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Abstract

The metriplectic framework, which allows for the formulation of an algebraic structure for dissipative systems, is applied to visco-resistive Magneto-Hydrodynamics (MHD), adapting what had already been done for non-ideal Hydrodynamics (HD). The result is obtained by extending the HD symmetric bracket and free energy to include magnetic field dynamics and resistive dissipation. The correct equations of motion are obtained once one of the Casimirs of the Poisson bracket for ideal MHD is identified with the total thermodynamic entropy of the plasma. The metriplectic framework of MHD is shown to be invariant under the Galileo Group. The metriplectic structure also permits us to obtain the asymptotic equilibria toward which the dynamics of the system evolves. This scheme is finally adapted to the two-dimensional incompressible resistive MHD, that is of major use in many applications.

INTRODUCTION

The impossibility of solving analytically the overwhelming majority of differential equations in Physics soon convinced physicists to investigate the properties of dynamical systems without searching for all the possible solutions. Integral properties of the solutions were then investigated, as conserved quantities, and not much more than the esthetic taste inspired theorists to formulate those shortcuts in a mathematically cleaner way: this is more or less the history of Action Principles [1, 2], beginning as acute observations on special problems, and soon generating the wonderful offspring of Lagrangian Dynamics (with its noble descendants of path integral representations [3]), and Hamiltonian Dynamics.

Algebrization of dynamical systems appears to be the final destination of that virtuous route [4]: in the Hamiltonian framework dynamics is turned into a bracket algebra of observable quantities, and then physical properties of systems, especially in terms of conserved quantities and symmetries [5], can be obtained without even the need of going back to the equations of motion [6]. Hamiltonian dynamics has, also, represented a huge breakthrough to Quantum Physics [7], that is exquisitely an algebraic formulation.

This cultural and methodological evolution, starting with some symmetry observations and ending up with the bracket algebræ, appears to be natural for *conservative systems*.

A very promising strategy to algebrize the dynamics of a dissipative system is the *metriplectic framework* [8, 9]. The system at hand must be *complete*, i.e. one must be able to keep trace of the total energy during the motion: typically, this means including all the energy exchanges in a conserved Hamiltonian. In other words, the metriplectic framework is applicable to closed systems.

Dissipation is generally understood as the interaction of dynamical variables of the otherwise Hamiltonian system with other *microscopic, statistically treated, degrees of freedom* (MSTDOF), giving rise to friction. The system is extended to include the MSTDOF, and this closes the system. The dynamics of the closed system with friction is then assigned by defining a symmetric extension of the Poisson bracket algebra, and an extension of the Hamiltonian to free energy. In order to extend the Hamiltonian to the free energy of the closed system, the entropy S of the MSTDOF will be used.

Hamiltonian dynamics evolves any quantity f as $\dot{f} = \{f, H\}$, being $\{f, g\}$ the Poisson bracket, while no-friction condition would imply no entropy production in the Hamiltonian system. Then, the entropy S must be conserved in the Hamiltonian limit of the dissipative system: $\{S, H\} = 0$. For noncanonical Hamiltonian systems, S is then expected to be expressable through *Casimir functionals of the Poisson bracket* $\{f, g\}$, i.e. quantities C such that

$$\{C, f\} = 0 \ \forall \ f. \tag{1}$$

If the, otherwise Hamiltonian, system were described via canonical Poisson bracket, (1) would simply mean that the MSTDOF, implemented through the entropy, are independent of the original dynamical variables, and must be summed directly to them to include dissipation. such examples, indicating the origin and the need of the Casimir nature of S, will be discussed in forthcoming papers.

The Hamiltonian is hence extended by defining the *free energy*

$$F = H + \lambda C. \tag{2}$$

The coefficient λ in (2) is a constant: under the hypothesis of thermal equilibrium for the MSTDOF and asymptotic equilibrium for the system, this λ will coincide with minus the temperature of the MSTDOF, but in general it should be understood just as an arbitrary constant coefficient left indicated.

The framework is completed by prescribing that the evolution of any quantity f is generated by F via an extension $\langle\langle f, g\rangle\rangle$ of the original Poisson bracket $\langle\langle f, g\rangle\rangle = \{f, g\} + (f, g)$, where the symbol (f, g) is symmetric, bilinear and semi-definite [10]. For instance, for the positive semi-definite case, we have:

$$(f, g) = (g, f)\,, \ (f, f) \geq 0 \ \forall \ f, g.$$

In a metriplectic framework the evolution is then generated as:

$$\dot{f} = \langle\langle f, F\rangle\rangle \tag{3}$$

(the symmetric bracket (f, g) will be defined so to cancel out the presence of the coefficient λ, defined in (2), removing it from the equations of motion).

The symmetric structure (f, g) is referred to as *metric* component of the motion, and is chosen so that H is conserved during the motion (3): due to (1) and (2), this can be realized by defining (f, g) so that

$$(H, f) = 0 \ \forall \ f. \tag{4}$$

With all these conditions, it's easy to observe the separation of the metriplectic motion (3) into a Hamiltonian component plus a metric one: $\langle\langle f, F\rangle\rangle = \{f, H\} + \lambda\,(f, C)$. The metriplectic evolution then reads:

$$\dot{f} = \{f, H\} + \lambda\,(f, C)\,. \tag{5}$$

While the Hamiltonian is conserved due to (5) and (4) (completeness of the system $\dot{H} = 0$), the Casimir C chosen in (2) to mimic the entropy undergoes a non-trivial evolution:

$$\dot{C} = \lambda\,(C, C)\,. \tag{6}$$

Due to the semi-definiteness of (f, g), $\dot{C}$ has a constant sign: constructing this C as suitably limited from above or below, it can be used as a Lyapunov quantity for the dynamics (3), admitting asymptotic equilibria, as it must be the case for dissipative systems. The entropic meaning of C will be discussed more deeply in forthcoming papers. Note, however, that its equation of motion (6) should be interpreted as an *H-Theorem for the MSTDOF* involved in dissipation: in this sense, the metriplectic scheme represents a simple strategy towards the *algebrization of irreversibility*.

METRIPLECTIC FORMULATION OF VISCO-RESISTIVE MHD

The system we want to deal with here is a fully ionized plasma interacting with the magnetic field generated by its own motion; dissipation takes place due to the *finite viscosity and resistivity* of the fluid [11]. More, *heat conductivity* is finite, hence nearby parcels of fluid tend to thermalize.

The configuration of the system is given by assigning the bulk velocity $\vec{v}$ of the fluid, the magnetic induction $\vec{B}$, the matter density ρ. Then, another field is introduced expressing the thermodynamic nature of the matter involved, e.g. the mass-specific entropy s. The resulting system of equations may be written in an $SO(3)$-covariant form as:

$$\begin{cases} \partial_t v_i = -v_k \partial^k v_i - \frac{1}{\rho}\partial_i p - \frac{1}{2\rho}\partial_i B^2 + \frac{1}{\rho}B_k \partial^k B_i - \partial_i \phi_{grav} + \frac{1}{\rho}\partial^k \sigma_{ik}, \\ \partial_t B_i = B_j \partial^j v_i - B_i \partial^j v_j - v_j \partial^j B_i + \mu \partial^2 B_i, \\ \partial_t \rho = -\partial^k (\rho v_k), \\ \partial_t s = -v_k \partial^k s + \frac{\sigma_{ik}}{\rho T}\partial^k v^i + \frac{\mu}{\rho T}\epsilon_{ikh}\epsilon^h{}_{mn}\partial^i B^k \partial^m B^n + \frac{\kappa}{\rho T}\partial^2 T, \\ \forall\ \vec{x} \in \mathbb{D},\ t \in I \end{cases} \tag{7}$$

(here $\mathbb{D} \subseteq \mathbb{R}^3$ is the space domain where the dynamical variables are defined and $I \subseteq \mathbb{R}$ is the time interval of interest). Local thermal equilibrium is assumed, so that the smooth field T may be defined. ϕ_{grav} is the gravitational potential to which the plasma undergoes. The stress tensor σ_{ik} is chosen to be linear in the gradient of the velocity:

$$\begin{cases} \sigma_{ik} = \Lambda_{ikmn}\partial^m v^n, \\ \Lambda_{ikmn} \stackrel{\text{def}}{=} \eta \left(\delta_{ni}\delta_{mk} + \delta_{nk}\delta_{mi} - \frac{2}{3}\delta_{ik}\delta_{mn}\right) + \zeta \delta_{ik}\delta_{mn}, \end{cases} \tag{8}$$

The addendum $\frac{\mu}{\rho T}\epsilon_{ikh}\epsilon^h{}_{mn}\partial^i B^k \partial^m B^n$ in the fourth equation of (7) is the entropy production rate $(\partial_t s)_B$ due to the Joule Effect and may be obtained through some considerations of elementary Thermodynamics. In that expression μ indicates the plasma resistivity. The system (7) is "closed" expressing the quantities p and T in terms of mass-specific internal energy of the fluid U:

$$p = \rho^2 \frac{\partial U}{\partial \rho},\ T = \frac{\partial U}{\partial s}. \tag{9}$$

In the system at hand, the fields $\vec{v}$, $\vec{B}$ and ρ may be intended as macroscopic, deterministically treated variables, while the Statistical Mechanics of the MSTDOF giving rise to dissipation is encoded in s.

The description of the isolated visco-resistive MHD as a complete system [8] is possible if the "total energy"

$$H = \int_{\mathbb{D}} \left[\frac{\rho}{2}v^2 + \rho\phi_{grav} + \frac{B^2}{2} + \rho U(\rho, s)\right] d^3x \tag{10}$$

is introduced. Thanks to the way in which the Joule Effect contribution appears in $\partial_t s$, it is possible to show that this H is a constant of motion for the equations (7), provided *suitably good boundary conditions* are given to the plasma. Indeed, along the motion (7) the quantity H changes only via a boundary term: $\dot{H} \overset{\partial}{=} 0$ ($a \overset{\partial}{=} b$ means that a and b only differ by a boundary term). The "suitable conditions" at $\partial\mathbb{D}$ are those rendering the magnetized plasma an isolated system.

H may be used as the Hamiltonian component of the free energy of the system which will metriplecticly generate the evolution (7).

Setting to zero the coefficients κ, μ, η and ζ, the ideal MHD is obtained:

$$\begin{cases} \partial_t v_i = -v_k \partial^k v_i - \frac{1}{\rho}\partial_i p - \frac{1}{2\rho}\partial_i B^2 + \frac{1}{\rho} B_k \partial^k B_i - \partial_i \phi_{grav}, \\ \\ \partial_t B_i = B_j \partial^j v_i - B_i \partial^j v_j - v_j \partial^j B_i, \\ \\ \partial_t \rho = -\partial^k \left(\rho v_k\right), \\ \\ \partial_t s = -v_k \partial^k s. \end{cases} \tag{11}$$

The functional H in (10) is the Hamiltonian for this field theory [12], with the noncanonical Poisson bracket [13]

$$\begin{aligned} \{f,g\} = -\int_{\mathbb{D}} d^3x \Bigg[& \frac{\delta f}{\delta \rho}\partial_i\left(\frac{\delta g}{\delta v_i}\right) - \frac{\delta g}{\delta \rho}\partial_i\left(\frac{\delta f}{\delta v_i}\right) - \frac{1}{\rho}\frac{\delta f}{\delta v_i}\epsilon_{ikj}\epsilon^{jmn}\frac{\delta g}{\delta v_k}\partial_m v_n + \\ & + \frac{1}{\rho}\frac{\delta f}{\delta v_i}\epsilon_{ijk}\epsilon^{kmn}B^j\partial_m\left(\frac{\delta g}{\delta B^n}\right) + \frac{\delta f}{\delta B_i}\epsilon_{ijk}\partial^j\left(\frac{1}{\rho}\epsilon^{kmn}B_m\frac{\delta g}{\delta v^n}\right) + \\ & + \frac{1}{\rho}\partial_i s\left(\frac{\delta f}{\delta s}\frac{\delta g}{\delta v_i} - \frac{\delta g}{\delta s}\frac{\delta f}{\delta v_i}\right)\Bigg]. \end{aligned} \tag{12}$$

Any quantity f is evolved along the motion (11) via the prescription $\dot{f} = \{f, H\}$. The Poisson bracket (12) has several Casimir observables, in particular we quote those of the form

$$C\left[\rho, s\right] = \int_{\mathbb{D}} \rho\varphi\left(s\right) d^3x, \tag{13}$$

among which one may recognize the total mass M and the total entropy S of the fluid:

$$M[\rho] = \int_{\mathbb{D}} \rho d^3x, \;\; S[\rho, s] = \int_{\mathbb{D}} \rho s d^3x. \tag{14}$$

M and S are conserved along the motion (11), because they have zero Poisson bracket with any quantity f, and in particular with H. The functionals C in (13) may be used to construct a metriplectic framework with H in (10), as prescribed in (2) and (3).

Other non-Casimir quantities, remarkably conserved by the motion (11), are all the space-time symmetries related to the Galileo transformation, i.e. the total momentum $\vec{P}$ of the system, the total angular momentum $\vec{L}$ and a quantity $\vec{G}$, which is the symplectic generator of Galileo's boosts. Their definitions

$$\begin{cases} P_h = \int_{\mathbb{D}} \rho v_h d^3x, \;\; L_h = \int_{\mathbb{D}} \rho \epsilon_{hij} x^i v^j d^3x, \\ \\ G_h = \int_{\mathbb{D}} \rho \left(x_h - v_h t\right) d^3x \end{cases} \tag{15}$$

plus the definition of H in (10) and of the Poisson bracket $\{f, g\}$ in (12) imply:

$$\{P_h, H\} \overset{\partial}{=} 0, \;\; \{L_h, H\} \overset{\partial}{=} 0, \;\; \{G_h, H\} \overset{\partial}{=} 0. \tag{16}$$

Let's turn back to the system with dissipation (7): the dissipative terms appearing there must be given by a suitable symmetric bracket (f, g) (still to be defined) of the dynamical variables at hand with the Casimir C to be used as in (5). The correct Casimir to be used is the plasma entropy $S[\rho, s]$ in (14).

The result presented here is the explicit expression of such bracket (f, g).

The dissipative terms in (7) are the 8 expressions

$$D_i^{(v)} = \frac{1}{\rho}\partial^k \sigma_{ik}, \;\; D_i^{(B)} = \mu \partial^2 B_i, \;\; D^{(\rho)} = 0, \;\; D^{(s)} = \frac{1}{\rho T}\left(\sigma_{ik}\partial^k v^i + \mu j^2 + \kappa \partial^2 T\right),$$

with self-evident meaning of the symbols. If these terms are collected in an 8-uple $\underline{D} = \left(\vec{D}^{(v)}, \vec{D}^{(B)}, D^{(\rho)}, D^{(s)}\right)$ and the dynamical variables in (7) are $\underline{\psi} = \left(\vec{v}, \vec{B}, \rho, s\right)$, then one aims to define the metric bracket (f, g) so that $\underline{D} = \lambda\left(\underline{\psi}, S\right)$.

Since the metriplectic scheme for a dissipative neutral fluid has been already worked out in [8], here (f, g) for the system (7) will be defined by generalizing the expressions of the metric part of dynamics to include the Joule effect dissipation. Considering (8), the dissipation element in the $\vec{v}$-equation and the corresponding entropy production due to the velocity gradients show a beautiful parallel with the same terms pertaining to the motion of $\vec{B}$:

$$\begin{cases} D_i^{(v)} = \frac{1}{\rho}\partial^k\left(\Lambda_{kimn}\partial^m v^n\right), \quad D_i^{(B)} = \partial^k\left(\Theta_{kimn}\partial^m B^n\right), \\ \\ \Theta_{jkmn} \stackrel{\text{def}}{=} \mu\epsilon_{jki}\epsilon^i{}_{mn}, \\ \\ \left(\partial_t s\right)_v = \frac{1}{\rho T}\Lambda_{jkmn}\partial^j v^k \partial^m v^n, \quad \left(\partial_t s\right)_B = \frac{1}{\rho T}\Theta_{jkmn}\partial^j B^k \partial^m B^n. \end{cases}$$

The system (7) may be re-written as:

$$\begin{cases} \partial_t v_i = -v_k\partial^k v_i - \frac{1}{\rho}\partial_i p - \frac{1}{2\rho}\partial_i B^2 + \frac{1}{\rho}B_k\partial^k B_i - \partial_i\phi_{grav} + \frac{1}{\rho}\partial^k\left(\Lambda_{kimn}\partial^m v^n\right), \\ \\ \partial_t B_i = B_j\partial^j v_i - B_i\partial^j v_j - v_j\partial^j B_i + \partial^k\left(\Theta_{kimn}\partial^m B^n\right), \\ \\ \partial_t\rho = -\partial^k\left(\rho v_k\right), \\ \\ \partial_t s = -v_k\partial^k s + \frac{1}{\rho T}\Lambda_{kimn}\partial^k v^i\partial^m v^n + \frac{1}{\rho T}\Theta_{kimn}\partial^k B^i\partial^m B^n + \frac{\kappa}{\rho T}\partial^2 T. \end{cases} \tag{17}$$

In both the cases of $\vec{v}$ and of $\vec{B}$, the dissipative term is given by the divergence of the contraction of a rank-4-tensor (Λ_{kimn} and Θ_{kimn} respectively) with the gradient of the local variable ($\partial^m v^n$ and $\partial^m B^n$ respectively); in both the cases, the contribution to the entropy production is *a quadratic form in the gradients of the field*, $\frac{1}{\rho T}\Lambda_{kimn}\partial^k v^i\partial^m v^n$ and $\frac{1}{\rho T}\Theta_{kimn}\partial^k B^i\partial^m B^n$ respectively (*quadratic dissipation*). In [8] the dissipative part of the motion of a viscous Navier-Stokes system is accounted for via

$$(f,g)_{\text{NS}} = \frac{1}{\lambda}\int_{\mathbb{D}} d^3x\left\{T\Lambda_{ikmn}\left[\partial^i\left(\frac{1}{\rho}\frac{\delta f}{\delta v_k}\right) - \frac{1}{\rho T}\partial^i v^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{1}{\rho}\frac{\delta g}{\delta v_n}\right) - \frac{1}{\rho T}\partial^m v^n\frac{\delta g}{\delta s}\right] + \right.$$
$$\left. +\kappa T^2\partial^k\left(\frac{1}{\rho T}\frac{\delta f}{\delta s}\right)\partial_k\left(\frac{1}{\rho T}\frac{\delta g}{\delta s}\right)\right\}: \tag{18}$$

the addendum linear in Λ_{ikmn} accounts for the dissipation as in the equations of motion of $\vec{v}$ and for the entropy production due to the viscosity. The other addendum describes the entropy variation due to the heat transport. The analogy between the quadratic dissipation for $\vec{v}$ and that for $\vec{B}$ suggests that the bracket for the dissipative MHD should be of the form:

$$(f,g)=\frac{1}{\lambda}\int_{\mathbb{D}} d^3x\left\{T\Lambda_{ikmn}\left[\partial^i\left(\frac{1}{\rho}\frac{\delta f}{\delta v_k}\right)-\frac{1}{\rho T}\partial^i v^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{1}{\rho}\frac{\delta g}{\delta v_n}\right)-\frac{1}{\rho T}\partial^m v^n\frac{\delta g}{\delta s}\right]+\right.$$

$$+T\Theta_{ikmn}\left[\partial^i\left(\frac{\delta f}{\delta B_k}\right)-\frac{1}{\rho T}\partial^i B^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{\delta g}{\delta B_n}\right)-\frac{1}{\rho T}\partial^m B^n\frac{\delta g}{\delta s}\right]+$$

$$\left.+\kappa T^2\partial^k\left(\frac{1}{\rho T}\frac{\delta f}{\delta s}\right)\partial_k\left(\frac{1}{\rho T}\frac{\delta g}{\delta s}\right)\right\}. \tag{19}$$

This bracket is shown to be the right one to produce the dissipative terms in (17) once the free energy is chosen as $F=H+\lambda S$, H being the Hamiltonian defined in (10) and S the total entropy given in (14), so that:

$$F\left[\vec{v},\vec{B},\rho,s\right]=\int_{\mathbb{D}}\left[\frac{\rho}{2}v^2+\rho\phi_{grav}+\frac{B^2}{2}+\rho U(\rho,s)+\lambda\rho s\right]d^3x. \tag{20}$$

The metric bracket (19) is shown to generate the dissipative part of $\partial_t\vec{v}$, because in the part concerning the velocity field this (f,g) is exactly the $(f,g)_{\rm NS}$ in (18); the addendum involving $\frac{\delta}{\delta\vec{B}}$ does not contribute to $\partial_t\vec{v}$. It contributes instead to the dissipative part of $\partial_t\vec{B}$, calculated as $\lambda(B_h,S)=\mu\partial^2B_h$.

The bracket in (19) is symmetric in the exchange $f\leftrightarrow g$, due to the property $\Lambda_{ikmn}=\Lambda_{mnik}$ and $\Theta_{ikmn}=\Theta_{mnik}$, and the self-evident symmetry of the addendum $\kappa T^2\partial^k\left(\frac{1}{\rho T}\frac{\delta f}{\delta s}\right)\partial_k\left(\frac{1}{\rho T}\frac{\delta g}{\delta s}\right)$. As far as its semi-definiteness is concerned, consider that it has been constructed by summing the bracket $(f,g)_{\rm NS}$ in (18) and the bracket

$$(f,g)_B=\frac{1}{\lambda}\int_{\mathbb{D}} d^3xT\Theta_{ikmn}\left[\partial^i\left(\frac{\delta f}{\delta B_k}\right)-\frac{1}{\rho T}\partial^i B^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{\delta g}{\delta B_n}\right)-\frac{1}{\rho T}\partial^m B^n\frac{\delta g}{\delta s}\right]. \tag{21}$$

The semi-definiteness of $(f,g)_{\rm NS}$ was proved in [8]. Now, one must do the same for the new Joule term $(f,g)_B$, for which one has:

$$\begin{cases}(f,f)_B=\dfrac{1}{\lambda}\displaystyle\int_{\mathbb{D}} d^3xT\Theta_{ikmn}T^{ik}(f)\,T^{mn}(f)\,,\\[2ex] T^{ab}(f)=\partial^a\left(\dfrac{\delta f}{\delta B_b}\right)-\dfrac{1}{\rho T}\partial^a B^b\dfrac{\delta f}{\delta s}.\end{cases}$$

$T^{ab}(f)$ can be subdivided into a symmetric part $S^{ab}(f) = \frac{1}{2}\left[T^{ab}(f) + T^{ba}(f)\right]$ plus an antisymmetric part $A^{ab}(f) = \frac{1}{2}\left[T^{ab}(f) - T^{ba}(f)\right]$, and, due to the symmtry properties of Θ_{ikmn}, $\Theta_{ikmn} = -\Theta_{kimn}$ and $\Theta_{ikmn} = -\Theta_{iknm}$, one can replace $T^{ab}(f)$ with its antisymmetric part $A^{ab}(f)$ only, since the symmetric parts will be canceled in the calculation of $(f,f)_B$:

$$(f,f)_B = \frac{2}{\lambda}\sum_{i,k}\int_{\mathbb{D}} \mu T A_{ik}^2(f)\, d^3x.$$

The sign of this expression is just that of λ for every functional f. The semi-definiteness of the whole $(f,g) = (f,g)_{\mathrm{NS}} + (f,g)_B$ is proved (so that S may be considered a good Lyapunov functional).

Last but not least, the metric algebra (19) generates exactly the local entropy production due to the mechanisms of dissipation and heat transport: $\lambda(s,S) = D^{(s)}$.

It is possible to show that the functional gradient of the Hamiltonian is a null mode of the metric algebra (19):

$$(H,f) = 0 \;\; \forall \;\; f.$$

Also, the metric part of the motion algebra keeps the quantities in (15) constant:

$$(P_h, S) = 0,\;\; (L_h, S) = 0,\;\; (G_h, S) = 0. \tag{22}$$

Equation (22), together with (16), renders the metriplectic motion of the non-ideal MHD *invariant under the transformations of the Galileo Group.*

The metriplectic bracket

$$\begin{aligned}
\langle\langle f,g\rangle\rangle = &-\int_{\mathbb{D}} d^3x \left[\frac{\delta f}{\delta\rho}\partial_i\left(\frac{\delta g}{\delta v_i}\right) + \frac{\delta g}{\delta\rho}\partial_i\left(\frac{\delta f}{\delta v_i}\right) - \frac{1}{\rho}\frac{\delta f}{\delta v_i}\epsilon_{ikj}\epsilon^{jmn}\frac{\delta g}{\delta v_k}\partial_m v_n + \right.\\
&+\frac{1}{\rho}\frac{\delta f}{\delta v_i}\epsilon_{ijk}\epsilon^{kmn}B^j\partial_m\left(\frac{\delta g}{\delta B^n}\right) + \frac{\delta f}{\delta B_i}\epsilon_{ijk}\partial^j\left(\frac{1}{\rho}\epsilon^{kmn}B_m\frac{\delta g}{\delta v^n}\right) + \\
&\left.+\frac{1}{\rho}\partial_i s\left(\frac{\delta f}{\delta s}\frac{\delta g}{\delta v_i} - \frac{\delta g}{\delta s}\frac{\delta f}{\delta v_i}\right)\right] + \frac{1}{\lambda}\int_{\mathbb{D}} d^3x T\left\{\kappa T\partial^k\left(\frac{1}{\rho T}\frac{\delta f}{\delta s}\right)\partial_k\left(\frac{1}{\rho T}\frac{\delta g}{\delta s}\right) + \right.\\
&+\Lambda_{ikmn}\left[\partial^i\left(\frac{1}{\rho}\frac{\delta f}{\delta v_k}\right) - \frac{1}{\rho T}\partial^i v^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{1}{\rho}\frac{\delta g}{\delta v_n}\right) - \frac{1}{\rho T}\partial^m v^n\frac{\delta g}{\delta s}\right] + \\
&\left.+\Theta_{ikmn}\left[\partial^i\left(\frac{\delta f}{\delta B_k}\right) - \frac{1}{\rho T}\partial^i B^k\frac{\delta f}{\delta s}\right]\left[\partial^m\left(\frac{\delta g}{\delta B_n}\right) - \frac{1}{\rho T}\partial^m B^n\frac{\delta g}{\delta s}\right]\right\},
\end{aligned} \tag{23}$$

obtained by putting together (12) and (19), has all the features required to govern the visco-resistive MHD, with the free energy defined in (20).

As suggested in [10], it is possible to determine the equilibrium configurations by studying the extrema of the free energy F. The functional derivatives of F read

$$\begin{cases} \dfrac{\delta F}{\delta \vec{v}} = \rho \vec{v}, \; \dfrac{\delta F}{\delta \vec{B}} = \vec{B}, \\ \\ \dfrac{\delta F}{\delta \rho} = \dfrac{v^2}{2} + \phi_{grav} + U + \rho \dfrac{\partial U}{\partial \rho} + \lambda s, \\ \\ \dfrac{\delta F}{\delta s} = \rho \dfrac{\partial U}{\partial s} + \lambda \rho, \end{cases}$$

so that, setting them to zero and considering the thermodynamic closure (9), the asymptotic equilibrium configuration is found to be:

$$\begin{cases} \vec{v}_{\text{eq}} = 0, \; \vec{B}_{\text{eq}} = 0, \; T_{\text{eq}} = -\lambda, \\ \\ p_{\text{eq}} = \rho_{\text{eq}} \left(Ts - U \right)_{\text{eq}} . \end{cases} \tag{24}$$

A configuration towards which the system may tend to relax (under suitable initial conditions) has neither bulk velocity, nor magnetic induction, while pressure equilibrates the thermodynamic free energy of the gas, and the temperature field matches everywhere minus the constant λ. At the equilibrium, the free energy of the metriplectic scheme really appears to be isomorphic to the expression known in traditional Thermodynamics $F = H - T_{\text{eq}} S$, being H the energy of the fluid and S its entropy.

As a corollary of the above results, one can obtain the metriplectic formulation of reduced MHD models [14], which are widely used when the dependence on one of the spatial coordinates can be ignored. This can be the case, form instance, of tokamak fusion devices, in which the presence of a strong toroidal component of the magnetic field $\vec{B}_0$ makes the dynamics essentially two-dimensional and taking place on the poloidal plane, perpendicular to the toroidal direction. Several such examples may be done both in astrophysical plasmas and fusion plasmas.

An incompressible 2D resistive MHD model, accounting for entropy production, may be obtained reducing the 3D system, taking the limit of zero viscosity and adopting magnetic potential, vorticity and entropy per unit mass, as dynamical variables [15]:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + [\phi, \psi] = \mu \partial_\perp^2 \psi, \\ \\ \dfrac{\partial \omega}{\partial t} + [\phi, \omega] + [\partial_\perp^2 \psi, \psi] = 0, \\ \\ \dfrac{\partial s}{\partial t} + [\phi, s] = \dfrac{\mu}{\rho_0 T} (\partial_\perp^2 \psi)^2. \end{cases} \tag{25}$$

In the above equations ψ is the poloidal magnetic flux, ϕ the stream function, $\omega = \partial_\perp^2 \phi$ the plasma vorticity, s the entropy per unit mass, μ the resistivity and $[a, b] = \partial_x a \partial_y b - \partial_y a \partial_x b$ is the canonical bracket in the x, y coordinates in the plane orthogonal to $\vec{B}_0$, the poloidal plane. $\vec{\partial}_\perp$ is the gradient along the poloidal plane, and $\partial_\perp^2$ is the corresponding Laplacian. All fields depend on x and y only. Consistently with the incompressibility assumption, the mass density ρ_0 is taken to be constant.

Although deprived of the terms depending on the fluid viscosity, the model (25) is a useful tool for investigating, for instance, the phenomenon of *magnetic reconnection* [16, 17], in which the dissipative term depending on the resistivity, allows for the change of topology of magnetic field line configurations, in addition to converting magnetic energy into heat.

The Hamiltonian component of the motions in (25), obtained in the limit $\mu = 0$, is generated by the Hamiltonian functional

$$H = \frac{1}{2} \int d^2x (|\vec{\partial}_\perp \psi|^2 + |\vec{\partial}_\perp \phi|^2) + \rho_0 \int d^2x U(s) \tag{26}$$

and by the Poisson bracket

$$\{f, g\} = \int d^2x \left(\psi([f_\psi, g_\omega] + [f_\omega, g_\psi]) + \omega[f_\omega, g_\omega] + s([f_s, g_\omega] + [f_\omega, g_s]) \right), \tag{27}$$

where subscripts indicate functional derivatives.

The last term on the right-hand side of (26) comes from the contribution of the internal energy U. In the constant density limit, however, such term is actually a Casimir of the bracket (27). The dissipative part of the system is generated with the help of a metric bracket $(,)$. In the incomplete case, with no entropy evolution, the symmetric bracket producing the resistive term in the Ohm's law in (25), had been presented in [18]. For the above complete system, the dissipative part is obtained from the $(,)_B$ metric bracket presented in (21), by applying the relation $\vec{\partial}_\perp \times f_{\vec{B}} = f_{\vec{A}}$, where $\vec{A}$ is the magnetic vector potential and $\vec{B}$ the magnetic induction, and then by projecting in 2D. The result is

$$(f,g) = \frac{\mu}{\lambda}\int d^2x \left(T f_\psi g_\psi + \frac{\partial_\perp^2 \psi}{\rho_0}(f_\psi g_s + f_s g_\psi) + \frac{(\partial_\perp^2\psi)^2}{\rho_0^2 T} f_s g_s \right). \tag{28}$$

For this reduced model, the properties characterizing the metric bracket can be shown with more immediacy. The bracket (28), indeed, is evidently symmetric. The relation

$$(H,g) = \frac{\mu}{\lambda}\int d^2x \left(T(-\partial_\perp^2\psi) g_\psi + \frac{\partial_\perp^2 \psi}{\rho_0}((-\partial_\perp^2\psi) g_s + \rho_0 T g_\psi) + \frac{(\partial_\perp^2\psi)^2}{\rho_0^2 T} \rho_0 T g_s \right) = 0$$

shows that the functional gradient of H is in the kernel of the metric bracket for any g. Concerning semi-definiteness one can see that

$$(f,f) = \frac{\mu}{\lambda}\int d^2x T \left(f_\psi + \frac{\partial_\perp^2\psi}{\rho_0 T} f_s \right)^2,$$

so that (f,f) has the same sign of λ. Finally, upon defining

$$F = H + \lambda\rho_0 \int s d^2x,$$

one can verify that (ψ, F), (ω, F) and (s, F) yield the desired dissipative terms:

$$(\psi, F) = \mu \partial_\perp^2 \psi, \qquad (\omega, F) = 0,$$

$$(s, F) = \frac{\mu}{\rho_0 T}(\partial_\perp^2\psi)^2.$$

CONCLUSIONS

The metriplectic formulation of the visco-resistive MHD equations has been derived. Such formulation is identified by a free energy functional, given by the sum of the Hamiltonian of ideal MHD with the entropy Casimir, and a bracket obtained by summing the Poisson bracket of ideal MHD with a new metric bracket giving rise to the dissipative terms. The metric bracket extends that derived in Ref. [8] for dissipative Navier-Stokes equations. In addition to yielding the desired dissipative terms, the bracket is shown to conserve the Hamiltonian of ideal MHD as well as other constants of motion, related to space-time symmetries. The dynamics governed by this metriplectic system is then shown to tend asymptotically in time toward states with no flow and no magnetic energy. From the general results on visco-resistive MHD, we obtained also the metriplectic formulation of a reduced resistive model for incompressible plasmas.

Concerning future directions, some equilibrium configuration less trivial than (24) should be investigated: the configuration (24) is “entropic death”, taking place when friction has dissipated all the bulk kinetic and magnetic energy into heat. The existence of the equilibrium configuration (24) is very intuitive, it is a configuration reachable from initial zero Galileo charges (15), but it represents only one possible final state. Actually, even if the free energy (20) seems to predict only this equilibrium configuration, other relaxation plasma states are known in nature, justifiable in this framework by generalizing the metric bracket and the functional F in (20) to some F', so to bring into the play constraints not considered here. An extremization of F conditioned to initial values of the quantities in (15) would, for instance, give a final $\vec{v}_{\rm eq}$ different from zero. Also, with suitable initial conditions, non-trivial configurations for $\vec{B}_{\rm eq}$ could be found.

Possibly, even more interesting would be the extension of F to expressions in which the Casimir functional C in (2) is not simply restricted to S, but involves the velocity and the magnetic degrees of freedom [19]. In general, however, the issue of the variety of final relaxed states, is related to the existance and number of attractors of the visco-resistive MHD equations.

All the conditioning schemes just mentioned appear to be very smart, but should better be deduced from a consistent “First Principle” of metriplectic Physics, which is not yet clear to the Authors.

As a second remark, we would like to underline that the temperature and the entropy of the MSTDOF have particular roles in this framework: considering equations (2) and (3), the equilibrium temperature coincides with minus the constant λ, while the Casimir C is the entropy S. Now, how does this framework adapt to systems in which the temperature is anisotropic, due to the anisotropy of viscosity and diffusivity, as in the Braginskii equations [20]? An adaptation of the present formalism to that context would maybe require the use of “anisotropic entropies”, with more than one Casimir involved, and represents a very interesting future investigation (even prior to that, a further necessary step would be of course the identification of a Hamiltonian structure for the Braginskii model in the non-dissipative limit).

A final important remark, is that the relationship between S in the evolution of the dissipative system, and its information theory interpretation should be investigated. Indeed, on the one hand, the relationship (5) renders S a piece of the functional F that *metriplectically generates the time translations*, so that the entropy is recognized as the quantity that is *fully responsible for dissipation*. On the other hand, S should quantify the lack of information about the precise state of the MSTDOF: in the metriplectic scheme, however, no mention to probability is done, it is apparently a fully deterministic dynamics, even if the proper Thermodynamics emerges clearly. The metriplectic framework could probably emerge in a natural way within the Physics of a *Hamiltonian system* interacting with *noise*, that represents the MSTDOF free to fluctuate stochastically [21]. Such a stochastic scenario is expected to be approximated by the deterministic dynamics (5) under suitable hypotheses. The theory of stochastic systems will be of great help in this line of research [22].

The metriplectic framework appears to the Authors as a natural extension of the Hamiltonian theory of dynamical systems to systems showing dissipation. The metriplectic framework regards the relationship between dissipative "forces" and entropic quantities of the MSTDOF under a new light, and points towards fundamental aspects of the friction-information relationship. Also, it is expected to lead to original predictions on more complicated dynamical systems given by a Hamiltonian system plus dissipation, and to simplify the derivation of known results to a great extent, due to its geometrical nature, in which symmetries of the theory emerge very easily.

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