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The disentangling power of unitaries

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We define the disentangling power of a unitary operator in a similar way as the entangling power defined by Zanardi, Zalka and Faoro [PRA, 62, 030301]. A general formula is derived and it is shown that both quantities are directly proportional. All results concerning the entangling power can simply be translated into similar statements for the disentangling power. In particular, the disentangling power is maximal for certain permutations derived from orthogonal latin squares. These permutations can therefore be interpreted as those that distort entanglement in a maximal way.

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I. INTRODUCTION

The entanglement of two quantum systems is generally recognised to be a valuable resource. The ability to create or destroy entanglement is therefore an important property of a unitary operator on the combined system. There are a number of ways to quantify this property; two such measures have been introduced in [1] and [2]. In the first of these, the entangling power of a unitary operator on $\mathcal{H}_1\otimes\mathcal{H}_2$ is defined to be the average entanglement created by the operator when acting on a separable state; in the second, the same phrase is used for the maximum increase in entanglement produced by the operator when acting on any state, with the possible use of ancillary systems. The first type of entangling power was investigated for the case $\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = d$ in [3], where it was found to be possible to identify maximally entangling unitary operators for all d except $d = 6$. The second type was investigated in [2], in which the notion of disentangling power was also introduced and it was shown that this need not be equal to the entangling power. It follows that this definition of the power of a unitary does not allow us to order unitary operators according to their capacity to affect entanglement.

In this note we examine this question in terms of the first definition of entangling power. We find that this, unlike the second definition, leads to the conclusion that the disentangling power of any bipartite unitary operator is proportional to its entangling power. Thus the unitaries with highest entangling power also have highest disentangling power; in [3] these were identified as permutations which correspond to orthogonal Latin squares.

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II. ENTANGLING POWER OF UNITARIES

In this section we briefly review the definitions and results of Ref. [1, 4] in the notations of Ref. [3].

Let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H} = \mathcal{H}_A\otimes\mathcal{H}_B$ be Hilbert spaces where $\dim\mathcal{H}_A = \dim\mathcal{H}_B = d$. As pure state entanglement measure we use the normalized linear entropy $S_L(\cdot)$ of the reduced density matrix. It is defined as

$$S_L(|\psi\rangle) := \frac{d}{d-1}(1-\operatorname{Tr}\rho^2), \quad \text{where} \quad \rho = \operatorname{Tr}_B|\psi\rangle\langle\psi|,$$

for $|\psi\rangle \in \mathcal{H}$. We define the entangling power $\epsilon(U)$ of a unitary $U \in \mathcal{U}(\mathcal{H}) \cong U(d^2)$ as the average amount of entanglement produced by U acting on a distribution of product states:

$$\epsilon(U) := \int_{\langle\psi_1|\psi_1\rangle=1}\int_{\langle\psi_2|\psi_2\rangle=1} S_L(U|\psi_1\rangle|\psi_2\rangle)d\psi_1 d\psi_2, \tag{1}$$

where $d\psi_1$ and $d\psi_2$ are normalized probability measures on unit spheres.

With each operator X on $\mathcal{H}\cong\mathcal{H}_1\otimes\mathcal{H}_3$ we can associate a state vector $|X\rangle$ in $\mathcal{H}\otimes\mathcal{H}\cong\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3\otimes\mathcal{H}_4$ as

$$|X\rangle_{A|B} = |X\rangle_{12|34} := (X_{13}\otimes I_{24})|\Psi_+\rangle_{13|24}, \tag{2}$$

where

$$|\Psi_+\rangle_{13|24} = \frac{1}{d}\sum_{i,j=1}^{d}|ij\rangle_{13}\otimes|ij\rangle_{24}$$

and I stands for the identity operator. It easily follows that $S_L(|I\rangle) = S_L(|S\rangle) = 1$, with $S = \sum_{ij}^{d}|ij\rangle\langle ji|$ the swap operator. This isomorphism allows us to rewrite equation (1) in a form that requires no averaging, as in the following theorem.

Theorem 1 (Zanardi [4]) *The entangling power of a unitary $U\in\mathcal{U}(\mathcal{H})$ is given by*

$$\epsilon(U) = \frac{d}{d+1}[S_L(|U\rangle) + S_L(|US\rangle) - S_L(|S\rangle)]. \tag{3}$$

It follows that $0\le\epsilon(U)\le\frac{d}{d+1}$.

The maximum $\epsilon(U) = d/(d+1)$ is reached for special permutations (except for $d = 2, 6$) constructed from orthogonal latin squares, see Ref. [3].

III. DISENTANGLING POWER OF UNITARIES

With this we define the *disentangling power* of a unitary $U \in \mathcal{U}(\mathcal{H}) \cong U(d^2)$ as

$$\delta(U) := 1 - \int_{V \in \mathcal{U}} \int_{W \in \mathcal{U}} S_L(U(V \otimes W)|\psi_+\rangle) dV dW, \quad (4)$$

where $V, W \in U(d)$, dV, dW are the Haar measure on $U(d)$ and $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. Thus, the disentangling power of a unitary is defined as the average decrease of the entanglement of the states obtained by applying the unitary on random maximally entangled states. Note that we could have chosen $V = I$, but for what follows, the above form is easier to work with.

Following a similar strategy as in [1, 4] we now present the analogue of Theorem 1 for the disentangling power.

Theorem 2 *The disentangling power of a unitary $U \in \mathcal{U}(\mathcal{H})$ is given by*

$$\delta(U) = \frac{1}{d-1}[S_L(|U\rangle) + S_L(|US\rangle) - S_L(|S\rangle)]. \quad (5)$$

Proof. (sketch) In a first step, we can rewrite Equation 4 in a similar form to Equation (3) from Ref. [1]. The method of doing so is completely analogous; one obtains

$$\delta(U) = \frac{1}{d-1}\left[d \operatorname{Tr}((U_{12} \otimes U_{34})\Omega(U_{12} \otimes U_{34})^\dagger (S_{13} \otimes I_{24})) - 1\right], \quad (6)$$

with

$$\Omega = \int_{V,W \in \mathcal{U}} (V_1 \otimes V_3 \otimes W_2 \otimes W_4) P^+_{13|24} (V_1 \otimes V_3 \otimes W_2 \otimes W_4)^\dagger dV dW. \quad (7)$$

Here, we have introduced four subsystems, and subscripts denote on which subsystem the operators act. We used $P^+_{13|24}$ to denote the maximally entangled state between subsystems 13 and 24. Integrals of this form can be evaluated using the fact that $V \otimes V$-invariant operators are linear combinations of I and S; see Ref. [5]. This particular integral was evaluated as (see Equation (27) in Ref. [5])

$$\begin{aligned}\Omega &= \frac{2}{d^3(d^2-1)}[(d-1)P^+_{13} \otimes P^+_{24} + (d+1)P^-_{13} \otimes P^-_{24}] \\ &= \frac{1}{d^2(d^2-1)}[I_{13} \otimes I_{24} + S_{13} \otimes S_{24}] \\ &- \frac{1}{d^3(d^2-1)}[S_{13} \otimes I_{24} + I_{13} \otimes S_{24}]. \quad (8)\end{aligned}$$

According to Equation 6 in Ref. [4] we have

$$S_L(|U\rangle) = \frac{d^2}{d^2-1}\big[1 - \frac{1}{d^4} \operatorname{Tr}((U_{12} \otimes U_{34}) \cdot (S_{13} \otimes I_{24}) \cdot (U_{12} \otimes U_{34})^\dagger S_{13} \otimes I_{24})\big]. \quad (9)$$

Substituting Equation (8) in Equation (6) and using the above expression for $S_L(|U\rangle)$ one obtains readily Equation (5). ■

From this theorem follows that

$$\delta(U) = \frac{d+1}{d(d-1)}\epsilon(U), \quad (10)$$

so that the entangling power is proportional to the disentangling power. With this in mind, all results for the entangling power can simply be translated into statements of the disentangling power. For instance we have the following analogue of Theorem 4 and its Corollary from Ref. [3].

Theorem 3 *The maximum value of the disentangling power $\delta(U)$ over all unitaries is achieved for the unitaries with maximum entangling power. For $d \neq 2, 6$ this maximum value is given by*

$$\delta(U) = \frac{1}{d-1}, \quad (11)$$

and can be attained by permutation matrices only.

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