) &:= 0. \end{aligned}$$

Soient $s > 0$ et $H^s([-1,1], \mathbb{R})$ l'espace de Sobolev classique, muni de la norme $\|.\|_s$.

Théorème 0.6 *Soit $f \in H^s([-1,1],\mathbb{R})$ pour un certain $s > 0$. Alors, il existe $\alpha > 0$ telle que*

$$\|\varphi_n - \varphi\| \leq \alpha[n^{1-s}\|T\varphi\|_{s-1} + n^{-s}\|f\|_s].$$

En deuxème lieu, nous présentons le problème d'approximation de Sloan:

Trouver $\varphi_n^S \in D$ telle que

$$\varphi_n^S + K\pi_n\varphi_n^S = A^{-1}f.$$

On trouve le système linéaire

$$(I_n + A_n)x_n = b_n,$$

où $x_n(j) := \langle \varphi_n^S, e_j \rangle$, et

$$\begin{aligned} A_n(i,j) &:= \sqrt{\frac{2j+1}{2}}\sqrt{\frac{2i+1}{2}}\int_{-1}^{1}\int_{-1}^{s}\oint_{-1}^{1}\frac{L_j(\tau)L_i(s)}{\tau - t}d\tau dt ds, \\ b_n(i) &:= \sqrt{\frac{2i+1}{2}}\int_{-1}^{1}\int_{-1}^{s} f(t)L_i(s)dtds. \end{aligned}$$

On pose

$$M := \sup_{n\geq N}\left\|(I + K\pi_n)^{-1}\right\|.$$

Théorème 0.7 *Soit $f \in H^s([-1,1],\mathbb{R})$ pour un certain $s > 0$. Il existe une constante positive $\beta > 0$, telle que*

$$\|\varphi_n^S - \varphi\| \leq M\beta\,\|K\|\,n^{-s}\|\varphi\|_s,$$

pour n assez grand.

Le dernier chapitre est consacré à une régularisation de l'équation intégrale

$$a\vartheta(x) + \frac{b}{\pi}\oint_{-1}^{1}\frac{\vartheta(t)}{t-x}dt - \mu\int_{-1}^{1}k(x,t)\vartheta(t)dt = g(x), \quad -1 < x < 1,$$

où a, b sont deux contantes, telles que $a^2 + b^2 = 1$. On suppose que $g \in C^{0,\alpha}([-1,1],\mathbb{C})$. Dans ce cas la solution s'écrit

$$\vartheta(x) = \omega(x)\varphi(x),$$

où la fonction φ vérifie la condition de Hölder, et ω est une fonction poids définie comme suit:

$$\omega(x) := (1-x)^{\alpha}(1+x)^{\beta},$$

où

$$\alpha := \frac{1}{2\pi i}\log\frac{a-ib}{a+ib} + \nu, \quad \beta := -\frac{1}{2\pi i}\log\frac{a-ib}{a+ib} + \nu', \quad -1 < \alpha, \beta < 1.$$

ν et ν' sont deux entiers liés à l'indice suivant:

$$\kappa := -(\alpha+\beta) = -(\nu+\nu').$$

On pose

$$K^0\phi(x) := a\omega(x)\phi(x) + \frac{b}{\pi}\oint_{-1}^{1}\frac{\omega(t)\phi(t)}{t-x}dt, \quad \phi \in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1,$$

$$K^1\phi(x) := \int_{-1}^{1} k(x,t)\omega(t)\phi(t)dt, \quad \phi \in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1.$$

$$K\phi(x) := a\omega^*(x)\phi(x) - \frac{b}{\pi}\oint_{-1}^{1}\frac{\omega^*(t)\phi(t)}{t-x}dt, \quad \phi \in C^{0,\alpha}([-1,1],\mathbb{C}), \quad -1<x<1,$$

$$\omega^*(x) := \frac{1}{\omega(x)},$$

$$h := g + \mu K^1\phi.$$

On trouve l'équation suivante:

$$\varphi - \mu K K^1 \varphi = Kg + \frac{b}{\pi}C,$$

que nous réécrivons

$$(T-\lambda I)\varphi = f,$$

où

$$f := \frac{-1}{\mu}(Kg + \frac{b}{\pi}C), \quad \lambda := \frac{1}{\mu}, \quad T := KK^1.$$

On suppose que $k \in C^1([-1,1]^2,\mathbb{C})$, $g \in C^1([-1,1],\mathbb{C})$, alors l'opérateur T est compact de $X := C^0([-1,1],\mathbb{C})$ dans X.

On pose

$$\psi := T\varphi = \lambda\varphi + f.$$

On trouve

$$P_n\psi = \frac{1}{\lambda}(P_nT\psi - P_nTf).$$

Soit ψ_n^P l'approximation de Kantorovich

$$\psi_n^P = \frac{1}{\lambda}(P_nT\psi_n^P - P_nTf).$$

Théorème 0.8 *L'opérateur inverse $(I - \frac{1}{\lambda}P_nT)^{-1}$ existe et est uniformément borné pour n assez grand. De plus il existe $M_0 > 0$ tel que*

$$\left\|\psi - \psi_n^P\right\| \le M_0 \left\|\psi - P_n\psi\right\|.$$

Notons

$$\tilde{\varphi}_n^P := \frac{1}{\lambda}(\psi_n^P - f).$$

Théorème 0.9

$$\left\|\tilde{\varphi}_n^P - \varphi\right\| \leq C_1 \left\|T - P_n T\right\| \ \ et \ \ \left\|\tilde{\varphi}_n^P - \varphi\right\| \leq \frac{M_0}{|\lambda|} \left\|\psi - P_n \psi\right\|.$$

Notons

$$\psi_n^S = \frac{1}{\lambda}(TP_n\psi_n^S - Tf).$$

Théorème 0.10 *Il existe $N \in \mathbb{N}$ tel que pour tout $n \geq N$, l'opérateur $(TP_n - \lambda I)^{-1}$ existe, est uniformément borné et*

$$\left\|\psi - \psi_n^S\right\| \leq M_1 \left\|T\right\| \left\|P_n\psi - \psi\right\|,$$

où

$$M_1 := \sup_{n \geq N} \left\|(TP_n - \lambda I)^{-1}\right\|.$$

Notons

$$\psi_n^G = \frac{1}{\lambda}(P_n T P_n \psi_n^G - P_n T f).$$

Théorème 0.11 *Il existe $N_0 \in \mathbb{N}$ tel que pour tout $n \geq N_0$, l'opérateur $\left(I - \frac{1}{\lambda}P_n T\right)^{-1}$ existe, est uniformément borné et*

$$\left\|\psi - \psi_n^G\right\| \leq \gamma \left\|P_n\psi - \psi\right\|,$$

où

$$\gamma := \sup_{n \geq N_0} \left\|(I - \frac{1}{\lambda}P_n T)^{-1}\right\|.$$

Les différentes méthodes proposées dans cette thèse sont illustrées par un ensemble conséquent d'expériences numériques.