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Abstract

We prove a Harnack inequality for distributional solutions to a type of degenerate elliptic PDEs in $N$ dimensions. The differential operators in question are related to the Kolmogorov operator, made up of the Laplacian in the last $N-1$ variables, a first-order term corresponding to a shear flow in the direction of the first variable, and a bounded measurable potential term. The first-order coefficient is a smooth function of the last $N-1$ variables and its derivatives up to certain order do not vanish simultaneously at any point, making the operators in question hypoelliptic.

1 Introduction

We prove a Harnack inequality for distributional solutions to the degenerate elliptic PDE

$$\Delta_y u + \beta(y)u_x + \gamma(x, y)u = 0 \quad (1.1)$$

in cylindrical domains in $\mathbb{R}^N$ with axes in the direction of the first variable $x$. Here $\gamma$ is bounded measurable and $\beta$ is a smooth function such that the operator

$$L = \sum_{n=1}^{N-1} X_n^2 + X_0 := \sum_{n=1}^{N-1} (\partial_{y_n})^2 + \beta(y)\partial_x = \Delta_y + \beta(y)\partial_x \quad (1.2)$$

satisfies Hörmander’s hypoellipticity condition. That is, vector fields $\{X_n\}_{n=0}^{N-1}$ and their commutators up to certain order span the whole tangent space $\mathbb{R}^N$ at each $(x, y)$. Moreover, $\beta$ changes sign so that $L$ is not parabolic, since then the “elliptic” Harnack inequality (1.4) below would not hold in general. These conditions on $\beta$ are equivalent to hypothesis (1.3) below and our result is then as follows:
Theorem 1.1 Let $D \subseteq \mathbb{R}^{N-1}$ be open connected and $u : (a, b) \times D \to [0, \infty)$ a bounded distributional solution of (1.1) with $\gamma$ bounded measurable and $\beta$ satisfying for some $r \in \mathbb{N}$,

$$\beta \in C^\infty(D), \quad \inf_D \beta < 0 < \sup_D \beta, \quad \text{and} \quad \sum_{0 \leq |\xi| \leq r} |D^\xi \beta(y)| > 0 \text{ for all } y \in D. \quad (1.3)$$

Then for each $[a', b'] \subseteq (a, b)$ and bounded open $D'$ with $\overline{D'} \subseteq D$, there is $C > 0$, depending only on $D$, $D'$, $\beta$ and an upper bound on $(a' - a)^{-1}, (b - b')^{-1}, b' - a'$, and $\|\gamma\|_\infty$, such that

$$\sup_{(a', b') \times D'} u \leq C \inf_{(a', b') \times D'} u. \quad (1.4)$$

Remark. We note that $\Delta_y$ could be replaced by any $x$-independent, uniformly elliptic in $y$ operator on $D$, but for the sake of simplicity we state the theorem with $\Delta_y$ instead.

This result is motivated by its application in our work [6] on large amplitude $A \to \infty$ asymptotics of traveling fronts in the $x$-direction, and their speeds, for the reaction-advection-diffusion equation

$$v_t + A \alpha(y) v_x = \Delta_{x,y} v + f(v) \quad (1.5)$$
on $\mathbb{R}^{N+1}$, with the first order term representing a shear flow in the $x$-direction and $f$ a non-negative reaction function vanishing at 0 and 1. The front speeds in question are proved to satisfy $\lim_{A \to \infty} c^*(A \alpha, f) / A = \kappa(\alpha, f)$ for some constant $\kappa(\alpha, f) \geq 0$, so after substituting the front ansatz $v(t, x, y) = u(x - c^*(A \alpha, f)t, y)$ into (1.5) and scaling by $A$ in the $x$ variable, one formally recovers (1.1) in the limit $A \to \infty$, with $\beta(y) := \kappa(\alpha, f) - \alpha(y)$ and $\gamma(x, y) := -f(u(x, y)) / u(x, y)$.

The study of hypoelliptic operators of the form

$$L = \sum_{n=1}^M X_n^2 + X_0$$

(where $X_n$ are first order differential operators with smooth coefficients), possibly with an additional potential term, has been systematically pursued since Hörmander’s fundamental paper [7]. Although various regularity and maximum principle results have been obtained soon thereafter (see, e.g., [2, 3, 4, 14, 19, 20]), Harnack inequalities and related heat kernel estimates for such operators have initially been proved only in the case when the tangent space at each point is spanned by the fields $\{X_n\}_{n=1}^M$ and their commutators, sometimes with $X_0$ either zero or a linear combination of $\{X_n\}_{n=1}^M$ [2, 9, 10, 12, 13].

More recently, Harnack inequalities have been obtained without this assumption for certain special classes of operators, not including (1.1) with general $\beta, \gamma$. Specifically, some operators with constant and linear coefficients, such as the Kolmogorov operator $L = \partial_{yy}^2 + y \partial_x - \partial_t$, were considered in [5, 16], and cases of more general coefficients satisfying somewhat rigid structural assumptions (see hypothesis [H.1] in [17]) were studied in [11, 17] and with a potential term in [18]. The domains involved in the obtained inequalities have to

\footnote{For $\zeta = (\zeta_1, \ldots, \zeta_{N-1}) \in \mathbb{N}^{N-1}$, we let $|\zeta| = \zeta_1 + \cdots + \zeta_{N-1}$ and $D^\zeta \beta(y) = \frac{\partial^{\zeta_1} \beta}{\partial y_1^{\zeta_1} \cdots \partial y_{N-1}^{\zeta_{N-1}}}(y)$.}
depend on the metrics associated to the operators rather than the Euclidean metric, as shows a counter-example to a Harnack inequality in [5]. This is related to the need for the sign-changing assumption on $\beta$ here. We also note that the operators considered in these papers involve the term $\partial_t$ and appropriate “parabolic-type” Harnack inequalities are obtained, but corresponding “elliptic” inequalities follow from these.

It was a mild surprise to us that we were not able to find in the literature a sufficiently general result which would include our case (1.1). It appears that Harnack inequalities and heat kernel estimates become much more involved when the field $X_0$ is required for Hörmander’s condition to be satisfied. One hint in this direction is the fact that the sign-changing hypothesis on $\beta$ is necessary for (1.4) to hold, so hypoellipticity of $L$ is in itself not a sufficient condition.

We therefore believe that our method of proof of Theorem 1.1 in the next section is itself also a valuable contribution to the problem of quantitative estimates for hypoelliptic operators. The proof is based on the Feynman-Kac formula for the stochastic process associated with the operator $L$, and uses the independence of $L$, and thus also of the stochastic process, on $x$. It is not immediately obvious whether this requirement can be lifted and replaced, for instance, by some assumption on the relation of the stochastic processes associated to $L$ and starting from two different points which can be connected by a path with tangent vector $X_0$ at each point. We leave this as an open problem.

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2 Proof of Theorem 1.1

Without loss we can assume $\inf_{D'} \beta < 0 < \sup_{D'} \beta$ and $D'$ connected, after possibly enlarging $D'$. We will also assume $a = -5$, $a' = 0$, $b = 6$, $D = B_3(0)$, $D' = B_1(0)$, and $\|\gamma\|_{\infty} \leq 1$, with $C$ then only depending on $\beta$, because the general case is analogous. We also note that [19, Theorem 18(c)] and boundedness of $u$ show that $u$ is actually continuous.

We first claim that for each $d > 0$ there is $C_{d, \beta} \geq 1$ such that

$$\sup_{[0,1] \times A_d} u \leq C_{d, \beta} \inf_{[0,1] \times B_1(0)} u,$$

with $A_d := A_d^+ \cup A_d^-$ and $A_d^+ := \{y \in B_1(0) \mid \pm \beta(y) > d \}$. Clearly it suffices to show this for all small enough $d$ such that $A_d^\pm \neq \emptyset$, which we shall assume.

To this end, note that parabolic regularity theory with $x$ as the time variable, applied on $[-1, 5] \times \{y \in B_2(0) \mid -\beta(y) > d/2 \}$, yields

$$\sup_{[0,1] \times A_d^+} u \leq C_{d, \beta} \inf_{[2,5] \times A_d^-} u,$$
where $C'_{d,\beta} > 0$ depends only on $d$ and $\beta$. Similarly, we obtain

$$\sup_{[3,4] \times A^+_d} u \leq C'_{d,\beta} \inf_{[-1,2] \times A^+_d} u, \quad (2.8)$$

Next, consider the stochastic process $(X^{x,y}_t, Y^{x,y}_t)$ starting at $(x, y) \in \mathbb{R} \times B_2(0)$ and satisfying the stochastic differential equation

$$(dX^{x,y}_t, dY^{x,y}_t) = (\beta(Y^{x,y}_t) dt, \sqrt{2} dB_t), \quad (X^{x,y}_0, Y^{x,y}_0) = (x, y).$$

Here $t$ is a new time variable and $B_t$ is a normalized Brownian motion on $\mathbb{R}^{N-1}$ with $B_0 = 0$ (defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$). We then have

$$(X^{x,y}_t, Y^{x,y}_t) = (X^{0,y}_0 + x, \sqrt{2}B_t + y). \quad (2.9)$$

for any $(x, y) \in \mathbb{R} \times B_2(0)$, in particular, $Y^{x,y}_t$ is independent from $x$. For any $y \in B_2(0)$ we also define the stopping time

$$\tau = \tau_y := \inf \{ t > 0 \mid Y^{x,y}_t \notin B_2(0) \}.$$

If $t \wedge \tau := \min\{t, \tau\}$, then by the Feynman-Kac formula, $\|\gamma\|_{\infty} \leq 1$, and the parabolic comparison principle, we have for each $t \geq 0$ and $(x, y) \in \mathbb{R} \times B_2(0),

$$e^{-t}\mathbb{E}(u(X^{x,y}_{t \wedge \tau}, Y^{x,y}_{t \wedge \tau})) \leq u(x, y) \leq e^{t}\mathbb{E}(u(X^{x,y}_{t \wedge \tau}, Y^{x,y}_{t \wedge \tau})). \quad (2.10)$$

(The Feynman-Kac formula is usually stated for $C^2$ functions so we provide a proof of (2.10) in Lemma 2.1 below.) Here

$$\mathbb{E}(u(X^{x,y}_{t \wedge \tau}, Y^{x,y}_{t \wedge \tau})) = \int_{\Omega} u(X^{x,y}_{t \wedge \tau}(\omega), Y^{x,y}_{t \wedge \tau}(\omega)) d\mathbb{P}(\omega) = \int_{\mathbb{R} \times B_2(0)} u(x', y') d\mu^{x,y}_t(x', y'), \quad (2.11)$$

with the probability measure $\mu^{x,y}_t$ on $\mathbb{R} \times B_2(0)$ such that $\mu^{x,y}_t(A) = \mathbb{P}(X^{x,y}_{t \wedge \tau}, Y^{x,y}_{t \wedge \tau} \in A)$ for Borel sets $A \subseteq \mathbb{R} \times B_2(0)$. Notice that $\mu^{x,y}_t$ is supported on $[x - \|\beta\|_{\infty}t, x + \|\beta\|_{\infty}t] \times B_2(0)$ and $\mu^{x,y}_t(\mathbb{R} \times \partial B_2(0)) = \mathbb{P}(\tau_y \leq t)$.

By (2.9), translation in $x$ equally translates the $\mu^{x,y}_t$, and the ($x$-independent) measure on $B_2(0)$ given by $\nu^{y}_t(A) = \mu^{x,y}_t(\mathbb{R} \times A)$ is just the law of $\sqrt{2}B_{t \wedge \tau_y} + y$, the Brownian motion on $B_2(0)$ starting at $y$, with stopping time $\tau_y$, and with time scaled by a factor of two. In particular for each $t > 0$ there is $h_t > 0$ such that for any $y_1, y_2 \in B_1(0)$ and any Borel sets $A_1 \subseteq B_1(0)$ and $A_2 \subseteq B_2(0),

$$h_t \nu^{y_1}_t(A_2) \leq \nu^{y_2}_t(A_2) \leq h_t^{-1} \nu^{y_1}_t(A_2) \quad \text{and} \quad h_t|A_1| \leq \nu^{y_1}_t(A_1) \leq h_t^{-1}|A_1|. \quad (2.12)$$

From this it follows for $t := \|\beta\|_{\infty}^{-1}$ that

$$\inf_{[0,1] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[-1,2] \times A^+_d} u \quad (2.13)$$

with $C''_{d,\beta} := e^{-t}h_t \min\{|A^+_d|, |A^-_d|\}$ and, similarly, we obtain

$$\inf_{[3,4] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[2,5] \times A^-_d} u. \quad (2.14)$$

\[\text{4}\]
Using (2.7), (2.14), (2.8), and (2.13) (in that order) yields
\[
\sup_{[0,1] \times A_d} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,
\]
with \( C_{d,\beta} > 0 \) depending only on \( d \) and \( \beta \). An analogous argument gives
\[
\sup_{[0,1] \times A_d^+} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,
\]
and (2.6) follows.

Next we let \( v(x, y) := \int_{-s}^z u(x + s, y)ds \) for some \( z \in (0, 1/3] \)
\[
\sup_{[0,1] \times B_1(0)} v \leq \tilde{C}_{z,\beta} \inf_{[0,1] \times B_1(0)} u
\tag{2.15}
\]
holds for some \( \tilde{C}_{z,\beta} \geq 1 \). Indeed, it follows from (2.9), (2.10), (2.11) that for each \((x, y) \in \mathbb{R} \times B_2(0)\),
\[
e^{-t} \int_{\mathbb{R} \times B_2(0)} u(x', y')d\mu_{t,x,y}^{x,y}(x', y') \leq v(x, y) \leq e^{t} \int_{\mathbb{R} \times B_2(0)} u(x', y')d\mu_{t,x,y}^{x,y}(x', y'),
\]
where \( \mu_{t,x,y}^{x,y}(x', y') = \mu_t^{x,y}(x', y') * (\chi_{[-z,z]}(x')dx') \). The above claims about \( \mu_t^{x,y} \) and the definition of \( \nu_t^y \) imply that
\[
\mu_t^{x,y}(x', y') \leq \kappa_t^{x,y}(x') \times \nu_t^y(y') \leq \sum_{m=-M}^M \mu_t^{x+2mz,y}(x', y'),
\]
where \( \kappa_t^{x,z} \) is the measure on \( \mathbb{R} \) with \( \kappa_t^{x,z}(B) = |B \cap [x - z - \|\beta\| \infty t, x + z + \|\beta\| \infty t]| \) for any Borel set \( B \subseteq \mathbb{R} \), and \( M \) is such that \((2M+1)z \geq 2(\|\beta\| \infty t)\), for instance, \( M := \lceil 1/2 + \|\beta\| \infty t/z \rceil \). This and the first claim in (2.12) means that
\[
v(x, y_1) \leq e^{2t}h_t^{-2} \sum_{m=-M}^M v(x + 2mz, y_2)
\tag{2.16}
\]
for any \( x \in \mathbb{R}, y_1, y_2 \in B_1(0) \) and \( t > 0 \).

Now we take any \( x \in [0,1], y_1 \in B_1(0), \) and \( y_2 \in A_d \) for some fixed \( d > 0 \) such that \( A_d^\pm \neq \emptyset \). Pick \( t := (2\|\beta\| \infty)^{-1}z \) and \( M = 1 \) to obtain using (2.16),
\[
v(x, y_1) \leq e^{2t}h_t^{-2} \int_{-3z}^{3z} u(x + s, y_2)ds \leq e^{2t}h_t^{-2} \int_{-1}^{2} u(x', y_2)dx'.
\]
Since (2.6) and its shifts in \( x \) give for \( c = -1,0,1, \)
\[
\sup_{[c-1,c] \times A_d} u \leq C_{d,\beta} \inf_{[0,1] \times A_d} u \leq C_{d,\beta} \sup_{[c,c+1] \times A_d} u,
\]
\[
\sup_{[c-1,c] \times A_d} u \leq C_{d,\beta} \inf_{(-1) \times A_d} u \leq C_{d,\beta} \sup_{[-2,c-1] \times A_d} u,
\]
we obtain (2.6) with \([0, 1]\) and \(C_{d, \beta}\) replaced by \([-1, 2]\) and \(C_{d, \beta}^3\). This proves (2.15). Similarly, (2.15) with \([-1, 0]\) and \([1, 2]\) in place of \([0, 1]\), together with (2.6), yield

\[
\sup_{[-1,2] \times B_1(0)} v \leq \tilde{C}_{z, \beta} C_{d, \beta} \inf_{[0,1] \times B_1(0)} u.
\]

In a similar way one can also obtain

\[
\sup_{[-1,2] \times B_2(0)} v \leq C_{z, d, \beta} \inf_{[0,1] \times B_2(0)} u.
\]

(2.17)

for some \(C_{z, d, \beta} > 0\) (recall that \(B_2(0) \subset D = B_3(0)\)).

We will now need to use (1.3) to finish the proof. This assumption makes the differential operator on the left-hand side of (1.1) hypoelliptic in the sense of Hörmander. It follows that for \(t > 0\), the measure \(\tilde{\mu}_t^{x,y}\) is absolutely continuous when restricted to \(\mathbb{R} \times B_2(0)\) and also to \(\mathbb{R} \times \partial B_2(0)\) (as an \((N-1)\)-dimensional measure in the latter case), with densities \(p_t(x, y, \cdot, \cdot), q_t(x, y, \cdot, \cdot) \geq 0\) such that

\[
p_t(x, y, x', y') = p_t(0, y, x' - x, y'),
\]

\[
q_t(x, y, x', y') = q_t(0, y, x' - x, y'),
\]

and \(p_t, q_t\) are bounded functions when restricted to \(y \in B_1(0)\) (with \(y' \in B_2(0)\) for \(p_t\) and \(y' \in \partial B_2(0)\) for \(q_t\)). For \(p_t\) this follows from the same claim for the corresponding measure \(\tilde{\mu}_t^{x,y}\) on \(\mathbb{R}^N\) given by (2.11) with \(t = 2\) in place of \(t \wedge \tau\) and \(\beta\) smoothly extended to a periodic function on \(\mathbb{R}^{N-1}\) (whose density is smooth in all arguments, [8, Theorem 3]). This is because \(\tilde{\mu}_t^{x,y}(A) \geq \tilde{\mu}_t^{x,y}(A)\) for any Borel set \(A \subseteq \mathbb{R} \times B_2(0)\).

For \(q_t\) this would follow from the same claim for the corresponding escape measure \(\tilde{\mu}_t^{x,y}\) on \(\mathbb{R} \times \partial B_2(0)\) given by (2.11) with \(\tau = \tau_y\) in place of \(t \wedge \tau\). We know of such a result for bounded domains only [1, Corollary 2.11] but since \(\tilde{\mu}_t^{x,y}\) is supported on a bounded cylinder, it applies in our case as well. Specifically, take any \(a_- < -\|\beta\|_{\infty} t\) and \(a_+ > \|\beta\|_{\infty} t\). There is a convex open domain \(G\) with a smooth boundary whose intersection with \([a_-, a_+] \times \mathbb{R}^{N-1}\) is \([a_-, a_+] \times B_2(0)\), and the intersection with \([(-\infty, a_-) \cup (a_+, \infty)] \times \mathbb{R}^{N-1}\) are two smooth “slanted” conical caps \(G_{\pm} \subseteq \mathbb{R} \times B_2(0)\) over the \((N-1)\)-dimensional balls \([a_{\pm}] \times B_2(0)\) with the two (rounded) tips at points with \(y'\) coordinates \(y'_\pm\) such that \(\pm \beta(y'_\pm) > 0\) and sufficiently long so that for any \((x', y') \in \partial G_{\pm} \cap \partial G\), the unit outer normal vector \(n(x', y')\) to \(\partial G_{\pm}\) at \((x', y')\) satisfies

\[
|n(x', y') \cdot (1, 0, \cdots, 0)| \leq \frac{1}{2}(\|\beta\|_{\infty}^{-1} + 1) \text{ whenever } \pm \beta(y') \leq 0.
\]

Then \(G\) satisfies the hypotheses of [1, Corollary 2.11] (it satisfies the escape condition and all points of \(\partial G\) are \(\tau'\)-regular). It follows that the escape measure \(\tilde{\mu}_t^{x,y}\) has a density \(\tilde{q}_t(x, y, \cdot, \cdot, \cdot)\) which is a continuous function of \((x, y, x', y') \in G \times \partial^* G\), where \(\partial^* G\) is the set of “good” points of \(\partial G\), that is, all \((x', y') \in \partial G\) except of the two cone tips, where \(n(x', y') = (\pm 1, 0, \cdots, 0)\). Thus \(\tilde{q}_t\) is bounded on \(S := \{0\} \times B_1(0) \times (a_-, a_+) \times \partial B_2(0)\). Since \(\{X^{0,y}_s\}_{s \leq \tau}\) almost surely stays in \((a_-, a_+)\), we obtain \(q_t \leq \tilde{q}_t\) on \(S\) and \(q_t = 0\) on \([\{0\} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)\] \(\setminus S\). Finally, \(q_t(x, y, x', y') = q_t(0, y, x' - x, y')\) shows that \(q_t\) is bounded on \(\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)\).
Let $d > 0$ be such that $A_d^+ \neq \emptyset$, let $z := 1/3$, $t := \|\beta\|^{-1}_\infty$, and

$$C_t := \max\{\sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times B_2(0)} p_t, \sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)} q_t\} < \infty.$$ 

Then $p_t(x, y, x', y')$, $q_t(x, y, x', y') \leq C_t \chi_{[x-1, x+1]}(x')$ because the measure $\mu_t^{x,y}$ is supported on $[x-1, x+1] \times B_2(0)$, so we obtain from (2.10) and (2.11)

$$\sup_{[0,1] \times B_1(0)} u \leq C_t e^t \int_{[-1,2] \times B_2(0)} u(x', y') dx' dy' + C_t e^t \int_{[-1,2] \times \partial B_2(0)} u(x', y') dx' dy'$$

$$\leq 10C_t e^t \sup_{[-1,2] \times B_2(0)} v$$

$$\leq 10C_t C_{z,d,\beta} e^t \inf_{[0,1] \times B_1(0)} u$$

by using $[-1, 2] = [-1, -1/3] \cup [-1/3, 1/3] \cup [1/3, 1] \cup [1, 5/3] \cup [4/3, 2]$ and (2.17). This is (1.4), so the theorem will be proved once we establish (2.10).

**Lemma 2.1** If $u, \beta, \gamma, X_t^{x,y}, Y_t^{x,y}, \tau_y$ are as in the proof of Theorem 1.1 (in particular, $\|\gamma\|_\infty \leq 1$), then (2.10) holds for $(x, y) \in \mathbb{R} \times B_2(0)$.

**Proof.** Let $Z_t^{x,y} = t$ so that $dZ_t^{x,y} = dt$ and $K := \Delta_y + \beta(y) \partial_x + \partial_z$ is the generator of the process $(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})$. If we let $v(x, y, z) := e^z u(x, y)$, then $Kv \geq 0$ on $\mathbb{R} \times B_3(0) \times \mathbb{R}$ in the sense of distributions, that is,

$$\int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} vK^* \phi \ dx dy dz \geq 0$$

for any $\phi \in C_c^\infty(\mathbb{R} \times B_3(0) \times \mathbb{R})$, with $K^* := \Delta_y - \beta(y) \partial_x - \partial_z$ the adjoint of $K$.

For any $\varepsilon > 0$ let $\delta_\varepsilon \in (0, 1/2\sqrt{N-1})$ be such that $|\beta(y) - \beta(y')| \leq \varepsilon^2$ whenever $y, y' \in B_{5/2}(0)$ and $|y - y'| \leq \sqrt{N-1}\delta_\varepsilon$. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth non-negative function supported in $[-1, 1]$, with $\int_{-1}^1 \eta(x') dx' = 1$ and $\|\eta'\|_\infty \leq 2$. For $\varepsilon > 0$ define the mollifier

$$\eta^\varepsilon(x, y, z) := \varepsilon^{-2}\delta_\varepsilon^{-N} \eta\left(\frac{x}{\varepsilon}\right) \eta\left(\frac{y}{\varepsilon}\right) \prod_{n=1}^{N-1} \eta\left(\frac{y_n}{\delta_\varepsilon}\right),$$

and let $v^\varepsilon := v * \eta^\varepsilon$ and $\phi^\varepsilon_{x,y,z}(x', y', z') := \eta^\varepsilon(x-x', y-y', z-z')$. For $\varepsilon \in (0, 1)$ the smooth function $v^\varepsilon$ then satisfies on $\mathbb{R} \times B_2(0) \times \mathbb{R}$

$$(Kv^\varepsilon)(x, y, z) = \int_{\mathbb{R} \times B_2(0) \times \mathbb{R}} vK^* \phi^\varepsilon_{x,y,z} \ dx' dy' dz'$$

$$+ \int_{\mathbb{R} \times B_2(0) \times \mathbb{R}} v(x', y', z')[\beta(y') - \beta(y)] \phi^\varepsilon_{x,y,z}(x', y', z') \ dx' dy' dz'.$$

The first integral is non-negative. The integrand in the second vanishes when $|x' - x| > \varepsilon$ or $|y' - y_n| > \delta_\varepsilon$ for some $n$ or $|z' - z| > \varepsilon$, and $|\phi^\varepsilon_{x,y,z}(x', y', z')| \leq 2\varepsilon^{-3}\delta_\varepsilon^{-N}$, so we have

$$(Kv^\varepsilon)(x, y, z) \geq -2^{N+2}\varepsilon e^{z+\varepsilon} \|u\|_\infty.$$
We next apply Dynkin’s formula [15, Theorem 7.4.1] to the smooth function \( v^\varepsilon \), the process \((X_t^x, Y_t^x, Z_t^x)\), and stopping time \( t \wedge \tau \) (with \( \tau = \tau_y \)), to obtain

\[
\mathbb{E}[v^\varepsilon(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})] = v^\varepsilon(x, y, 0) + \mathbb{E}\left[ \int_0^{t \wedge \tau} (Kv^\varepsilon)(X_s^{x,y}, Y_s^{x,y}, Z_s^{x,y})ds \right]
\geq v^\varepsilon(x, y, 0) - 2N+2\varepsilon e^{t+\varepsilon}\|u\|_\infty t.
\]

Since \( v^\varepsilon \to v \) uniformly on \([x - \|\beta\|_\infty t, x + \|\beta\|_\infty t] \times B_2(0) \times [0, t]\) as \( \varepsilon \to 0 \) (by continuity of \( v \)) and \( Z_t^{x,y} \leq t \), it follows that

\[
e^t \mathbb{E}[u(X_t^{x,y}, Y_t^{x,y})] \geq \mathbb{E}[v(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})] \geq u(x, y).
\]

This is the second inequality in (2.10). The first inequality is obtained in the same way, this time with \( v(x, y, z) := e^{-z}u(x, y) \), so that \( Kv \leq 0 \) on \( \mathbb{R} \times B_3(0) \times \mathbb{R} \) and

\[
(Kv^\varepsilon)(x, y, z) \leq 2N+2\varepsilon e^{-z+\varepsilon}\|u\|_\infty.
\]

\[\square\]

References


