Distances between classes of sphere-valued Sobolev maps

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Abstract

We introduce an equivalence relation on $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ involving the topological degree, and we evaluate the distances (in the usual sense and in the Hausdorff sense) between the equivalence classes. In some special cases we even obtain exact formulas. Next we discuss related issues for $W^{1,p}(\Omega;\mathbb{S}^1)$.

Résumé

Distances entre classes d'applications de Sobolev à valeurs dans une sphère

On introduit une relation d'équivalence sur $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ liée au degré topologique et on présente des estimées pour les distances (au sens usuel et au sens de Hausdorff) entre les classes d'équivalence. Dans certains cas particuliers il s'agit même de formules exactes. On considère ensuite des questions semblables pour $W^{1,p}(\Omega;\mathbb{S}^1)$.

1 Introduction

We report on two recent works [4] and [5] concerning distances between classes of Sobolev maps taking their values in $\mathbb{S}^N$, in two different settings. In the first part we deal with distances in $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ ([5]). The classes in this case correspond to the equivalence relation $f \sim g$ if and only if $\deg f = \deg g$. As we will recall below, the topological degree makes sense not only for continuous maps, but also for VMO-maps, and in particular for maps in $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ with $sp \geq N$. In the second part we consider distances between classes of $W^{1,p}(\Omega;\mathbb{S}^1)$, where Ω is a smooth bounded simply connected domain in $\mathbb{R}^N$, $N \geq 2$ and $p \in [1,2)$. In contrast with the first setting, the decomposition into classes is not due to the nontrivial topology of the domain ($\mathbb{S}^N$ in the first setting) but instead it is related to the location and topological degree of the singularities of the maps in each class. More precisely, the classes are defined according to an equivalence relation: $u \sim v$ if and only if there exists $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ such that $u = e^{\imath\varphi} v$. This definition is analogous to the one used in the first part when $N = 1$, see Remark 1 below. This

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is an indication of a deep connection between the two parts. Actually, we are going to see (in Remark 5) that the results concerning $W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$ in Section 2 can be viewed as special cases of analogous results about $W^{1,1}(\Omega;\mathbb{S}^1)$ in Section 3.

2 Distances between homotopy classes of $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$

The first author and L. Nirenberg [6] developed a concept of topological degree for map in $\mathrm{VMO}(\mathbb{S}^N;\mathbb{S}^N)$, $N \geq 1$, which applies in particular to the (integer or fractional) Sobolev spaces $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ with

$$s > 0,\ 1 \leq p < \infty \text{ and } sp \geq N. \tag{1}$$

We will make assumption (1) throughout this section. We define an equivalence relation on $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ by $f \sim g$ if and only if $\deg f = \deg g$. It is known that the homotopy classes of $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ are precisely the equivalence classes for the relation $f \sim g$ and are given by

$$\mathscr{E}_d := \{f \in W^{s,p}(\mathbb{S}^N;\mathbb{S}^N);\, \deg f = d\} \ \text{ where } d \in \mathbb{Z}; \tag{2}$$

these classes depend not only on d, but also on s and p, but in order to keep notation simple we do not mention the dependence on s and p.

Remark 1 When $N = 1$ there is an alternative description of the equivalence relation $f \sim g$. Given $f,g \in W^{s,p}(\mathbb{S}^1;\mathbb{S}^1)$ we have (see [2])

$$f \sim g \text{ if and only if } f = e^{\imath\varphi} g \text{ for some } \varphi \in W^{s,p}(\mathbb{S}^1;\mathbb{R}). \tag{3}$$

Therefore, it makes sense to denote also $\mathscr{E}_d = \mathscr{E}(f)$ when $\deg f = d$. We shall use this notation in Remark 2 below.

Our purpose is to investigate the usual distance and the Hausdorff distance (in $W^{s,p}$) between the classes $\mathscr{E}_d$. For that matter we introduce the $W^{s,p}$-distance between two maps $f,g \in W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$ by

$$d_{W^{s,p}}(f,g) := |f - g|_{W^{s,p}}, \tag{4}$$

where for $h \in W^{s,p}(\mathbb{S}^N;\mathbb{R}^{N+1})$ we let

$$|h|_{W^{s,p}} := \left\| h - \fint_{\mathbb{S}^N} h \right\|_{W^{s,p}},$$

and $\|\ \|_{W^{s,p}}$ is any one of the standard norms on $W^{s,p}$. Let $d_1 \neq d_2$ and define the following two quantities:

$$\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) := \inf_{f\in\mathscr{E}_{d_1}} \inf_{g\in\mathscr{E}_{d_2}} d_{W^{s,p}}(f,g), \tag{5}$$

and

$$\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) := \sup_{f\in\mathscr{E}_{d_1}} \inf_{g\in\mathscr{E}_{d_2}} d_{W^{s,p}}(f,g). \tag{6}$$

It is conceivable that

$$\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) = \mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_2},\mathscr{E}_{d_1}), \forall\, d_1, d_2 \in \mathbb{Z}, \tag{7}$$

but we have not been able to prove this equality (see Open Problem 1 below). Therefore we consider also the symmetric version of (6), which is nothing but the Hausdorff distance between the two classes:

$$H-\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) = \max\left\{\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}),\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_2},\mathscr{E}_{d_1})\right\}. \tag{8}$$

The usual distance $\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2})$ in certain (non-fractional) Sobolev spaces was investigated in works by J. Rubinstein and I. Shafrir [8], when $s=1$, $p\geq N=1$, and S. Levi and I. Shafrir [7], when $s=1$, $p\geq N\geq 2$. In particular, they obtained exact formulas for the distance (see [8, Remark 2.1], [7, Theorem 3.4]) and tackled the question whether this distance is achieved (see [8, Theorem 1], [7, Theorem 3.4]).

We pay special attention to the case where $N=1$ and $s=1$. In this case, we have several sharp results when we take

$$d_{W^{1,p}}(f,g) = |f-g|_{W^{1,p}} := \left(\int_{\mathbb{S}^1} |\dot{f}-\dot{g}|^p\right)^{1/p}. \tag{9}$$

The following result was obtained in [8] (see also [5]).

$$\begin{aligned}\mathrm{dist}_{W^{1,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) &= \inf_{f\in\mathscr{E}_{d_1}}\ \inf_{g\in\mathscr{E}_{d_2}}\ d_{W^{1,p}}(f,g)\\ &= \left(\frac{2}{\pi}\right)\min_{h\in\mathscr{E}_{d_1-d_2}}\left(\int_{\mathbb{S}^1}|\dot{h}|^p\right)^{1/p}\\ &= 2^{(1/p)+1}\pi^{(1/p)-1}\,|d_1-d_2|.\end{aligned} \tag{10}$$

In particular,

$$\mathrm{dist}_{W^{1,1}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}) = 4\,|d_1-d_2|. \tag{11}$$

It is natural to ask whether, given $d_1\neq d_2$, the infimum in (10) is achieved. The answer is given by the following result, proved in [8] when $p=2$.

Theorem 2.1 *Let $N=1$. Let $d_1,d_2\in\mathbb{Z}$, $d_1\neq d_2$.*

1. *When $p=1$, the infimum in* (10) *is always achieved.*
2. *When $1<p<2$, the infimum in* (10) *is achieved if and only if $d_2=-d_1$.*
3. *When $p\geq 2$, the infimum in* (10) *is not achieved.*

For $s=1$, $N\geq 2$, $p\geq N$, and for the semi-norm $|f-g|_{W^{1,p}}=\|\nabla f-\nabla g\|_{L^p}$, the exact value of the $W^{1,p}$ distance $\mathrm{dist}_{W^{1,p}}$ between the classes $\mathscr{E}_{d_1}$ and $\mathscr{E}_{d_2}$, $d_1\neq d_2$, has been computed by S. Levi and I. Shafrir [7]. By contrast with (10) this distance does not depend on d_1 and d_2, but only on p (and N).

We now turn to the case $s\neq 1$ and $N\geq 1$. Here, we will only obtain the order of magnitude of the distances $\mathrm{dist}_{W^{s,p}}$, and thus our results are not sensitive to the choice of a specific distance among various equivalent ones. When $0<s<1$ a standard distance is associated with the Gagliardo $W^{s,p}$ semi-norm

$$d_{W^{s,p}}(f,g) := \left(\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{|[f(x)-g(x)]-[f(y)-g(y)]|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{1/p}. \tag{12}$$

We start with $\mathrm{dist}_{W^{s,p}}$.

Theorem 2.2 *We have*

1. *If* $N \geq 1$ *and* $1 < p < \infty$, *then*

$$\operatorname{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 0, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{13}$$

2. *If* $N = 1$, $s > 0$, $1 \leq p < \infty$ *and* $sp > 1$, *then*

$$C'_{s,p} |d_1 - d_2|^s \leq \operatorname{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \leq C_{s,p} |d_1 - d_2|^s. \tag{14}$$

3. *If* $N \geq 2$, *[*$1 < p < \infty$ *and* $s > N/p$*] or [*$p = 1$ *and* $s \geq N$*], then*

$$C'_{s,p,N} \leq \operatorname{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \leq C_{s,p,N}, \ \forall d_1, d_2 \in \mathbb{Z} \text{ such that } d_1 \neq d_2. \tag{15}$$

In the above, C, C' are positive constants independent of d_1, d_2.

We now turn to $\operatorname{Dist}_{W^{s,p}}$.

Theorem 2.3 *We have*

1. *If* $N = 1$, $s = 1$ *and* $p = 1$, *then*

$$\operatorname{Dist}_{W^{1,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = 2\pi |d_1 - d_2|, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{16}$$

2. *If* $N \geq 1$ *and* $1 \leq p < \infty$, *then*

$$\operatorname{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \leq C_{p,N} |d_1 - d_2|^{1/p}, \ \forall d_1, d_2 \in \mathbb{Z}. \tag{17}$$

3. *If* $N \geq 1$ *and* $sp > N$, *then*

$$\operatorname{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \infty, \ \forall d_1, d_2 \in \mathbb{Z} \text{ such that } d_1 \neq d_2. \tag{18}$$

The detailed proofs appeared in [5]. We call the attention of the reader to a new idea which yields inequality "≥" in (16) in a "more uniform" way; this will become clear in Remark 5 below.

Remark 2 *For later use it is convenient to reformulate* (11) *and* (16) *as follows. Assume* $N = 1, s = 1$ *and* $p = 1$. *Then* $\forall f, g \in W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$ *we have*

$$\operatorname{dist}_{W^{1,1}}(\mathscr{E}(f), \mathscr{E}(g)) = 4 |\deg(f\bar{g})| \tag{19}$$

and

$$\operatorname{Dist}_{W^{1,1}}(\mathscr{E}(f), \mathscr{E}(g)) = 2\pi |\deg(f\bar{g})|. \tag{20}$$

In particular, $\forall f \in W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$ *we have*

$$4 |\deg f| \leq \operatorname{dist}_{W^{1,1}}(f, \mathscr{E}_0) \leq 2\pi |\deg f|. \tag{21}$$

Moreover the constants 4 *and* 2π *in* (21) *are optimal.*

Here are two natural questions that we could not solve.

Open Problem 1 *Is it true that for every $d_1, d_2 \in \mathbb{Z}$, $N \geq 1$, $s > 0$, $1 \leq p < \infty$,*

$$\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) = \mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_2}, \mathscr{E}_{d_1})? \tag{22}$$

Or even better:

$$\textit{Does } \mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \textit{ depend only on } |d_1 - d_2| \textit{ (and } s, p, N\textit{)?} \tag{23}$$

There are several cases where we have an explicit formula for $\mathrm{Dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ and in all such cases (23) holds. We may also ask questions similar to (23) for $\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ and for $H-\mathrm{dist}_{W^{s,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ (assuming the answer to (23) is negative). A striking special case still open when $N = 1$ is: does $\mathrm{dist}_{W^{2,1}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2})$ depend only on $|d_1 - d_2|$?

Open Problem 2 *Is it true that for every $N \geq 1$ and every $1 \leq p < \infty$, there exists some $C'_{p,N} > 0$ such that*

$$H - \mathrm{dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq C'_{p,N} |d_1 - d_2|^{1/p}, \ \forall\, d_1, d_2 \in \mathbb{Z}? \tag{24}$$

Even better, do we have

$$\mathrm{Dist}_{W^{N/p,p}}(\mathscr{E}_{d_1}, \mathscr{E}_{d_2}) \geq C'_{p,N} |d_1 - d_2|^{1/p}, \ \forall\, d_1, d_2 \in \mathbb{Z}? \tag{25}$$

Some partial answers to these open problems are presented in [5].

3 Distances between classes in $W^{1,p}(\Omega;\mathbb{S}^1)$

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. In order to simplify the presentation we assume throughout that Ω is simply connected; however many of the results remain valid without this assumption (see, e.g., Remark 5 below). In this section we decompose $W^{1,p}(\Omega;\mathbb{S}^1)$ into equivalence classes and study their distances. We start with the case $p = 1$ and recall two basic "negative" facts originally discovered by F. Bethuel and X. Zheng [1] (see also [2] for an updated and more detailed presentation).

<u>Fact 1</u>. Maps u of the form $u = e^{\imath\varphi}$ with $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ belong to $W^{1,1}(\Omega;\mathbb{S}^1)$. However they do not exhaust $W^{1,1}(\Omega;\mathbb{S}^1)$: there exist maps in $W^{1,1}(\Omega;\mathbb{S}^1)$ which *cannot* be written as $u = e^{\imath\varphi}$ for some $\varphi \in W^{1,1}(\Omega;\mathbb{R})$. We set

$$X = \{u \in W^{1,1}(\Omega;\mathbb{S}^1);\, u = e^{\imath\varphi} \text{ for some } \varphi \in W^{1,1}(\Omega;\mathbb{R})\}. \tag{26}$$

<u>Fact 2</u>. Maps in $C^\infty(\overline{\Omega};\mathbb{S}^1)$ are *not* dense in $W^{1,1}(\Omega;\mathbb{S}^1)$. In fact (see e.g. [2]) we have

$$X = \overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{1,1}}. \tag{27}$$

We now introduce an equivalence relation in $W^{1,1}(\Omega;\mathbb{S}^1)$:

$$u \sim v \text{ if and only if } u = e^{\imath\varphi} v \text{ for some } \varphi \in W^{1,1}(\Omega;\mathbb{R}). \tag{28}$$

We denote by $\mathscr{E}(u)$ the equivalence class of an element $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. In particular, if $u = 1$ then $\mathscr{E}(u) = X$.

A useful device for constructing maps in the same equivalence class is the following (see [4]). Let $T \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ be a map of degree one. Then

$$T \circ u \sim u \quad \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{29}$$

To each $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ we associate a number $\Sigma(u) \geq 0$ defined by

$$\Sigma(u) = \inf_{v \in \mathscr{E}(u)} \int_\Omega |\nabla v|. \tag{30}$$

Note that

$$\Sigma(u) = \inf_{\psi \in W^{1,1}(\Omega;\mathbb{R})} \int_\Omega |u \wedge \nabla u - \nabla \psi|. \tag{31}$$

This follows from the identities

$$(uv) \wedge \nabla(uv) = u \wedge \nabla u + v \wedge \nabla v \qquad \forall\, u, v \in W^{1,1}(\Omega;\mathbb{S}^1), \tag{32}$$

$$e^{\imath\varphi} \wedge \nabla(e^{\imath\varphi}) = \nabla\varphi \qquad \forall\, \varphi \in W^{1,1}(\Omega;\mathbb{R}), \tag{33}$$

$$\bar{u} \wedge \nabla\bar{u} = -u \wedge \nabla u \qquad \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{34}$$

The quantity $\Sigma(u)$ was originally introduced in [3] when $N = 2$. It plays an extremely important role in many questions involving $W^{1,1}(\Omega;\mathbb{S}^1)$ (see [2]). In some sense it measures how much a given $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ “deviates" from X. By (30) we have, $u \sim v \implies \Sigma(u) = \Sigma(v)$. Moreover we have (see [2]):

$$u \sim 1 \iff \Sigma(u) = 0, \tag{35}$$

and

$$\Sigma(u\bar{v}) \geq |\Sigma(u) - \Sigma(v)| \quad \forall\, u, v \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{36}$$

Given $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that u_0 is *not* equivalent to v_0, it is of interest to consider the distance of u_0 to $\mathscr{E}(v_0)$ defined by

$$d_{W^{1,1}}(u_0, \mathscr{E}(v_0)) = \inf_{v \sim v_0} \int_\Omega |\nabla u_0 - \nabla v|,$$

and define, analogously to (5)–(6),

$$\begin{aligned} \mathrm{dist}_{W^{1,1}}(\mathscr{E}(u_0), \mathscr{E}(v_0)) &:= \inf_{u \sim u_0} d_{W^{1,1}}(u, \mathscr{E}(v_0)) \\ &= \inf_{u \sim u_0} \inf_{v \sim v_0} \int_\Omega |\nabla(u - v)|, \end{aligned} \tag{37}$$

$$\begin{aligned} \mathrm{Dist}_{W^{1,1}}(\mathscr{E}(u_0), \mathscr{E}(v_0)) &:= \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathscr{E}(v_0)) \\ &= \sup_{u \sim u_0} \inf_{v \sim v_0} \int_\Omega |\nabla(u - v)|. \end{aligned} \tag{38}$$

The next theorem provides explicit formulas for these two quantities.

Theorem 3.1 *For every $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have*

$$\mathrm{dist}_{W^{1,1}}(\mathscr{E}(u_0), \mathscr{E}(v_0)) = \frac{2}{\pi}\Sigma(u_0\overline{v}_0) \tag{39}$$

and

$$\mathrm{Dist}_{W^{1,1}}(\mathscr{E}(u_0), \mathscr{E}(v_0)) = \Sigma(u_0\overline{v}_0). \tag{40}$$

The two assertions in Theorem 3.1 look very simple but the proofs are quite tricky (see [4]). Note in particular that it follows from (40) that $\mathrm{Dist}_{W^{1,1}}$ is symmetric, which is not clear from its definition (compare with Open Problem 1).

Remark 3 There is an alternative point of view on the equivalence relation $u \sim v$ using the Jacobian of u. For every $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ consider the antisymmetric matrix with coefficients in $\mathscr{D}'$ defined by

$$Ju := \frac{1}{2}\left[\frac{\partial}{\partial x_i}\left(u \wedge \frac{\partial u}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(u \wedge \frac{\partial u}{\partial x_i}\right)\right].$$

One can show (see [2]) that $\forall\, u,v \in W^{1,1}(\Omega;\mathbb{S}^1)$,

$$u \sim v \text{ if and only if } J(u\bar{v}) = Ju - Jv = 0. \tag{41}$$

Remark 4 In order to have a feeling for the equivalence relation $u \sim v$ it is instructive to understand what it means, when $N = 2$ and Ω is simply connected, for $u,v \in \mathscr{R}$ where

$$\mathscr{R} = \{u \in W^{1,1}(\Omega;\mathbb{S}^1);\, u \text{ is smooth except at a finite number of points}\}.$$

The class $\mathscr{R}$ plays an important role since it is dense in $W^{1,1}(\Omega;\mathbb{S}^1)$ (see [1, 2]). If $u \in \mathscr{R}$ then $Ju = \pi \sum k_j \delta_{a_j}$ where a_j are the singular points of u and $k_j = \deg(u,a_j)$. In particular, when $u,v \in \mathscr{R}$, then $u \sim v$ if and only if u and v have the same singularities *and the* same degree *for each singularity.*

A special case of interest is the distance of a given $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ to the class $\mathscr{E}(1) = X = \overline{C^\infty(\overline{\Omega};\mathbb{S}^1)}^{W^{1,1}}$ (see (26)–(27)) that we denote for convenience

$$d(u,X) = d_{W^{1,1}}(u,X) = \inf\left\{\int_\Omega |\nabla u - \nabla(e^{\imath\varphi})|\,;\, \varphi \in W^{1,1}(\Omega;\mathbb{R})\right\}. \tag{42}$$

An immediate consequence of Theorem 3.1 is that for every $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have

$$\frac{2}{\pi}\Sigma(u) \le d(u,X) \le \Sigma(u), \tag{43}$$

and the bounds are optimal in the sense that

$$\sup_{\Sigma(u)>0} \frac{d(u,X)}{\Sigma(u)} = 1, \tag{44}$$

and

$$\inf_{\Sigma(u)>0} \frac{d(u,X)}{\Sigma(u)} = \frac{2}{\pi}. \tag{45}$$

The proof of (40) actually provides an explicit recipe for constructing "maximizing sequences" for $\mathrm{Dist}_{W^{1,1}}$. In order to describe it we first introduce, for each $n \ge 3$, a map $T_n \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg T_n = 1$ by $T_n(e^{\imath\theta}) = e^{\imath\tau_n(\theta)}$, with τ_n defined on $[0,2\pi]$ by setting $\tau_n(0) = 0$ and

$$\tau_n'(\theta) = \begin{cases} n, & \theta \in [2j\pi/n^2),(2j+1)\pi/n^2] \\ -(n-2), & \theta \in ((2j+1)\pi/n^2),(2j+2)\pi/n^2] \end{cases},\ j = 0,1,\ldots,n^2-1. \tag{46}$$

A basic ingredient in the proof of (40) in Theorem 3.1 is the following

Theorem 3.2 *For every $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that $u_0 \not\sim v_0$ we have*

$$\lim_{n\to\infty} \frac{d_{W^{1,1}}(T_n \circ u_0, \mathscr{E}(v_0))}{\Sigma(u_0\overline{v}_0)} = 1 \tag{47}$$

and the limit is uniform *over all such u_0 and v_0.*

From (29) it is clear that Theorem 3.2 implies inequality "≥" in (40). The inequality "≤" in (40) is an immediate consequence of the following result established in [4]:

$$d_{W^{1,1}}(u, \mathscr{E}(v_0)) \le \Sigma(u_0\overline{v}_0), \quad \forall\, u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1),\ \forall\, u \in \mathscr{E}(u_0). \tag{48}$$

We now discuss briefly the proof of Theorem 3.2. Inequality "≤" in (47) is a consequence of (48). The heart of the proof of inequality "≥" in (47) is the next lemma.

Lemma 3.3 *For each $\delta > 0$ there exists $n_1 = n_1(\delta)$ such that for every $u, v \in W^{1,1}(\Omega;\mathbb{S}^1)$ and $n \ge n_1$ there holds*

$$\int_\Omega |\nabla(T_n \circ u) - \nabla v| \ge (1-\delta)\Sigma(u\overline{v}).$$

Remark 5 Note the similarity between the definitions of $f \sim g$ in $W^{s,p}(\mathbb{S}^1;\mathbb{S}^1)$ (see (3)*) and $u \sim v$ in $W^{1,1}(\Omega;\mathbb{S}^1)$ (see* (28)*) and also the analogy between* (19)–(20) *and Theorem 3.1 where $|\deg f|$ plays a role similar to $\Sigma(u)$. In fact, the analogy goes beyond the formal resemblance of the formulas. In the above we could replace Ω by a manifold $\mathscr{M}$ (with or without boundary, simply connected or not). Theorem 3.1 holds as is and this is the case also for Theorem 3.2 and Lemma 3.3. Choosing $\mathscr{M} = \mathbb{S}^1$ one sees easily that $\Sigma(u) = 2\pi|\deg u|$, for all $u \in W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$. Indeed, denoting $k = \deg u$, we have on the one hand, for $w(z) = z^k$, $\int_{\mathbb{S}^1} |\dot{w}| = \int_{\mathbb{S}^1} |w \wedge \dot{w}| = 2\pi|k|$, and on the other hand, for all $v \in \mathscr{E}(u)$,*

$$\int_{\mathbb{S}^1} |\dot{v}| = \int_{\mathbb{S}^1} |v \wedge \dot{v}| \ge \left| \int_{\mathbb{S}^1} v \wedge \dot{v} \right| = 2\pi|k|.$$

Hence some results from Section 2 about $W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$ become special cases of Theorem 3.1. It is interesting to write explicitly the statement of Theorem 3.2 for the special case $\Omega = \mathbb{S}^1$. It improves upon [5, Lemma 3.1] by providing a "more uniform" estimate:
For all $f, g \in W^{1,1}(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg f \ne \deg g$ *we have*

$$\lim_{n\to\infty} \frac{d_{W^{1,1}}(T_n \circ f, \mathscr{E}(g))}{2\pi|\deg f - \deg g|} = 1$$

and the limit is uniform *over all such f and g.*

There are many challenging open problems concerning the question whether the supremum and the infimum in various formulas above are achieved. Here are some brief comments, restricted to the case $N = 2$; we refer to [2, 4] for further discussions.

(i) The question whether the infimum in (30) is achieved is extensively studied in [2]. The answer is delicate and depends heavily on Ω and u.

(ii) Concerning the infimum in (42) the answer is positive when Ω is the unit disc and $u(x) = \dfrac{x}{|x|}$, and in some other cases satisfying $d(u, X) = (2/\pi)\Sigma(u)$ (see [4]). In general the question is widely open.

(iii) Concerning the infimum in (45), the answer seems to depend on the shape of Ω. We know that when Ω is the unit disc, the infimum in (45) is achieved by $u(x) = \frac{x}{|x|}$. On the other hand, it seems plausible that if Ω is the interior of a non circular ellipse, then the infimum in (45) is not achieved.

(iv) The question whether the supremum in (44) is achieved is widely open. We suspect that the supremum in (44) is achieved in *every* domain, but we do not know *any* domain in which the supremum is achieved.

Finally, we turn to the classes in $W^{1,p}(\Omega;\mathbb{S}^1)$, $1 < p < 2$, defined in the same way as in the $W^{1,1}$-case. The distances between the classes are defined analogously to (37)–(38) by

$$\mathrm{dist}_{W^{1,p}}(\mathscr{E}(u_0),\mathscr{E}(v_0)) := \inf_{u\sim u_0}\inf_{v\sim v_0}\|\nabla(u-v)\|_{L^p(\Omega)} \tag{49}$$

and

$$\mathrm{Dist}_{W^{1,p}}(\mathscr{E}(u_0),\mathscr{E}(v_0)) := \sup_{u\sim u_0}\inf_{v\sim v_0}\|\nabla(u-v)\|_{L^p(\Omega)}. \tag{50}$$

We first establish a lower bound for $\mathrm{dist}_{W^{1,p}}$:

Proposition 3.1 *For every $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$ we have*

$$\mathrm{dist}_{W^{1,p}}(\mathscr{E}(u_0),\mathscr{E}(v_0)) \geq \left(\frac{2}{\pi}\right)\inf_{w\sim u_0\bar{v}_0}\|\nabla w\|_{L^p(\Omega)}. \tag{51}$$

Remark 6 For $p > 1$ the infimum on the R.H.S. of (51) *is actually a minimum, see [2].*

We do not know whether the lower bound in (51) is optimal:

Open Problem 3 *Is there equality in* (51) *for every $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$?*

We suspect that the answer might be negative in general. We are able to prove that the answer is positive in the case of the distance to smooth maps:

Theorem 3.4 *For every $u_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$, $p \in (1,2)$, we have*

$$\mathrm{dist}_{W^{1,p}}(\mathscr{E}(u_0),\mathscr{E}(1)) = \left(\frac{2}{\pi}\right)\inf_{w\sim u_0\bar{v}_0}\|\nabla w\|_{L^p(\Omega)}. \tag{52}$$

By analogy with item *3* in Theorem 2.3 we have the following result:

Theorem 3.5 *For every $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$ such that $u_0 \not\sim v_0$ we have*

$$\mathrm{Dist}_{W^{1,p}}(\mathscr{E}(u_0),\mathscr{E}(v_0)) = \infty. \tag{53}$$

The detailed proofs of the results on $W^{1,p}(\Omega;\mathbb{S}^1)$, $p \in [1,2)$, will appear in [4].

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