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SETS OF UNIQUENESS FOR DIRICHLET-TYPE SPACES

KARIM KELAY

Abstract. We study the uniqueness sets on the unit circle for weighted Dirichlet spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $\mathbb{T} = \partial \mathbb{D}$ be the unit circle. Let $H^2$ denote the Hardy space of analytic functions on $\mathbb{D}$. If $\mu$ is a positive Borel measure on the unit circle $\mathbb{T}$, the Dirichlet-type space $D(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$D_{\mu}(f) := \int_{\mathbb{D}} |f'(z)|^2 P\mu(z) dA(z) < \infty,$$

where $dA(z) = dx dy / \pi$ stands for the normalized area measure in $\mathbb{D}$ and $P\mu$ is the Poisson integral of $\mu$

$$P\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The space $D(\mu)$ is endowed with the norm

$$\|f\|_{\mu}^2 := \|f\|_{H^2}^2 + D_{\mu}(f).$$

Since $D(\mu) \subset H^2$, every function $f \in D(\mu)$ has non-tangential limits almost everywhere on $\mathbb{T}$. We denote by $f(\zeta)$ the non-tangential limit of $f$ at $\zeta \in \mathbb{T}$ if it exists. It turns out that there is a useful formula for expressing the norm of the Dirichlet-type space in terms of the local Dirichlet integral

$$D_{\zeta}(f) := \int_{\mathbb{T}} |f(e^{it}) - f(\zeta)|^2 \frac{dt}{|e^{it} - \zeta|^2}.$$ 

For a proof of this see [13, Proposition 2.2]. Note that if $d\mu(e^{it}) = dt / 2\pi$, the normalized arc measure on $\mathbb{T}$, then the space $D(\mu)$ coincides with the classical space of functions with finite Dirichlet integral. These spaces were introduced by Richter [11] and generalized by Aleman [1] for nonnegative finite Borel measure on $\overbar{\mathbb{D}}$. The spaces $D(\mu)$ were studied in [1, 11, 12, 13, 14, 15, 17].

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Let $\mathcal{D}^h(\mu)$ be the harmonic version of $\mathcal{D}(\mu)$ given by
\[ \mathcal{D}^h(\mu) := \{ f \in L^2(\mathbb{T}) : D_\mu(f) < \infty \}. \]
We define the capacity $C_\mu$ of a set $E \subset \mathbb{T}$ by
\[ C_\mu(E) := \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E \}, \]
see [4, 5]. If $C_\mu(E) = 0$, then $E$ has Lebesgue measure zero. Indeed, if $C_\mu(E) = 0$, then there exists a sequence $(f_i) \in \mathcal{D}^h(\mu)$ such that $\|f_i\|_\mu \leq 2^{-i}$ and $|f_i| \geq 1$ a.e. on a neighborhood of $E$. Then $f = \sum_i f_i \in \mathcal{D}^h(\mu)$ and $|f| = \infty$ on $E$. We have $\infty > \mathcal{D}_\mu(f) \geq \int_E D_\xi(f) d\mu(\xi)$ and this forces $E$ to have measure zero. We say that a property holds $C_\mu$-quasi-everywhere ($C_\mu$-q.e.) if it holds everywhere outside a set of zero $C_\mu$ capacity. Note that $C_\mu$-q.e implies a.e. We have
\[ C_\mu(E) := \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ } C_\mu \text{-q.e on } E \}. \]
see [6, Theorem 4.2]. Every function $f \in \mathcal{D}(\mu)$ has non-tangential limits $C_\mu$-quasi-everywhere on $\mathbb{T}$ [4, Theorem 2.1.9]. Let $E$ be a subset of $\mathbb{T}$. The set $E$ is said to be a uniqueness set for $\mathcal{D}(\mu)$ if, for each $f \in \mathcal{D}(\mu)$ such that it non-tangential limit $f = 0$ on $E$, we have $f = 0$.

In order to state our main result, we define some notions. Given $E \subset \mathbb{T}$, we write $|E|$ for the Lebesgue measure of $E$. For $w \in L^1(\mathbb{T})$, we denote by $I(w)$ the mean of $w$ over $I$
\[ I[w] = \frac{1}{|I|} \int_I w(\zeta) d\zeta. \]
A nonnegative function $w$ is a Muckenhoupt $A_2$-weight if for all arc $I \subset \mathbb{T}$
\[ \sup_{I \subset \mathbb{T}} I[w] I[w^{-1}] < +\infty. \]

**Theorem 1.1.** Let $\mu$ be an absolutely continuous measure with respect to the Lebesgue measure on $\mathbb{T}$, $d\mu(\zeta) = w(\zeta) |d\zeta|$ and $w$ is a Muckenhoupt $A_2$-weight. Let $E$ be a Borel subset of $\mathbb{T}$ of Lebesgue measure zero. We assume that there exists a family of pairwise disjoint open arcs $(I_n)$ of $\mathbb{T}$ such that $E \subset \bigcup_n I_n$. Suppose
\[ \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} = -\infty; \]
then $E$ is a uniqueness set for $\mathcal{D}(\mu)$.

The case of the Dirichlet space, $d\mu(\zeta) = |d\zeta|/2\pi$, was obtained by Khavin and Maz’ya [9]; see also [2, 3, 8]. In [10], we give the generalization of their result in the Dirichlet spaces $\mathcal{D}_s$, $0 < s \leq 1$, which consist of all analytic functions $f \in H^2$ such that
\[ \mathcal{D}_s(f) := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(\zeta) - f(\xi)|^2 |d\zeta| |d\xi|}{|\zeta - \xi|^{1+s}} 2\pi = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-s} dA(z). \]
The remaining of the note is devoted to proof of the theorem.
2. Proof

To prove our theorem, we use the following lemmas,

**Lemma 2.1.** Let $w$ be a Muckenhoupt $A_2$-weight and let $d\mu(\zeta) = w(\zeta)|d\zeta|$, then

(a) If $I$ is an arc of $\mathbb{T}$ and $\xi_I$ its center, then

$$|I| \int_{T \setminus I} \frac{d\mu(\zeta)}{|\zeta - \xi_I|^2} \leq c|I|w,$$

for some positive constant $C$ independent of $I$.

(b) for all nonnegative function $g$ and all arcs $I$ of $\mathbb{T}$

$$\left(\frac{1}{|I|} \int_I g(\zeta)|d\zeta|\right)^2 \leq \frac{1}{\mu(I)} \int_I g(\zeta)^2 d\mu(\zeta).$$

(c) for all open arcs $I$

$$\mu(I) \geq \frac{\mu(\mathbb{T})}{\pi c} |I|^2 \left(\log \frac{2\pi}{|I|}\right)^2.$$

**Proof.** For a proof of (a), see [7, Lemma 1] and [16] p.200 for (b). Let now to prove (c).

By (b), we have

$$|T \setminus I|^2/|T|^2 \leq \mu(T \setminus I)/\mu(T).$$

By (a) and (b) we get

$$\mu(I) \geq \frac{\mu(T)}{\pi c} |I|^2 \left(\log \frac{2\pi}{|I|}\right)^2.$$

Let $I$ be an open arc of $\mathbb{T}$ and $f$ be a function. We set

$$D_{I,\mu}(f) := \int_I \int_I \frac{|f(z) - f(w)|^2 |dz|}{|z - w|^2} 2\pi d\mu(w) \quad \text{and} \quad m_I(f) := \frac{1}{|I|} \int_I |f(\zeta)||d\zeta|.$$

**Lemma 2.2.** Let $d\mu = wdm$ be a measure such that $w \in (A_2)$. Suppose that $0 < \gamma < 1$. Let $E \subset \mathbb{T}$ and $f \in D(\mu)$ be such that $|f|E = 0$. Then, for any open arc $I \subset \mathbb{T}$ with $|I| \leq \gamma \pi$

$$m_I(f)^2 \leq \frac{\kappa D_{I,\mu}(f)}{C_\mu(E \cap I)},$$

where $\kappa$ depending only on $\gamma$.

**Proof.** Without loss generality, we assume that $I = (e^{-i\theta}, e^{i\theta})$ with $\theta < \gamma \pi/2$. Let $J = (e^{-2i\theta/(1+\gamma)}, e^{2i\theta/(1+\gamma)})$ and $\tilde{f}$ be such that

$$\tilde{f}(e^{it}) = \begin{cases} f(e^{it}), & e^{it} \in I, \\ f(e^{i\frac{\theta - \theta}{1+\gamma}}), & e^{it} \in J \setminus I. \end{cases}$$

Then by a change of variable, we get

$$D_{I,\mu}(f) \asymp D_{J,\mu}(\tilde{f}) \quad \text{and} \quad m_I(f) \asymp m_J(\tilde{f}),$$

where the implied constants depend only on $\gamma$, see [10].
Let $I_\gamma = (e^{-\theta_\gamma}, e^{\theta_\gamma})$ with $\theta_\gamma = \frac{3+\gamma}{2(1+\gamma)} \theta$. Note that $I \subset I_\gamma \subset J$. Let $\phi$ be a positive function on $\mathbb{T}$, $0 \leq \phi \leq 1$, such that supp $\phi = I_\gamma$, $\phi = 1$ on $I$ and

$$|\phi(z) - \phi(w)| \leq \frac{c_1}{|J|}|z - w|, \quad z, w \in \mathbb{T}.$$

where $c_1$ depending only on $\gamma$.

Now, we consider the function

$$F(z) = \phi(z) \left| 1 - \frac{|\tilde{f}(z)|}{m_J(f)} \right|, \quad z \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ $C_\mu$-q.e on $E \cap I$. Therefore,

$$C_\mu(E \cap I) \leq \|F\|_\mu^2. \quad (2)$$

We claim that

$$\|F\|_\mu^2 \leq \kappa \frac{D_{I,\mu}(f)}{m_I(f)^2} \quad (3)$$

where $\kappa$ depending only on $\gamma$. The Lemma 2.2 follows from (2) and (3).

Now, we prove the claim (3). We have

$$\|F\|_\mu^2 = \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2 |d\zeta| d\mu(\xi)}{2\pi}$$

$$\leq \frac{1}{m_J(f)^2} \int_{J} |m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 \frac{|d\zeta|}{2\pi} + \int_{J} \int_{J} \frac{|F(\zeta) - F(\xi)|^2 |d\zeta| d\mu(\xi)}{2\pi}$$

$$+ \frac{1}{m_J(f)^2} \int_{\mathbb{T} \setminus J} \int_{\mathbb{T} \setminus J} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 |d\zeta| d\mu(\xi)}{2\pi}$$

$$+ \frac{1}{m_J(f)^2} \int_{\mathbb{T} \setminus J} \int_{\mathbb{T} \setminus J} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 |d\zeta| d\mu(\xi)}{2\pi}$$

$$= \frac{A}{2\pi m_J(f)^2} + \frac{B}{4\pi^2} + \frac{C}{4\pi^2 m_J(f)^2} + \frac{D}{4\pi^2 m_J(f)^2}. \quad (4)$$

Note that, by Lemma 2.1 (b)

$$|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 \leq \left( \frac{1}{|J|} \int_{J} |\tilde{f}(\xi) - \tilde{f}(\zeta)||d\xi| \right)^2 \leq \frac{1}{\mu(J)} \int_{J} |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 d\mu(\xi).$$
Hence by (1) and Lemma 2.1 (c)

\[
A := \int_J |m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2 d\zeta |
\leq \frac{1}{\mu(J)} \int_J \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 d\mu(\xi) d\zeta |
\leq \frac{|J|^2}{\mu(J)} \int_J \int_J \frac{|\tilde{f}(\xi) - \tilde{f}(\zeta)|^2}{|\xi - \zeta|^2} d\mu(\xi) d\zeta |
\leq c_2 D_{I,\mu}(f),
\]

(5)

where \(c_2\) depending only on \(\gamma\).

Let us now estimate \(B\). If \((\zeta, \xi) \in J \times J\), then we write

\[
|F(\zeta) - F(\xi)| = |\phi(\zeta)\left(1 - \frac{|\tilde{f}(\zeta)|}{m_J(f)}\right) - (\phi(\zeta) - \phi(\xi))1 - \frac{|\tilde{f}(\xi)|}{m_J(f)}| + \frac{c_1}{m_J(f)} |\zeta - \xi| |m_J(f) - |\tilde{f}(\xi)||.
\]

(6)

Note that, by Cauchy-Schwarz,

\[
|m_J(\tilde{f}) - \tilde{f}(\xi)|^2 \leq \frac{1}{|J|} \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 d\eta |.
\]

(7)

So, by (6), (7) and (1)

\[
B := \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta||d\mu(\xi)
\leq \frac{2}{m_J(f)^2} \int_J \int_J \frac{|\tilde{f}(\zeta) - \tilde{f}(\xi)|^2}{|\zeta - \xi|^2} |d\zeta||d\mu(\xi)
+ \frac{2c_1^2}{m_J(f)^2 |J|^3} \int_J \int_J \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 |d\eta||d\zeta||d\mu(\xi)
\leq \frac{2 + 2c_1^2}{m_J(f)^2} \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\eta||d\mu(\xi)
\leq c_3 \frac{D_{I,\mu}(f)}{m_I(f)^2}.
\]

(8)

where \(c_3\) depending only on \(\gamma\).

Next, using again (7) and (1)
\[ C := \int_{\zeta \in T \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2}{|\zeta - \xi|^2}|d\zeta|d\mu(\xi) \]
\[ \leq \int_{\zeta \in T \setminus J} \frac{|d\zeta|}{d(\zeta, I_\gamma)^2} \int_{\xi \in I_\gamma} |m_J(\tilde{f}) - |\tilde{f}(\xi)||^2|d\mu(\xi) \]
\[ \leq \frac{c_4}{|J|^2} \int_J \int_J |f(\eta) - \tilde{f}(\xi)|^2|\eta|d\mu(\xi) \]
\[ \leq c_4 \int_J \int_J \frac{|f(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2}|d\eta|d\mu(\xi) \]
\[ \leq c_5 D_{I,\mu}(f), \tag{9} \]

where \( c_4, c_5 \) depend only on \( \gamma \).

Finally, by Lemma (2.1) (a) and (b) and (1)

\[ D := \int_{\zeta \in T \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2}{|\zeta - \xi|^2}|d\zeta|d\mu(\xi) \]
\[ \leq \int_{\zeta \in T \setminus J} \frac{d\mu(\xi)}{d(\zeta, I_\gamma)^2} \int_{\xi \in I_\gamma} |m_J(\tilde{f}) - |\tilde{f}(\xi)||^2|d\zeta| \]
\[ \leq c_6 \frac{\mu(J)}{|J|^2} \int_{\zeta \in I_\gamma} \frac{1}{\mu(J)} \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2|d\mu(\eta)|d\zeta| \]
\[ \leq c_6 \frac{\mu(J)}{|J|^2} \int_J \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2|\eta|d\mu(\eta) \]
\[ \leq c_6 \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\zeta)|^2}{|\eta - \zeta|^2}|d\eta|d\mu(\zeta) \]
\[ \leq c_7 D_{I,\mu}(f), \tag{10} \]

where \( c_6, c_7 \) depend only on \( \gamma \).

By (5), (8), (9) and (10) we get (3) and the proof is complete. \( \Box \)

Proof of Theorem 1.1. Since \(|E| = 0\), we can assume that \( \sup_n |I_n| \leq \gamma \pi \) with \( \gamma \in (0, 1) \).

Let \( f \in D(\mu) \) be such that \( f|E = 0 \). We set \( \ell = \sum_n |I_n| \). By Lemma 2.2 and Jensen's
inequality
\[
\int \bigcup I_n \log |f(\xi)| d\xi = \sum_n |I_n| \frac{1}{|I_n|} \int_{I_n} \log |f(\xi)| d\xi \\
\leq \sum_n |I_n| \log \frac{1}{|I_n|} \int_{I_n} |f(\xi)| d\xi \\
\leq \sum_n |I_n| \log \left( \frac{\kappa D_{I_n,\mu}(f)}{C_{\mu}(E \cap I_n)} \right) \\
= \sum_n |I_n| \log \frac{|I_n|}{C_{\mu}(E \cap I_n)} + \frac{\ell}{\ell} \sum_n |I_n| \log \left( \frac{\kappa D_{I_n,\mu}(f)}{|I_n|} \right) \\
\leq \sum_n |I_n| \log \frac{|I_n|}{C_{\mu}(E \cap I_n)} + \ell \log \left( \frac{\kappa}{\ell} \sum_n D_{I_n,\mu}(f) \right) \\
\leq \sum_n |I_n| \log \frac{|I_n|}{C_{\mu}(E \cap I_n)} + \ell \log \left( \frac{\kappa}{\ell} D_{\mu}(f) \right) = -\infty.
\]

By Fatou’s Theorem we obtain \( f = 0 \) and the proof is complete.

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