Equations hyperboliques scalaires à flux discontinu

Florence Bachmann

UNIVERSITÉ AIX-MARSEILLE I - PROVENCE
U.F.R. de Mathématiques, Informatique et Mécanique

THÈSE

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ AIX-MARSEILLE I
Discipline : Mathématiques

École Doctorale Mathématiques et Informatique de Marseille
E.D. numéro 184

présentée et soutenue publiquement
par

Florence Bachmann

le *7 décembre 2005*

Equations hyperboliques scalaires à flux discontinu

Directeur de thèse : **M. Thierry Gallouët**

Rapporteurs :

Yann Brenier	Directeur de recherche CNRS à l'Université de Nice Sophia-Antipolis
François James	Professeur à l'Université d'Orléans

Jury :

Yann Brenier	Directeur de recherche CNRS à l'Université de Nice Sophia-Antipolis
Thierry Gallouët	Professeur à l'Université de Provence
Olivier Guès	Professeur à l'Université de Provence
Jean-Marc Hérard	Ingénieur EDF Départment MFTT à Chatou
François James	Professeur à l'Université d'Orléans
Julien Vovelle	Chargé de recherche CNRS à l'ENS Cachan (Antenne de Bretagne)

Remerciements

Tout d'abord, je tiens à remercier vivement Thierry Gallouët. Son enthousiasme communicatif et son envie de comprendre les problèmes qui se posent à lui, sont des valeurs qu'on ne peut qu'apprécier. Il a su motiver chaque étape de mon travail par des remarques pertinentes et a su me faire progresser dans mes recherches. Tout au long de ce travail, j'ai très fortement apprécié ses qualités de pédagogue et de chercheur. C'est à ses côtés et avec beaucoup d'admiration que j'ai appris à avoir une démarche mathématique sincère et sérieuse.

Julien Vovelle m'a très chaleureusement accueillie en stage de DEA. Il a partagé ses idées sur le sujet et suivi ce travail durant ces trois années. Travailler à ses côtés est très enrichissant et formateur. Sa rigueur et son enthousiasme ont été pour moi un chemin que je ne peux qu'essayer de suivre. Je souhaite qu'il trouve ici l'expression de ma reconnaissance et l'en remercie une nouvelle fois.

Je tiens à remercier Yann Brenier et François James, rapporteurs de ce travail, pour l'attention qu'ils ont portée à ce document. C'est un grand honneur pour moi.

Merci à Jean-Marc Hérard d'avoir accepté et de s'être libéré pour être membre du jury. Pour l'intérêt qu'il a porté à mon travail et son soutien, je l'en remercie.

Que Olivier Guès, par son enthousiasme et son soutien, trouve ici mes sincères remerciements.

Que Franck Boyer et Florence Hubert acceptent toute ma reconnaissance et mon amitié. Tous deux m'ont toujours très chaleureusement accueillie dans leur bureau pour un conseil, une aide, mais aussi pour une pause café. Durant ces années, ils ont été d'un grand soutien et notamment lors de la finalisation de cette thèse. Leur aide et amitié ont été précieuses.

Je remercie toute l'équipe d'analyse appliquée qui après m'avoir eue en tant qu'élève m'a très chaleureusement intégrée dans ce groupe, merci notamment à Yves Dermenjian, Thierry Gallouët, Raphaële Herbin, Anne Nouri, Philippe Tchamitchan. Merci aussi à Jérôme Le Rousseau d'avoir eu la patience de me redonner quelques bases en anglais.

Je souhaite remercier Nicolas Seguin pour les discussions que nous avons pu partager et notamment pour l'aide qu'il m'a apportée dans mon apprentissage des schémas numériques.

Je tiens aussi à remercier Marie-Christine Tort et Sylvie Blanc qui sont toujours présentes pour nous dépatouiller des petits problèmes administratifs.

Durant ces années, j'ai rencontré des doctorants, avec qui j'ai pu discuter, évoluer et me détendre. Je pense tout particulièrement à Nicolas, Alexandre, Murielle, Rémi, Stéphane, Isabelle, Céline, Franck, Philippe. J'ai aussi une pensée toute particulière pour Julie et Léa pour leur soutien et leur amitié.

J'en viens maintenant à remercier mes proches sans qui ce travail n'aurait pas pu aboutir. Tout particulièrement, mes parents qui tout au long de ma scolarité m'ont fait confiance et m'ont soutenue.

Enfin, c'est à Jérôme que le plus grand mérite revient pour son soutien quotidien, sa confiance et son immense patience. Merci d'être là ! Je ne peux finir que par notre petit Paul qui est un bonheur chaque jour !

Table des matières

Introduction

L'objectif de cette thèse est d'étudier les lois de conservation scalaires à flux discontinu. Ces équations interviennent, par exemple, lors de la modélisation d'un écoulement unidimensionnel d'un fluide composé de deux phases (par exemple eau/huile) dans un milieu poreux hétérogène soumis à la gravitation. Ces équations sont de la forme :

$$\begin{cases} \partial_t u(t,x) + \partial_x\big(g(x,u(t,x))\big) = 0, \quad t \in \mathbb{R}_+,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R}). \end{cases} \tag{1}$$

où $u : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ est l'inconnue et g la fonction flux discontinue par rapport à x et lipschitzienne par rapport à u.

L'étude de l'équation (1) consiste à donner un sens mathématique à la solution u, à obtenir un résultat d'existence et d'unicité d'une solution et enfin à mettre en oeuvre des schémas numériques qui complètent l'étude théorique et qui permettent de valider le modèle.

Rappels historiques sur les lois de conservation

Dès les années 50, les lois de conservation scalaires du premier ordre, qui sont des équations aux dérivées partielles de la forme

$$\partial_t u + \partial_x\big(F(t,x,u)\big) = 0,$$

ont été étudiées pour des flux F de classe C^1. Pour analyser les comportements des solutions de cette équation, la notion de courbe caractéristique a été introduite. On va supposer pour simplifier la présentation que $F(t,x,u) = f(u)$ où f est une fonction régulière de $\mathbb{R}$ à valeurs dans $\mathbb{R}$. L'équation devient :

$$\partial_t u + \partial_x\big(f(u)\big) = 0. \tag{2}$$

Pour étudier le problème de Cauchy associé à l'équation (2), on se donne également une condition initiale $u_0 : \mathbb{R} \to \mathbb{R}$. Maintenant, admettons que l'on cherche une solution régulière du problème (2) : on suppose que u est de classe C^1 et bornée sur $[0,T]\times\mathbb{R}$ pour $T>0$. On veut alors calculer la solution u au point $(t_1,x_1)\in[0,T]\times\mathbb{R}$. On note X la solution de l'équation différentielle suivante :

$$\begin{cases} \dfrac{dX}{dt}(t) = f'(u(t,X(t))), \quad 0 \le t \le t_1, \\ \\ X(t_1) = x_1. \end{cases}$$

D'après le théorème de Cauchy-Lipschitz d'une part, et parce que $f'(u)$ est bornée d'autre part, il existe une solution X définie sur $[0,t_1]$. On pose $a(t)=u(t,X(t))$, sachant que u est supposée de classe C^1, on a alors :

$$\begin{aligned} \frac{da}{dt}(t) &= \partial_t u(t,X(t)) + \frac{dX}{dt}(t)\partial_x\big(u(t,X(t))\big) \\ &= \partial_t\big(u(t,X(t))\big) + f'(u(t,X(t)))\partial_x\big(u(t,X(t))\big) \\ &= \partial_t\big(u(t,X(t))\big) + \partial_x\big(f(u)\big)(t,X(t)) \\ &= 0. \end{aligned}$$

On obtient que a est constante et $u(t_1,x_1)=a(t_1)=a(0)=u_0(X(0))$. De plus $\frac{dX}{dt}=f'(a(t))=cste$, donc X est affine : $X(t)=f'(a(0))(t-t_1)+x_1$. On peut alors conclure que u est constante le long de la droite de pente $1/f'(u_0(X(0)))$ passant par $(0,X(0))$ dans le plan (t,x) et sa valeur le long de cette droite est $u_0(X(0))$. Le graphe de X est la courbe caractéristique de u associée au point (t_1,x_1).
On remarque que si deux caractéristiques se rencontrent en un point (t^*,x^*), la solution u ne peut pas être régulière en ce point de rencontre (t^*,x^*) (elle prendrait deux valeurs distinctes). Donc la notion de solution régulière n'est pas adaptée au problème (2). En effet, si l'on considère l'équation de Burgers avec une condition initiale décroissante :

$$\begin{cases} \partial_t u + \partial_x\big(u^2/2\big) = 0, \\ \\ u_0(x) = -x \end{cases} \tag{3}$$

Les courbes caractéristiques sont des droites satisfaisant $X(t)=X(0)+t\,u_0(X(0))$. Les courbes caractéristiques associées au point $(0,-1)$ et $(0,1)$ se croisent en $(0,1)$. Donc aucune solution régulière en tout temps du problème (3) ne peut exister.

D'où la nécessité d'introduire la notion de solution faible associée au problème (2) :

Définition 1. *Une fonction $u\in L^\infty(\mathbb{R}_+\times\mathbb{R})$ est une solution faible du problème (2), avec pour donnée initiale $u_0\in L^\infty(\mathbb{R})$, si $\forall\varphi\in\mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R})$:*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}[u(t,x)\partial_t\varphi(t,x) + f(u(t,x))\partial_x\varphi(t,x)]\,dt\,dx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)\,dx = 0. \tag{4}$$

On montre que le problème (2) admet une solution faible mais cette solution faible n'est pas unique comme le montre l'exemple suivant. On considère l'équation de Burgers avec une condition initiale constante par morceaux, comme suit :

$$\begin{cases} \partial_t u + \partial_x\big(u^2/2\big) = 0, \\ u_0(x) = \begin{cases} -1 & \text{si} \quad x < 0 \\ 1 & \text{si} \quad x \geq 0. \end{cases} \end{cases} \tag{5}$$

On peut construire au moins deux solutions faibles du problème (5). La première est celle qui reste égale à u_0 pour tout $t > 0$, c'est à dire :

$$u(t,x) = \begin{cases} -1 & \text{si} \quad x < 0 \\ 1 & \text{si} \quad x \geq 0. \end{cases}$$

En effet, $u \in L^\infty((0,T)\times\mathbb{R})$ et on a :

$$\int_0^T \int_{-\infty}^0 (u\,\partial_t\varphi + f(u)\partial_x\varphi)dxdt + \int_{-\infty}^0 u_0(x)\,\varphi(0,x)dx = \int_0^T f(-1)\varphi(t,0)dt,$$

$$\int_0^T \int_0^{+\infty} (u\,\partial_t\varphi + f(u)\partial_x\varphi)dxdt + \int_0^{+\infty} u_0(x)\,\varphi(0,x)dx = -\int_0^T f(1)\varphi(t,0)dt,$$

En sommant ces deux égalités, on obtient que u est solution faible puisque $f(1) = f(-1)$.
La deuxième solution est donnée par :

$$u(t,x) = \begin{cases} -1 & \text{si} \quad x/t < -1 \\ x/t & \text{si} \quad -1 \leq x/t \leq 1 \\ 1 & \text{si} \quad x/t \geq 1. \end{cases}$$

On vérifie que v est aussi solution faible du problème de Burgers (5).
La notion de solution faible ne suffit donc pas à déterminer la solution physiquement observée car elle n'est pas unique.

Un critère d'origine physique (les conditions d'entropies) a été introduit pour sélectionner une unique solution faible au problème d'évolution (2). Sous sa forme la plus générale, il a été écrit par Kruzhkov [Kru70] comme suit :

Définition 2. *Soit $u_0 \in L^\infty(\mathbb{R};[0,1])$. Une fonction u appartenant à $L^\infty(\mathbb{R}_+ \times \mathbb{R};[0,1])$ est une solution entropique du problème (2) si elle satisfait les inégalités entropiques suivantes : pour tout $\kappa \in [0,1]$, pour toute fonction positive $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\int_0^\infty \int_{\mathbb{R}} |u(t,x)-\kappa|\,\partial_t\varphi(t,x)\,dx\,dt + \int_0^\infty \int_{\mathbb{R}} \Phi(u(t,x),\kappa)\,\partial_x\varphi(t,x)\,dx\,dt$$
$$+ \int_{\mathbb{R}} |u_0(x)-\kappa|\,\varphi(0,x)\,dx \geq 0, \tag{6}$$

où Φ est le flux entropique associé aux entropies de Kruzhkov,

$$\Phi(u,\kappa) = sgn(u-\kappa)(g(u)-g(\kappa)).$$

Remarque 1. *On rappelle la définition de la fonction $sgn : \mathbb{R} \to \mathbb{R}$*

$$sgn(x) = \begin{cases} -1 & si \quad x < 0, \\ 0 & si \quad x = 0, \\ 1 & si \quad x > 0. \end{cases}$$

En supposant le flux f localement lipschitzien, Kruzhkov a démontré l'existence et l'unicité d'une solution entropique pour le problème (2) [Kru70].

On peut alors se demander d'où vient cette définition. D'un point de vue physique, si on prend en compte en plus des phénomèmes d'échange les phénomènes de diffusion, le nouveau problème nous conduit à l'étude d'une équation du type

$$\partial_t u^\varepsilon + \partial_x \big(f(u^\varepsilon)\big) - \varepsilon\, \partial_{xx} u^\varepsilon = 0, \tag{7}$$

où $\varepsilon > 0$ est un paramètre petit devant les autres grandeurs. On remarque que si $\varepsilon = 0$, l'équation (7) donne l'équation (2).

L'étude des équations paraboliques étant connue dans les années 70, on sait que pour $u_0 \in BV(\mathbb{R})$ [1], pour tout $\varepsilon > 0$, il existe une unique solution faible du problème (7) u^ε. De plus, l'équation parabolique a un effet régularisant sur la solution : pour tout $\varepsilon > 0$, u^ε est infiniment dérivable sur $(0, T) \times \mathbb{R}$. Enfin, on sait que la famille (u^ε) converge vers u dans $L^1_{loc}((0, T) \times \mathbb{R})$.

Maintenant, pourquoi la fonction u satisfait-elle les inégalités (6) ? Soit η une fonction convexe de classe C^2 sur $\mathbb{R}$, soit $\Phi' = \eta' f'$. Multiplions l'équation (7) par $\eta'(u^\varepsilon)$, on obtient :

$$\partial_t\big(\eta(u^\varepsilon)\big) + \partial_x\big(\Phi(u^\varepsilon)\big) - \varepsilon\partial_{xx}\big(\eta(u^\varepsilon)\big) = -\varepsilon\eta''(u^\varepsilon)|\partial_x(u^\varepsilon)|^2 \leq 0.$$

En multipliant cette dernière équation par une fonction test positive $\varphi \in \mathcal{C}^\infty_c([0, T) \times \mathbb{R})$, puis en faisant une intégration par partie, on obtient :

$$\int_0^T \int_{\mathbb{R}} \eta(u^\varepsilon)\, \partial_t\varphi + \Phi(u^\varepsilon)\, \partial_x\varphi \, dt\, dx + \int_{\mathbb{R}} \eta(u_0(x))\, dx \geq \int_0^T \int_{\mathbb{R}} \varepsilon\partial_x\big(\eta(u^\varepsilon)\big)\, \partial_x\varphi\, dt\, dx.$$

En faisant tendre ε vers 0, formellement, la fonction u satisfait :

$$\int_0^T \int_{\mathbb{R}} \eta(u)\, \partial_t\varphi + \Phi(u)\, \partial_x\varphi\, dt\, dx + \int_{\mathbb{R}} \eta(u_0(x))\, dx \geq 0.$$

Cette dernière inégalité est vraie pour n'importe quelle fonction convexe C^2 et par approximation des fonctions $s \to |s - \kappa|$, $\kappa \in \mathbb{R}$, on obtient que u satisfait les inégalités (6).

Voilà donc les principales caractéristiques de l'étude des lois de conservation de la forme (2). Je vais maintenant présenter les spécificités des lois de conservation à flux discontinu.

[1] Une fonction $v \in L^1_{loc}(\mathbb{R})$ est à variation bornée, c.a.d. $v \in BV(\mathbb{R})$ si $|v|_{BV(\mathbb{R})} = \sup\big\{ \int_R v(x)\varphi_x(x)\, dx,\ \varphi \in \mathcal{C}^1_c(\mathbb{R}, \mathbb{R}), |\varphi(x)| \leq 1, \forall x \in \mathbb{R}\big\} < +\infty$.

Loi de conservation à flux discontinu

L'étude des lois de conservation à flux discontinu est un sujet récent. Les travaux ont commencé il y a une dizaine d'années. Les comportements de schémas numériques ont été étudiés dans ([Tow00, AJV04]), puis ont été introduites des notions de solutions ([Tow00, KRT02b]), et l'existence de ces solutions a été établie par passage à la limite sur les schémas numériques. Toutefois, dans tous les travaux traitant d'existence de solution ou d'unicité de solution, la fonction flux était supposée convexe ou concave et/ou vraiment non linéaire.
On notera que les équations de transport linéaire à flux discontinu avaient été étudiées sous la forme non conservative :

$$\partial_t u + a(t,x)\partial_x u = 0,$$

avec a bornée discontinue dans [BJ98]. En supposant la fonction a continue par morceaux, ces auteurs établissent l'existence d'une solution ainsi que la description de telles solutions le long des lignes de discontinuité.
Toutefois, en ce qui concerne les lois de conservation à flux discontinu telles que le problème (1), aucun résultat d'existence et d'unicité de solution dans un même espace fonctionnel n'avait été établi jusqu'en 2003 [SV03], malgré une notion de solution entropique introduite par [Tow00]. Essayons de comprendre pourquoi.

Le premier constat à faire est que le problème linéaire est mal posé. En effet, considérons l'équation $\partial_t u + \partial_x(k(x)u) = 0$ avec $k(x) = sgn(x)$. Alors, à l'aide des caractéristiques, on voit que u n'est pas unique.

Modèle physique

En fait, pour obtenir un problème bien posé, il faut partir du modèle. On considère l'écoulement unidimensionnel d'un fluide dans un milieu hétérogène (sable/argile) composé de deux phases, par exemple eau et huile, soumis à la force gravitationnelle (voir Fig. 1). Si on suppose que ce fluide suit la loi de Darcy, on obtient le système d'équations suivant :

$$\Gamma(x)\partial_t u - \partial_x(k\lambda_w(\partial_x p + \rho_w G)) = 0, \tag{8}$$

$$\Gamma(x)\partial_t(1-u) - \partial_x(k\lambda_o(\partial_x p + \rho_o G)) = 0. \tag{9}$$

On a choisi un système cartésien de coordonnées tel que la force gravitationnelle soit dirigée dans la direction des x positifs. On note u la saturation de l'eau et $1-u$ la saturation de l'huile, ρ_w et ρ_o les densités respectives de l'eau et de l'huile, p la pression des fluides, $\lambda_w = k_w/(\rho_w\mu_w)$ et $\lambda_o = k_o/(\rho_o\mu_o)$ où μ_w et μ_a sont les viscosités respectives des deux fluides, k_w et k_o sont les perméabilités relatives des deux phases eau et huile. Enfin, on note G l'accélération gravitationnelle, k la perméabilité absolue du milieu et Γ la porosité.

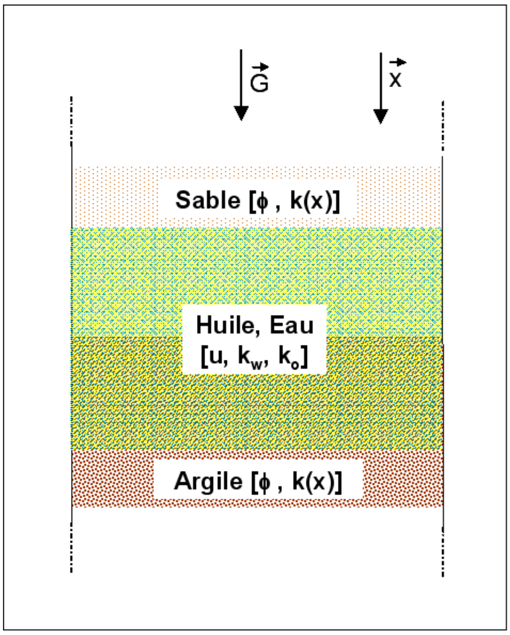

Fig. 1 – Modélisation de l'écoulement Huile/Eau en milieu hétérogène

En ajoutant les équations (8) et (9), on obtient $\partial_x Q = 0$ où

$$Q = k(\lambda_w + \lambda_o)\partial_x p + k\lambda_w \rho_w G + k\lambda_o \rho_o G$$

est le flux total. En supposant que l'écoulement est stationnaire, on exprime $\partial_x p$ comme une fonction de Q. On remplace alors $\partial_x p$ dans (8), et on obtient :

$$\Gamma(x)\partial_t u + \partial_x\Big(k\frac{\lambda_w\lambda_o}{\lambda_w+\lambda_o}(\rho_o-\rho_w)G + \frac{\lambda_w}{\lambda_w+\lambda_o}(-Q)\Big) = 0.$$

Pour le modèle Γ est une fonction discontinue en x, mais on va supposer pour simplifier l'analyse mathématique que $\Gamma \equiv cst \equiv 1$ (en réalité cela ne change pas fondamentalement l'analyse). On obtient alors la loi de conservation suivante :

$$\partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0,$$

avec

$$\begin{cases} g(u) := (\rho_o - \rho_w)\dfrac{\lambda_w(u)\lambda_o(u)}{\lambda_w(u)+\lambda_o(u)}G\,, \\ f(u) := (-Q)\dfrac{\lambda_w(u)}{\lambda_w(u)+\lambda_o(u)}. \end{cases}$$

Comme le flux Q est constant, il a un signe et on suppose, par exemple, qu'il est négatif. Enfin, les fonctions λ_w et λ_o sont les perméabilités relatives des deux phases et satisfont les hypothèses suivantes (cf. [GMT96]] :

1. $\lambda_w \in C^1([0,1])$ est croissante et satisfait $\lambda_w(0) = 0$,
2. $\lambda_o \in C^1([0,1])$ est décroissante et satisfait $\lambda_o(1) = 0$,
3. Il existe $\alpha > 0$ tel que $\lambda_w(u) + \lambda_o(u) > \alpha$ pour tout $u \in [0,1]$.

Finalement, on obtient le problème de Cauchy suivant à étudier :

$$\begin{cases} \partial_t u + \partial_x \big(k(x)g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R};[0,1]). \end{cases} \tag{10}$$

Les fonctions f, g et k satisfont les hypothèses suivantes :
(H1) $g \in C^1([0,1])$, est positive et $g(0) = g(1) = 0$,
(H2) $f \in C^1([0,1])$, est croissante et $f(0) = 0$,
(H3) k est la fonction discontinue définie par :
$$k(x) = \begin{cases} k_L & \text{si } x < 0 \\ k_R & \text{si } x > 0 \end{cases} \quad \text{avec } k_L, k_R > 0 \text{ et } k_L \neq k_R\,.$$

L'hypothèse sur les valeurs des fonctions flux en 0 et 1 est nécessaire (et suffisante) pour que le problème (10) soit bien posé. Notamment, cette hypothèse assure que les deux fonctions constantes respectivement égale à 0 ou à 1, sont solutions du problème alors que les constantes, en général ne le sont pas à cause de la discontinuité de la fonction k. Du point de vue physique, cela traduit le fait que si au temps initiale le fluide n'est composé que d'eau ($u_0 \equiv 1$) ou que d'huile ($u_0 \equiv 1$), au cours du temps, la saturation du fluide reste constante (le fluide est toujours composé que d'huile ou que d'eau.) Le modèle permet donc de poser correctement le problème.

Premier résultat d'existence et d'unicité

Le premier résultat d'existence et d'unicité de solution du problème (10) est dû à N. Seguin et J. Vovelle [SV03]. Ils ont introduit une notion de solution entropique pour ce problème dans la cas $g(u) = u(1-u)$ et $f = 0$, notion équivalente à celle introduite par Towers [Tow00].
Pour établir l'existence d'une solution entropique, ils obtiennent une estimation BV sur $\Phi(u^\varepsilon, 1/2)$, où u^ε est la solution entropique d'un problème approché du problème (10). Puis, du fait que $\Phi(., 1/2)$ soit une fonction de Temple [Tem82], ils obtiennent la convergence d'une sous-famille de (u^ε) vers une solution entropique du problème (10).
La preuve d'unicité comprend deux étapes. Tout d'abord, grâce aux résultats de Kruzhkov sur la comparaison de deux solutions entropiques d'une loi de conservation à flux lipschitzien, ils peuvent comparer deux solutions en dehors d'un compact contenant $\{x = 0\}$ (la fonction k est constante en dehors d'un tel compact). Dans un deuxième temps, par passage à la limite sur les fonctions tests, en utilisant l'existence de traces pour une solution entropique le long de la ligne de discontinuité de k, la comparaison de deux solutions entropiques est déduite. L'unicité est une conséquence immédiate. L'existence de ces fonctions traces le long de la ligne $\{x = 0\}$ pour une

solution entropique est obtenue en utilisant un résultat dû à Vasseur [Vas01]. Toutefois, ce résultat n'est valable que si g est vraiment non linéaire en u, c'est à dire que la dérivée seconde de g ne s'annule pas sur un intervalle ouvert de $[0,1]$, (ce qui est le cas pour la fonction $u \to u(1-u)$).

Cette première étude est à la base du travail de ma thèse.

Schéma volumes finis

Un deuxième point de la thèse a été consacré à l'étude de certains schémas numériques pour le problème (10). La méthode des volumes finis est une méthode d'analyse numérique et adaptée aux lois de conservation parce qu'elle conserve les flux numériques.

Le principe général de cette méthode sur la loi de conservation (2) est le suivant. Le domaine $\mathbb{R}$ est découpé en volume de contrôle $K_i =]x_{i-1/2}, x_{i+1/2}[$ où $(x_{i+1/2})_{i\in\mathbb{Z}}$ est une suite de réels strictement croissante, et le segment $[0,T]$ est partitionné comme suit :

$$0 = t^0 < t^1 < ... < t^{N-1} < t^N = T.$$

Pour la suite, on note $h_i = x_{i+1/2} - x_{i-1/2}$ et $k_n = t^{n+1} - t^n$.
Si on intègre l'équation (2) sur $[t^n, t^{n+1}] \times K_i$, on obtient

$$\int_{K_i} \big(u(t^{n+1},x) - u(t^n,x)\big)\, dx + \int_{t^n}^{t^{n+1}} \big(f(u(t,x_{i+1/2})) - f(u(t,x_{i-1/2}))\big)\, dt = 0. \quad (11)$$

On note u_i^n la valeur approchée de $u(t^n, x_i)$ (où x_i est un point de K_i), valeur construite par le schéma. Si on remplace dans (11) $u(t^n,x)$ et $u(t^{n+1},x)$ par u_i^n et u_i^{n+1} dans la première intégrale et le flux $f(u(t,x_{i+1/2}))$ par le flux numérique $\varphi_{i+1/2}^n$, on obtient :

$$h_i(u_i^{n+1} - u_i^n) + k_n(\varphi_{i+1/2}^n - \varphi_{i-1/2}^n) = 0.$$

Remarque 2. *Par construction, les schémas volumes finis conservent le flux car $f(u(t,x_{i+1/2}^+)$ est approché comme $f(u(t,x_{i+1/2}^-))$. Ces schémas sont donc adaptés aux lois de conservation.*

Le choix de $\varphi_{i+1/2}^n$ détermine la méthode des volumes finis employée. Par exemple pour une méthode explicite monotone à trois points, on suppose que $\varphi_{i+1/2}^n = F(u_i^n, u_{i+1}^n)$ où $F : \mathbb{R}^2 \to \mathbb{R}$ est croissante par rapport à sa première variable et décroissante par rapport à sa seconde.

Dans ce cas, le schéma s'écrit :

$$u_i^{n+1} = u_i^n - \frac{k_n}{h_i}(F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)) = 0. \tag{12}$$

On remarque alors que connaissant les trois valeurs u_{i-1}^n, u_i^n et u_{i+1}^n, on construit u_i^{n+1}.
L'étude de ce type de schémas a fait l'objet de nombreux travaux (cf. [EGH00] pour référence).

Après avoir situé le cadre général de mon travail de thèse, je propose maintenant une introduction de chaque chapitre de ce manuscrit.

Chapitre 1

Comme on l'a vu précédemment, l'existence et l'unicité d'une solution entropique pour le problème (10) n'ont été établies que si la fonction g est concave et vraiment non linéaire (avec un seul maximum sur $[0,1]$), comme $g(u) = u(1-u)$). On notera que les preuves d'existence et d'unicité proposées dans [SV03] utilisent fortement ces hypothèses sur g.

L'objectif est donc de généraliser ce travail pour une fonction g ne satisfaisant que l'hypothèse (H1) (cf page 6) et d'ajouter la fonction f dans le flux.

Tout d'abord, une bonne notion de solution entropique a dû être introduite.

Définition 3. *Soit $u_0 \in L^\infty(\mathbb{R};[0,1])$. Une fonction u appartenant à $L^\infty(\mathbb{R}_+ \times \mathbb{R};[0,1])$ est une solution entropique du problème (10) si elle satisfait les inégalités entropiques suivantes : pour tout $\kappa \in [0,1]$, pour toute fonction positive $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\, \partial_t \varphi(t,x)\, dx\, dt \\ + & \int_0^\infty \int_{\mathbb{R}} \big(k(x)\, \Phi(u(t,x),\kappa) + \Psi(u(t,x),\kappa)\big)\, \partial_x \varphi(t,x)\, dx\, dt \\ + & \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx\, + |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq 0, \end{aligned} \tag{13}$$

où Φ et ψ sont est les flux entropiques associés aux entropies de Kruzhkov,

$$\Phi(u,\kappa) = sgn(u-\kappa)(g(u) - g(\kappa)),\ \Psi(u,\kappa) = sgn(u-\kappa)(f(u) - f(\kappa)).$$

Cette définition est celle donnée par J.D. Towers dans le cas d'une fonction g convexe et $f = 0$. Elle est aussi valable pour une fonction g quelconque, ni convexe, ni concave. Le premier résultat obtenu généralise donc cette notion de solution entropique pour le problème (10).
L'existence et l'unicité d'une solution entropique ont été établies pour une fonction flux g vraiment non linéaire satisfaisant (H1) et pour une fonction f satisfaisant (H2). Ces résultats sont présentés dans le chapitre 1 :

Théorème 1. *Soit $u_0 \in L^\infty(\mathbb{R})$ telle que $0 \leq u_0 \leq 1$ p.p. sur $\mathbb{R}$. En supposant g vraiment non linéaire, il existe une solution entropique $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ du problème (10).*

Théorème 2. *Soient u_0, v_0 appartenant à $L^\infty(\mathbb{R})$ telles que $0 \leq u_0, v_0 \leq 1$ p.p. On suppose que $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ et $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ sont deux solutions entropiques du problème (10), avec comme conditions initiales respectives u_0 et v_0. Alors, en supposant g vraiment non linéaire, pour tout $R, T > 0$, on a l'inégalité suivante :*

$$\int_0^T \int_{-R}^R |u(t,x) - v(t,x)| \, dx \, dt \leq T \int_{-R-TM}^{R+TM} |u_0(x) - v_0(x)| \, dx, \tag{14}$$

où $M = \max(k_L, k_R) \sup_{u \in [0,1]} |g'(u)| + \sup_{u \in [0,1]} |f'(u)|$.

Remarque 3. *L'hypothèse "g vraiment non linéaire" n'est pas donnée par le modèle. Nous en avons eu besoin pour des raisons techniques.*

Pour établir l'existence d'une solution entropique, on introduit un problème (1_ε) qui est une loi de conservation avec la fonction k régularisée. Grâce aux résultats dûs à Kruzhkov, on obtient l'existence et l'unicité de u^ε, solution entropique du problème (1_ε). Une estimation BV sur $\gamma(u^\varepsilon)$ est alors établie, où $\gamma(s) = \int_0^s |g'(s)| ds$. Il existe alors une sous-suite de $\gamma(u^\varepsilon)$ qui converge presque partout. Pour obtenir la convergence d'une sous-suite de u^ε, on utilise une conséquence de "g vraiment non linéaire" qui nous donne que γ inversible.
On remarque que pour une loi de conservation avec la fonction k constante, si on a une condition initiale dans $BV(\mathbb{R})$, alors $u \in BV([0,T] \times \mathbb{R})$ pour tout $T > 0$. Par contre pour le problème (10), aucune estimation BV n'a pu être obtenue sur la solution entropique directement, le problème reste ouvert.
Finalement, le cheminement de preuve consistant à utiliser la compacité dans l'espace BV utilise l'hypothèse sur g vraiment non linéaire.

Pour l'unicité, dans un premier temps, on a voulu adapter au problème (10) la preuve d'unicité de Kruzhkov par dédoublement de variables. Pour cela, en remarquant que la fonction k est constante en dehors d'un compact contenant $\{x = 0\}$, la comparaison de deux solutions entropiques est connue [Kru70]. En passant alors à la limite sur les fonctions tests, la comparaison entre deux solutions entropiques du problème (10) est alors possible, mais il faut pour cela connaître les limites des solutions entropiques en $x = 0^-$ et $x = 0^+$. On utilise un résultat dû à Vasseur [Vas01], sur l'existence de traces fortes des solutions, mais ce résultat n'est valable que si g est vraiment non linéaire. Finalement, par ce schéma de preuve, pour établir la comparaison entre deux solutions entropiques et par là même l'unicité d'une telle solution, l'hypothèse sur g vraiment non linéaire est nécessaire.

En conclusion, on a obtenu l'existence et l'unicité d'une solution entropique pour le problème (10) sans hypothèse de convexité ou concavité sur g. Toutefois, l'hypothèse sur g vraiment non linéaire ne semble que technique et non liée au problème.

Ce travail a été publié dans la revue Advances in Differential Equation ([Bac04]).

Chapitre 2

L'objectif de ce travail, en collaboration avec Julien Vovelle, est de lever l'hypothèse de vraiment non linéarité faite sur g. L'existence et l'unicité d'une solution entropique ont ainsi été obtenues sous les seules hypothèses du modèle (H1), (H2) et (H3).

Théorème 3. *Soit $u_0 \in L^\infty(\mathbb{R})$ telle que $0 \leq u_0 \leq 1$ p.p. sur $\mathbb{R}$. Alors, il existe une unique solution entropique $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ du problème (10).*

Ces résultats ont nécessité de considérer le problème comme un problème nouveau et de s'écarter des méthodes et des outils du premier chapitre. Pour l'existence, on a introduit la notion de solution processus entropique pour le problème (10) (ou notion à valeurs mesures [DiP85]). Cette notion a été introduite par R. Eymard, T. Gallouët et R. Herbin [EGH00], pour une loi de conservation à flux lipschitzien. On a donc étendue cette notion au problème (10) comme suit :

Définition 4. *Soit $u_0 \in L^\infty(\mathbb{R}; [0,1])$. Une fonction u appartenant à $L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1); [0,1])$ est une solution entropique du problème (10) si elle satisfait les inégalités entropiques suivantes : pour tout $\kappa \in [0,1]$, pour toute fonction positive $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned}
&\int_0^1 \int_0^\infty \int_{\mathbb{R}} |u(t,x,\alpha) - \kappa|\, \partial_t \varphi(t,x)\, dx\, dt\, d\alpha \\
+ &\int_0^1 \int_0^\infty \int_{\mathbb{R}} \big(k(x)\, \Phi(u(t,x,\alpha),\kappa) + \Psi(u(t,x,\alpha),\kappa)\big)\, \partial_x \varphi(t,x)\, dx\, dt\, d\alpha \\
+ &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \ + |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq 0.
\end{aligned}$$

Pourquoi introduire cette notion ? Cette notion est une notion *à priori* plus faible que celle de solution entropique. Cet outil est essentiel dans l'étude de la convergence d'approximations ayant peu de propriétés de compacité. En effet, on peut justifier le passage à la limite dans une équation approchée, en se basant uniquement sur une estimation L^∞. L'existence d'une solution processus entropique est alors obtenue pour g qui peut être constante sur des intervalles ouvert de $[0,1]$.

D'autre part, pour établir la comparaison entre deux solutions processus entropiques, le problème ne pouvait plus être considéré comme dans le premier chapitre. En effet, comme on l'a vu, cette approche nécessite l'existence de traces des solutions le long de la ligne $\{x = 0\}$. Or, l'existence des traces des solutions n'est pas établie pour g quelconque. L'existence de traces n'est assurée que si g est vraiment non linéaire. De plus, dans le cas où $k_L = k_R$ et g non vraiment non linéaire, Oleinik, en 1957, a montré que si on regarde les problèmes sur $\mathbb{R}_+ \times \mathbb{R}_-$ et sur $\mathbb{R}_+ \times \mathbb{R}_+$, il n'y a pas existence de traces de la solution entropique sur les bords si la donnée initiale est seulement dans L^∞ ([Ole56, Ole57]).

Il a donc fallu aborder le problème différemment. On a essayé d'adapter la preuve de dédoublement de variables mais aucun résultat satisfaisant n'a été obtenu. On s'est alors intéressé à la notion de solution cinétique. Cette notion est a été introduite dans plusieurs travaux ([Bre83, GM83, LPT94]) pour une loi de conservation à flux lipschitzien. Une preuve d'unicité d'une solution cinétique a été établie par B. Perthame [Per98]. On a alors étendu cette notion à celle de solution processus cinétique pour le problème (10), notion équivalente à celle de solution processus entropique :

Définition 5. *Soient a et b les fonctions dérivées des fonctions flux :*

$$a(\xi) := g'(\xi)\,, \qquad b(\xi) := f'(\xi)\,, \quad \xi \in \mathbb{R}\,.$$

Soit $u_0 \in L^\infty(\mathbb{R};[0,1])$ et $u \in L^\infty(Q \times (0,1))$.
Soient $h_\pm$ et $h^0_\pm$ les fonctions d'équilibre associées à u et u_0 :

$$h_\pm(t,x,\alpha,\xi) = \mathrm{sgn}_\pm(u(t,x,\alpha) - \xi), \quad h^0_\pm(x,\xi) = \mathrm{sgn}_\pm(u_0(x) - \xi)\,.$$

La fonction u est une solution processus cinétique du problème (10) s'il existe $m_\pm \in \mathcal{C}(\mathbb{R}_\xi; w * -\mathcal{M}_+(\bar{Q}))$ telle que $m_+(\cdot,\xi)$ qui s'annule pour ξ grand (resp. $m_-(\cdot,\xi)$ qui s''annule pour $-\xi$ grand) et telle que pour toute fonction $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^3)$,

$$\int_0^1 \int_{Q\times\mathbb{R}_\xi} h_\pm(\partial_t + (k(x)a(\xi) + b(\xi))\partial_x)\varphi\, dt\, dx\, d\xi\, d\alpha + \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h^0_\pm \varphi_{|t=0}\, dx\, d\xi$$
$$- \quad (k_L - k_R)^\pm \int_{\Sigma\times\mathbb{R}_\xi} a(\xi)\varphi_{|x=0}\, dx\, d\xi = \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \varphi dm_\pm. \tag{15}$$

Ces outils ont été adaptés au problème (10) et ont permis d'obtenir l'unicité d'une solution processus entropique et l'équivalence entre les notions de solution entropique, processus entropique et processus cinétique.
La comparaison de deux solutions processus cinétiques et celle de deux solutions processus entropiques sont liées. En effet, soient u et v deux solutions entropiques du problème (16) et $h_\pm$, $j_\pm$, respectivement, les fonctions d'équilibre associées. On a alors :

$$\int_{\mathbb{R}} h_\pm(t,x,\lambda,\xi)\, j_\mp(t,x,\alpha,\xi)\, d\xi = (u(t,x,\lambda) - v(t,x,\alpha))^\pm \quad \forall t,\ x,\ \lambda,\ \alpha.$$

Comparer u et v revient donc à comparer $h_\pm$ et $j_\mp$. Comme l'équation (15) est linéaire, les fonctions d'équilibre d'une solution processus cinétique admettent des fonctions traces le long de la ligne de discontinuité de k, pour une fonction flux g quelconque. On peut alors comparer $h_\pm$ et $j_\mp$ en dehors d'un compact contenant $\{x = 0\}$ (k est constante en dehors d'un tel compact), puis par passage à la limite et en utilisant les fonctions traces des fonctions d'équilibre, la comparaison est établie sur $\mathbb{R}$.

Théorème 4. *Soit $u_0 \in L^\infty(\mathbb{R}; [0, 1])$. Sous les hypothèses $(H1)$, $(H2)$ et $(H3)$, il existe une unique solution entropique $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0, 1])$ du problème (10).*

Ainsi dans ce travail, nous avons établi l'existence et l'unicité d'une solution entropique pour le problème (10). Ce problème est donc bien posé sous les seules hypothèses du modèle. De plus, nous avons introduit de nouvelles notions de solution toutes équivalentes à celle de solution entropique.

Ce travail a donné lieu à un article accepté pour une publication dans la revue Communications in Partial Differential Equations [BV05].

Chapitre 3

L'objectif de cette troisième partie est de présenter un schéma volume fini adapté au problème (10) et d'établir la convergence du schéma vers la solution entropique. En effet, cette preuve de convergence n'a jamais été faite et pourtant de nombreux industriels utilisent des schémas de ce type pour étudier le comportement de leur modèle.

En effet, depuis une dizaine d'années, plusieurs travaux présentaient des schémas numériques pour les lois de conservation à flux discontinu ([Tow00, Tow01, KRT02b, AJV04]). Dans ([Tow00, Tow01]), l'auteur présente un schéma à maillage décalé pour la fonction k et établit la convergence d'une sous-suite, mais pas la convergence du schéma.

Notre analyse inclut les schémas "scheme 1", "scheme 2" et "Godunov scheme" présentés dans [SV03], et les preuves peuvent être adaptées pour un maillage décalé sur la fonction k comme le proposait Towers. De fait, on montre que les schémas utilisés par les industriels et le schéma de Godunov, qui a de meilleures propriétés de convergence, convergent tous vers la solution entropique du problème (10).

Théorème 5. *On se donne une suite de maillages $\mathcal{T}_n$ dont la taille tend vers zéro lorsque n tend vers l'infini. Pour chaque maillage, on note u_n la fonction constante par maille construite par le schéma. Sous les hypothèses $(H1)$, $(H2)$ et $(H3)$, la suite $(u_n)_{n\in\mathbb{N}}$ converge vers l'unique solution entropique du problème (10), dans tout les espaces $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ pour $p \in [1, +\infty[$.*

Cette analyse est complétée par quelques essais numériques. Il est présenté trois types de schémas : le schéma de Godunov, le schéma VFRoe-ncv et un troisième nommé God/VFRoe-ncv. Ces trois schémas sont testés pour deux fonctions g. Le premier test est effectué avec une fonction g ni convexe, ni concave qui admet deux maximums locaux et un minimum local. Le deuxième test concerne une fonction linéaire par morceaux. Ce deuxième test a deux objectifs. Tout d'abord, cela met en évidence que l'hypothèse sur g vraiment non linéaire (qui est imposée dans tous les travaux antérieurs) n'est pas nécessaire.
Deuxièmement, l'étude de lois de conservation à flux discontinu peut être vue comme un travail préliminaire à l'étude des systèmes résonnants [IT86]. Le problème (10) peut être introduit comme un système résonnant (système dont le caractère hyperbolique peut être mis en défaut). En effet, l'équation (10) (si on suppose $f = 0$ pour simplifier) peut se réécrire :

$$\partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \quad \partial_t k = 0,$$

dont la matrice caractéristique est :

$$\begin{pmatrix} kg'(u) & g(u) \\ 0 & 0 \end{pmatrix}$$

On remarque alors que cette matrice n'est pas diagonalisable pour les valeurs où g' s'annule.

Pour les deux séries de tests, on observe un comportement similaire des trois schémas et une convergence, en norme L^1 discrète, à l'ordre 1. Enfin, on remarquera que cette estimation d'erreur n'est que numérique, aucune analyse n'a permis à l'heure actuelle d'obtenir une estimation d'erreur.

Ce travail a fait l'objet d'une présentation avec acte au colloque Fourth Finite Volume for Complex Applications, en juillet 2005. De plus, une version complétée a été soumise pour publication.

Chapitres 4 et 5

Les chapitres 4 et 5 concernent l'analyse de la loi de conservation à flux discontinu suivante :

$$\begin{cases} \partial_t u + \partial_x\big(g(x,u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & u_0 \in L^\infty(\mathbb{R};[0,1]). \end{cases} \tag{16}$$

Les fonctions f et g satisfaisant les hypothèses suivantes :

(H4) g est la fonction discontinue définie par
$$g(x,u) = \begin{cases} g_L(u) & \text{si } x < 0 \\ g_R(u) & \text{si } x > 0 \end{cases} \quad \text{avec } g_L \neq g_R,$$
$g_L, g_R \in \mathrm{Lip}([0,1])$ et $g_L(0) = g_R(0) = g_L(1) = g_R(1) = 0$,

(H5) $f \in \mathrm{Lip}([0,1])$.

On remarque que cette loi de conservation est une généralisation du problème (10). Pour cela, il suffit de prendre comme fonction $g_L = k_L g$ et $g_R = k_R g$. Dans ([KR01, AJV04]), les auteurs présentent une analyse de convergence, mais aucune preuve n'est établie. De même que pour le problème (10), aucun résultat d'existence et d'unicité de solution n'a été obtenu sous les hypothèses (H4) et (H5). On notera toutefois que parallèlement et indépendamment du travail présenté, une première preuve de convergence du schéma Lax-Friedrichs a été établie en supposant que le flux était vraiment non linéaire [KT04].
Dans ces deux chapitres, je présente un résultat d'existence et d'unicité de solution entropique, et une preuve de convergence d'un schéma volume fini. Pour cela, de nouveaux points d'analyse sont nécessaires.

Tout d'abord, si on revient au modèle décrit précédemment pour le problème (10), on suppose que la perméabilité absolue du milieu k dépend de x et les perméabilités relatives ne dépendent que de u. Dans ce nouveau problème, on suppose que la perméabilité absolue dépend de x, que les perméabilités relatives dépendent du milieu et du fluide (donc de x et de u), et que le flux total est nul, en rappelant que u est la saturation de l'eau (et $1-u$ la saturation de l'huile), d'où la forme de g.

On remarque que le signe des fonctions g_L et g_R est quelconque. Par contre, les fonctions g_L et g_R s'annulent en 0 et 1 (comme pour la fonction g dans les trois premiers chapitres). Toutefois, cette hypothèse est cohérente avec le modèle car elle traduit le fait que lors de l'écoulement, en présence d'un seul des deux fluides, le flux est indépendant de x. De plus, cette hypothèse est suffisante pour que le problème soit bien posé dans les deux modèles.

La notion de solution processus entropique a dû être généralisée au nouveau problème (16) :

Définition 6. *Soit $u_0 \in L^\infty(\mathbb{R})$ telle que $0 \le u_0 \le 1$ p.p. sur $\mathbb{R}$. Une fonction $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1); [0,1])$ est une solution processus entropique du problème (16) si elle satisfait les inégalités entropiques suivantes : pour tout $\kappa \in [0,1]$, pour toute* *fonction positive* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{aligned} & \int_0^1 \int_0^\infty \int_{\mathbb{R}} (u(t,x,\alpha) - \kappa)^\pm \, \partial_t \varphi(t,x) \, dt \, dx \, d\alpha \\ + & \int_0^1 \int_0^\infty \int_{\mathbb{R}} \big(\Phi^\pm(x, u(t,x,\alpha), \kappa) + \Psi^\pm(u(t,x,\alpha), \kappa)\big) \, \partial_x \varphi(t,x) \, dx \, dt \, d\alpha \\ + & \int_{\mathbb{R}} (u_0(x) - \kappa)^\pm \, \varphi(0,x) \, dx \; + \int_0^\infty (g_L(\kappa) - g_R(\kappa))^\pm \, \varphi(t,0) \, dt \geq 0, \end{aligned} \tag{17}$$

avec $\Phi^\pm$ *et* $\Psi^\pm$ *les flux entropiques associées aux entropies de Kruzhkov,*

$$\begin{aligned} \Phi^\pm(x,u,\kappa) &= \operatorname{sgn}_\pm(u-\kappa)(g(x,u) - g(x,\kappa)), \\ \Psi^\pm(u,\kappa) &= \operatorname{sgn}_\pm(u-\kappa)(f(u) - f(\kappa)). \end{aligned}$$

Dans ce cadre, l'unicité d'une solution processus entropique du problème (16) a été établie sous la forme suivante :

Théorème 6. *Soient* u *(resp.* $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$*) une solution entropique du problème (16), associée à la condition initiale* $u_0 \in L^\infty(\mathbb{R};[0,1])$ *(resp.* $v_0 \in L^\infty(\mathbb{R};[0,1])$*). Alors, sous les hypothèses (H4) et (H5), pour tout* $R, T > 0$*, on a*

$$\int_0^1 \int_0^1 \int_0^T \int_{-R}^R (u(t,x,\alpha) - v(t,x,\lambda))^\pm \, dx \, dt \, d\alpha \, d\lambda \leq T \int_{-R-CT}^{R+CT} (u_0(x) - v_0(x))^\pm dx,$$

avec $C := \max\{\mathrm{Lip}(g_L,)\mathrm{Lip}(g_R)\} + \mathrm{Lip}(f)$.

La preuve d'unicité a nécessité de prendre en compte le fait que les courbes de g_L et de g_R, sur $[0,1]$, pouvaient se croiser, ce qui n'était pas le cas pour les fonctions $k_L g$ et $k_R g$ (l'une étant toujours au dessus de l'autre suivant le signe de $k_L - k_R$). Mais un théorème de comparaison des solutions processus entropiques, puis des solutions entropiques du problème (16) a tout de même été obtenu. De plus, on en déduit qu'une solution processus entropique est en fait une solution entropique.

Dans le chapitre 5, je propose un schéma volume fini explicite pour le problème (16). Le schéma considéré est toujours monotone. Le point délicat est de définir un flux à l'interface $\{x = 0\}$. On a alors introduit un critère de monotonie qui est satisfait par les schémas déjà étudiés au chapitre 3 pour le problème (10).
L'analyse de convergence est faite en plusieurs étapes. Tout d'abord, la monotonie du schéma et les inégalités entropiques discrètes, satisfaites par la solution approchée construite par le schéma, sont obtenues. Puis, une estimation BV faible sur la solution approchée est obtenue. Cette estimation est à la base de la preuve de convergence et peut-être vu formellement comme suit :

Remarque 4. *Approcher une solution du problème (16) par une méthode volumes finis (pour simplifier à pas constants h et k) est équivalent à approcher une solution du problème (16) par une solution du problème parabolique suivant :*

$$\partial_t u + \partial_x \big(g^\varepsilon(x,u) + f(u)\big) - \varepsilon \partial_{xx} u = 0 \tag{18}$$

où $\varepsilon = (h-k)/2$ *sous une condition CFL.*
On suppose alors que u est assez régulière, qu'elle et sa fonction dérivée admettent des limites nulles lorsque $x \to \pm\infty$ *et que* g^ε *est une fonction régulière qui approche g (quand* ε *tend vers zéro) telle que* $g^\varepsilon(x,u) = g(x,u)$ *pour* $|x| > \varepsilon$*,* $u \in [0,1]$ *et* $g^\varepsilon(x,u) \in [g_L(u), g_R(u)]$ *ou* $\in [g_L(u), g_R(u)]$ *pour* $|x| \leq \varepsilon$*,* $u \in [0,1]$.
Si on multiplie (18) par u et qu'on intègre sur $(0,T)\times\mathbb{R}$*, on obtient :*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}} u^2(T,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} u^2(0,x)\,dx + \int_0^T \int_{\mathbb{R}} \varepsilon(\partial_x u)^2(t,x)\,dx\,dt \\ + \quad &\int_0^T \int_{\mathbb{R}} \partial_x\big(g^\varepsilon(x,u) + f(u)\big)\,u\,dx\,dt = 0. \end{aligned}$$

En utilisant la régularité de u on a :

$$\int_0^T \int_{\mathbb{R}} \partial_x(f(u))\,u\,dx\,dt = 0.$$

On a aussi $\partial_x(g^\varepsilon(x,u))\,u = g_x^\varepsilon(x,u)u + \partial_u \bar{g}^\varepsilon$ *avec* $\bar{g}^\varepsilon$ *une fonction régulière définie par :* $\partial_u \bar{g}^\varepsilon = u\partial_u(g^\varepsilon)(x,u)$*. On en déduit :*

$$\Big|\int_0^T \int_{\mathbb{R}} \partial_x g^\varepsilon(x,u)\,u\,dx\,dt\Big| \leq ||u||_\infty ||\partial_x g^\varepsilon||_1 \leq C_1.$$

Finalement, on a montré, formellement, que pour T suffisamment grand :

$$\int_0^T \int_{\mathbb{R}} \varepsilon(\partial_x u)^2(t,x)\,dx\,dt \leq C_2$$

avec C_2 *qui ne dépend que de g, f et* u_0*. C'est la variante continue de l'estimation BV-faible obtenue sur les solutions discrètes dans le chapitre 5.*

Enfin, à l'aide de l'estimation BV faible, l'existence d'une solution processus entropique est déduite.

Théorème 7. *Soit* $u_0 \in L^\infty(\mathbb{R};[0,1])$*. On se donne une suite de maillage* $\mathcal{T}_n$ *dont la taille tend vers zéro lorsque n tend vers l'infini. Pour chaque maillage, on note* u_n *la fonction constante par maille construite par le schéma. Sous les hypothèses* $(H4)$ *et* $(H5)$*, la suite* $(u_n)_{n\in\mathbb{N}}$ *admet une sous-suite convergente vers* $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R};[0,1])$ *pour la topologie faible-*$\star$ *non linéaire. De plus, la fonction v est une solution processus entropique du problème (16).*

L'existence est donc obtenue différemment que celle démontrée dans le chapitre 2. Toutefois, la même démarche de preuve que dans le chapitre 2 est aussi valable et présentée. Finalement, la solution processus entropique s'avère être la solution entropique (par les résultats présentés dans le chapitre 4), ce qui entraîne la convergence de la méthode des volumes finis pour la topologie non linéaire faible-$\star$ dans L^∞, puis à l'aide de la non linéarité, la convergence est établie dans tous les espaces $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ pour $p \in [1, +\infty[$. Ce résultat est résumé dans le théorème suivant :

Théorème 8. *On se donne une suite de maillages $\mathcal{T}_n$ dont la taille tend vers zéro lorsque n tend vers l'infini. Pour chaque maillage, on note u_n la fonction constante par maille construite par le schéma. Sous les hypothèses $(H4)$ et $(H5)$, la suite $(u_n)_{n\in\mathbb{N}}$ converge vers l'unique solution entropique du problème (16), dans tous les espaces $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ pour $p \in [1, +\infty[$.*

Ce travail a été soumis pour publication.

Bibliographie

[AJV04] Adimurthi, J. Jaffré et G.D. Veerappa Gowda. *Godunov-type methods for conservation laws with a flux function discontinuous in space time*, SIAM J. Numer Anal., **42**, (2004), no. 1, 179-208.

[Bac04] F. Bachmann. *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations (nov-dec 2004), no. 11-12, 1317–1338.

[BV05] F. Bachmann et J. Vovelle. *Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients*, to appear in Communications in Partial Differential Equations.

[BJ98] F. Bouchut et F. James. *One-dimensionnel transport equations with discontinuous coefficients*, Nonlinear Anal., **32**, (1998), no. 7, 891-933.

[Bre83] Y. Brenier, *Résolution d'équations d'évolution quasilinéaires en dimension N d'espaces à l'aide d'équations linéaires en dimension N+1*, J. Differential Eq uations, **50**, (1983), 375-390.

[DiP85] R.J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223-270.

[GMT96] G. Gagneux et M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Mathématiques et Applications, Springer-Verlag, Berlin, **22**, (1996). With a preface by Charles-Michel Marle.

[GM83] Y. Giga et T. Miyakawa, *A kinetic construction of global solutions of first-order quasilinear equations*, Duke Math. J., **50**, (1983), 505-515.

[EGH00] R. Eymard, T. Gallouët, et R. Herbin. *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, (2000), 713-1020.

[IT86] E.I. Isaacson et J.B. Temple. *Analysis of a singular hyperbolic system of conservation law*, J. Differential Equations, **65**, (1986), no. 2, 250-268.

[KR01] K.H. Karlsen et N.H. Risebro. *Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients*, M2AN Math. Model. Numer. Anal., **35**, (2001), no. 2, 239-269.

[KRT02] K.H. Karlsen, N.H. Risebro et J.D Towers. On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient. *Electron J. Differential Equations*, **93**, (2002), 23 pp (electronic)

[KT04] K.H. Karlsen et J.D. Towers. *Convergence of the Lax-Friedrichs Scheme and Stability for Conservation Laws with a Discontinuous Space-Time Dependent Flux* preprint (http ://www.math.ntnu.no/conservation/2004/005.html) (2004).

[KR95] K.H. Karlsen et N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness and asymptotic behavior*, Comm. Partial Differential Equations, **20**(11-12), (1995), 1959-1990.

[Kru70] S.N. Kruzhkov. *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.), **81**, (1970), (123), 228-255.

[LPT94] P.-L. Lions, B. Perthame, et E. Tadmor. *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc. **7** (1994), no. 1, 169-191.

[LTW95a] L.W. Lin, J.B. Temple et J.H. Wang. *A comparison of convergence rates for Godunov's method and Glimm's method in resonant nonlinear systems of conservation laws*, SIAM J. Numer. Anal., **32**, (1995), no. 3, 824-840.

[LTW95b] L.W. Lin, J.B. Temple et J.H. Wang. *Suppression of oscillations in Godunov's method for a resonant non-strictly hyperbolic system*, SIAM J. Numer. Anal., **32**, (1995), no. 3, 841-864.

[Ole56] O.A. Oleinik. *The problem of Cauchy for non-linear differential equations of the first order with discontinuous initial conditions*, Trudy Moskov. Mat. Obsc., **5**, (1996), 433-454.

[Ole57] O.A. Oleinik. *Discontinuous solutions of nonlinear differential equations*, Usephi Mat. Nauk (N.S.), **12**, (1957), 3-73.

[Per98] B. Perthame. *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures Appl. (9), **77**, (1998), no. 10, 1055–1064.

[SV03] N. Seguin et J. Vovelle. *Analysis and approximation of a scalar law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221-257.

[Tem82] B. Temple. *Global solution of the Cauchy problem for a class of* 2×2 *nonstrictly hyperbolic conservation laws.* Adv. in Appl. Math., **3**, (1982), no. 3.

[Tow00] J.D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), 681-698.

[Tow01] J.D. Towers. *A difference scheme for conservation laws with a discontinuous flux : the non convex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197-1218.

[Vas01] A. Vasseur. *Strong traces for weak solutions to multidimensional conservation laws*, Arch. of Rat. Mech. Anal., **160**, (2001), no. 3, 181-193.

[Vol67] A.I. Volpert. *Spaces BV and quasilinear equations*, Mat. sb. (N.S.), **73**, (1967), 255-302.

Chapter 1

Analyse d’une loi de conservation à flux discontinu vraiment non linéaire

1.1 Introduction

We study here a model of conservation law with a flux function with discontinuous coefficients, namely the equation $\partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0$. We prove the existence and the uniqueness of an entropy solution in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ for u_0, the initial condition, in $L^\infty(\mathbb{R})$. We provide some physical background for the study of this equation. In particular, g is not assume to be convex nor concave and k is a discontinuous function.

The issues of existence, uniqueness and entropy conditions for hyperbolic conservation laws with discontinuous coefficients are investigated. The Cauchy problem writes:

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with initial value $u_0 \in L^\infty(\mathbb{R})$ and a.e. $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $0 \le u(t,x) \le 1$.

The functions f, g and k are supposed to satisfy the following hypotheses:

(H1) $g \in C^1([0,1])$, is non-negative and $g(0) = g(1) = 0\,,$

(H2) $f \in C^1([0,1])$, is non-decreasing and $f(0) = 0\,,$

(H3) $g \equiv 0$ or g is genuinely nonlinear, i.e., if I is a non-empty open interval in $[0,1]$, $g'' = 0$ on $I \Rightarrow g \equiv 0$,

(H4) k is a discontinuous function defined by $k(x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x > 0 \end{cases}$ with $k_L, k_R > 0$ and $k_L \neq k_R\,.$

What underlies these hypotheses is our aim at giving and analyzing a model for a two-phase flow in porous media with distinct permeabilities (e.g. sand/clay). Let us precise this fact [GMT96]: we consider the flow (vertical and one-dimensional) of two immiscible fluids. We assume that the flow of both fluids can be adequately described by Darcy's law and that capillarity forces can be neglected. Then, from the equations of conservation of mass we deduce the following system:

$$\Gamma(x)\partial_t u - \partial_x(k\lambda_w(\partial_x p + \rho_w G)) = 0, \tag{1.2}$$

$$\Gamma(x)\partial_t (1-u) - \partial_x(k\lambda_o(\partial_x p + \rho_o G)) = 0. \tag{1.3}$$

Here, we choose a Cartesian system of coordinates such that the gravitational force points in the positive x-direction. We denote by u the saturation of the phase w, so that $1-u$ is the saturation of the phase o. Herein, ρ_w and ρ_o respectively denote the fluid densities of the phases w and o, p is the pressure of the fluids, $\lambda_w = k_w/(\rho_w\mu_w)$ and $\lambda_o = k_o/(\rho_o\mu_o)$ where μ_w and μ_o are the respective viscosities and k_w, k_o are the relative permeabilities of the phases w and o. We denote by G the acceleration of gravity, by k the absolute permeability and by Γ the porosity.
The addition of (1.2) and (1.3) gives $\partial_x Q = 0$ where

$$Q = k(\lambda_w + \lambda_o)\partial_x p + k\lambda_w\rho_w G + k\lambda_o\rho_o G \tag{1.4}$$

is the total flow. We suppose that this flow is stationary. It is thus constant with respect to x and t. By (1.4), $\partial_x p$ can be written as a function of Q which, plugged in (1.2) leads to

$$\Gamma(x)\partial_t u + \partial_x\Big(k\frac{\lambda_w\lambda_o}{\lambda_w+\lambda_o}(\rho_o-\rho_w)G + \frac{\lambda_w}{\lambda_w+\lambda_o}(-Q)\Big) = 0. \tag{1.5}$$

The function Γ is a discontinuous function of x but assuming $\Gamma \equiv cst \equiv 1$ does not change the mathematical analysis of the problem, without loss of generality we hence assume $\Gamma = cst$.
We thus obtain the form of equation (1.1) with

$$\begin{cases} g(u) := (\rho_o - \rho_w)\dfrac{\lambda_w(u)\lambda_o(u)}{\lambda_w(u)+\lambda_o(u)}G\,, \\ f(u) := (-Q)\dfrac{\lambda_w(u)}{\lambda_w(u)+\lambda_o(u)}. \end{cases} \tag{1.6}$$

The total flow being constant, it has a constant sign. We suppose for example $-Q \geq 0$. Like relative permeabilities, the functions λ_w and λ_o satisfy ([GMT96])

1. $\lambda_w \in C^1([0,1])$ is a non-decreasing function such that $\lambda_w(0) = 0$,

2. $\lambda_o \in C^1([0,1])$ is a non-increasing function such that $\lambda_o(1) = 0$,

3. there exists $\alpha > 0$ such as $\lambda_w(u) + \lambda_o(u) > \alpha$ for every $u \in [0,1]$.

Functions f and g defined in hence (1.6) satisfy hypotheses (H1) and (H2).
Hypothesis (H3) has no physical basis. However, we need it to prove existence of strong traces for an entropy solution to problem (1.1): we use a result of Vasseur [Vas01], which allows to show the uniqueness of an entropy solution to problem (1.1). Hypothesis (H3) is also necessary in the proof of Theorem 1.2.
Notice that we take care to study the equation $\partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0$ on a physical background. This is a way to ensure that the Cauchy problem (1.1) makes sense. Indeed, this is not always the case. Consider for example the problem $\partial_t u + \partial_x(k(x)u) = 0$ with $k(x) = -\text{sgn}(x)$: the computation of the solution of the Cauchy problem along the characteristic lines shows that this one cannot be specified in the domain $\{t > 0\,, |x| < t\}$.
The interpretation of hypothesis (H4) for the model of a two-phase flow is the following: the fluids move in different porous media (e.g. in sand for $x < 0$ and clay for $x > 0$) which permeabilities are distinct. Let us recall that our interest lies in the analysis of the conservation law $u_t + (k(x)g(u)+f(u))_x = 0$ where k is a discontinuous function and that, from that point of view, the elementary case where k is piecewise constant is relevant: the main features of conservation laws with discontinuous coefficients stand out. In particular, the question of the entropy condition satisfied by the potential solution on the line $\{x = 0\}$, line of discontinuity of the function k, arises. In the analysis of problem (1.1), this question is probably the first to require an answer, insofar as it rules the admissibility of solutions. Indeed, in the case where the function k is regular, or constant on $\mathbb{R}$, entropy conditions have been specified and proved to constitute accurate admissibility criteria for solutions of (1.1) [Ole57, Kru70]. In the study of problem (1.1) with the discontinuous function k defined in (H4), the admissible solutions are of course subject to entropy conditions away from $\{x = 0\}$ and the point is to identify conditions on $\{x = 0\}$ that should be satisfied. Since the beginning of the 80's, several answers have been given to this issue. The account of these solutions first requires some mathematical setting: suppose that the function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ under consideration is a weak solution of (1.1), i.e. solution in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$, and admits some traces $\gamma u^+(t)$ and $\gamma u^-(t) \in L^\infty(\mathbb{R}_+)$ at $x = 0+$ and $x = 0-$ respectively (this the case if $u|_{Q^\pm} \in L^\infty \cap BV(Q^\pm)$, where $Q^+ = (0, +\infty) \times (0, +\infty)$ and $Q^- = (0, +\infty) \times (-\infty, 0)$ for example). Then, from the fact that u is a weak solution of (1.1) the Rankine-Hugoniot relation

$$k_L g(\gamma u^-) + f(\gamma u^-) = k_R g(\gamma u^+) + f(\gamma u^+) \tag{1.7}$$

is easily deduced. This relation is still too weak to constitute an accurate criterion of selection. Additional criteria have been given in the case $f = 0$, $g'' < 0$ and $g'(u^\star) = 0$, first by Isaacson and Temple [IT86, IT92] and Temple [Tem82]. The authors give the following geometrical condition: in the (u, k) plane, a state $(\gamma u^-, k_L)$ being given, define

$$\mathcal{S}_0 = \{(\gamma u^+, k_R)\,;\ k_L g(\gamma u^-) + f(\gamma u^-) = k_R g(\gamma u^+) + f(\gamma u^+)\}. \tag{1.8}$$

The condition then reads:

$$\mathcal{S}_0 \cap \{(u, k)\,; u = u^\star\} = \emptyset. \tag{1.9}$$

In [KR95], Klingenberg and Risebro give the wave criterion, inspired of Oleinik conditions [Ole56, Ole57],

$$\partial_x(k(x)g'(u)) \leq C(1+\frac{1}{t}).$$

In [Tow00], Towers gives the analytical condition

$$[g'(\gamma u^+)]_+ \cdot [g'(\gamma u^-)]_- = 0 \tag{1.10}$$

(where $[a]_\pm$ is, respectively, the positive and negative part of a real function a). Notice that this condition is equivalent to the non crossing condition given by Isaacson and Temple in [IT86, IT92], at least when the initial condition of (1.1) is a Riemann data:

$$u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0, \end{cases}$$

(Indeed, if one takes into account the fact that the state u_R (resp. u_L) has to be linked to the state γu^+ (resp. γu^-) by (classical) admissible waves, then the non crossing condition of Isaacson and Temple is equivalent to the condition of Towers.)
An other important feature of problem (1.1) is highlighted in [Tem82, IT86]. Indeed, by adding the equation $k_t = 0$ to (1.1), Isaacson and Temple actually solve the Riemann problem associated to the system

$$\partial_t U + \partial_x F(U) = 0$$

where $U = (u,k)^T$ and $F(U) = (kg(u) + f(u), 0)^T$. This system is viewed as a prototype of *resonant hyperbolic systems*, that is a system for which eigenvalues can coincide. Indeed, the derivative of the flux, $DF(U)$, has two eigenvalues $kg'(u)+f'(u)$ and 0 which can coincide and, in that case, $DF(U) = \begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix}$ is not diagonalizable on $\mathbb{R}$. The connection between scalar conservation laws with discontinuous coefficients and resonant hyperbolic systems has not only a theoretical interest, it also has applications to their numerical approximations, see [LTW95a, Tow01, SV03]. We refer to [Tem82, IT86, IT92, GL03] for further references and results on the Riemann problem for resonant hyperbolic systems.
In fact, in [Tem82, IT86, LTW95a, Tow01, SV03], one takes $f = 0$, so that the first eigenvalue $kg'(u)$ of $DF(U)$ vanishes if and only if $g'(u) = 0$. Besides, g' has only one point of cancellation, $u^\star$, which value governs the entropy condition at $\{x = 0\}$: the pertinence of this additional entropy condition is therefore thoroughly related to the occurrence of resonance.
Regarding problem (1.1), we have (briefly) discussed the questions of entropy conditions and resonant hyperbolic systems. An other issue at stake here is the possibility to give global weak entropy conditions, i-e, entropy conditions in $\mathcal{D}'$, as (1.11) for example, in opposition to local entropy conditions (1.9), (1.10). To clarify our terminology, let us come back on the historical progression in the investigation of

entropy conditions for (classical) scalar conservation laws $u_t + (h(u))_x = 0$ with, say, $h'' \geq \alpha > 0$: in 1956-1957 [Ole56, Ole57], Oleĭnik first gave the entropy condition

$$\frac{u(t,x') - u(t,x)}{x' - x} \leq \frac{1}{\alpha t},$$

which we qualify as "local" while the work of Volpert and Kruzhkov [Vol67, Kru70] led to the well-known "global" entropy condition $\partial_t|u-\kappa|+\partial_x[\mathrm{sgn}(u-\kappa)(h(u)-h(\kappa))] \leq 0$ in $\mathcal{D}'$.

This point is of importance as the validity of local entropy conditions for approximations (in particular numerical approximations) of the solution u can be very difficult to check, whereas global entropy conditions are much simpler to evaluate for an approximate solution. Global entropy conditions are therefore a powerful and essential tool in the analysis of the convergence of approximations of (1.1). The elaboration of the definition of a weak entropy solution is a considerable advance in the analysis of scalar conservation laws with discontinuous coefficient. In the case where $f = 0$ and g has a unique local maximum, this step was accomplished by Towers in [Tow00]. For general functions f and g satisfying hypotheses (H1) to (H4), we show that this definition remains accurate (see Definition 1.11), as, first, a L^1 contraction property for such entropy solutions holds (Theorem 1.1) and, second, such entropy solutions exist (Theorem 1.3) and are limit of the approximation trough the regularization of the coefficient k (Theorem 1.2). The proof of the convergence of this approximation relies on the use of a Temple function as introduced by Temple in [Tem82].

Indeed, given an initial datum $u_0 \in L^\infty \cap BV(\mathbb{R})$, the BV semi-norm $||u(\cdot, 0+)||_{BV}$ of the entropy solution u of problem (1.1) at time $t = 0+$ may not be bounded [LTW95b]. It is known that, in the case where the function k is regular, the BV semi-norm of entropy solutions with initial data in $L^\infty \cap BV(\mathbb{R})$ remains bounded with time. Consequently, still in the case where the function k is regular, approximations of entropy solutions have also bounded BV semi-norm and this provides a criterion of compactness in L^1. For discontinuous k, BV bounds are satisfied not by the entropy solution u but by the new unknown $H(u)$ to be defined in the sequel, usually called a Temple function. Such functions have a wide range of applications: the study of the Riemann problem for resonant hyperbolic systems [IT86], the analysis of the convergence of approximations given by the Glimm scheme, Godunov method [LTW95b, LTW95a]; the design and the analysis of a front tracking algorithm [KR95], the convergence of numerical schemes [Tow00, Tow01]; the convergence of the approximation by regularization of the coefficient k in [SV03] and in the present paper (see the definitions (1.40) and (1.41) of the functions F^- and F^+ in the proof of Theorem 1.2).

As already mentioned, the analysis of problem (1.1) (or analysis of related problems as in [KRT02b]) is usually performed under the restrictive assumptions $f = 0$ and $g'' < 0$ (or g has a single local maximum) and that $g \equiv 0$ or g is genuinely nonlinear. These assumptions are not satisfied by the present model. Here, we solely assume that g is genuinely nonlinear which seems in our opinion, technical and is used essentially to obtain existence of traces for the solution on $\{x = 0\}$ (see [Vas01] below).

The paper is structured as follows. In Section 1.2, we give the definition of entropy solution. In Section 1.3, the uniqueness of such solutions is proved. In Section 1.4, the convergence of the approximation of (1.1) is analyzed which follows from the regularization of coefficient k: *via* the use of a Temple function, we prove the convergence of this approximation (and, at the same time, the existence of an entropy solution) when u_0 is BV. Existence in the case $u_0 \in L^\infty(\mathbb{R})$ then follows.
Notice that, independently of our work, an analysis of the equation $\partial_t u + \partial_x f(\gamma(x), u) = \partial_{xx} A(u)$, where γ is a discontinuous function and A is a non-decreasing Lipschitz continuous function, was performed by Karlsen, Risebro and Towers [KRT03]. In this work, they improved the range of admissible functions f for which uniqueness of the entropy solution can be addressed.

1.2 Definition of an entropy solution

Definition 1.1. *Let $u_0 \in L^\infty(\mathbb{R})$, with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. A function u in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is said to be an entropy solution of problem (1.1) if it satisfies $0 \le u \le 1$ a.e. and the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\, \partial_t \varphi(t,x) \\ + \quad & \int_0^\infty \int_{\mathbb{R}} \big(k(x)\, \Phi(u(t,x),\kappa) + \Psi(u(t,x),\kappa)\big)\, \partial_x \varphi(t,x)\, dx\, dt \\ + \quad & \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \\ + \quad & |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \ge 0, \end{aligned} \tag{1.11}$$

where respectively Φ and Ψ denote the entropy flux associated with the Kruzhkov entropy,

$$\Phi(u,\kappa) = sgn(u-\kappa)(g(u) - g(\kappa)),$$
$$\Psi(u,\kappa) = sgn(u-\kappa)(f(u) - f(\kappa)).$$

Remark 1.1. *An entropy solution of (1.1) is a weak solution of (1.1), i.e., for all non-negative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} u(t,x)\, \partial_t \varphi(t,x) \;+\; (k(x)\, g(u(t,x) + f(u(t,x))\, \partial_x \varphi(t,x)\, dx\, dt \\ + \quad & \int_{\mathbb{R}} u_0(x)\, \varphi(0,x)\, dx = 0. \end{aligned}$$

1.3 Uniqueness of an entropy solution

Theorem 1.1. *Let u_0, v_0 in $L^\infty(\mathbb{R})$ such that $0 \le u_0, v_0 \le 1$ a.e. We suppose that $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ are two entropy solutions of problem*

(1.1), with initial conditions u_0 and v_0, respectively. Then, for every $R, T > 0$, the following estimate holds :

$$\int_0^T \int_{-R}^{R} |u(t,x) - v(t,x)|\, dx\, dt \le T \int_{-R-TM}^{R+TM} |u_0(x) - v_0(x)|\, dx, \tag{1.12}$$

where $M = \max(k_L, k_R) \sup_{u\in[0,1]} |g'(u)| + \sup_{u\in[0,1]} |f'(u)|$.

Proof of Theorem 1.1 :

The classical proof of uniqueness of Kruzhkov applies without changes to prove that, if u and v are two entropy solutions of problem (1.1), if φ is non-negative function of $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ which vanishes in a neighborhood of the line $\{x = 0\}$ when k is discontinuous, then the following inequality holds

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \varphi(t,x) \\ +\;& \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x), v(t,x)) \;+\; \Psi(u(t,x), v(t,x)))\partial_x \varphi(t,x)\, dx\, dt \\ +\;& \int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi(0,x)\, dx \ge 0. \end{aligned} \tag{1.13}$$

Now, consider any non-negative function ψ in $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ and, for $\varepsilon > 0$, set $\varphi(t,x) = \psi(t,x)(1 - \omega_\varepsilon(x))$ where the function ω_ε is defined by

$$\omega_\varepsilon(x) = \begin{cases} 0 & \text{si } 2\varepsilon < |x|, \\ \frac{-|x|+2\varepsilon}{\varepsilon} & \text{si } \varepsilon \le |x| \le 2\varepsilon, \\ 1 & \text{si } |x| < \varepsilon. \end{cases}$$

By use of Lebesgue dominated convergence theorem and passing to the limit $\varepsilon \to 0$ in the inequality (1.13), we obtain

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \psi(t,x) \\ +\;& \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x), v(t,x)) + \Psi(u(t,x), v(t,x)))\partial_x \psi(t,x)\, dx\, dt \\ +\;& \int_{\mathbb{R}} |u_0(x) - v_0(x)| \psi(0,x)\, dx - J \ge 0, \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} J &:= \limsup_{\varepsilon\to 0} J_\varepsilon \\ &= \limsup_{\varepsilon\to 0} \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u,v) + \Psi(u,v))\psi(t,x)\omega'_\varepsilon(x)\, dx\, dt. \end{aligned}$$

We now prove that J is non-negative.
First, we evaluate J. In fact, J_ε can explicitly computed :

$$\begin{aligned} J_\varepsilon &= \frac{1}{\varepsilon}\int_0^\infty \int_{-2\varepsilon}^{-\varepsilon} (k_L\Phi(u,v)+\Psi(u,v))\psi(t,x)\,dx\,dt \\ &- \frac{1}{\varepsilon}\int_0^\infty \int_{\varepsilon}^{2\varepsilon} (k_R\Phi(u,v)+\Psi(u,v))\psi(t,x)\,dx\,dt. \end{aligned} \tag{1.15}$$

In order to estimate this term J_ε, we use the result of existence of strong traces for solutions of non-degenerate conservations laws by Vasseur [Vas01]. Here, hypothesis (H3) on g is used. We however believe that hypothesis (H3) is superfluous and that the estimate (1.12) is true without such an hypothesis. The proofs uses the following lemma which is adopted from lemma in Vasseur [Vas01]:

Lemma 1.1. *Let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ be an entropy solution to problem (1.1) with initial condition $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. Then the function u admits strong traces on the line $\{x = 0\}$, that is : there exists two functions γu^- and γu^+ in $L^\infty(0,+\infty)$ such that, for every compact K of $(0,+\infty)$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon}\int_K |u(t,x)-\gamma u^-(t)|\,dx\,dt = 0, \tag{1.16}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\int_K |u(t,x)-\gamma u^+(t)|\,dx\,dt = 0. \tag{1.17}$$

From (1.16) and (1.17), we will deduce :

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\infty \int_{-2\varepsilon}^{-\varepsilon} (k_L\Phi(u,v)+\Psi(u,v))\psi(t,x)\,dx\,dt \\ &= \int_0^\infty (k_L\Phi(\gamma u^-,\gamma v^-)+\Psi(\gamma u^-,\gamma v^-))\psi(t,0)\,dt, \end{aligned} \tag{1.18}$$

Let K a compact such that supp $\psi \subset K \times \mathbb{R}$, let $\varepsilon > 0$ then

$$\begin{aligned} & |\frac{1}{\varepsilon}\int_0^\infty \int_{-2\varepsilon}^{-\varepsilon} (k_L\Phi(u,v)+\Psi(u,v))\psi(t,x)\,dx\,dt \\ - & \int_0^\infty (k_L\Phi(\gamma u^-,\gamma v^-)+\Psi(\gamma u^-,\gamma v^-))\psi(t,0)\,dt| \\ \le\ & k_L \int_K \int_{-2\varepsilon}^{-\varepsilon} |\Phi(u,v)\psi(t,x)-\Phi(\gamma u^-,\gamma v^-)\psi(t,0)|\,dx\,dt \\ + & \int_K \int_{-2\varepsilon}^{-\varepsilon} |\Psi(u,v)\psi(t,x)-\Psi(\gamma u^-,\gamma v^-)\psi(t,0)\,|dx\,dt \\ \le\ & k_L \int_K \alpha_\varepsilon(t)\,dt + \int_K \beta_\varepsilon(t)\,dt \end{aligned} \tag{1.19}$$

where

$$\alpha_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon} |\Phi(u,v)\psi(t,x) - \Phi(\gamma u^-, \gamma v^-)\psi(t,0)|\,dx$$

and

$$\beta_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon} |\Psi(u,v)\psi(t,x) - \Psi(\gamma u^-, \gamma v^-)\psi(t,0)|\,dx.$$

Remark 1.2. *By noting* $M_g^1 = \max_{u\in[0,1]} |g'(u)|$*, we have* $|\Phi(a,b) - \Phi(a',b)| \le M_g^1|a-a'|$ *and* $|\Phi(a,b) - \Phi(a,b')| \le M_g^1|b-b'|$*, for all* $a, a', b, b' \in [0,1]$.

Then, there exists $C > 0$ such that:

$$\begin{aligned}
& \int_K |\Phi(u,v)\psi(t,x) - \Phi(\gamma u^-, \gamma v^-)\psi(t,0)|\,dt \\
\le\ & \int_K |\Phi(u,v) - \Phi(\gamma u^-, v)|\psi(t,x)\,dt \\
+\ & \int_K |\Phi(\gamma u^-, v)||\psi(t,x) - \psi(t,0)|\,dt \\
+\ & \int_K |\Phi(\gamma u^-, v) - \Phi(\gamma u^-, \gamma v^-)|\psi(t,0)\,dt \\
\le\ & C\Big(\int_K |u(t,x) - \gamma u^-(t)|\,dt + \int_K |\psi(t,x) - \psi(t,0)|\,dt \\
& + \int_K |v(t,x) - \gamma v^-(t)|\,dt\Big)
\end{aligned}$$

By using theorem of Fubini-Tonelli, lemma 1.1 and the regularity of function ψ, we get:

$$\int_K \alpha_\varepsilon(t)\,dt \longrightarrow_{\varepsilon\to 0} 0.$$

In the same way, this yields:

$$\int_K \beta_\varepsilon(t)\,dt \longrightarrow_{\varepsilon\to 0} 0.$$

Finally, we obtain (1.18).
In the same way, we have :

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\infty\int_\varepsilon^{2\varepsilon} (k_R\Phi(u,v) + \Psi(u,v))\psi(t,x)\,dx\,dt \\
=\ & \int_0^\infty (k_R\Phi(\gamma u^+, \gamma v^+) + \Psi(\gamma u^+, \gamma v^+))\psi(t,0)\,dt. \qquad (1.20)
\end{aligned}$$

Then, J is well defined and this yields:

$$\begin{aligned}
J = \int_0^\infty & [k_L\Phi(\gamma u^-, \gamma v^-) + \Psi(\gamma u^-, \gamma v^-) \\
& - k_R\Phi(\gamma u^+, \gamma v^+) - \Psi(\gamma u^+, \gamma v^+)]\psi(t,0)dt. \qquad (1.21)
\end{aligned}$$

With this formula, we can actually determine the sign of J, $J \geq 0$: if we replace φ by $\varphi\omega_\varepsilon$ as the test function in (1.11) it yields:

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} |u(t,x)-\kappa|\partial_t\varphi(t,x)\omega_\varepsilon(x)dxdt \\
+ & \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x),\kappa)+\Psi(u(t,x),\kappa)\partial_x\varphi(t,x)\omega_\varepsilon(x)dxdt \\
+ & \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x),\kappa)+\Psi(u(t,x),\kappa)\varphi(t,x)\omega'_\varepsilon(x)dxdt \\
+ & \int_{\mathbb{R}} |u_0(x)-\kappa|\varphi(0,x)\omega_\varepsilon(x)dx \\
+ & |k_L-k_R| \int_0^\infty g(\kappa)\varphi(t,0)dt \geq 0.
\end{aligned} \tag{1.22}$$

Since $\omega_\varepsilon(x) \rightarrow_{\varepsilon\rightarrow 0} 0$, by Lebesgue dominated convergence theorem both the first and the forth terms tend to 0 when $\varepsilon \rightarrow 0$. Furthermore, with Lemma 1.1, we also have:

$$\begin{aligned}
& \lim_{\varepsilon\rightarrow 0} \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x),\kappa)+\Psi(u(t,x),\kappa))\varphi(t,x)\omega'_\varepsilon(x)dxdt \\
= & \int_0^\infty (k_L\Phi(\gamma u^-,\kappa)-k_R\Phi(\gamma u^+,\kappa)+\Psi(\gamma u^-,\kappa)-\Psi(\gamma u^+,\kappa))\varphi(t,0)dt.
\end{aligned}$$

Eventually, passing to the limit $\varepsilon \rightarrow 0$ in (1.22), we obtain:

$$\begin{aligned}
& \int_0^\infty (k_L\Phi(\gamma u^-,\kappa)-k_R\Phi(\gamma u^+,\kappa)+\Psi(\gamma u^-,\kappa)-\Psi(\gamma u^+,\kappa))\varphi(t,0)dt \\
+ & |k_L-k_R| \int_0^\infty g(\kappa)\varphi(t,0)dt \geq 0, \quad \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R}),\ \varphi \geq 0.
\end{aligned}$$

Consequently, for every $\kappa \in [0,1]$, u entropy solution of problem (1.1), the following inequality holds:

$$\begin{aligned}
I_u(\kappa) := \quad & k_L\Phi(\gamma u^-,\kappa)+\Psi(\gamma u^-,\kappa)-k_R\Phi(\gamma u^+,\kappa)-\Psi(\gamma u^+,\kappa) \\
+ \quad & |k_L-k_R|g(\kappa) \geq 0.
\end{aligned} \tag{1.23}$$

By choosing $\kappa = 0$ in inequality (1.23), we have:

$$k_Lg(\gamma u^-)+f(\gamma u^-)-(k_Rg(\gamma u^+)+f(\gamma u^+)) \geq 0,$$

since $g(0)=0$, $f(0)=0$, and since $\gamma u^+ \geq 0$, $\gamma u^- \geq 0$ a.e.
Similarly, choosing $\kappa = 1$ in inequality (1.23) leads to:

$$\begin{aligned}
& -k_Lg(\gamma u^-)-(f(\gamma u^-)-f(1))-[-k_Rg(\gamma u^+)-(f(\gamma u^+)-f(1))] \geq 0 \\
& \qquad \Longrightarrow k_Lg(\gamma u^-)+f(\gamma u^-)-(k_Rg(\gamma u^+)+f(\gamma u^+)) \leq 0.
\end{aligned}$$

Therefore, the following Rankine-Hugoniot relation holds

$$k_L g(\gamma u^-) + f(\gamma u^-) = k_R g(\gamma u^+) + f(\gamma u^+). \tag{1.24}$$

Let us now prove, using (1.24) and (1.23), that $J \geq 0$. Suppose for example that $k_L > k_R$. We must discern several cases:

- $\underline{sgn(\gamma u^- - \gamma v^-) = sgn(\gamma u^+ - \gamma v^+) = s}$, then

$$\begin{aligned} J &= s[k_L g(\gamma u^-) + f(\gamma u^-) - k_L g(\gamma v^-) - f(\gamma v^-)] \\ &- s[k_R g(\gamma u^+) - k_R g(\gamma v^+) + f(\gamma u^+) - f(\gamma v^+)] \\ &= 0 \quad by\,(1.24) \text{ for } u \text{ and } v. \end{aligned}$$

- $\underline{\gamma u^- \geq \gamma v^- \text{ and } \gamma u^+ < \gamma v^+}$, then by (1.24)

$$\begin{aligned} J &= 2k_L(g(\gamma u^-) - g(\gamma v^-)) + 2(f(\gamma u^-) - f(\gamma v^-)) \\ &= 2k_R(g(\gamma u^+) - g(\gamma v^+)) + 2(f(\gamma u^+) - f(\gamma v^+)). \end{aligned}$$

  1. $\gamma v^+ \leq \gamma u^-$ implies $\gamma u^+ < \gamma v^+ \leq \gamma u^-$, by choosing $\kappa = \gamma v^+$ in I_u we have

$$2(k_R(g(\gamma u^+) + f(\gamma u^+)) - 2(k_R g(\gamma v^+) + f(\gamma v^+)) \geq 0 \Longrightarrow J \geq 0,$$

  2. $\gamma v^+ > \gamma u^-$ implies $\gamma v^- \leq \gamma u^- < \gamma v^+$, by choosing $\kappa = \gamma u^-$ in I_v we obtain

$$2(k_L g(\gamma u^-) + f(\gamma u^-)) - 2(k_L g(\gamma v^-) + f(\gamma v^-)) \geq 0 \Longrightarrow J \geq 0,$$

- $\underline{\gamma u^- < \gamma v^- \text{ and } \gamma u^+ \geq \gamma v^+}$, and by (1.24)

$$\begin{aligned} J &= 2k_L(g(\gamma v^-) - g(\gamma u^-)) + 2(f(\gamma v^-) - f(\gamma u^-)) \\ &= 2k_R(g(\gamma v^+) - g(\gamma u^+)) + 2(f(\gamma v^+) - f(\gamma u^+)). \end{aligned}$$

  1. $\gamma v^- \leq \gamma u^+$ implies $\gamma u^- < \gamma v^- \leq \gamma u^+$, by choosing $\kappa = \gamma v^-$ in I_u we obtain

$$2(k_L g(\gamma v^-) + f(\gamma v^-)) - 2(k_L g(\gamma u^-) + f(\gamma u^-)) \geq 0 \Longrightarrow J \geq 0,$$

  2. $\gamma v^- > \gamma u^+$ implies $\gamma v^+ \leq \gamma u^+ < \gamma v^-$, by choosing $\kappa = \gamma u^+$ in I_v we obtain

$$2(k_R g(\gamma v^+) + f(\gamma v^+)) - 2(k_R g(\gamma u^+) + f(\gamma u^+)) \geq 0 \Longrightarrow J \geq 0.$$

Finally, for all non-negative $\psi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, we reach

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} |u(t,x) - v(t,x)| \psi_t(t,x) \\ + & \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(u(t,x), v(t,x)) + \Psi(u(t,x), v(t,x)))\psi_x(t,x)\, dx\, dt \\ + & \int_{\mathbb{R}} |u_0(x) - v_0(x)| \psi(0,x)\, dx \geq 0. \end{aligned}$$

It is then classical to derive inequality (1.12). This concludes the proof of Theorem 1.1.

1.4 Existence of an entropy solution

To prove the existence of an entropy solution to problem (1.1), we use the existence of an entropy solution in the case where the function k is regularized and then pass to the limit. Let us consider a sequence $(k_\varepsilon)_\varepsilon$ of regular functions converging to the function k. We suppose that $\forall \varepsilon > 0$ the function k_ε is regular monotone non-decreasing, or non-increasing according to the sign of $k_R - k_L$, and satisfies

$$\begin{cases} k_\varepsilon(x) = k_L & \text{if } x \leq -\varepsilon, \\ k_\varepsilon(x) = k_R & \text{if } x \geq \varepsilon. \end{cases}$$

Then, by the result of Kruzhkov [Kru70], we know that for any initial condition $u_0 \in L^\infty(\mathbb{R}, [0,1])$ there exists an unique entropy solution u_ε to problem (1.25) :

$$\begin{cases} \partial_t u + \partial_x \big(k_\varepsilon(x) g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \\ u(0,x) = u_0(x). \end{cases} \tag{1.25}$$

The solution u_ε of problem (1.25) satisfies: for all non-negative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ and for all $\kappa \in [0,1]$

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} |u_\varepsilon(t,x) - \kappa| \partial_t \varphi(t,x) \\ + & \int_0^{+\infty} \int_{\mathbb{R}} \big(k_\varepsilon(x)\Phi(u_\varepsilon(t,x),\kappa) + \Psi(u_\varepsilon(t,x),\kappa)\big)\partial_x \varphi(t,x) dt dx \\ + & \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(0,x) dx \\ - & \int_0^{+\infty} \int_{\mathbb{R}} k_\varepsilon'(x) \operatorname{sgn}(u_\varepsilon(t,x) - \kappa) g(\kappa) \varphi(t,x) dx dt \geq 0. \end{aligned}$$

Remark 1.3. *The solution of (1.25) satisfies* $0 \leq u_\varepsilon \leq 1$ *a.e. in* $\mathbb{R}_+ \times \mathbb{R}$.

1.4.1 BV Estimates

In this paragraph, we assume that $u_0 \in BV(\mathbb{R})$ and we show that the sequence $(u_\varepsilon)_\varepsilon$ converges in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ to a function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

Lemma 1.2. *Assume $u_0 \in BV(\mathbb{R})$ and $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Then the solution u_ε of problem (1.25) satisfies the following BV estimate : for any $T > 0$, for any $\kappa \in [0,1]$, there exists $C > 0$ depending only on T, k_L, k_R such that*

$$|k_\varepsilon \Phi(u_\varepsilon, \kappa) + \Psi(u_\varepsilon, \kappa)|_{BV((0,T)\times\mathbb{R})} \leq C(|u_0|_{BV(\mathbb{R})} + |k_L - k_R|). \tag{1.26}$$

Definition 1.2. *A function $v \in L^1_{loc}(I)$ is of bounded variation, i.e. $v \in BV(I)$, if*

$$|v|_{BV(I)} = \sup\left\{\int_I v\, div\varphi,\ \varphi \in \mathcal{C}^1_c(I), ||\varphi||_\infty \leq 1\right\} < +\infty.$$

Proof of Lemma 1.2.
First we assume that $u_0 \in \mathcal{C}^\infty_c(\mathbb{R}, [0,1])$.
Let v^μ denote the solution of the viscous approximation of problem (1.25), that is

$$\begin{cases} \partial_t v + \partial_x\big(k_\varepsilon(x) g(v) + f(v)\big) - \mu \partial_{xx} v = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ v(0,x) = u_0(x). \end{cases} \tag{1.27}$$

Then, as (1.27) is a parabolic equation, there exists a unique solution v^μ which is smooth. Moreover, it satisfies the following properties:

Lemma 1.3.
- *i). Let w^μ be an other smooth solution of (1.27) with initial condition w_0, such that $g(w(t, \pm\infty)) = 0$ and $f(w(t, \pm\infty)) = 0$. Then,*

$$\int_{\mathbb{R}} (v^\mu(t,x) - w^\mu(t,x))^- \, dx \leq \int_{\mathbb{R}} (u_0(x) - w_0(x))^- \, dx, \quad \forall t \geq 0. \tag{1.28}$$

- *ii). The solution v^μ satisfies $0 \leq v^\mu \leq 1$ in $\mathbb{R}_+ \times \mathbb{R}$.*
- *iii). For all $R, T > 0$, there exists a constant $C_{T,R}$ such as*

$$\mu \int_0^T \int_{-R}^R |\partial_x v^\mu|^2 \, dx \, dt \leq C_{T,R}. \tag{1.29}$$

For the proof of the first point, let η_α denote a smooth approximation of the function $v \mapsto v^-$ defined by

$$\eta_\alpha(v) = \begin{cases} \alpha - v & \text{if } v \leq -2\alpha, \\ v^2/4\alpha & \text{if } -2\alpha \leq v \leq 0, \\ 0 & \text{if } 0 \leq v. \end{cases}$$

Multiplying the equation

$$\partial_t(v^\mu - w^\mu) + \partial_x\big(k_\varepsilon g(v^\mu) - k_\varepsilon g(w^\mu) + f(v^\mu) - f(w^\mu)\big) = \mu \partial_{xx}(v^\mu - w^\mu)$$

by $\eta_\alpha^{'}(v^\mu - w^\mu)$ and denoting

$$\begin{aligned} A^\mu &= \partial_t \eta_\alpha(v^\mu - w^\mu) \\ &\quad + \partial_x[\eta_\alpha^{'}(v^\mu - w^\mu)(k_\varepsilon g(v^\mu) - k_\varepsilon g(w^\mu) + f(v^\mu) - f(w^\mu))] \\ &\quad - \eta_\alpha^{''}(v^\mu - w^\mu)[k_\varepsilon g(v^\mu) - k_\varepsilon g(w^\mu) + f(v^\mu) - f(w^\mu)], \end{aligned}$$

we have

$$A^\mu = \mu \partial_{xx} \eta_\alpha(v^\mu - w^\mu) - \mu \eta_\alpha^{''}(v^\mu - w^\mu)[\partial_x(v^\mu - w^\mu)]^2 \leq \mu \partial_{xx}\eta_\alpha(v^\mu - w^\mu).$$

We integrate this last inequality over $(0,t) \times \mathbb{R}$. We note that $g(v(t, \pm\infty)) = 0$ because $v(t,.)$ decreases rapidly to zero when $x \to \pm\infty$. Note furthermore that $g(w(t,\pm\infty)) = 0$, $f(w(t,\pm\infty)) = 0$ by hypothesis. We hence obtain

$$\begin{aligned} & \int_{\mathbb{R}} \eta_\alpha(v^\mu - w^\mu)\, dx - \int_{\mathbb{R}} \eta_\alpha(u_0 - w_0)\, dx \\ \leq\ & \int_0^t \int_{\mathbb{R}} \eta_\alpha^{''}(v^\mu - w^\mu)[k_\varepsilon g(v^\mu) - k_\varepsilon g(w^\mu) + f(v^\mu) - f(w^\mu)] \\ & \partial_x(v^\mu - w^\mu)\, dx\, dt \\ \leq\ & C \int_0^t \int_{\mathbb{R}} \eta_\alpha^{''}(v^\mu - w^\mu)[k_\varepsilon|v^\mu - w^\mu| + |v^\mu) - w^\mu|]|\partial_x(v^\mu - w^\mu)|\, dx\, dt, \end{aligned}$$

where $C = \max(\max\limits_{u\in[0,1]} |g'(u)|, \max\limits_{u\in[0,1]} |f'(u)|)$.
Letting α tend to zero yields (1.28).

We use inequality (1.28) to estimate a lower bound on the solution in L^∞. As $g(0) = 0$ and $f(0) = 0$, the constant function 0 is a solution to (1.27) with initial condition 0, therefore one has

$$\int_{\mathbb{R}} (v^\mu)^- dx \leq \int_{\mathbb{R}} (u_0)^- dx = 0.$$

Consequently, $v^\mu \geq 0$ a.e. in $\mathbb{R}_+ \times \mathbb{R}$.
To prove that $v^\mu \leq 1$, we use the following equalities

$$\partial_t(v^\mu - 1) = \partial_t v^\mu, \quad \partial_{xx}(v^\mu - 1) = \partial_{xx} v^\mu, \quad \partial_x\big(f(v^\mu) - f(1)\big) = \partial_x\big(f(v^\mu)\big).$$

Since g(1)=0, we obtain

$$\begin{aligned} & \partial_t(v^\mu - 1) + \partial_x\Big(k_\varepsilon(x)(g(v^\mu) - g(1)) + f(v^\mu) - f(1)\Big) \\ - \quad & \mu\partial_{xx}(v^\mu - 1) = 0 \\ \Longrightarrow \quad & \partial_t(v^\mu - 1) + \partial_x\Big(k_\varepsilon(x)(g(v^\mu) - g(1)) + f(v^\mu) - f(1)\Big) = \partial_{xx}(v^\mu - 1). \end{aligned}$$

We can then derive $\int_{\mathbb{R}} (v^\mu - 1)^+\, dx \leq \int_{\mathbb{R}} (u_0 - 1)^+ dx$ following the same procedure that yields inequality (1.28). This yields $v^\mu \leq 1$ a.e. in $\mathbb{R}_+ \times \mathbb{R}$.

We have hence proved ii) of Lemma 1.3.

To prove the last point of Lemma 1.3, multiply (1.27) by v^μ, integrate over $[0,T]\times\mathbb{R}$. This yields

$$\int_0^T \frac{1}{2}\partial_t \int_{\mathbb{R}} v^{\mu 2}\,dxdt + \int_0^T \int_{\mathbb{R}} \partial_x\big(k_\varepsilon g(v^\mu) + f(v^\mu)\big)v^\mu dxdt \\ -\mu \int_0^T \int_{\mathbb{R}} v^\mu \partial_{xx} v^\mu\,dxdt = 0.$$

Since $v^\mu(\pm\infty) = 0$ and $\partial_x v^\mu(\pm\infty) = 0$

$$\int_0^T \frac{1}{2}\partial_t \int_{\mathbb{R}} v^{\mu 2}\,dxdt = \frac{1}{2}\int_{\mathbb{R}} v^{\mu 2}(T,x)dx - \frac{1}{2}\int_{\mathbb{R}} u_0^2(x)dx.$$

We have

$$\frac{1}{2}\int_{\mathbb{R}} v^{\mu 2}(T,x)dx - \frac{1}{2}\int_{\mathbb{R}} u_0^2(x)dx + \int_0^T \int_{\mathbb{R}} \partial_x\big(k_\varepsilon g(v^\mu) + f(v^\mu)\big)v^\mu dxdt \\ -\mu \int_0^T \int_{\mathbb{R}} \partial_x v^{\mu 2} dxdt = 0,$$

and therefore obtain

$$\int_0^T \int_{\mathbb{R}} \partial_x\big(k_\varepsilon g(v^\mu) + f(v^\mu)\big)v^\mu dx + \mu \int_0^T \int_{\mathbb{R}} \partial_x v^{\mu 2} \le \frac{1}{2}\int_{\mathbb{R}} u_0^2(x)dx < +\infty. \quad (1.30)$$

We now prove the existence of a constant $C > 0$ such that

$$I := \int_0^T \int_{\mathbb{R}} \partial_x\big(k_\varepsilon g(v^\mu) + f(v^\mu)\big)v^\mu dxdt \le TC|k_\varepsilon|_{BV(\mathbb{R})}. \quad (1.31)$$

Since $v^\mu(\pm\infty) = 0$, we have

$$I = -\int_0^T \int_{\mathbb{R}} \big(k_\varepsilon g(v^\mu) + f(v^\mu)\big)\partial_x v^\mu dxdt$$

and, setting $G(u) = \int_0^u g(s)ds$ and $F(u) = \int_0^u f(s)ds$,

$$I = -\int_0^T \int_{\mathbb{R}} k_\varepsilon \partial_x G(v^\mu)dxdt - \int_0^T \int_{\mathbb{R}} \partial_x F(v^\mu)dxdt.$$

On the one hand,

$$\int_0^T \int_{\mathbb{R}} \partial_x F(v^\mu)dxdt = 0$$

because $F(v^\mu(\pm\infty)) = F(0) = 0$.
On the other hand, since $G(0) = 0$, we have

$$\begin{aligned} I &= -\int_0^T \int_{\mathbb{R}} k_\varepsilon \partial_x G(v^\mu) dx dt \\ &= \int_0^T \int_{\mathbb{R}} k'_\varepsilon G(v^\mu) dx dt \\ &\leq T \max_{v\in[0,1]} |G(v)| |k_\varepsilon|_{BV(\mathbb{R})} \\ &\leq T \max_{v\in[0,1]} |G(v)| |k_L - k_R| \end{aligned}$$

which proves (1.31). Using inequalities (1.30) and (1.31), inequality (1.29) follows.

We now turn to the BV-estimate and, to this purpose, we first give a bound on the L^1-norm of v_t^μ. For $h > 0$, the function $w^\mu(t,x) = v^\mu(t+h,x)$ is a solution of equation (1.27) with initial condition $v^\mu(h,.)$. Using the result of comparison (1.28) with $s \mapsto |s|$ instead of $s \mapsto s^-$, we obtain

$$\int_{\mathbb{R}} |v^\mu(t+h,x) - v^\mu(t,x)| dx \leq \int_{\mathbb{R}} |v^\mu(h,x) - u_0(x)| dx, \quad \text{for every } t \geq 0.$$

Dividing this inequality by h and letting h tend to 0^+ yields

$$\int_{\mathbb{R}} |\partial_t v^\mu(t,x)| dx \leq \int_{\mathbb{R}} |\partial_t v^\mu(0,x)| dx, \quad \text{for every } t \geq 0.$$

We denote by $M_g = \sup_{u\in[0,1]} |g(u)|$, $M_g^1 = \sup_{u\in[0,1]} |g'(u)|$, $M_f^1 = \sup_{u\in[0,1]} |f'(u)|$.
Since, $v_t^\mu(0,x) = -k_\varepsilon(x) g(u_0(x)) - k_\varepsilon(x) g'(u_0(x)) u_0' - f'(u_0(x)) u_0' + \mu u_0''(x)$ and $|k_\varepsilon(x)|_{BV(\mathbb{R})} = \int_{\mathbb{R}} |k'_\varepsilon(x)| dx$, we obtain a bound on the L^1-norm

$$\begin{aligned} \int_{\mathbb{R}} |\partial_t v^\mu(t,x)| dx &\leq M_g |k_\varepsilon(x)|_{BV(\mathbb{R})} + (\max(k_L,k_R) M_g^1 + M_f^1) |u_0|_{BV(\mathbb{R})} \\ + \quad \mu \int_{\mathbb{R}} |u_0''(x)| dx. \end{aligned} \tag{1.32}$$

We can now prove estimate (1.26). Let $\kappa \in [0,1]$, multiplying equation (1.27) by $sgn(v^\mu - \kappa)$ yields

$$\partial_x (k_\varepsilon \Phi(v^\mu, \kappa) + \Psi(v^\mu, \kappa)) \leq S_1^\mu + S_2^\mu + S^\mu$$

in $\mathcal{D}'((0,T) \times \mathbb{R})$, with

$$S_1^\mu = -\partial_t |v^\mu - \kappa|, \quad S_2^\mu = k_\varepsilon' sgn(v^\mu - \kappa) g(\kappa) \quad \textit{and} \quad S^\mu = \mu \partial_{xx} |v^\mu - \kappa|.$$

We evaluate each distribution on a test function $\varphi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{R})$ such that $0 \leq \varphi \leq 1$. From L^1 estimate of $\partial_t v^\mu$ (1.32), we deduce that

$$< S_1^\mu, \varphi > \leq T\left(M_g|k_\varepsilon|_{BV(\mathbb{R})} + M|u_0|_{BV(\mathbb{R})} + \mu\int_{\mathbb{R}} |u_0^{''}(x)|dx \right) \tag{1.33}$$

where $M := \max(k_L, k_R)M_g^1 + M_f^1$. Moreover, we have

$$< S_2^\mu, \varphi > \leq \frac{T}{4}|k_\varepsilon|_{BV(\mathbb{R})}. \tag{1.34}$$

From energy estimate (1.29) and from the Cauchy-Schwartz inequality, we deduce that

$$< S^\mu, \varphi > \leq C_\varphi ||\partial_x\varphi||_{L^2(\mathbb{R})}\sqrt{\mu}, \tag{1.35}$$

where C_φ depends on the support of φ.

Now, it is known that $\lim_{\mu\to 0} v^\mu = u_\varepsilon$ in $L^1_{loc}((0,+\infty)\times\mathbb{R})$ [Kru70].
Therefore, S_1^μ, S_2^μ, S^μ converge in $\mathcal{D}'((0,T)\times\mathbb{R})$. From (1.35), we deduce $S^\mu \to 0$, so that $\partial_x(k_\varepsilon\Phi(u_\varepsilon,\kappa) + \Psi(u_\varepsilon,\kappa)) \leq S_1 + S_2$ holds with

$$\begin{aligned} < S_1, \varphi > &\leq T\left(M_g|k_\varepsilon|_{BV(\mathbb{R})} + M|u_0|_{BV(\mathbb{R})}\right), \\ < S_2, \varphi > &\leq \frac{T}{4}|k_\varepsilon|_{BV(\mathbb{R})}, \end{aligned}$$

for every $\varphi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{R})$ such that $0 \leq \varphi \leq 1$.
Hence for all φ in $\mathcal{C}_c^\infty((0,T)\times\mathbb{R})$,

$$\begin{aligned} & | < \partial_x\big(k_\varepsilon\phi(u_\varepsilon) + \Psi(u_\varepsilon)\big), \phi > | \\ \leq \; & T\left(4M_g|k_\varepsilon|_{BV(\mathbb{R})} + 2M|u_0|_{BV(\mathbb{R})}\right) \\ \leq \; & T\left(4M_g|k_L - k_R| + 2M|u_0|_{BV(\mathbb{R})}\right). \end{aligned} \tag{1.36}$$

Finally, as $u_0, k_\varepsilon \in BV(\mathbb{R})$ and $|k_\varepsilon|_{BV(\mathbb{R})} \leq |k_L - k_R|$, we know that $(u_\varepsilon)_t$ are measures uniformly bounded with respect to ϵ. Moreover as $g, f \in \mathcal{C}^1([0,1])$, we deduce that $(k_\varepsilon\Phi(u_\varepsilon,\kappa)+\Psi(u_\varepsilon,\kappa))_t$ are measures on $[0,T]\times\mathbb{R}$ uniformly bounded with respect to ϵ. Therefore, there exists C depending only on f, g, T, such as for all $\varphi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{R})$, $0 \leq \varphi \leq 1$,

$$| < \partial_t\big(k_\varepsilon\Phi(u_\varepsilon) + \Psi(u_\varepsilon)\big), \varphi > | \leq C|u_0|_{BV(\mathbb{R})}. \tag{1.37}$$

From (1.36) and (1.37), we see that $(k_\varepsilon\Phi(u_\varepsilon,\kappa) + \Psi(u_\varepsilon,\kappa)) \in BV((0,T)\times\mathbb{R})$ and

$$\begin{aligned} & |k_\varepsilon\Phi(u_\varepsilon,\kappa) + \Psi(u_\varepsilon,\kappa)|_{BV((0,T)\times\mathbb{R})} \\ \leq \; & T\left(4M_g|k_L - k_R| + 2M|u_0|_{BV(\mathbb{R})}\right) + C|u_0|_{BV(\mathbb{R})}. \end{aligned} \tag{1.38}$$

We thus obtain a uniform BV-estimate on the sequence
$(k_\varepsilon\Phi(u_\varepsilon,\kappa) + \Psi(u_\varepsilon,\kappa))_\varepsilon$ if $u_0 \in \mathcal{C}_c^\infty(\mathbb{R})$. The same result, for $u_0 \in BV(\mathbb{R})$ is obtained by a density argument.

1.4.2 Existence for u_0 in $BV(\mathbb{R})$

Theorem 1.2. *Let $u_0 \in BV(\mathbb{R})$ be such that $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Then there exists an unique entropy solution u to problem (1.1) in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$.*

Proof of Theorem 1.2:
Let η be a regular function such that $\eta(x) = \text{sgn}(x)$, if $|x| > 1$. Let $T > 0$.
Let

$$\begin{aligned} H^-(u) = & \int_0^1 k_L \partial_\sigma \eta(g'(\sigma))(\Phi(u,\sigma) - g(\sigma))\, d\sigma \\ + & \int_0^1 \partial_\sigma \eta(f'(\sigma))(\Psi(u,\sigma) - f(\sigma))\, d\sigma. \end{aligned} \tag{1.39}$$

With this function H^-, we build a Temple function F^- such that $(F^-(u_\varepsilon))_\epsilon$ will is bounded in $\in BV((0,T) \times \mathbb{R}_-)$. After one integration by parts, we obtain

$$\begin{aligned} H^-(u) = & \quad 2F^-(u) + k_L \eta(g'(1))g(u) + \eta(f'(1))f(u) \\ - & \quad k_L \eta(g'(0))g(u) + \eta(f'(0))f(u) \end{aligned}$$

where,

$$F^-(u) = 2\int_0^u k_L \eta(g'(\sigma))g'(\sigma) + \eta(f'(\sigma))f'(\sigma) d\sigma. \tag{1.40}$$

As g is genuinely non linear (Hypothesis H3) and f is non-decreasing, F^- is an invertible function. But, physically, we can suppose that f is strictly increasing and in such case function F^- is an invertible function too, even if g' cancel or not and then hypothesis (H3) is not necessary.
On the one hand, we have $k_L\eta(g'(1))g(u)+\eta(f'(1))f(u)$ and $k_L\eta(g'(1))g(u)+\eta(f'(1))f(u)$ in $BV(\mathbb{R}_+ \times \mathbb{R}_-^*)$ with (1.38).
On the other hand, we get $H^-(u_\varepsilon) \in BV((0,T) \times \mathbb{R}_-^*)$. Let $\varphi \in \mathcal{C}_c^\infty((0,T) \times \mathbb{R})$, $x < 0$. There exists $C > 0$ such that, for ε sufficiently small,

$$\begin{aligned} | < H^-(u_\varepsilon), \partial_t \varphi + \partial_x \varphi > | & \leq C(\int_0^1 |\partial_\sigma \eta(g'(\sigma))| + |\partial_\sigma \eta(f'(\sigma))| d\sigma) ||\varphi||_\infty \\ & < +\infty. \end{aligned}$$

Finally, $(F^-(u_\varepsilon))_\varepsilon$ is bounded in $BV((0,T) \times \mathbb{R}_-^*)$. By Helly's Theorem, there exists a subsequence of $(F^-(u_\varepsilon))_\varepsilon$ that converges in $L^1_{loc}((0,T) \times \mathbb{R}_-^*)$ and a.e. to $U^- := F^-(u^-)$ when ε tends to zero.

In the same way, we build F^+, with

$$F^+(u) = 2\int_0^u k_R \eta(g'(\sigma))g'(\sigma) + \eta(f'(\sigma))f'(\sigma) d\sigma, \tag{1.41}$$

such that and $(F^+(u_\varepsilon))_\varepsilon$ is bounded in $BV((0,T)\times\mathbb{R}_+^*)$. There exists a subsequence of $(F^+(u_\varepsilon))_\varepsilon$ that converges in $L^1_{loc}((0,T)\times\mathbb{R}_+^*)$ and a.e. to $U^+ := F^+(u^+)$.
Consequently, we define a function v in $L^\infty((0,T)\times\mathbb{R})$ by:

$$v := u^- \text{ if } x<0 \text{ and } v := u^+ \text{ if } x>0.$$

With F^- and F^+ invertible, we see that the sequence $(u_\varepsilon)_\varepsilon$ converges to v in $L^1_{loc}((0,T)\times\mathbb{R})$ and a.e. in $(0,T)\times\mathbb{R}$.
On the other hand, from as v^μ converges to u_ε a.e. as μ tends to 0, we deduce from Lemma 1.3 that the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $L^\infty((0,T)\times\mathbb{R})$. Hence there exists a subsequence of $(u_\varepsilon)_\varepsilon$ that converges to u for the weak star topology.

Finally, as $(u_\varepsilon)_\varepsilon$ converges to v a.e. in $(0,T)\times\mathbb{R}$, we can claim that $u=v$ a.e.
We conclude that $(u_\varepsilon)_\varepsilon$ converges to u a.e. on $(0,T)\times\mathbb{R}$ and $u\in L^\infty((0,T)\times\mathbb{R})$. Moreover, as $0\le u_\varepsilon\le 1$ a.e., we have the following inequality

$$0\le u\le 1 \quad \text{a.e. in } (0,T)\times\mathbb{R}.$$

We now show that u is the entropy solution of problem (1.1). Let $\kappa\in[0,1]$ and φ be a non-negative function of $\mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R})$. Let T be such that $\varphi(t,x)=0$ for every $(t,x)\in[T,+\infty)\times\mathbb{R}$. For every $\varepsilon>0$, the function u_ε satisfies the following entropy inequality :

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} |u_\varepsilon(t,x)-\kappa|\,\partial_t\varphi(t,x)\\
+\;&\int_0^\infty\int_{\mathbb{R}} (k_\varepsilon(x)\,\Phi(u_\varepsilon(t,x),\kappa)+\Psi(u_\varepsilon(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt\\
+\;&\int_{\mathbb{R}} |u_0(x)-\kappa|\,\varphi(0,x)\,dx\\
-\;&\int_0^\infty\int_{\mathbb{R}} k'_\varepsilon(x)\mathrm{sgn}(u_\varepsilon(t,x)-\kappa)g(\kappa)\,\varphi(t,x)\,dx\,dt\ge 0,
\end{aligned}\tag{1.42}$$

As u_ε converges to u a.e. on $[0,T]\times\mathbb{R}$, u_ε converges to u in $L^1_{loc}([0,T]\times\mathbb{R})$, so the first term in inequality (1.42) converges to the first term in inequality (1.11) as ε tends to 0. The estimate $|\mathrm{sgn}(u_\varepsilon(t,x)-\kappa)|\le 1$ implies that $|\int_0^\infty\int_{\mathbb{R}} k'_\varepsilon(x)\mathrm{sgn}(u_\varepsilon(t,x)-\kappa)g(\kappa)\,\varphi(t,x)\,dx\,dt|\le I_\varepsilon$, where
$I_\varepsilon=\int_0^\infty\int_{\mathbb{R}}|k'_\varepsilon(x)|\,g(\kappa)\,\varphi(t,x)\,dx\,dt$. Hence, we obtain

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} |u_\varepsilon(t,x)-\kappa|\,\partial_t\varphi(t,x)\\
+\;&\int_0^\infty\int_{\mathbb{R}} (k_\varepsilon(x)\,\Phi(u_\varepsilon(t,x),\kappa)+\Psi(u_\varepsilon(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt\\
+\;&\int_{\mathbb{R}} |u_0(x)-\kappa|\,\varphi(0,x)\,dx\;-\;I_\varepsilon\ge 0,
\end{aligned}\tag{1.43}$$

To conclude, we use the fact that the monotony of the function k_ε is set by the sign of $k_L - k_R$. Several integrations by parts yield

$$\begin{aligned} I_\varepsilon &= sgn(k_R - k_L) \int_0^\infty \int_{\mathbb{R}} k'_\varepsilon(x)\, g(\kappa)\, \varphi(t,x)\, dx\, dt, \\ &= -sgn(k_R - k_L) \int_0^\infty \int_{\mathbb{R}} k_\varepsilon(x)\, g(\kappa)\, \partial_x \varphi(t,x)\, dx\, dt, \end{aligned}$$

and we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I_\varepsilon &= -sgn(k_R - k_L) \int_0^\infty \int_{\mathbb{R}} k(x)\, g(\kappa)\, \partial_x \varphi(t,x)\, dx\, dt \\ &= |k_R - k_L| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt. \end{aligned} \tag{1.44}$$

Remark 1.4. *This last results justifies the definition of an entropy solution we gave in Definition 1.1. The entropy inequality is obtained as the limit of the inequality entropy in the regular case.*

If ε tends to 0 in inequality (1.43), with (1.44), we obtain inequality (1.11). Eventually, we see that u is an entropy solution.

Remark 1.5. *In the proof we saw that if the initial condition u_0 is in $BV(\mathbb{R})$, then the sequence $(u_\varepsilon)_\varepsilon$ is compact in $L^1_{loc}([0,T]\times\mathbb{R})$ for every $T > 0$ and has at least one limit point value, which is an entropy solution of problem (1.1). It is the uniqueness of an entropy solution that ensures the whole sequence $(u_\varepsilon)_\varepsilon$ converges to u.*

1.4.3 Existence for u_0 in $L^\infty(\mathbb{R})$

Theorem 1.3. *Suppose that $u_0 \in L^\infty(\mathbb{R})$ such that $0 \le u_0 \le 1$ a.e. in $\mathbb{R}$. Then there exists an unique entropy solution u of problem (1.1) in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$.*

Proof of Theorem 1.3. Let $(\rho_n)_{n\in\mathbb{N}}$ be a classical sequence of mollifiers, such that is $\rho_n = n\rho(n\cdot)$ with $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$ such that ρ is non-negative, $\int_{\mathbb{R}} \rho(x)dx = 1$ and $\text{supp}(\rho) \subset [-1,1]$. Define the sequence

$$u_0^n = \rho_n * (\chi_{(-n,n)} u_0).$$

We have the classical result:

Lemma 1.4. *For every $n \in \mathbb{N}$, $u_0^n \in L^\infty(\mathbb{R},[0,1]) \cap BV(\mathbb{R})$ and $\lim_{n\to+\infty} u_0^n = u_0$ in $L^1_{loc}(\mathbb{R})$.*

Proof of Lemma 1.4: Let $n \in \mathbb{N}^*$,

$$\begin{aligned} u_0^n(x) &= \int_{\mathbb{R}} \rho_n(x-y)\chi_{[-n,n]} u_0(y) dy \\ &\le \int_{-n}^n \rho_n(x-y) dy, \quad \text{because } 0 \le u_0 \le 1 \text{ a.e.} \\ &\le \int_{\mathbb{R}} \rho_n(x-y) dy = 1. \end{aligned}$$

then $0 \leq u_0^\alpha \leq 1$. Showing $u_0^n \in BV(\mathbb{R})$:

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u_0^n(x)| dx &\leq \int_{-1/\alpha}^{1/\alpha} |u_0(y)| \int_{\mathbb{R}} n^2 |\rho'(\frac{x-y}{\alpha})| dx dy \\ &\leq C\, n ||u_0||_{L^\infty} \times 2n < +\infty\end{aligned}$$

As $\partial_x u_0^n \in L^1(\mathbb{R})$, $u_0^n \in BV(\mathbb{R})$.
For the last point, let K a compact of $\mathbb{R}$, by using the theorem of Fubini-Tonelli, this yields:

$$\begin{aligned}\int_K |u_0^n(x) - u_0(x)| dx &\leq \int_{-n}^{n} \rho_n(y) \int_K |u_0(x-y) - u_0(x)| dx dy \\ &\leq \int_{-1}^{1} \rho(y) \int_K |u_0(x - y/n) - u_0(x)| dx dy \\ &\longrightarrow_{n\to+\infty} 0.\end{aligned}$$

Therefore, if u^n denotes the entropy solution corresponding to the initial data u_0^n, then the sequence $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1_{loc}((0,T)\times\mathbb{R})$ by the estimate (1.12)

$$\int_0^T \int_{-R}^{R} |u^n(t,x) - u^m(t,x)|\, dx\, dt \leq T \int_{-R-MT}^{R+MT} |u_0^n(x) - u_0^m(x)|\, dx,$$

for every $R, T > 0$. Consequently, the sequence $(u^n)_{n\in\mathbb{N}}$ is convergent. Denote by u its limit in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$. The function u is an entropy solution of problem (1.1). This concludes the proof.

Bibliography

[GMT96] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Mathématiques et Applications, **22**, Springer-Verlag, Berlin, (1996), with a preface by Charles-Michel Marle.

[GL03] P. Goatin and P. Lefloch. *The Riemann problem for a class of resonant hyperbolic systems of balance laws*, preprint

[IT92] E. Isaacson and B. Temple. *Nonlinear resonance in systems of conservation laws*, SIAM J. Appl. Math., **52**, (1992), no. 5, 1260-1278.

[IT86] E. Isaacson and B. Temple. *Analysis of a singular hyperbolic system of conservation laws*, J. Differential Equations, **65**, (1986), no. 2, 250-268.

[KRT02] K.H. Karlsen, N.H. Risebro and J.D Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron J. Differential Equations, (2002), **93**, 23 pp (electronic).

[KRT03] K.H. Karlsen, N.H. Risebro and J.D Towers. *L1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, www.math.ntnu.no/conservation/2003/022.html, (2003).

[KR95] K.H. Karlsen and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness ans asymptotic behaviour*, Comm. Partial Differential Equations, **20**, (1995),1959-199.

[Kru70] S.N. Kruzkov. *First order quasilinear equations with several independent variables* Mat. Sb. (N.S.),**81**, (1970), 228-255.

[LTW95a] L.W. Lin, J.B. Temple and J.H. Wang. *A comparison of convergence rates for Godunov's method and Glimm's method in resonant nonlinear systems of conservation laws*, SIAM J. Numer. Anal., **32**, (1995), no. 3, 824-840.

[LTW95b] L.W. Lin, J.B. Temple and J.H. Wang. *Suppression of oscillations in Godunov's method for a resonnant non-strictly hyperbolic system*, SIAM J. Numer. Anal., **32**, (1995), no. 3, 841-864.

[Ole56] O.A. Oleinik. *The problem of Cauchy for non-linear differential equations of the first order with discontinuous initial conditions*, Trudy Moskov. Mat. Obsc., *5*, (1956), 433-454.

[Ole57] O.A. Oleinik. *Discontinuous solutions of nonlinear differential equations*, Usephi Mat. Nauk (N.S.), **12**, (1957), 3-73.

[SV03] N. Seguin and J. Vovelle. *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221-257.

[Tem82] B. Temple. *Global solution of the Cauchy problem for a class of* 2×2 *non-strictliy hyperbolic conservation laws.* Adv. in Appl. Math., **3**, (1982), no. 3.

[Tow00] J.D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681-698.

[Tow01] J.D. Towers. *A difference scheme for conservation laws with a discontinuous flux: the non convex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197-1218.

[Vas01] A. Vasseur. *Strong traces for weak solutions to multidimensional conservation laws*, Arch. of Rat. Mech. Anal., **160**, (2001), no; 3, 181-193.

[Vol67] A.I. Volpert. *Spaces BV and quasilinear equations*, Mat. sb. (N.S.), **73**, (1967), 255-302.

Chapter 2

Analyse d'une loi de conservation à flux discontinu

Ce travail a été fait en collaboration avec Julien Vovelle.

2.1 Introduction

In this paper, we investigate the issue of existence, uniqueness and entropy conditions for hyperbolic conservation laws with discontinuous coefficients. We consider the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \\ u(0,x) = u_0(x)\,, \end{cases} \tag{2.1}$$

with initial value $u_0 \in L^\infty(\mathbb{R};[0,1])$. The functions f, g and k are supposed to satisfy the following hypotheses:

(H1) $g \in C^1([0,1])$ is non-negative and $g(0) = g(1) = 0$,
(H2) $f \in C^1([0,1])$ and $f(0) = 0$,
(H3) k is the discontinuous function defined by

$$k(x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x > 0 \end{cases} \quad \text{with } k_L, k_R > 0 \text{ and } k_L \neq k_R\,.$$

We introduce the time - space domain $Q := (0,+\infty) \times \mathbb{R}$ and the curve (a line here) of discontinuity of the function k in the time - space domain, $\Sigma := (0,+\infty) \times \{0\}$.

The particular shape of the functions f, g and k described through the hypotheses (H1), (H2), (H3) is given by a model for two-phase flow in porous media with distinct permeabilities. We refer to [GMT96] for the description of the model and to [Bac04] for the adaptation to our notation. Let us just claim here that, in this context, the hypotheses on f, g and k are natural. We also remark that no hypothesis of convexity or genuine non-linearity on g is assumed, which is a new point in comparison with

all the preceding works on the subject (see in particular [KR95, Kaa99, KRT02b, KRT02b, KRT03, Tow01, SV03, KT04, Bac04]).
To justify further the hypotheses (H1), (H2), (H3), notice that we take care to study the equation $\partial_t u + \partial_x(k(x)g(u) + f(u)) = 0$ on a physical background. This is a way to ensure that Cauchy problem (2.1) makes sense. Indeed, this is not always the case. Consider for example the problem $\partial_t u + \partial_x(k(x)u) = 0$ with $k(x) = -\text{sgn}(x)$: the computation of the solution of the Cauchy problem along the characteristic lines shows that this one cannot be specified in the domain $\{t > 0\,, |x| < t\}$. Other physical models lead to the study of problem (2.1) (sedimentation process, traffic flow... [BKRT04]); the analysis of problem (2.1) has also to be related to the study of resonant hyperbolic systems of conservation laws (systems of the form $\partial_t u + \partial_x F(u) = 0$, in $1D$, for which the matrix $DF(u)$ has real eigenvalues whose multiplicity happen to vary at some state u^*). Indeed, adding the equation $k_t = 0$ to the equation $\partial_t u + \partial_x(kg(u) + f(u)) = 0$, one gets a 2×2 systems with resonance at a state u^* such that $kg'(u^*) + f'(u^*) = 0$.

Since the beginning of the 80's, problem (2.1) (and, more generally, the Cauchy problem associated to the conservation law $\partial_t u + \partial_x(A(x,u)) = 0$ with a flux function A possibly discontinuous with respect to the x-variable) has been the subject of various works of analysis (definition of solution, existence, uniqueness, properties of solutions, convergence of approximations...). We refer to the introductory part of [KRT02b, KRT02b, Bac04] for description of this latter, and also to the introductory part of the recent paper [AP04] as regards the issue of uniqueness of solutions in particular.
Indeed, the question of uniqueness of solutions to (2.1) may have been not completely settled yet, and the purpose of this present work is the investigation of the question of uniqueness (stability) for problem (2.1). Indeed, we believe that a result of uniqueness should satisfy the two following points:

- (R1) no particular structural hypothesis on the data or on the solution is required;
- (R2) the proof is the most algebraic as possible.

In item (R1), by reference to some 'particular structural hypothesis', we have in mind hypotheses as : "g genuinely non-linear", or "the solution admits strong traces aside Σ", additional hypotheses that were present in all the preceding works dealing with uniqueness for problem (2.1) (see for example [KRT03, KT04, Bac04]). In item (R2), we have in mind the proof of the L^1-contraction property for problem (2.1) in the case where k is constant, proof of Kruzhkov by the technique of the doubling of variables, which is completely algebraic. It is this algebraic character which makes this proof so adaptable to the proof of error estimates (Kuznetsov, [Kuz76]). Our investigations of a proof of uniqueness respecting the two preceding requirements has led us to introduce various equivalent formulation of solution. In section 2.2, we recall the definition of entropy solution to problem (2.1) given by Towers [Tow00]. In section 2.3.2, we explain why this notion is not exactly appropriate for the proof of a uniqueness result (very shortly : "the constants are not solutions of (2.1)") and, inspired by the works of Portilheiro [Por03a, Por03b] and Perthame, Souganidis [PS03], we introduce the equivalent notion of so-called "χ-compared solution". We explain why this notion is also not well adapted to our goal and, in section 2.3.3, we introduce the equivalent notion of kinetic solution. Kinetic solutions for scalar conservation laws (problem (2.1) with k constant) have been introduced by Lions, Perthame, Tadmor [LPT94] and a proof of uniqueness of entropy solutions of such problems which rests on the use of this tool has been given by Perthame in [Per98]. We adapt this notion to the case where k is discontinuous and prove the following theorem (entropy solutions to Problem 2.1 are defined in Definition 2.1):

Theorem 2.1. *Under Hypotheses (H1)-(H2)-(H3), L^1 comparison holds for the Cauchy problem (2.1): if u and $v \in L^\infty(Q)$ are two entropy solutions of problem (2.1), associated to the initial data u_0 and $v_0 \in L^\infty(\mathbb{R};[0,1])$ respectively and $R,T>0$, then*

$$\int_0^T \int_{-R}^{R} |u(t,x)-v(t,x)|\,dx\,dt \le T \int_{-R-CT}^{R+CT} |u_0(x)-v_0(x)|\,dx$$

with $C := \max\{k_L,k_R\}\ \max\{|g'(u)|; 0\le u\le 1\} + \max\{|f'(u)|; 0 \le u \le 1\}$.

Therefore, we manage to satisfy first point (R1) of the two requirements concerning the proof of uniqueness. However, it turns out that, despite our different attempts, second point (R2) is not completely satisfied (see Remark 2.5 for discussion of this aspect).

Our attempts to develop a proof of uniqueness for problem (2.1) which satisfies the two preceding requirement (R1) and (R2) were also motivated by our will to develop the tools of nonlinear analysis that are the notions of "generalized weak entropy solutions" (introduced by DiPerna with the measure-valued solutions [DiP85], extensively used by Eymard, Gallouët, Herbin for the study of the Finite Volume Method under the form of entropy process solution [EGH00]). These tools are essential in the analysis of several approximations to nonlinear first-order hyperbolic problems. In subsection 2.2.2, we explain how these tools are involved in the proof of the convergence of approximations to nonlinear scalar conservation laws, how they allow to compensate the possible weak compactness estimates on the approximate solutions by a robust result of comparison for entropy solutions (from which follows the importance of the proof of uniqueness for entropy solutions) and illustrate our work by proving the existence of weak entropy solutions *via* the proof of the convergence of the solution of problem (2.1) with a regularized coefficient k (Theorem 2.4). Notice that, as a by-product of this technique for the analysis of the convergence of approximations, no use of "Temple functions" or singular mapping technique is required here. Simple (and natural) L^∞ estimates are sufficient to prove the convergence. Let us also insist on the fact that the tools of nonlinear analysis that we will discuss in this work in subsection 2.2.2 are *not* related to the technique of compensated compactness, in particular no superfluous (with regard to the question of the convergence of approximation) hypothesis as "g completely nonlinear" will be required in the proof of the convergence of approximation.

2.2 Entropy solution - Entropy process solution

Generically, the discontinuity of k enforces the instantaneous apparition of discontinuities in the solution of problem (2.1) (whatever the regularity of the initial datum may be), therefore weak solutions have to be considered, with the additional property to satisfy entropy inequality to ensure uniqueness (selection of shocks, or physical discontinuities, among possible discontinuities). The definition of entropy solutions for problem (2.1) has been given by Towers [Tow00].

2.2.1 Entropy solution

Definition 2.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$.*

1. *A function $u \in L^\infty(Q)$ is said to be an entropy subsolution (resp. entropy supersolution) of problem (2.1) if it satisfies the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$
\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (u(t,x) - \kappa)^\pm \, \partial_t \varphi(t,x) \, dt \, dx \\
+ \quad & \int_0^\infty \int_{\mathbb{R}} \big(k(x) \, \Phi^\pm(u(t,x),\kappa) + \Psi^\pm(u(t,x),\kappa) \big) \, \partial_x \varphi(t,x) \, dx \, dt \\
+ \quad & \int_{\mathbb{R}} (u_0(x) - \kappa)^\pm \, \varphi(0,x) \, dx \\
+ \quad & (k_L - k_R)^\pm \int_0^\infty g(\kappa) \, \varphi(t,0) \, dt \geq 0,
\end{aligned} \tag{2.2}
$$

where respectively $\Phi^\pm$ and $\Psi^\pm$ denote the entropy flux associated with the Kruzhkov entropy,

$$
\begin{aligned}
\Phi^\pm(u,\kappa) &= \mathrm{sgn}_\pm(u-\kappa)(g(u) - g(\kappa)), \\
\Psi^\pm(u,\kappa) &= \mathrm{sgn}_\pm(u-\kappa)(f(u) - f(\kappa)).
\end{aligned}
$$

2. *The function $u \in L^\infty(Q)$ is an* entropy solution *of the problem (2.1) if it is both an entropy subsolution and an entropy supersolution.*

Remark 2.1. *Let $u \in L^\infty(Q)$ be an entropy subsolution. By choosing $\kappa = 1$ in (2.2), it is easy to see that $u \leq 1$ a.e. Similarly, if $u \in L^\infty(Q)$ is an entropy supersolution then $u \geq 0$ a.e. Therefore, if $u \in L^\infty(Q)$ is an entropy solution then $0 \leq u \leq 1$ a.e. (which is expected, owing to the physical origin of the equation, in which the unknown u is typically a saturation) and $g(u), f(u)$ and also $\Phi(u,\kappa), \Psi(u,\kappa)$ are well defined. To let the definition of entropy subsolution and entropy supersolution make sense, we implicitly continue the functions f and g on $\mathbb{R}$, for example by setting $g = 0$ on $\mathbb{R} \setminus [0,1]$, $f = 0$ on $(-\infty, 0)$, $f = f(1)$ on $(1, +\infty)$. Also notice that an entropy solution of (2.1) is a weak solution of (2.1), ie: for all non-negative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$*

$$
\begin{aligned}
& \int_0^{+\infty} \int_{\mathbb{R}} u(t,x) \partial_t \varphi(t,x) + \big(k(x) g(u(t,x)) + f(u(t,x)) \big) \partial_x \varphi(t,x) \, dt \, dx \\
+ \quad & \int_{\mathbb{R}} u_0(x) \varphi(0,x) \, dx = 0.
\end{aligned}
$$

This equality is a consequence of the two inequalities obtained, first by developing the entropy inequality written for subsolution with $\kappa = 0$*, second by developing the entropy inequality written for supersolution written with* $\kappa = 1$ *on the basis of the bound* $0 \leq u \leq 1$ *a.e. Similarly, if a function* $u \in L^\infty(Q)$ *satisfies* $0 \leq u \leq 1$ *a.e. on the one hand and the entropy inequalities with classical Kruzhkov entropies: for all* $\kappa \in [0,1]$*, for all non-negative* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$*,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\, \partial_t \varphi(t,x)\, dt\, dx \\ + \quad &\int_0^\infty \int_{\mathbb{R}} \big(k(x)\, \Phi(u(t,x),\kappa) + \Psi(u(t,x),\kappa)\big)\, \partial_x \varphi(t,x)\, dx\, dt \\ + \quad &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \; + \; |k_R - k_L| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq 0, \end{aligned} \tag{2.3}$$

on the other hand, then u *is a weak solution of problem (2.1) and therefore satisfies (2.2) (indeed* $u^+ = (|u| + u)/2$ *and* $u^- = (|u| - u)/2$*). Conversely, by adding the two inequalities of (2.2), we see that* $u \in L^\infty(Q)$ *is an entropy solution to problem (2.1) if, and only if, it satisfies* $0 \leq u \leq 1$ *a.e. and (2.3).*

It has been proved in different works ([Tow00, KR01, KRT02b, SV03, Bac04]) that this notion of entropy solution is the accurate notion of solution for problem (2.1) (existence, uniqueness, consistence with approximations... holds).

2.2.2 Approximation of problem (2.1)

Consider the problem of the approximation of problem (2.1) (by the finite volume method [Tow00, Tow01, SV03, KT04], by viscous regularization, or by regularization of the coefficient k – this last situation will be considered here). More precisely, we suppose that we are in the following position : (u^ε) is a sequence of solutions of approximate problems to problem (2.1). Specifically, we suppose that the approximate problem under consideration is consistent enough with problem (2.1) to ensure, first, the existence of approximate entropy inequalities: for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (u^\varepsilon(t,x) - \kappa)^\pm\, \partial_t \varphi(t,x)\, dt\, dx \\ + \quad &\int_0^\infty \int_{\mathbb{R}} (k(x)\, \Phi^\pm(u^\varepsilon(t,x),\kappa) + \Psi^\pm(u^\varepsilon(t,x),\kappa))\, \partial_x \varphi(t,x)\, dx\, dt \\ + \quad &\int_{\mathbb{R}} (u_0(x) - \kappa)^\pm\, \varphi(0,x)\, dx \\ + \quad &(k_L - k_R)^\pm \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq -\eta_\varepsilon(\varphi), \end{aligned} \tag{2.4}$$

where $\lim\limits_{\varepsilon \to 0} \eta_\varepsilon(\varphi) = 0$, and, second, the respect of the natural bounds for the solution of (2.1):

$$0 \leq u^\varepsilon \leq 1 \quad \text{a.e.} \tag{2.5}$$

To prove the convergence of (u^ε) to the entropy solution u of problem (2.1), a possible approach consists in i) proving the strong (in $L^1_{\rm loc}$ for example) convergence of (u^ε) to a function $\hat{u}$, ii) passing to the limit in (2.4) to prove that $\hat{u}$ is an entropy solution, iii) using the uniqueness of entropy solution to show $\hat{u} = u$.
Notice that, in step ii), one has to pass to the limit in the non-linear terms $\Phi^\pm(u^\varepsilon, \kappa)$ and $\Psi^\pm(u^\varepsilon, \kappa)$, which seemingly requires a result of *strong* convergence in step i). Unfortunately, such a result of strong convergence is itself deduced from a result of strong compactness, which requires in some way or other, a priori estimates on the derivatives of the solution u^ε. These estimates are particularly difficult to obtain for approximate solutions to problem (2.1) (in fact, it is not possible to get such BV estimates on u^ε: one proves BV estimates on $G(u^\varepsilon)$, G being an accurate invertible function : a Temple function (also called "singular mapping") [Tow01, Bac04]). A possible way to bypass these difficulties is to use the method of the compensated compactness. This has been done in [KRT02b, KT04]. By the method of compensated compactness, one can show a regularizing effect of conservation laws, since it yields a result of strong convergence of the solution with an hypothesis of weak convergence of the data (see also the application of kinetic formulation of conservation laws by Perthame on that point [Per02]). However this requires an hypothesis of complete nonlinearity of the flux function which is superfluous when the mere question of the convergence of approximations of the problem is analysed. Let us detail our approach to this question:
A second approach to the proof of the convergence of (u^ε) to u is i') use the simple estimate (2.5) to deduce the convergence of u^ε in an very weak sense (to be precised) to a function $\hat{u}$, ii') pass to the limit in (2.4) to prove that $\hat{u}$ is a kind of entropy solution to problem (2.1), iii') use a reinforced principle of uniqueness for problem (2.1) to show that $\hat{u} = u$ (u entropy solution of problem (2.1)) and that the convergence of u^ε is strong.
Let us be more specific about points i'), ii') and iii'). The weak convergence of u^ε which is alluded to in point i) is the so-called nonlinear weak-$*$ convergence:

Definition 2.2. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$, $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges towards u in the "nonlinear weak-$\star$ sense" if*

$$\int_\Omega g(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega g(u(x,\alpha))\psi(x)dxd\alpha, \ \ \textit{as } n \to +\infty \qquad \forall \psi \in L^1(\Omega),\ \forall g \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{2.6}$$

Otherwise speaking, the sequence (u_n) converges to $u \in L^\infty(\Omega \times (0,1))$ in the nonlinear weak-$\star$ sense if, for every $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$, the nonlinear expression $g(u_n)$ converges in $L^\infty(\Omega)$ weak-$*$ to a limit which has the structure $\displaystyle\int_0^1 g(u(\cdot,\alpha))d\alpha$. The fact is, that any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$*$ sense:

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence converging in the nonlinear weak-$\star$ sense.*

This result is established in [EGH00]. It can be proved via the introduction of the concept of measured-valued solutions, which makes reference to the original work of DiPerna [DiP85].

Remark 2.2. *If the nonlinear weak-$*$ limit u of a bounded sequence (u_n) of $L^\infty(\Omega)$ (Ω open bounded subset of $\mathbb{R}^N$) happen to not depend on the variable α, the convergence is strong : indeed, the choice of the nonlinearity $g(u) = u^2$ shows that $||u_n||_2$ converges to $||u||_2$ and therefore that (u_n) converges to u in the Hilbert space $L^2(\Omega)$. The strong convergence of (u_n) to u in $L^p(\Omega)$ $(1 \leq p < +\infty)$ then follows from the L^∞ bound on (u^n).*

In view of Theorem 2.2 and 2.5, we see that the function $\hat{u}$ considered in i') and ii') is what we call an entropy process solution, according to the following definition:

Definition 2.3. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$. Let $u \in L^\infty(Q \times (0,1))$.*

1. *The function u is a* weak entropy process subsolution *(resp.* weak entropy process supersolution*) of problem (2.1) if for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\varphi \geq 0$,*

$$\begin{aligned}
&\int_0^1 \int_Q (u(t,x,\lambda) - \kappa)^\pm \partial_t \varphi(t,x) dt dx d\lambda \\
+\quad &\int_0^1 \int_Q \left[k(x)\Phi^\pm(u(t,x,\lambda),\kappa) + \Psi^\pm(u(t,x,\lambda),\kappa)\right] \partial_x \varphi(t,x) dt dx d\lambda \\
+\quad &\int_{\mathbb{R}} (u_0(x) - \kappa)^\pm \varphi(0,x) dx \\
+\quad &(k_L - k_R)^\pm \int_0^{+\infty} g(\kappa)\varphi(t,0) dt \geq 0. \qquad (2.7)
\end{aligned}$$

2. *The function u is a* weak entropy process solution *of (2.1) if it is both a weak entropy process subsolution and a weak entropy process supersolution.*

A remark analogous to remark 2.1 applies to this definition of entropy process solution. In particular, any entropy process solution satisfies $0 \leq u \leq 1$ a.e.

In point iii'), we speak of a reinforced principle of uniqueness insofar as we have in mind a result of the kind: if u and $v \in L^\infty(Q \times (0,1))$ are entropy process solutions of problem (2.1), then $u(t,x,\lambda) = v(t,x,\zeta)$ for a.e. $(t,x,\lambda,\zeta) \in Q \times (0,1) \times (0,1)$. Integrating this last equality with respect to $\zeta \in (0,1)$ (resp. $\lambda \in (0,1)$) shows that u (resp. v) actually does not depend on the additional variable λ (resp. ζ) and therefore is a real entropy solution to problem (2.1). This result also yields the uniqueness of entropy solutions (for, if u and $v \in L^\infty(Q)$ are entropy solutions then the functions

$(t,x,\lambda) \mapsto u(t,x)$ and $(t,x,\zeta) \mapsto v(t,x)$ are entropy process solution). Consequently, by this "reinforced principle of uniqueness", the limit of (a subsequence of) u^ε, which is known to be an entropy process solution turns out to be an entropy solution, and in fact *the* entropy process solution of problem (2.1). By remark 2.2, the convergence is strong in $L^p_{\mathrm{loc}}(Q)$ for any $1 \le p < +\infty$ and, by uniqueness of the limit, the whole sequence u^ε converges.

The advantages of this approach to the proof of the convergence of approximations is that it relaxes the need for a priori estimates on the approximate solutions and focus the difficulties on the comparison of solution (or comparison of 'generalized solutions'). This explains in part why we insist on the proof of uniqueness for entropy (process) solution in this present work. Before coming to this very proof of uniqueness, let us state our main result and, then, an application of the method previously described:

Theorem 2.3 (Comparison). *Assume hypotheses (H1), (H2), (H3). Let u (resp. $v \in L^\infty(Q \times (0,1))$) be an entropy process subsolution (resp. entropy process supersolution) of problem* (2.1)*, associated to the initial conditions $u_0 \in L^\infty(\mathbb{R};[0,1])$ (resp. $v_0 \in L^\infty(\mathbb{R};[0,1])$). Then, given $R, T > 0$, one has*

$$\int_0^1\int_0^1\int_0^T\int_{-R}^{R}(u(t,x,\lambda)-v(t,x,\zeta))^+ dxdtd\lambda d\zeta \le \\ T\int_{-R-CT}^{R+CT}(u_0(x)-v_0(x))^+dx, \tag{2.8}$$

where $C := \max\{k_R, k_L\}\mathrm{Lip}(g) + \mathrm{Lip}(f)$.

In particular, if $u_0 = v_0$, we obtain $u(t,x,\lambda) = v(t,x,\zeta)$ for a.e. $(t,x,\lambda,\zeta) \in Q \times (0,1) \times (0,1)$ as desired. Note also that Theorem 2.1 is an easy consequence of Theorem 3.6.

Consider now the approximation of problem (2.1) by regularization of the coefficient k:

$$\begin{cases} \partial_t u + \partial_x\big(k_\varepsilon(x)g(u^\varepsilon) + f(u^\varepsilon)\big) = 0 & (t,x) \in Q \\ u^\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}\,, \end{cases} \tag{2.9}$$

where $(k_\varepsilon)_\varepsilon$ is a sequence of regular functions converging to the function k such that: $\forall\varepsilon > 0$, the function k_ε is regular, monotone non-decreasing or non-increasing, according to the sign of $k_R - k_L$ and satisfies

$$\begin{cases} k_\varepsilon(x) = k_L & \text{if } x \le -\varepsilon, \\ k_\varepsilon(x) = k_R & \text{if } x \ge \varepsilon. \end{cases}$$

Results of Kruzhkov [Kru70] ensure that there exists a unique entropy solution $u^\varepsilon \in$

$L^\infty(Q)$ to problem (2.9), which, besides, satisfies the following entropy inequalities:

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (u^\varepsilon(t,x)-\kappa)^\pm\, \partial_t\varphi_t(t,x)\,dt\,dx \\
+ & \int_0^\infty \int_{\mathbb{R}} (k(x)\,\Phi^\pm(u^\varepsilon(t,x),\kappa) + \Psi^\pm(u^\varepsilon(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt \\
+ & \int_{\mathbb{R}} (u_0(x)-\kappa)^\pm\,\varphi(0,x)\,dx \\
- & \int_0^\infty \int_{\mathbb{R}} k_\varepsilon'(x)\mathrm{sgn}_\pm(u^\varepsilon-\kappa)g(\kappa)\varphi\,dx\,dt \geq 0.
\end{aligned} \tag{2.10}$$

From this entropy inequalities, the choice $\kappa = 0$ or 1 and an appropriate choice of test-function, follows (2.5) for u^ε.

- In fact, by choosing $\kappa = 0$ in (2.10) (with the semi-entropies $u \mapsto (u-k)^-$). Since $(u_0-\kappa)^- = 0$ a.e. and $g(0) = 0$, we have

$$\int_{\mathbb{R}}\int_0^\infty (u^\varepsilon)^-\partial_t\varphi + (k(x)\Phi^-(u^\varepsilon,0) + \Psi^-(u^\varepsilon,0))\partial_x\varphi\,dx\,dt \geq 0\,. \tag{2.11}$$

  Let $R, T > 0$, let $r \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ be such that: r is non-increasing, $r \equiv 1$ on $[0, R+ŁT]$, $r \equiv 0$ on $[R+ŁT+1, +\infty)$ with $Ł = \max\{1, k_L, k_R\}\mathrm{Lip}(g)+\mathrm{Lip}(f)$. The choice $\varphi(x,t) = \dfrac{T-t}{T}\chi_{(0,T)}(t)r(|x|+Łt)$ in (2.11) gives

$$\begin{aligned}
-\frac{1}{T} & \int_{\mathbb{R}}\int_0^T (u^\varepsilon)^- r(|x|+\omega t)\,dx\,dt \\
+ & \int_{\mathbb{R}}\int_0^T \frac{T-t}{T} r'(|x|+Łt) \\
& \Big(Ł(u^\varepsilon)^- + \mathrm{sgn}(x)\big(K(x)\Phi^-(u^\varepsilon,0) + \Psi^-(u^\varepsilon,0)\big)\Big) \geq 0\,.
\end{aligned}$$

  Since $|\Phi^-(u^\varepsilon,0)| \leq \mathrm{Lip}(g)(u^\varepsilon)^-$, $|\Psi^-(u^\varepsilon,0)| \leq \mathrm{Lip}(f)(u^\varepsilon)^-$ and since $r'(|x|+Łt) \leq 0$ the second term of the left hand-side of the previous inequality is non-negative. Since $r(|x|+Łt) = 1\,,\ \forall (x,t) \in (-R,R)\times(0,T)$ and since $r \geq 0$, the first term is upper bounded by $-\frac{1}{T}\int_{-R}^R\int_0^T (u^\varepsilon)^-\,dx\,dt$ which is, by consequent, non-negative. Therefore, we have $(u^\varepsilon)^- = 0$ on $(-R,R)\times(0,T)$. Letting $R, T \to +\infty$, we have $u^\varepsilon \geq 0$ a.e.

- Similarly, by choosing $\kappa = 1$ in (2.10) (with the semi-entropies $u \mapsto (u-1)^+$), we prove $u^\varepsilon \leq 1$ a.e.

On the other hand, notice that, since $g(\kappa)\varphi \geq 0$, we have

$$-k_\varepsilon'(x)\mathrm{sgn}_\pm(u^\varepsilon-\kappa)g(\kappa)\varphi \leq [k_\varepsilon'(x)]^\mp g(\kappa)\varphi\,.$$

As the function k_ε is monotone non-decreasing or non-increasing according to the sign of $k_R - k_L$, we have $[k'_\varepsilon(x)]^\mp = \mathrm{sgn}_\mp(k_L - k_R)k'_\varepsilon(x)$. Therefore, the last term in equality (2.10) admits the bound

$$\begin{aligned}
-\int_0^\infty \int_{\mathbb{R}} k'_\varepsilon(x)\mathrm{sgn}_\pm(u^\varepsilon - \kappa)g(\kappa)\varphi\,dx\,dt \\
\leq \mathrm{sgn}_\mp(k_L - k_R)\int_0^\infty \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon(x)g(\kappa)\varphi\,dx\,dt \\
= \mathrm{sgn}_\pm(k_L - k_R)\int_0^\infty g(\kappa)(k_L\varphi(t,-\varepsilon) - k_R\varphi(t,\varepsilon))dt + \eta^1_\varepsilon(\varphi)
\end{aligned}$$

where

$$\begin{aligned}
\eta^1_\varepsilon(\varphi) &:= \left|\mathrm{sgn}_\pm(k_L - k_R)\int_0^\infty \int_{-\varepsilon}^{\varepsilon} k_\varepsilon(x)g(\kappa)\partial_x\varphi\,dx\,dt\right| \\
&\leq 2g(\kappa)\max\{k_L, k_R\}\int_0^\infty \max\{|\partial_x\varphi(t,x)|\,;\ x \in \mathbb{R}\}dt\,\varepsilon\,. \qquad (2.12)
\end{aligned}$$

This simple estimate shows that (2.4) holds true, with $\eta_\varepsilon(\varphi) := \eta^1_\varepsilon(\varphi) + \eta^2_\varepsilon(\varphi)$ where

$$\begin{aligned}
\eta^2_\varepsilon(\varphi) := (k_L - k_R)^\pm \int_0^\infty g(\kappa)\varphi(t,0)dt \\
- \mathrm{sgn}_\pm(k_L - k_R)\int_0^\infty g(\kappa)(k_L\varphi(t,-\varepsilon) - k_R\varphi(t,\varepsilon))dt
\end{aligned}$$

tends to 0 with ε. The application of the method previously described (points i'), ii'), iii')) then shows that these simple estimates ensure that sequence (u^ε) converges in $L^p_{\mathrm{loc}}(Q)$ for every $1 \leq p < +\infty$ to the entropy solution of problem (2.1). In particular, we have the following result:

Theorem 2.4. *Assume Hypotheses (H1), (H2), (H3); let $u_0 \in L^\infty(\mathbb{R};[0,1])$. There exists a unique entropy solution to problem (2.1).*

In addition to the concept of entropy process solution and to the result of Theorem (2.2), the heart of this method is Theorem (3.6); in what follows, we now explain our attempts to give a proof of uniqueness likely to be generalized to a proof of theorem (3.6), a proof satisfying also requirements (R1) and (R2) discussed in the introduction.

2.3 Several concepts of solutions

We present different concept of solutions to problem (2.1). Each one brings a new point of view and a possible new proof of uniqueness. This point is discussed and, of course, we prove that the new concept introduced is equivalent to notion of entropy solution.

2.3.1 Entropy solution

Entropy solution with regular entropies

For problem (2.1) in the case k Lipschitz, it is a classical fact that the formulation with Kruzhkov entropies is equivalent to the formulation with regular entropies. Here, the term corresponding to the term related to the discontinuity of k in the entropy formulation with regular entropies is not clear at first sight and we detail this relation in the following proposition:

Proposition 2.1. *Let $u_0 \in L^\infty(\mathbb{R};[0,1])$ and $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfying $0 \le u \le 1$ a.e. on $\mathbb{R}_+ \times \mathbb{R}$. The function u is an entropy solution of problem (2.1) if and only if u satisfies the following inequalities: for all convex function $\eta \in \mathcal{C}^2([0,1])$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \eta(u(t,x))\partial_t\varphi(t,x) \;+\; (k(x)\,\Phi(u(t,x)) + \Psi(u(t,x)))\,\partial_x\varphi(t,x)\,dx\,dt \\ +\; &\int_{\mathbb{R}} \eta(u_0(x))\,\varphi(0,x)\,dx \\ +\; &[(k_L-k_R)^-\Phi(0) - (k_L-k_R)^+\Phi(1)] \int_0^\infty \varphi(t,0)\,dt \ge 0, \end{aligned} \tag{2.13}$$

where respectively Φ, Ψ denote the entropy flux associated with the entropy η, i.e. $\Phi' = \eta' g'$ and $\Psi' = \eta' f'$.

The entropy inequalities (2.2) follow from (2.13) by approximation of the semi Kruzhkov entropies by regularized entropies, while, conversely, (2.13) is a consequence of (2.2) and the following lemma:

Lemma 2.1. *For every convex function $\eta \in \mathcal{C}^2([0,1])$, there exists a sequence $(\eta_n)_{n\in N^*}$ of functions of the form $\eta_n(s) = c_n s + d_n + \sum_{i=1}^{I_n} \alpha_i^n (u-\kappa_i^n)^+$ with $\alpha_i^n \ge 0$ for all $i = 1..n$, c_n, $d_n \in \mathbb{R}$ such that $(\eta_n')_{n\in\mathbb{N}^*}$ and $(\eta_n)_{n\in N^*}$ converge locally uniformly to η' and η respectively.*

Proof of Lemma 2.1:
Let $n \in \mathbb{N}^*$, we define for all $i = 1..n$,

$$\alpha_i = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \eta''(t)dt,$$

and

$$\eta_n(s) = \eta'(0)s + \sum_{i=1}^n \alpha_i (u-\kappa_i)^+.$$

We have $\eta_n'(s) = \eta'(0) + \sum_{i=1}^n \alpha_i sgn_+(s-\kappa_i)$. For $s \in [0,1]$, there exists $i \in [1,n]$, such as $i/n \le n \le (i+1)/n$. Then

$$\eta_n'(s) = \eta'(0) + \sum_{i \le nt} \alpha_i = \eta'(0) + \int_0^{i/n} \eta''(t)dt = \eta'(i/n),$$

so

$$\eta_n'(s) - \eta'(s) = \eta'(i/n) - \eta'(s).$$

Finally, we get for all $s \in [0,1]$, $|\eta_n'(s) - \eta'(s)| \leq ||\eta''||_\infty \frac{1}{n}$, so $(\eta_n')_n$ converge uniformly to η'.
Moreover, $\eta_n(0) = 0 = \eta(0)$, so $(\eta_n)_n$ converge uniformly to η.

Proof of Proposition 2.1:
I. We suppose that u satisfies (2.13). Let $\varphi \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R})$.
We take $\eta(u) = (u-\kappa)^+$, with $\kappa \in \mathbb{R}$. Let η_α denote the smooth approximation of the function $v \mapsto v^+$ defined by

$$\eta_\alpha(v) = \begin{cases} 0 & \text{if } v \leq 0, \\ v^2/4\alpha & \text{if } 0 \leq v \leq 2\alpha, \\ v - \alpha & \text{if } 2\alpha \leq v. \end{cases}$$

By use Lebesgue dominated convergence theorem, and passing to the limit in (2.13) when $\alpha \to 0$, we have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} (u(t,x)-\kappa)^+ \partial_t \varphi(t,x)\, dt\, dx \\ + & \int_0^\infty \int_{\mathbb{R}} k(x)\, sgn_+(u(t,x)-\kappa)(g(u(t,x)) - g(\kappa)) \partial_x \varphi(t,x)\, dt\, dx \\ + & \int_0^\infty \int_{\mathbb{R}} sgn_+(u(t,x)-\kappa)(f(u(t,x)) - f(\kappa))\, \partial_x \varphi(t,x)\, dt\, dx \\ + & \int_{\mathbb{R}} (u_0(x)-\kappa)^+\, \varphi(0,x)\, dx \\ + & [(k_R - k_L)^+ \Phi(1) - (k_R-k_L)^- \Phi(0)] \int_0^\infty \varphi(t,0)\, dt \geq 0, \end{aligned} \tag{2.14}$$

with $\Phi(1) = \text{sgn}_+(1-\kappa)(g(1)-g(\kappa)) = -g(\kappa)$ and $\Phi(0) = \text{sgn}_+(0-\kappa)(g(0)-g(\kappa)) = 0$. So it yields,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} (u(t,x)-\kappa)^+ \partial_t \varphi(t,x)\, dx\, dt \\ + & \int_0^\infty \int_{\mathbb{R}} k(x)\, sgn_+(u(t,x)-\kappa)(g(u(t,x)) - g(\kappa))\, \partial_x \varphi(t,x)\, dt\, dx \\ + & \int_0^\infty \int_{\mathbb{R}} sgn_+(u(t,x)-\kappa)(f(u(t,x)) - f(\kappa))\, \partial_x \varphi(t,x)\, dt\, dx \\ + & \int_{\mathbb{R}} (u_0(x)-\kappa)^+\, \varphi(0,x)\, dx\ + [-(k_R-k_L)^+ g(\kappa)] \int_0^\infty \varphi(t,0)\, dt \geq 0, \end{aligned}$$

but $-(k_R - k_L)^+ = (k_R - k_L)^-$. Then following:

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (u(t,x)-\kappa)^+ \partial_t \varphi(t,x)\,dxdt \\
+ \; & \int_0^\infty \int_{\mathbb{R}} k(x)\, sgn_+(u(t,x)-\kappa)(g(u(t,x)) - g(\kappa)))\,\partial_x \varphi(t,x)\,dt\,dx \\
+ \; & \int_0^\infty \int_{\mathbb{R}} sgn_+(u(t,x)-\kappa)(f(u(t,x)) - f(\kappa)))\,\partial_x \varphi(t,x)\,dx\,dt \\
+ \; & \int_{\mathbb{R}} (u_0(x)-\kappa)^+ \,\varphi(0,x)\,dx \\
+ \; & (k_R - k_L)^- \int_0^\infty g(\kappa)\,\varphi(t,0)\,dt \geq 0. \qquad (2.15)
\end{aligned}$$

In the same way with the entropy $u \mapsto (u-\kappa)^-$, we obtain

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (u(t,x)-\kappa)^- \partial_t \varphi(t,x)\,dxdt \\
+ \; & \int_0^\infty \int_{\mathbb{R}} k(x)\, sgn_-(u(t,x)-\kappa)(g(u(t,x)) - g(\kappa))\,\partial_x \varphi(t,x)\,dt\,dx \\
+ \; & \int_0^\infty \int_{\mathbb{R}} sgn_-(u(t,x)-\kappa)(f(u(t,x)) - f(\kappa)))\,\partial_x \varphi(t,x)\,dx\,dt \\
+ \; & \int_{\mathbb{R}} (u_0(x)-\kappa)^- \,\varphi(0,x)\,dx \,+ (k_R - k_L)^+ g(\kappa) \int_0^\infty \varphi(t,0)\,dt \geq 0. \qquad (2.16)
\end{aligned}$$

Then with (2.15) and (2.16), we have shown that u is an entropy solution because it satisfies two inequalities (2.1).

II. We suppose that u is an entropy solution.
Let η a convex function, $\eta = \eta(0) + \bar{\eta}$ with $\bar{\eta}$ convex such as $\bar{\eta}(0) = 0$. With Lemma 2.1, let $(\bar{\eta}_n)_n$ a sequence which converge uniformly to $\bar{\eta}$ and $(\bar{\eta}_n')_n$ to $\bar{\eta}'$ with $\bar{\eta}_n = \bar{\eta}'(0)s + \sum_{i=1}^n \alpha_i (u-\kappa_i)^+$, $\alpha_i \in \mathbb{R}^+$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ non-negative.
By adding the inequality (2.1) for all κ_i, $i = 1 \cdots n$ and by use u is a weak solution, we obtain:

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}^+} \bar{\eta}_n(u(t,x))\partial_t \varphi(t,x) + k(x)\bar{\Phi}_n(u(t,x)) + \bar{\Psi}_n(u(t,x))\partial_x \varphi(t,x)\,dt\,dx \\
+ \; & \int_{\mathbb{R}} \bar{\eta}_n(u_0(x))\varphi(0,x)\,dx \\
+ \; & [(k_R - k_L)^+ \bar{\Phi}_n(0) - (k_R - k_L)^- \bar{\Phi}_n(1)] \int_{\mathbb{R}} \varphi(t,0)\,dt \geq 0.
\end{aligned}$$

Moreover, η_n' converges to $\bar{\eta}'$, so $\bar{\Phi}_n'$ converges to $\bar{\Phi}'$ and $\bar{\Psi}_n'$ converges to $\bar{\Psi}'$.
On the other hand, we can choose for all n, $\bar{\Phi}_n(0) = \bar{\Phi}(0)$ and $\bar{\Psi}_n(0) = \bar{\Psi}(0)$, so $\bar{\Phi}_n$ and $\bar{\Psi}_n$ respectively converge uniformly to $\bar{\Phi}$ and $\bar{\Psi}$.

By use Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^+} \bar{\eta}(u(t,x))\partial_t\varphi(t,x) + k(x)\bar{\Phi}(u(t,x)) + \bar{\Psi}(u(t,x))\partial_x\varphi(t,x)dtdx \\
+ &\int_{\mathbb{R}} \bar{\eta}(u_0(x))\varphi(0,x)dx \\
+ &[(k_R - k_L)^+\bar{\Phi}(0) - (k_R - k_L)^-\bar{\Phi}(1)] \int_{\mathbb{R}} \varphi(t,0)dt \geq 0,
\end{aligned}$$

with $\bar{\Phi}' = \bar{\eta}'g' = \eta'g' = \Phi$ and $\bar{\Psi}' = \bar{\eta}'f' = \eta'f' = \Psi'f'$.
This yields:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^+} \bar{\eta}(u(t,x))\partial_t\varphi(t,x) + k(x)\Phi(u(t,x)) + \Psi(u(t,x))\partial_x\varphi(t,x)dtdx \\
+ &\int_{\mathbb{R}} \bar{\eta}(u_0(x))\varphi(0,x)dx \\
+ &[(k_R - k_L)^+\Phi(0) - (k_R - k_L)^-\Phi(1)] \int_{\mathbb{R}} \varphi(t,0)dt \geq 0. \qquad (2.17)
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} \eta(0)\partial_t\varphi(t,x) + \int_{\mathbb{R}} \eta(0)\varphi(0,x)dx = 0. \qquad (2.18)$$

By add (2.17) and (2.18), we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^+} \eta(u(t,x))\partial_t\varphi(t,x) + k(x)\Phi(u(t,x)) + \Psi(u(t,x))\partial_x\varphi(t,x)dtdx \\
+ &\int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x)dx \\
+ &[(k_R - k_L)^+\Phi(0) - (k_L - k_R)^+\Phi(1)] \int_{\mathbb{R}} \varphi(t,0)dt \geq 0. \qquad (2.19)
\end{aligned}$$

Finally, u satisfies (2.13).

Uniqueness for entropy solution

The technique of the doubling of variable introduced by Kruzhkov [Kru70] to prove the uniqueness of entropy solutions in the case where k is constant (or Lipschitz continuous at least) satisfies the points (R1) and (R2) discussed in the introduction. Unfortunately, we do not know how to adapt this technique to the case considered here (k discontinuous). It is possible to use this technique to compare two entropy solutions away from the line of discontinuity Σ of k, but then, in order to compare the two solutions in the vicinity of Σ, one has to ensure the existence of traces of the functions, existence which rests on superfluous hypotheses (e.g. finite number of discontinuities in the solution [Tow00], g genuinely nonlinear [Bac04]...).

The technical obstruction to the efficiency of the doubling of variables method is the fact that, if ρ_ε is an approximation of the unit, then $(k_R\partial_x + k_L\partial_y)\rho_\varepsilon(x-y)$ vanishes only if $k_L = k_R$. But, fundamentally, the copy of the technique of the doubling of variable of Kruzhkov is inefficient because *constants are not solutions* of problem (2.1) (to be solution, a constant κ should at least satisfies the Rankine-Hugoniot relation $k_L g(\kappa) = k_R g(\kappa)$ on Σ, i.e. $g(\kappa) = 0$).

Indeed, the basis of the technique of Kruzhkov (in the case where k is a constant function) is to start from a result of comparison between any entropy solution and the particular entropy solution that is a constant function, to deduce comparison between any two entropy solutions.

Here, one should therefore start from a formulation of solution which already contains a result of comparison between the solution and a particular class of solution $\mathcal{C}$. This is what we do in the next subsection, where, after Portilheiro [Por03a] and Perthame and Souganidis [PS03], we introduce a notion of "χ-compared solution" for problem (2.1). In that case, the particular class of functions $\mathcal{C}$ under consideration is the one of regular functions $\chi(x)$ $\big($such functions are solutions of the equation with source term $\partial_t\chi + \partial_x\big(k(x)g(\chi) + f(\chi)\big) = S$, where $S := \partial_x\big(k(x)g(\chi) + f(\chi)\big)\big)$.

However, a natural choice for $\mathcal{C}$ would be the class of stationary solutions to problem (2.1). This choice turns to be accurate as soon as the class $\mathcal{C}$ so defined is large enough. Unfortunately, this is not the case in the context of our analysis, i.e. under hypotheses (H1), (H2), (H3), because certain values of the interval $[0,1]$, in which entropy solutions take their values, are out of reach by means of stationary solution.

Let us be more specific on this point by considering the example $g(u) = u(1-u)$, $f = 0$, $k_L > k_R$. If w is a stationary entropy solution to problem (2.1) then w is constant aside Σ ($w(x) = w_\pm$ if $x > 0$, resp. $x < 0$), and the Rankine-Hugoniot relation $k_L g(w_-) = k_R g(w_+)$ (together with an additional entropy condition) holds. This Rankine-Hugoniot relation implies $w_- \notin (a, 1-a)$ where $a \in (0, 1/2)$ is such that $k_L g(a) = k_R \max g = k_R g(1/2)$. Therefore, on $x < 0$, the values in $(a, 1-a)$ are not reachable by stationary solutions while an entropy solution may well take these values (at least for $x < 0$ far from 0).

Yet, notice that, in a different context (and therefore on a different physical background) such a program of comparison of solution via the use of comparison with the particular class of stationary solutions has been realized by Audusse and Perthame in a recent work [AP04]. This yields a very interesting proof of comparison of solutions.

To conclude, let us add that one may also consider the class $\mathcal{C}$ of stationary solutions to problem (2.1) with *non-homogeneous* source term. Such an idea is in connection with the ideas developed in the theory of mild solutions to evolution problems [Bén72]. We do not know how to deduce a result of comparison (satisfying (R1) and (R2)) from such considerations.

2.3.2 χ-compared solution

Definition, equivalence with entropy solution

We introduce the notion of"χ-compared solution" briefly described in the previous subsection.

Definition 2.4. *Let $u_0 \in L^\infty(\mathbb{R};[0,1])$. Let $u \in L^\infty(Q)$.*

- *The function u is a"χ-compared subsolution " (resp."χ-compared supersolution") of problem (2.1) if for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ non-negative, $\kappa \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $0 \le \kappa + \chi(x) \le 1$, we have*

$$\begin{aligned}
& \int_0^{+\infty}\int_{\mathbb{R}} (u(t,x)-\kappa-\chi(x))^{\pm}\partial_t\varphi(t,x)dtdx \\
+ & \int_0^{+\infty}\int_{\mathbb{R}} [k(x)\Phi^{\pm}(u(t,x),\kappa+\chi(x)) \\
& \qquad +\Psi^{\pm}(u(t,x),\kappa+\chi(x))]\partial_x\varphi(t,x)dtdx \\
- & \int_0^{+\infty}\int_{\mathbb{R}} sgn_{\pm}(u-\kappa-\chi(x))k(x)\partial_x\big(g(\kappa+\chi(x))\big)\varphi(t,x)dtdx \\
+ & \int_0^{+\infty}\int_{\mathbb{R}} sgn_{\pm}(u-\kappa-\chi(x))\partial_x\big(f(\kappa+\chi(x))\big)\varphi(t,x)dtdx \\
+ & (k_L-k_R)^{\pm}\int_0^{+\infty} g(\kappa+\chi(0))\varphi(t,0)dt \\
+ & \int_{\mathbb{R}}(u_0(x)-\kappa-\chi(x))^{\pm}\varphi(0,x)dx \ge 0.
\end{aligned} \tag{2.20}$$

- *The function u is a χ-compared solution of (2.1) if it is both a χ-compared subsolution and supersolution.*

Of course, a χ-compared solution u to the problem (2.1) is also an entropy solution (take $\chi = 0$), in particular $0 \le u \le 1$ a.e. The converse is true:

Theorem 2.5. *Assume Hypotheses (H1), (H2), (H3). Let $u_0 \in L^\infty(\mathbb{R};[0,1])$. If $u \in L^\infty(Q)$ is an entropy solution to problem (2.1), then u is a χ-compared solution of problem (2.1).*

Proof of Theorem 2.5:
Let $\varphi, \theta \in \mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$, $\Psi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}_+)$ such that supp$(\Psi)\subset [-1,1]$ and $\int_{\mathbb{R}} \Psi(x)dx = 1$.
Let η_α be a regularization of the function $s \mapsto s^+$. It is the entropy function.
Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}^+)$, $\kappa \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $0 \le \chi(y)+\kappa \le 1$. We know that u satisfies the inequalities (2.13).
For $\varepsilon > 0$, choose the test function $\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})\varphi(t,y)$ in (2.13) and integrate the result with respect to y to get:

$$I_1 + I_2 + I_3 + I_4 + I_5 \ge 0.$$

In a first step, we study each term when ε tends to zero.
1. We study I_1:

$$I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{+\infty} \eta_\alpha(u(t,x) - \kappa - \chi(y)) \partial_t \varphi(t,y) \frac{1}{\varepsilon} \Psi(\frac{x-y}{\varepsilon}) dx dy dt.$$

Let $z = \frac{x-y}{\varepsilon}$,

$$\begin{aligned} & \int_{\mathbb{R}} \eta_\alpha(u(t,x) - \kappa - \chi(y)) \partial_t \varphi(t,y) \frac{1}{\varepsilon} \Psi(\frac{x-y}{\varepsilon}) dy \\ = & \int_{\mathbb{R}} \eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) \Psi(z) dz. \end{aligned}$$

Then

$$\begin{aligned} & | \int_{\mathbb{R}} \eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) \Psi(z) dz \\ & - \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) | \\ = & | \int_{\mathbb{R}} \big(\eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) \\ & \quad - \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) \big) \Psi(z) dz | \\ \leq & \int_{\{|z| \leq 1\}} | \eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) \\ & \qquad - \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) | \Psi(z) dz, \end{aligned}$$

but

$$\eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) - \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) \to_{\varepsilon \to 0} 0,$$

and

$$\begin{aligned} & | \eta_\alpha(u(t,x) - \kappa - \chi(x - \varepsilon z)) \partial_t \varphi(t, x - \varepsilon z) - \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) | \\ & \qquad \leq (||u||_\infty + \kappa + ||\chi||_\infty) \Theta(t,x), \end{aligned}$$

where $\Theta(t,x) = \max_{x-1 \leq z \leq x+1} |\varphi(t,z)|$ and $\Theta \in \mathcal{C}_c(\mathbb{R}^+ \times \mathbb{R})$, so by use Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon \to 0} I_1 = \int_{\mathbb{R}} \int_0^{+\infty} \eta_\alpha(u(t,x) - \kappa - \chi(x)) \partial_t \varphi(t,x) dx dt. \tag{2.21}$$

2. We study I_2:

$$I_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{+\infty} k(x) \Phi_\alpha(u(t,x), \kappa - \chi(y)) \varphi(t,y) \frac{1}{\varepsilon^2} \Psi'(\frac{x-y}{\varepsilon}) dx dy dt, \tag{2.22}$$

with $\Phi_\alpha(u,\kappa)=\int_\kappa^u \eta'_\alpha(s-\kappa)g'(s)ds$.

$$\begin{aligned} I_2 &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(y))\varphi(t,y)\frac{1}{\varepsilon}\partial_y[\Psi(\frac{x-y}{\varepsilon})]dxdydt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(y))\chi'(y)\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdydt \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(y))\partial_x\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdydt, \end{aligned} \tag{2.23}$$

with an integration by parts.
We study each term of the previous equality. First, we have

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(y))\chi'(y)\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdydt \\ =&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(x-\varepsilon z))\chi'(x-\varepsilon z) \\ &\qquad\qquad \varphi(t,y)\Psi(z)dxdzdt, \end{aligned}$$

with $\partial_\kappa\Phi_\alpha(u,\kappa)=-\int_\kappa^u \eta''_\alpha(s-\kappa)g'(s)ds$ so that

$$\begin{aligned} &\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(x-\varepsilon z))\chi'(x-\varepsilon z)\varphi(t,x-\varepsilon z) \\ \to_{\varepsilon\to 0}\ &\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(x))\chi'(x)\varphi(t,x), \end{aligned}$$

and

$$\begin{aligned} &|\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(x-\varepsilon z))\chi'(x-\varepsilon z)\varphi(t,x-\varepsilon z)\Psi(z)| \\ &\qquad\qquad \leq \Theta(t,x)||\chi'||_\infty||g'||_\infty\Psi(z), \end{aligned}$$

with the left term is in $L^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)$.
By use of the Lebesgue dominated convergence theorem we obtain:

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(y))\chi'(y)\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdydt \\ &\qquad = \int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa\Phi_\alpha(u(t,x),\kappa+\chi(x))\chi'(x)\varphi(t,y)dxdt. \end{aligned}$$

In the same way, the second term of (2.23) yields:

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(y))\partial_x\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdydt \\ =&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(x-\varepsilon z))\partial_x\varphi(t,x-\varepsilon z)\Psi(z)dxdzdt \\ \to_{\varepsilon\to 0}&\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(x))\partial_x\varphi(t,x)dxdt. \end{aligned}$$

Finally, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2 = &\int_{\mathbb{R}}\int_0^{+\infty} k(x)\partial_\kappa \Phi_\alpha(u(t,x),\kappa+\chi(x))\chi'(x)\varphi(t,x)dxdt \\ + &\int_{\mathbb{R}}\int_0^{+\infty} k(x)\Phi_\alpha(u(t,x),\kappa+\chi(x))\partial_x\varphi(t,x)dxdt\,. \end{aligned} \tag{2.24}$$

3. <u>We study I_3</u>:

$$I_3 = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty} \Psi_\alpha(u(t,x),\kappa-\chi(y))\varphi(t,y)\frac{1}{\varepsilon^2}\Psi'(\frac{x-y}{\varepsilon})dxdydt,$$

with $\Psi_\alpha(u,\kappa) = \int_\kappa^u \eta'(s-\kappa)f'(s)ds$.
In the same way that for I_2, replacing g by f , we show that:

$$\begin{aligned}\lim_{\varepsilon\to 0} I_3 \;=\; &\int_{\mathbb{R}}\int_0^{+\infty} \partial_\kappa\Psi_\alpha(u(t,x),\kappa+\chi(x))\chi'(x)\varphi(t,x)dxdt \\ + &\int_{\mathbb{R}}\int_0^{+\infty} \Psi_\alpha(u(t,x),\kappa+\chi(x))\partial_x\varphi(t,x)dxdt. \end{aligned} \tag{2.25}$$

4. <u>We study I_4</u>:
By use Lebesgue dominated convergence theorem, we show:

$$\begin{aligned} I_4 = &\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\alpha(u_0(x)-\kappa-\chi(y))\varphi(0,y)\frac{1}{\varepsilon}\Psi(\frac{x-y}{\varepsilon})dxdy \\ \to_{\varepsilon\to 0} &\int_{\mathbb{R}} \eta_\alpha(u_0(x)-\kappa-\chi(x))\varphi(0,x)dx. \end{aligned} \tag{2.26}$$

5. <u>We study I_5</u>:
In the same way, the term I_5 becomes:

$$\begin{aligned} I_5 = &\int_{\mathbb{R}}\int_0^{+\infty} (k_L-k_R)^-\Phi_\alpha(0,\kappa+\chi(y))\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{-y}{\varepsilon})dydt \\ - &\int_{\mathbb{R}}\int_0^{+\infty} (k_L-k_R)^+\phi_\alpha(1,\kappa+\chi(y))\varphi(t,y)\frac{1}{\varepsilon}\Psi(\frac{-y}{\varepsilon})dydt \\ \to_{\varepsilon\to 0} &\int_0^{+\infty} (k_L-k_R)^-\Phi_\alpha(0,\kappa+\chi(0))\varphi(t,0)dt \\ - &\int_0^{+\infty} (k_L-k_R)^+\phi_\alpha(1,\kappa+\chi(0))\varphi(t,0)dt. \end{aligned} \tag{2.27}$$

We conclude with (2.21), (2.24), (2.25) (2.26) and (2.27), for all $\alpha > 0$:

$$\begin{aligned}
& \int_{\mathbb{R}} \int_0^{+\infty} \eta_\alpha(u(t,x) - \kappa - \chi(x))\partial_t \varphi(t,x) dxdt \\
+ & \int_{\mathbb{R}} \int_0^{+\infty} k(x)\partial_\kappa \Phi_\alpha(u(t,x), \kappa + \chi)\chi'(x)\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}} \int_0^{+\infty} \partial_\kappa \Psi_\alpha(u(t,x), \kappa + \chi(x))\chi'(x)\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}} \int_0^{+\infty} k(x)\Phi_\alpha(u(t,x), \kappa + \chi(x))\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}} \int_0^{+\infty} \Psi_\alpha(u(t,x), \kappa + \chi(x))\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}} \eta_\alpha(u_0(x), \kappa + \chi(x))\varphi(0,x)dx \\
+ & \int_0^{+\infty} \big((k_L - k_R)^- \Phi_\alpha(0, \kappa + \chi(0)) - (k_L - k_R)^+ \Phi_\alpha(1, \kappa + \chi(0))\big) \\
& \qquad \varphi(t,0)dt \geq 0.
\end{aligned}$$

In a second step, we pass to the limit when α tends to zero. We study each term separately.

- The first term is

$$I_1^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} \eta_\alpha(u(t,x) - \kappa - \chi(x))\partial_t \varphi(t,x) dxdt\,. \tag{2.28}$$

  Since $|\eta_\alpha(u(t,x) - \kappa - \chi(x))\partial_t\varphi(t,x)| \leq 2|\partial_t\varphi(t,x)|$ which is integrable on $\mathbb{R}_+ \times \mathbb{R}$ and $\eta_\alpha(u(t,x) - \kappa - \chi(x))\partial_t\varphi(t,x) \to_{\alpha\to 0} sgn^+(u(t,x) - \kappa - \chi(x))\partial_t\varphi(t,x)$, we have:

$$\lim_{\alpha\to 0} I_1^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} (u(t,x) - \kappa - \chi(x))^+ \partial_t\varphi(t,x)dxdt. \tag{2.29}$$

- We study the term

$$I_2^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} k(x)\partial_\kappa \Phi_\alpha(u(t,x), \kappa + \chi(x))\chi'(x)\varphi(t,x)dxdt.$$

  We have $\partial_\kappa \Phi_\alpha(u,\kappa) = -\int_\kappa^u \eta''_\alpha(s-\kappa)g'(s)ds$. After discussion of the respective positions of κ and s :

  1. If $u \leq \kappa$, $\partial_\kappa \Phi_\alpha(u,\kappa) = 0$

2. If $u > \kappa$, then there exists α_0 such as $\forall \alpha \leq \alpha_0$, $2\alpha < u - \kappa$ and then

$$\partial_\kappa \Phi_\alpha(u, \kappa) = -\frac{1}{2\alpha} \int_\kappa^{2\alpha} g'(s) ds \to_{\alpha \to 0} -g'(\kappa),$$

because g' is continuous.

This follows

$$\partial_\kappa \Phi_\alpha(u, \kappa) \to_{\alpha \to 0} -sgn^+(u - \kappa) g'(\kappa).$$

Moreover there exists $M > 0$ such that $|k(x) \partial_\kappa \Phi_\alpha(u(t, x), \kappa + \chi(x)) \chi'(x) \varphi(t, x)| \leq M \varphi(t, x)$, so by use of the Lebesgue dominated convergence theorem, this yields:

$$\lim_{\alpha \to 0} I_2^\alpha = - \int_{\mathbb{R}} \int_0^{+\infty} sgn_+(u(-\kappa - \chi(x)) k(x) \partial_x \big(g(\kappa + \chi(x)) \big) \varphi(t, x) \, dx dt. \tag{2.30}$$

- We study the term

$$I_3^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} \partial_\kappa \Psi_\alpha(u(t, x), \kappa + \chi(x)) \chi'(x) \varphi(t, x) dx dt.$$

Replacing g by f in the study of term I_2, we obtain:

$$\lim_{\alpha \to 0} I_3^\alpha = - \int_{\mathbb{R}} \int_0^{+\infty} sgn_+(u(t, x) - \kappa - \chi(x)) \partial_x \big(f(\kappa + \chi(x)) \big) \varphi(t, x) dx dt. \tag{2.31}$$

- We study the term

$$I_4^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} k(x) \Phi_\alpha(u(t, x), \kappa + \chi(x)) \varphi(t, x) dx dt.$$

For the term I_4^α, we have

$$\Phi_\alpha(u, \kappa) \to_{\alpha \to 0} sgn_+(u - \kappa)(g(u) - g(\kappa)) = \Phi^+(u, \kappa). \tag{2.32}$$

By using Lebesgue dominated convergence theorem, we obtain:

$$\lim_{\alpha \to 0} I_4^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} k(x) \Phi^+(u(t, x), \kappa + \chi(x)) \varphi(t, x) dx dt \tag{2.33}$$

- In the same way

$$I_5^\alpha = \int_{\mathbb{R}} \int_0^{+\infty} k(x) \Phi_\alpha(u(t, x), \kappa + \chi(x)) \varphi(t, x) dx dt.$$

We have

$$\Psi_\alpha(u,\kappa) \to_{\alpha\to 0} sgn_+(u-\kappa)(f(u)-f(\kappa)). \tag{2.34}$$

By using Lebesgue dominated convergence theorem, we have:

$$\lim_{\alpha\to 0} I_5^\alpha = \int_{\mathbb{R}}\int_0^{+\infty} \Psi^+(u(t,x),\kappa+\chi(x))\varphi(t,x)dxdt. \tag{2.35}$$

- The limit of the term

$$I_6^\alpha = \int_{\mathbb{R}} \eta_\alpha(u_0(x),\kappa+\chi(x))\varphi(0,x)dx$$

is obtained by using Lebesgue dominated convergence theorem:

$$\lim_{\alpha\to 0} I_6^\alpha = \int_{\mathbb{R}} sgn_+(u_0-\kappa-\chi(x))\varphi(0,x)dx. \tag{2.36}$$

- Eventually, by (2.32), (2.34), we have $\lim_{\alpha\to 0}\Phi_\alpha(0,\kappa+\chi(x))=0$ and $\lim_{\alpha\to 0}\Phi_\alpha(1,\kappa+\chi(0))=-g(\kappa+\chi(0))$ so that

$$\lim_{\alpha\to 0} I_7^\alpha = \int_0^{+\infty}(k_L-k_R)^+ g(\kappa+\chi(0))\varphi(t,0)dt. \tag{2.37}$$

We conclude with (2.29),(2.30),(2.31),(2.33), (2.35), (2.36) and (2.37) :

$$\begin{aligned}
& \int_{\mathbb{R}}\int_0^{+\infty} (u(t,x)-\kappa-\chi(x))^+\partial_t\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}}\int_0^{+\infty} sgn_+(u-\kappa-\chi(x)) \\
& \quad (k(x)(g(u)-g(\kappa))+(f(u)-f(\kappa))\varphi(t,x)dxdt \\
- & \int_{\mathbb{R}}\int_0^{+\infty} sgn_+(u-\kappa-\chi(x)) \\
& \quad [k(x)\partial_x(g(\kappa+\chi(x)))+\partial_x(f(\kappa+\chi(x)))]\varphi(t,x)dxdt \\
+ & \int_{\mathbb{R}}(u_0-\kappa-\chi(x))^+\varphi(0,x)dx \\
+ & (k_L-k_R)^+ g(\kappa+\chi(0))\int_0^{+\infty}\varphi(t,0)dt \ge 0.
\end{aligned}$$

We do the same reasoning with the non-positive semi entropy to conclude that u is a χ-compared solution.

Uniqueness via χ-compared solution

Contrary to the derivative ∂_x, which is weighted by the function $k(x)$, the derivative ∂_t has the constant coefficient 1 in the weak form of conservation law (2.1), and it is therefore possible to show by the method of the doubling of variable that, if $u \in L^\infty(Q)$ is a χ-compared solution to problem (2.1), then u can be compared to any regular (say $\mathcal{C}^1(Q)$) solution ψ of the equation $\psi_t + (k(x)g(\psi) + f(\psi))_x = S$ (where, by definition, the source term S is the first member of the equation): for every nonnegative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned} & \int_0^{+\infty}\int_{\mathbb{R}} (u-\psi)^\pm \partial_t\varphi dtdx \\ + & \int_0^{+\infty}\int_{\mathbb{R}} [k(x)\Phi^\pm(u,\psi) + \Psi^\pm(u,\psi)]\partial_x\varphi dtdx \\ - & \int_0^{+\infty}\int_{\mathbb{R}} sgn_\pm(u-\psi)k(x)\partial_x(g(\psi)) + \partial_x(f(\psi))\varphi dtdx \\ + & (k_L - k_R)^\pm \int_0^{+\infty} g(\psi(t,0))\varphi(t,0)dt \\ + & \int_{\mathbb{R}} (u_0(x) - \psi(0,x))^\pm \varphi(0,x)dx \geq 0. \end{aligned} \tag{2.38}$$

The next step on the way of Theorem 2.1 would then be the deduction from (2.38) of the comparison between two entropy process solutions (without requiring one of them to be regular). The way to proceed is an unsolved question for the moment. Notice that, even in the case where k is constant, and as emphasized by Portilheiro, this remains an unsolved problem. In particular, and in this last case k constant, one could think to approximate any entropy solution v by a sequence of regular function ψ^ε in $L^1_{loc}(Q)$, such that

$$\partial_t\psi^\varepsilon + \partial_x(g(\psi^\varepsilon)) = S^\varepsilon, \tag{2.39}$$

with $||S^\varepsilon||_{L^1_{\text{loc}}(Q)}$ small with respect to ε (then, passing to the limit in the equation

$$\begin{aligned} & \int_0^{+\infty}\int_{\mathbb{R}} (u-\psi^\varepsilon)^\pm \partial_t\varphi + \Phi^\pm(u,\psi^\varepsilon)\partial_x\varphi dtdx \\ - & \int_0^{+\infty}\int_{\mathbb{R}} sgn_\pm(u-\psi^\varepsilon)S^\varepsilon\varphi dtdx \\ + & \int_{\mathbb{R}} (u_0(x) - \psi^\varepsilon(0,x))^\pm \varphi(0,x)dx \geq 0, \end{aligned} \tag{2.40}$$

the analogous to (2.38) in the case k constant, $f = 0$, one would obtain the comparison between u and v.) Let us highlight the fact that such a result of approximation is wrong: in case the entropy solution v has a shock, it requires a source term of strength $\mathcal{O}(1)$ (with respect to ε) in (2.39) to approach v by ψ^ε regular. The reason is that, in presence of a shock, the entropy solution actually dissipates entropy while the solution ψ^ε of (2.39) dissipates entropy with an order $||S^\varepsilon||_{L^1_{\text{loc}}(Q)}$. We justify this assertion on the example of the Burgers-Hopf equation with a stationary shock : $g(u) = u^2/2$ here and we let v_0, the initial datum, be equal to $x \mapsto -\mathrm{sgn}(x)$ on the interval $[-2,2]$. Suppose also $-1 \leq v_0 \leq 1$ a.e. and v_0 with compact support; then, by finite speed of propagation, the entropy solution v of the equation $v_t + (g(v))_x = 0$ with initial condition $v|_{t=0} = v_0$, equals $x \mapsto -\mathrm{sgn}(x)$ for $(x,t) \in [-1,1] \times [0,T_1]$, $T_1 = 1$. Let $\eta(u) = u^2$ and let $\Phi : u \mapsto u^3/3$ be the corresponding entropy flux. Let φ be the test function defined by $\varphi(x,t) = \theta(x)\mu(t)$, where

$$\theta(x) = \begin{cases} -|x| + 1 & \text{if} \quad x \in [-1,1], \\ 0 & \text{else,} \end{cases} \qquad \mu \in \mathcal{C}_c^1(0,T_1)\,, \quad \int_0^{T_1} \mu = 1\,.$$

We compute

$$\int_0^{+\infty} \int_{\mathbb{R}} \eta(v)\partial_t\varphi + \Phi(v)\partial_x\varphi dx\, dt = 2/3\,. \tag{2.41}$$

On the other hand, if ψ^ε is a regular function which solves (2.39) with, say, S^ε with compact support, $||S^\varepsilon||_{L^1(Q)}$ as well as $\psi^\varepsilon|_{t=0} - v_0$ small with respect to ε then (by comparison of entropy solutions of non-homogeneous scalar conservation laws) ψ^ε is close to v in $L^1_{\mathrm{loc}}(Q)$. Suppose also the sequence (ψ^ε) uniformly bounded with respect to ε in $L^\infty(Q)$ (this is consistent with the maximum principle for scalar conservation laws), then, by the dominated convergence theorem, $\int_0^{+\infty}\int_{\mathbb{R}} \eta(\psi^\varepsilon)\partial_t\varphi + \Phi(\psi^\varepsilon)\partial_x\varphi dx\, dt$ is close to the left hand-side of (2.41) for ε small. But this former quantity can also be computed by multiplying (2.39) by $\eta'(\psi^\varepsilon)$, using the chain-rule for derivative of composed functions, multiplying the result by φ and then integrating by parts (operation which are licit for ψ^ε is required to be regular). Therefore $2/3$ is close to the quantity $-\int_0^{+\infty}\int_{\mathbb{R}} \eta'(\psi^\varepsilon)S^\varepsilon\varphi dx\, dt$ which is bounded by $C||S^\varepsilon||_{L^1(Q)}$, and this yields the contradiction with the hypothesis $||S^\varepsilon||_{L^1(Q)}$ small with respect to ε.

2.3.3 Kinetic solution

The concept of χ-compared solution did not bring technical facilities to the proof of Theorem 2.1, and, regarding this problem, the technique of the doubling of variable is partly unsuccessful. Therefore, on the basis of the works of Lions, Perthame and Tadmor [LPT94] and Perthame [Per98] we have introduced a concept of kinetic solution to problem (2.1), and then proved Theorem 2.1. In fact, this proof can be adapted to show Theorem 3.6 and we have directly defined a notion of kinetic process solution. We define it and show its equivalence with the notion of entropy process solution in subsections 2.3.3, 2.3.3, 2.3.3, then prove Theorem 3.6 in subsection 2.3.4.

Kinetic and equilibrium functions

If Ω is a subset of $\mathbb{R}^m$ $(m \geq 1)$ and $u : \Omega \to \mathbb{R}$ is measurable, the equilibrium function χ_u associated to u is the function $\Omega \times \mathbb{R} \ni (x,\xi) \mapsto \mathrm{sgn}_+(u(x) - \xi) + \mathrm{sgn}_-(\xi)$. Notice that χ_u is measurable and that $\chi_u \in L^\infty(\Omega \times \mathbb{R}; [-1,1])$. More generally, a kinetic function is a function $h \in L^\infty(\Omega \times \mathbb{R})$ such that:

$$\begin{aligned} &0 \leqslant h(x,\xi)\mathrm{sgn}(\xi) \leqslant 1, \\ &\partial_\xi h(x,\xi) = \delta(\xi) - \nu_x(\xi) \end{aligned} \tag{2.42}$$

where ν is a Young measure. For an equilibrium function, $\nu_x(\xi) = \delta(\xi - u(x))$. In the following, we also consider two functions associated with any kinetic one:

$$\begin{aligned} h_+(x,\xi) &= h(x,\xi) - \mathrm{sgn}_-(\xi), \\ h_-(x,\xi) &= h(x,\xi) - \mathrm{sgn}_+(\xi). \end{aligned}$$

For equilibrium functions, we have, for a.e. $\xi \in \mathbb{R}$:

$$\begin{aligned} h_+(x,\xi) &= \mathrm{sgn}_+(u(x) - \xi), \\ h_-(x,\xi) &= \mathrm{sgn}_-(u(x) - \xi). \end{aligned}$$

For X a locally compact Hausdorff space, $\mathcal{M}_+(X)$ denotes the set of positive Borel measures on X which are finite on compact subsets of X or, equivalently (by Riesz representation theorem) the cone of nonnegative linear form on $\mathcal{C}_c(X)$. Therefore $m \in \mathcal{C}(\mathbb{R}_\xi; w* - \mathcal{M}_+(X))$ means $m(\xi) \in \mathcal{M}_+(X)$ for every $\xi \in \mathbb{R}$ and, for every $\varphi \in \mathcal{C}_c(X)$,

$$\xi \mapsto \int_X \varphi dm(\xi)$$

is continuous.

Kinetic solution

Denote by a and b the derivatives of the flux functions:

$$a(\xi) := g'(\xi)\,, \qquad b(\xi) := f'(\xi)\,, \quad \xi \in \mathbb{R}\,.$$

We recall that, since f and g have been continued by constants outside the interval $[0,1]$, these functions a and b are defined everywhere, except possibly at 0 and 1 and that they vanish outside $[0,1]$.

Definition 2.5. *Let $u_0 \in L^\infty(\mathbb{R};[0,1])$ and $u \in L^\infty(Q \times (0,1))$.*

1. *Let h and h^0 be the equilibrium functions associated with u and u_0:*

$$h(t,x,\lambda,\xi) = \chi_{u(t,x,\lambda)}(\xi)\,, \quad h^0(x,\xi) = \chi_{u_0(x)}(\xi)\,.$$

The function u is a kinetic process subsolution *(resp.* kinetic process supersolution*) of (2.1) if there exists $m_\pm \in \mathcal{C}(\mathbb{R}_\xi; w* - \mathcal{M}_+(\bar{Q}))$ such that $m_+(\cdot,\xi)$ vanishes for large ξ (resp. $m_-(\cdot,\xi)$ vanishes for large $-\xi$) and such that for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$,*

$$\begin{aligned} &\int_0^1 \int_{Q\times\mathbb{R}_\xi} h_\pm(\partial_t + (k(x)a(\xi) + b(\xi))\partial_x)\varphi + \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h^0_\pm \varphi_{|t=0} \\ - \quad &(k_L - k_R)^\pm \int_{\Sigma\times\mathbb{R}_\xi} a(\xi)\varphi_{|x=0} = \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \varphi dm_\pm. \end{aligned} \tag{2.43}$$

2. *The function u is a* kinetic process solution *of (2.1) if it is both a kinetic process subsolution and process supersolution.*

Remark 2.3. *In the right hand-side of* (2.43), *$m_\pm$ is defined by:*

$$\forall \phi \in \mathcal{C}_c(\bar{Q} \times \mathbb{R}_\xi)\,, \int_{\bar{Q}\times\mathbb{R}_\xi} \phi dm_\pm := \int_{\mathbb{R}} d\xi \int_{\bar{Q}} \phi(\cdot,\xi) dm_\pm(\xi)$$

and, since $m_\pm \in \mathcal{C}(\mathbb{R}_\xi; w*-\mathcal{M}_+(\bar{Q}))$, we have $m_\pm \in \mathcal{M}_+(\bar{Q}\times\mathbb{R}_\xi)$.

Remark 2.4. *Notice that the class of admissible test-functions in* (2.43) *is larger than $\mathcal{C}_c^\infty(\mathbb{R}^3)$. Indeed, since $m_+(\xi)$ vanishes for large ξ, any function $\phi \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that* $\mathrm{supp}(\phi) \subset K \times [-R,+\infty)$ *for K a compact of $\bar{Q}$ and $R \in \mathbb{R}$ is admissible in* (2.43) *written for subsolutions. Similarly, any function $\phi \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that* $\mathrm{supp}(\phi) \subset K \times (-\infty,R)$ *for K a compact of $\bar{Q}$ and $R \in \mathbb{R}$ is admissible in* (2.43) *written for supersolutions.*

Equivalence of the two notions

Theorem 2.6. *The notions of weak entropy process and kinetic process semi-solutions are equivalent.*

We prove that the notions of weak entropy process and kinetic process subsolutions are equivalent. Let $u \in L^\infty(Q\times(0,1))$ be a weak entropy process subsolution of (2.1). For $\kappa \in \mathbb{R}$, define the linear form m_+^κ on $\mathcal{C}_c^\infty(\bar{Q})$ by:

$$\begin{aligned} m_+^\kappa(\varphi) &= \int_0^1 \int_Q (u(t,x,\lambda)-\kappa)^+ \partial_t\varphi(t,x)\,dtdxd\lambda \\ &+ \int_0^1 \int_Q (k(x)\Phi^+(u(t,x,\lambda),\kappa) + \Psi(u(t,x,\lambda),\kappa))\partial_x\varphi(t,x)\,dtdxd\lambda \\ &+ \int_{\mathbb{R}} (u_0(x)-\kappa)^+\varphi(0,x)dx + (k_L-k_R)^+ \int_\Sigma g(\kappa)\varphi(t,0)dt \end{aligned} \tag{2.44}$$

Let $\kappa \in \mathbb{R}$ be fixed. Since u is a weak entropy process subsolution (resp. weak entropy process supersolution), m_+^κ is nonnegative. It therefore induces a nonnegative linear form on $\mathcal{C}_c(\bar{Q})$ which can be represented by a Borel measure, still denoted m_+^κ. We set $m^+(\xi) = m_+^\xi$ ($\xi \in \mathbb{R}$). For K a compact subset of $\bar{Q}$, there exists a nonnegative $\varphi \in \mathcal{C}_c^\infty(\bar{Q})$ such that $\varphi(t,x) = 1$ for all $(t,x) \in K$. If $|\kappa| \le R$ ($R > 0$) we thus have, by (2.44):

$$m_+^\kappa(K) \le m_+^\kappa(\varphi) \le C_R \tag{2.45}$$

where the constant C_R depends on R (and φ) only. This yields $m_+ \in \mathcal{C}(\mathbb{R}_\xi; w*-\mathcal{M}_+(\bar{Q}))$. Indeed, if (ξ_n) is a sequence of real numbers converging to $\xi \in \mathbb{R}$, then there exists $R > 0$ such that $|\xi_n| \le R$ for every n and, by (2.45), $m^+(\varphi,\xi_n)$ is

bounded and nonnegative for every nonnegative $\varphi \in \mathcal{C}_c(\bar{Q})$. By the Banach-Steinhaus theorem, there exists $m_\infty^+ \in \mathcal{M}_+(\bar{Q})$ such that $m^+(\varphi, \xi_n) \to m_\infty^+(\varphi)$ as $n \to +\infty$ for every $\varphi \in \mathcal{C}_c(\bar{Q})$. By (2.44), we have $m_\infty^+(\varphi) = m^+(\varphi, \xi)$ for every $\varphi \in \mathcal{C}_c^\infty(\bar{Q})$, this remains true for $\varphi \in \mathcal{C}_c(\bar{Q})$ by density: therefore $m_+ \in \mathcal{C}(\mathbb{R}_\xi; w*-\mathcal{M}_+(\bar{Q}))$. Besides, from (2.44), and the fact that $u \le A$ a.e. for an $A \in \mathbb{R}$ ($u \in L^\infty(Q \times (0,1))$ by hypotheses, it appears that $m_+(\xi)$ vanishes for $\xi > A$, in particular for large ξ.
Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. We compute with an integration by part:

$$
\begin{aligned}
& \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \phi(t,x,\xi) dm_+(t,x,\xi) \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} (u-\xi)^+ \partial_t \partial_\xi \phi + (k(x)\Phi^+(u,\xi) + \Psi(u,\xi))\partial_x \partial_\xi \phi \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} (u_0 - \xi)^+ \partial_\xi \phi_{|t=0} + (k_L - k_R)^+ \int_\Sigma g(\xi) \partial_\xi \phi_{|x=0} \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} \mathrm{sgn}_+(u-\xi)(\partial_t \phi + (k(x)a(\xi) + b(\xi))\partial_x \phi) \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} \mathrm{sgn}_+(u_0-\xi)\phi_{|t=0} - (k_L - k_R)^+ \int_\Sigma a(\xi)\phi_{|x=0} \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} h_+ \left(\partial_t \phi + (k(x)a(\xi) + b(\xi))\, \partial_x \phi\right) \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} h_+^0 \phi_{|t=0} - (k_L - k_R)^+ \int_\Sigma a(\xi)\phi_{|x=0}.
\end{aligned}
$$

Therefore u is a kinetic process subsolution.

Conversely, suppose that $u \in L^\infty(Q \times (0,1))$ is a kinetic process subsolution. For $\kappa \in \mathbb{R}$, let $\xi \mapsto E_n(\xi)$ be a smooth and convex approximation of $\xi \mapsto (\xi - \kappa)^+$ such that $|E_n'(\xi)| \le 1$ for any positive integer n. Let Ψ be a smooth function with support in $[-2,2]$, values in $[0,1]$ and that equals 1 on $[-1,1]$. Next, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, and define $\Psi_n(\xi) = \Psi(\xi/n)$. Now apply (2.43) to the test function $\phi(t,x,\xi) = \varphi(t,x)\Psi_n(\xi)E_n'(\xi)$:

$$
\begin{aligned}
& \int_0^1 \int_Q \left[\int_{\mathbb{R}_\xi} \Psi_n E_n' h_+\right] \partial_t \varphi + \left[\int_{\mathbb{R}_\xi} (k(x)a(\xi) + b(\xi))\Psi_n E_n' h_+\right] \partial_x \varphi \\
+ & \int_{\mathbb{R}} \left[\int_{\mathbb{R}_\xi} \Psi_n E_n' h_+^0\right] \varphi_{|t=0} - (k_L - k_R)^+ \int_\Sigma \left[\int_{\mathbb{R}_\xi} \Psi_n E_n' a(\xi)\right] \varphi_{|x=0} \\
= & \int_{\bar{Q}\times\mathbb{R}_\xi} \varphi[\Psi_n' E_n' + \Psi_n E_n''] dm_+.
\end{aligned}
$$

If moreover φ is assumed to be nonnegative, then $\int_{\bar{Q}\times\mathbb{R}_\xi} \varphi \Psi_n E_n'' dm_+ \ge 0$ and letting

$n \to +\infty$, we get:

$$\begin{aligned}
&\int_0^1 \int_Q (u(t,x,\lambda)-\kappa)^+ \partial_t \varphi(t,x) \\
+ &\int_0^1 \int_Q (k(x)\Phi^+(u(t,x,\lambda),\kappa) + \Psi^+(u(t,x,\lambda),\kappa))\partial_x \varphi(t,x) dtdxd\lambda \\
+ &\int_{\mathbb{R}} (u_0(x)-\kappa)^+ \varphi(0,x)dx + (k_L - k_R)^+ \int_0^{+\infty} g(\kappa)\varphi(t,0)dt \geq 0,
\end{aligned}$$

which is (4.6).

2.3.4 Proof of Theorem 3.6

Kinetic traces

We introduce two functions: regularization and cut-off function. Let $\rho \in \mathcal{C}_c^\infty(0,1)$ be a nonnegative function with mass 1. For a small parameter ε, we define the regularizing kernel ρ_ε by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$$

and the cut-off function ω_ε by

$$\omega_\varepsilon(x) = \int_0^{|x|} \rho_\varepsilon(\sigma)d\sigma\,.$$

Proposition 2.2. *Let $h \in L^\infty(Q \times (0,1) \times \mathbb{R}_\xi)$ satisfy* (2.43). *Then there exists two functions $h_\pm^{\tau_0} \in L^\infty(Q \times (0,1) \times \mathbb{R}_\xi)$ and $\Upsilon_\pm \in L^\infty(\Sigma \times (0,1) \times \mathbb{R}_\xi)$ such that, up to subsequences:*

$$\lim_{\eta \to 0^+} \int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} \left[\int_0^{+\infty} h_\pm(t)\omega_\eta'(t)dt\right]\theta = \int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_\pm^{\tau_0}\theta, \tag{2.46}$$

$$\lim_{\eta \to 0^+} \int_0^1 \int_{\Sigma \times \mathbb{R}_\xi} \left[\int_{\mathbb{R}} (k(x)a(\xi)+b(\xi))h_\pm(x)\omega_\eta'(x)dx\right]\psi = \int_0^1 \int_{\Sigma \times \mathbb{R}_\xi} \Upsilon_\pm \psi \tag{2.47}$$

for any $\theta \in L_c^1(\mathbb{R}_x \times (0,1) \times \mathbb{R}_\xi)$ and any $\psi \in L_c^1(\Sigma \times (0,1) \times \mathbb{R}_\xi)$ (the subsequences with respect to η are independent of θ and ψ respectively). Besides, denoting by $m_\pm^{\tau_0}$ (resp. $\overline{m}_\pm$) the restriction of $m_\pm$ to $\{0\} \times \mathbb{R}_x \times [0,1] \times \mathbb{R}_\xi$ (resp. $[0,+\infty) \times \{0\} \times [0,1] \times \mathbb{R}_\xi$), we have: $\forall \theta \in \mathcal{C}_c^\infty(\mathbb{R}_x \times \mathbb{R}_\xi)$, $\forall \psi \in \mathcal{C}_c^\infty([0,+\infty) \times \mathbb{R}_\xi)$

$$\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_\pm^{\tau_0}\theta = \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_\pm^0 \theta - \int_{\mathbb{R}_x \times \mathbb{R}_\xi} \partial_\xi \theta dm_\pm^{\tau_0} \tag{2.48}$$

and

$$\int_0^1 \int_{\Sigma \times \mathbb{R}_\xi} \Upsilon_\pm \psi = -(k_L - k_R)^\pm \int_0^1 \int_{\Sigma \times \mathbb{R}_\xi} a(\xi)\psi - \int_{[0,+\infty) \times \mathbb{R}_\xi} \partial_\xi \psi d\overline{m}_\pm\,. \tag{2.49}$$

The existence of $h_\pm^{\tau_0}$ and $\Upsilon_\pm$ follows from the local uniform boundedness of $\int_0^{+\infty} h_\pm(t)\omega'_\eta(t)dt$ and $\int_\mathbb{R}(a(\xi)k(x)+b(\xi))h_\pm(x)\omega'_\eta(x)dx$ in $L^\infty(\mathbb{R}_x\times(0,1)\times\mathbb{R}_\xi)$ and $L^\infty(\Sigma\times(0,1)\times\mathbb{R}_\xi)$ respectively. To prove (2.48), replace ϕ in (2.43) by the function $(t,x,\xi)\mapsto\theta(x,\xi)(1-\omega_\eta)(t)$, for $\theta\in\mathcal{C}_c^\infty(\mathbb{R}_x\times\mathbb{R}_\xi)$ and pass to the limit on η in the equation thus obtained. Similarly, use the test function $(t,x,\xi)\mapsto\psi(t,\xi)(1-\omega_\eta)(x)$ in (2.43) to get (2.49).

Regularization, comparison

Suppose $k_L<k_R$. Let h and j denote the equilibrium functions associated with u and v respectively and denote by m_+ and q_- the associated entropy defect measure. For $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}&\int_0^1\int_{Q\times\mathbb{R}_\xi} h_+(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\phi+\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0\phi_{|t=0}\\ -\quad&(k_L-k_R)\int_{\Sigma\times\mathbb{R}_\xi} a(\xi)\phi_{|x=0}=\int_{\bar{Q}\times\mathbb{R}_\xi}\partial_\xi\phi dm_+\end{aligned}\tag{2.50}$$

and

$$\begin{aligned}&\int_0^1\int_{Q\times\mathbb{R}_\xi} j_-(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\phi+\int_{\mathbb{R}_x\times\mathbb{R}_\xi} j_-^0\phi_{|t=0}\\ =\quad&\int_{\bar{Q}\times\mathbb{R}_\xi}\partial_\xi\phi dq_-.\end{aligned}\tag{2.51}$$

Let $\theta\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ be a test function *with compact support in* $\mathbb{R}_t^*\times\mathbb{R}_x^*\times\mathbb{R}_\xi$ (θ vanishes in a neighborhood of $\mathbb{R}_t\times\{0\}\times\mathbb{R}_\xi$ and in a neighborhood of $\{0\}\times\mathbb{R}_x\times\mathbb{R}_\xi$). Denote by $\rho_{\beta,\nu,\sigma}$ the function $(t,x)\mapsto\rho_\beta(-t)\rho_\nu(x)\rho_\sigma(\xi)$ and by $\gamma_{\beta,\nu,\sigma}$ the function $(t,x)\mapsto\rho_{\beta,\nu,\sigma}(-t,-x,-\xi)$. For ν small enough, the function $(t,x)\mapsto\theta\star\gamma_{\beta,\nu,\sigma}$ still vanishes on $\mathbb{R}_t\times\{0\}\times\mathbb{R}_\xi$ and $\{0\}\times\mathbb{R}_x\times\mathbb{R}_\xi$ so that we can specify this test function in (2.51) to obtain

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)\star\rho_{\beta,\nu,\sigma}\partial_t\theta\\ +\quad&\int_0^1\int_{\mathbb{R}^3}\big(a(\xi)(\mathrm{sgn}_+(t)k(x)j_-)+b(\xi)(\mathrm{sgn}_+(t)j_-)\big)\star\rho_{\beta,\nu,\sigma}\partial_x\theta\\ =\quad&\int_{\mathbb{R}^3}\partial_\xi\theta\; d(\mathrm{sgn}_+(t)q_-)\star\rho_{\beta,\nu,\sigma}.\end{aligned}$$

Still for ν small enough, we have $(\mathrm{sgn}_+(t)k(x)j_-)\star\rho_{\beta,\nu,\sigma}\partial_x\theta=k(x)(\mathrm{sgn}_+(t)j_-)\star\rho_{\beta,\nu,\sigma}\partial_x\theta$ and therefore get the regularized equation

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^3} j_-^{\beta,\nu,\sigma}(t,x,\zeta,\xi)(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\theta(t,x,\xi)\\ =\quad&\int_{\mathbb{R}^3}\partial_\xi\theta(t,x,\xi)\; dq_-^{\beta,\nu,\sigma}(t,x,\xi),\end{aligned}\tag{2.52}$$

where $j_-^{\beta,\nu,\sigma} = (\mathrm{sgn}_+(t)j_-) \star \rho_{\beta,\nu,\sigma}$, $q_-^{\beta,\nu,\sigma} = (\mathrm{sgn}_+(t)q_-) \star \rho_{\beta,\nu,\sigma}$. Similarly, with obvious notations, the following regularized kinetic equation is satisfied by h_+ (for ε small enough):

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^3} h_+^{\alpha,\varepsilon,\delta}(t,x,\lambda,\xi)(\partial_t + (k(x)a(\xi)+b(\xi))\partial_x)\theta(t,x,\xi) \\ = &\int_{\mathbb{R}^3} \partial_\xi \theta(t,x,\xi)\, dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi)\,. \end{aligned} \tag{2.53}$$

Let $\varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function with compact support in $\mathbb{R}_t^* \times \mathbb{R}_x^* \times \mathbb{R}_\xi$. We apply (2.52) to the test function $\theta = -j_-^{\beta,\nu,\sigma}\varphi$ and integrate the result with respect to $\zeta \in [0,1]$ (notice that, although this test function does not vanish for large ξ, it is admissible by Remark 2.4), apply (2.53) to the test function $\theta = -h_+^{\alpha,\varepsilon,\delta}\varphi$ and integrate the result with respect to $\lambda \in [0,1]$ (admissible by Remark 2.4) and sum the two resulting equations to get

$$\begin{aligned} &\int_0^1 \int_0^1 \int_{\mathbb{R}^3} \varphi_1(\partial_t + (k(x)a(\xi)+b(\xi))\partial_x)(-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma}) \\ + \;& 2\int_{\mathbb{R}^3} (-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma})(\partial_t + (k(x)a(\xi)+b(\xi))\partial_x)\varphi_1 \\ = \;& \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1 \partial_\xi(-j_-^{\beta,\nu,\sigma}) dm_+^{\alpha,\varepsilon,\delta} \\ + \;& \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1 \partial_\xi(-h_+^{\alpha,\varepsilon,\delta}) dq_-^{\beta,\nu,\sigma}\,. \end{aligned} \tag{2.54}$$

Since $m_+^{\alpha,\varepsilon,\delta}, q_-^{\beta,\nu,\sigma} \geq 0$ and $\partial_\xi(-j_-^{\beta,\nu,\sigma}), \partial_\xi(-h_+^{\alpha,\varepsilon,\delta}) \geq 0$, the right hand-side of (2.54) is nonnegative. We integrate by parts with respect to (t,x) in the left hand-side (an operation which is admissible since φ_1 vanishes in the vicinity of the line of discontinuity of the coefficient k) to get

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma})(\partial_t + (k(x)a(\xi)+b(\xi))\partial_x)\varphi_1 \geq 0$$

and letting $\alpha, \varepsilon, \delta$ tends to zero, we have:

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-^{\beta,\nu,\sigma})(\partial_t + (k(x)a(\xi)+b(\xi))\partial_x)\varphi_1 \geq 0\,. \tag{2.55}$$

Let us now remove the condition imposed on the test function: let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function, replace φ_1 by $(t,x) \mapsto \varphi(t,x)\omega_\eta(t)\omega_{\widetilde{\eta}}(x)$ in (2.55), use

Proposition 2.2 and pass to the limit on accurate subsequences on η and $\widetilde{\eta}$ to get

$$\begin{aligned} &\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\varphi \\ +&\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\ +&\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_+(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)\,dt\,d\xi\,d\lambda\,d\zeta \\ &\geq 0\,. \end{aligned} \tag{2.56}$$

By (2.49), and since $k_L<k_R$, we have

$$\begin{aligned} &\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_+(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)\,dt\,d\xi\,d\lambda\,d\zeta \\ =\;& -\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\lambda\,d\zeta\leq 0\,. \end{aligned}$$

Similarly, by (2.48), we get

$$\begin{aligned} &\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\ \leq\;& \int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\ =\;& \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta \end{aligned}$$

and

$$\begin{aligned} &\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\varphi \\ +&\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta\geq 0\,. \end{aligned}$$

We let $\nu,\sigma\to 0$ in this last inequality, to get

$$\begin{aligned} &\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta})(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\varphi \\ +&\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} f_+^{0}(x,\xi)(h_-^{\beta}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta\geq 0 \end{aligned}$$

and now compute the limit as $\beta\to 0$ of the remaining terms. First,

$$\begin{aligned} &\lim_{\beta\to 0}\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta})(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\varphi \\ &=\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-)(\partial_t+(k(x)a(\xi)+b(\xi))\partial_x)\varphi\,, \end{aligned}$$

second,

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^\beta(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta \\
= &\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} \int_0^\infty h_+^0(x,\xi)(-j_-(s,x,\zeta,\xi))\rho_\beta(s)\varphi(0,x)\,dx\,d\xi\,d\zeta \\
= &\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} \int_0^\infty h_+^0(x,\xi)(-j_-(s,x,\zeta,\xi))\omega_\beta'(s)\varphi(0,x)\,dx\,d\xi\,d\zeta
\end{aligned}$$

and therefore, for an appropriate subsequence, we have by Proposition 2.2,

$$\begin{aligned}
&\lim_{\beta\to 0}\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^\beta(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta \\
= &\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\tau_0}(x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta.
\end{aligned}$$

The trace $j_-^{\tau_0}$ satisfy the identity

$$j_-^{\tau_0} = j_-^0 + \partial_\xi q_-^0$$

from which we deduce

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\tau_0}(x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,\zeta \\
\leq &\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^0(x,\xi))\varphi(0,x)\,dx\,d\xi\,.
\end{aligned}$$

Collecting the previous results, and using the identities

$$\begin{aligned}
&\int_{\mathbb{R}} h_+(-j_-)d\xi = (u-v)^+\,, \qquad \int_{\mathbb{R}} h_+^0(-j_-^0)d\xi = (u_0-v_0)^+\,, \\
&\int_{\mathbb{R}} a(\xi)h_+(-j_-)d\xi = \Phi^+(u,v)\,, \qquad \int_{\mathbb{R}} b(\xi)h_+(-j_-)d\xi = \Psi^+(u,v)
\end{aligned}$$

eventually leads to the inequality

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^\infty\int_{\mathbb{R}} (u-v)^+\partial_t\varphi \\
+ &\int_0^1\int_0^1\int_0^\infty\int_{\mathbb{R}} (k(x)\Phi^+(u,v) + \Psi^+(u,v))\partial_x\varphi\,dx\,dt\,d\lambda\,d\zeta \\
+ &\int_{\mathbb{R}} (u_0-v_0)^+\varphi(0,x)dx \geq 0
\end{aligned}$$

from which (2.8) is classically deduced. This concludes the proof of Theorem 3.6.

Remark 2.5. *Notice that, insofar as the proof of* (2.8) *uses the weak notion of traces introduced in subsection 2.3.4, it has not a complete algebraic character. Although this weak notion of traces – natural in the context of kinetic solution – allows us to satisfy the first requirement (R1) discussed in the introduction, the requirement (R2) is not satisfied. This could be an obstacle to the possible analysis of error estimates for approximations of problem (2.1).*

Bibliography

[AP04] E. Audusse and B. Perthame. *Uniqueness for a scalar conservation law with discontinuous flux via adapted entropies*, Research report Inria RR-5261 (2004).

[Bac] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations (nov-dec 2004), no. 11-12, 1317-1338.

[Bén72] P. Bénilan, *Equations d'évolution dans un espace de Banach quelconque et applications*, Thèse d'Etat, Orsay (1972).

[BKRT04] R. Bürger, K. H. Karlsen, N. H. Risebro, and J. D. Towers, *Well-posedness in* BV_t *and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units*, Numer. Math., **97**, (2004), no. 1, 25-65.

[DiP85] R.J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223–270.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin, *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, (2000), 713-1020.

[GMT96] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Springer-Verlag, Berlin, (1996), with a preface by Charles-Michel Marle.

[Kaa99] E. F. Kaasschieter, *Solving the Buckley-Leverett equation with gravity in a heterogeneous porous medium*, Comput. Geosci., **3**, (1999), no. 1, 23-48.

[KR95] C. Klingenberg and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness and asymptotic behavior*, Comm. Partial Differential Equations, **20**, (1995), no. 11-12, 1959-1990.

[KR01] K.H. Karlsen and N.H. Risebro. *Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients*, M2AN Math. Model. Numer. Anal., **35**, (2001), no. 2, 239-269.

[KRT02a] K. H. Karlsen, N. H. Risebro, and J. D. Towers. *Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient*, IMA J. Numer. Anal., **22**, (2002), no. 4, 623-664.

[KRT02b] K.H. Karlsen, N.H. Risebro, and J.D. Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient*, Electron. J. Differential Equations (2002), **93**, 23 pp. (electronic).

[KRT03] K. H. Karlsen, N. H. Risebro, and J. D. Towers. *L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vidensk. Selsk. (2003), no. 3, 1-49.

[Kru70] S. N. Kruzhkov. *First order quasilinear equations with several independent variables.*, Mat. Sb. (N.S.) **81 (123)** (1970), 228-255.

[KT04] K.H. Karlsen and J.D. Towers. *Convergence of the lax-friedrichs scheme and stability for conservation laws with a discontinuous space-time dependant flux*, www.math.ntnu.no/conservation/2004/005.html (2004).

[Kuz76] N. N. Kuznetsov. *The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation*, Ž. Vyčisl. Mat. i Mat. Fiz. **16** (1976), no. 6, 1489-1502, 1627.

[LPT94] P.-L. Lions, B. Perthame, and E. Tadmor. *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc. **7** (1994), no. 1, 169-191.

[Per98] B. Perthame. *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures Appl. (9) **77** (1998), no. 10, 1055-1064.

[Per02] B. Perthame. *Kinetic formulation of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, vol. 21, Oxford University Press, Oxford, (2002).

[Por03a] M. Portilheiro. *Weak solutions for equations defined by accretive operators. I*, Proc. Roy. Soc. Edinburgh Sect., A **133**, (2003), no. 5, 1193-1207.

[Por03b] M. Portilheiro. *Weak solutions for equations defined by accretive operators. II. Relaxation limits*, J. Differential Equations, **195**, (2003), no. 1, 66-81.

[PS03] B. Perthame and P. E. Souganidis. *Dissipative and entropy solutions to non-isotropic degenerate parabolic balance laws*, Arch. Ration. Mech. Anal., **170**, (2003), no. 4, 359–370.

[SV03] N. Seguin and J. Vovelle, *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221-257.

[Tow00] J.D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681-698.

[Tow01] J.D. Towers. *A difference scheme for conservation laws with a discontinuous flux: the nonconvex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197-1218.

Chapter 3

Schéma Volume Fini pour une loi de conservation à flux discontinu

3.1 Introduction

The notion of entropy solution, and the convergence of finite volume scheme are presented for the following hyperbolic conservation law:

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{3.1}$$

with initial value $u_0 \in L^\infty(\mathbb{R};[0,1])$. And finally, several numerical results are introduced.
The functions f, g and k are supposed to satisfy the following hypotheses :

(H1) $g \in \mathrm{Lip}([0,1])$ is non-negative and $g(0) = g(1) = 0$,
(H2) $f \in \mathrm{Lip}([0,1])$,
(H3) k is the discontinuous function defined by
$$k(x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x > 0 \end{cases} \quad \text{with } k_L, k_R > 0 \text{ and } k_L \neq k_R\,.$$

The particular shape of the functions f, g and k described through the hypotheses (H1), (H2), (H3) is given by a model for two-phase flow in porous media with distinct permeabilities (see [Bac04]). Let us just claim here that, in this context, the hypotheses on f, g and k are natural.

No hypothesis of convexity or genuine non-linearity on g is assumed, which is a new point in comparison with all the preceding works on the subject (see by example [Tow00, Tow01, KT04, SV03, Bac04]). Indeed, these preceding works assume that the entropy solution must have traces along the line $\{x = 0\}$. To guarantee the existence of these traces, they impose that g is genuinely non linear. Without the hypothesis on

g genuinely non linear, these traces of function can not be considered. A new difficulty is introduced. Indeed, problem (3.1) can not be considered as two conservation laws with Lipschitz continuous flux on each side of the line $\{x = 0\}$, because this approach seems to need the trace of the solution (see by example [KRT03, SV03, Bac04]).

Moreover, in [Tow00] and in [Tow01], it is only proved that a subsequence of the approximation function, build with the scheme, converges to an entropy solution. In [KT04], authors prove the convergence of the Lax-Friedrichs scheme without extraction of a subsequence but they still assume that g is genuinely non linear. In fact, they need g genuinely non linear to show the uniqueness of entropy solution, and the uniqueness permits to conclude that the whole sequence converges to the entropy solution. Recently, in [AV03, AJV04], the authors present some studies for a generalized problem for the following hyperbolic conservation law:

$$\begin{cases} \partial_t u + \partial_x(g(x,u)) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{3.2}$$

with

$$g(x,u) = \begin{cases} g_L(u) & \text{if} \quad x < 0, \\ g_R(u) & \text{if} \quad x > 0, \end{cases}$$

such that $g_L(0) = g_R(0) = g_R(1) = g_R(0)$. Assuming g_L and g_R convexs, an explicit formula of the solution to problem (3.2) is given.

To begin, the notion of entropy solution to problem (3.1) is recalled. Generically, the discontinuity of k enforces the instantaneous apparition of discontinuities in the solution to problem (3.1) (whatever the regularity of the initial value may be). In order to ensure uniqueness, weak solutions satisfying entropy inequalities have to be considered.

Definition 3.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$. A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is said to be an entropy solution to problem (3.1) if it satisfies the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\, \partial_t \varphi(t,x)\, dt\, dx \\ + &\int_0^\infty \int_{\mathbb{R}} (k(x)\, \Phi(u(t,x),\kappa) + \Psi(u(t,x),\kappa))\, \partial_x \varphi(t,x)\, dx\, dt \\ + &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \, + \, |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq 0, \end{aligned} \tag{3.3}$$

where respectively Φ and Ψ denote the entropy flux associated with the Kruzhkov entropy:

$$\begin{aligned} \Phi(u,\kappa) &= \mathrm{sgn(u - \kappa)(g(u) - g(\kappa))}, \\ \textit{and} \quad \Psi(u,\kappa) &= \mathrm{sgn(u - \kappa)(f(u) - f(\kappa))}. \end{aligned}$$

Remark 3.1. *Let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ be an entropy solution. Choosing $\kappa = 1$ in inequality (3.3) and using $g(1) = 0$, it is easy to see that $u \leq 1$ a.e.. Similarly, choosing $\kappa = 0$ in inequality (3.3), we obtain $u \geq 0$ a.e.. Then $0 \leq u \leq 1$ a.e.. This property will also be satisfied by the approximated solution given by the scheme (see Lemma 3.1).*

This paper is divided into two main parts. First, the convergence of the scheme is established. In section 3.2, the scheme is presented : this scheme is Euler explicit in time and finite volume in space. Both discretizations (in time and in space) are of first order. The aim of subsection 3.2.3 is to establish a stability property which is verified by the approximate solution given by the scheme (see Theorem 3.1). In subsection 3.2.5, some discrete entropy inequalities which are satisfied by the approximated solution are established. In particular, the monotonicity of the scheme is introduced in subsection 3.2.2 is used.
By use of stability property of the scheme, in section 3.3, the existence of entropy process solution is established. This notion appears as a generalization of entropy solution. The convergence of a subsequence to an entropy process solution is proved. Finally, using the theorem of comparison between two entropy process solutions established in [BV05], the equivalence of entropy solution and entropy process solution and the uniqueness of entropy solution are deduced. Then, the convergence of the scheme to the unique entropy solution to problem (3.1) is obtained.

Secondly, some numerical results are presented. On the one hand, the behaviour of Godunov scheme and VFRoe-ncv scheme is studied with g nor concave neither convex. The approximated function build with this scheme converges to the entropy solution. We observe, numerically, a first order convergence.
On the other hand, problem (3.1), setting $f = 0$, is equivalent to the 2×2 resonant system :

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \\ \partial_t k = 0. \end{cases} \tag{3.4}$$

This system is resonant for all values where g' is equal to zero. In fact, if the function g is constant on an interval I included in $[0, 1]$, system (3.4) is resonant on I. However, for such a function g, we show that problem (3.1) is well posed. Godunov, VFRoe-ncv schemes are presented. The convergence of these schemes is observed although the VFRoe-ncv scheme is not monotone. Moreover, all these schemes have the same behaviour.

3.2 Finite volume scheme

3.2.1 Presentation of the scheme

Definition 3.2. *An admissible mesh $\mathcal{T}$ of $\mathbb{R}$ is given by an increasing sequence of real values $(x_{i+1/2})_{i\in\mathbb{Z}}$, such that $\mathbb{R} = \bigcup_{i\in\mathbb{Z}}[x_{i-1/2}, x_{i+1/2}]$. The mesh $\mathcal{T}$ is the set of*

$\mathcal{T} = \{K_i,\, i \in \mathbb{Z}\}$ of subsets of $\mathbb{R}$ defined by $K_i = (x_{i-1/2}, x_{i+1/2})$ for all $i \in \mathbb{Z}$. The length of K_i is denoted by h_i, and set $h = \mathrm{size}(\mathcal{T}) = \sup_{i\in\mathbb{Z}} h_i$.

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.2 and let $\Delta t \in \mathbb{R}_+^*$ be the time step. To fix the notation, one assumes that $x_{1/2} = 0$.
In the general case, the finite volume scheme for the discretization of problem (3.1) can be written: $\forall i \in \mathbb{Z}$, $\forall n \in \mathbb{N}$

$$\begin{cases} \dfrac{h_i}{\Delta t}(u_i^{n+1} - u_i^n) + H(u_i^n, u_{i+1}^n, k_i, k_{i+1}) - H(u_{i-1}^n, u_i^n, k_{i-1}, k_i) = 0, \\ \\ u_i^0 = \dfrac{1}{h_i}\displaystyle\int_{K_i} u_0(x)dx, \quad k_i = \dfrac{1}{h_i}\displaystyle\int_{K_i} k(x)dx, \end{cases} \tag{3.5}$$

where u_i^n is expected to be an approximation of u at time $t_n = n\Delta t$ in cell K_i. The quantity $H(u_i^n, u_{i+1}^n, k_i, k_{i+1})$ is the numerical flux at point $x_{i+1/2}$ and time t_n associated to the function $k(x)g(u) + f(u)$.
The formulation (3.5) is equivalent to:

$$u_i^{n+1} \quad = \quad G(u_{i-1}^n, u_i^n, u_{i+1}^n, k_{i-1}, k_i, k_{i+1}). \tag{3.6}$$

The approximated finite volume solution is defined by

$$u_{\mathcal{T},\Delta t}(x,t) = u_i^n \text{ for } x \in K_i \text{ and } t \in [nk, (n+1)k). \tag{3.7}$$

The flux functions satisfy the following hypotheses :

(H4) **Flux adapted to the function g:** $\forall u, v \in [0,1]$, $H(u,v,k_L,k_L) = H_L(u,v)$, $H(u,v,k_R,k_R) = H_R(u,v)$, $H(0,0,k_L,k_R) = H(1,1,k_L,k_R) = 0$ and $H_L(0,0) = H_L(1,1) = H_R(0,0) = H_R(1,1) = 0$.

(H5) **Regularity:** The function H is locally Lipschitz continuous from $\mathbb{R}^4$ to $\mathbb{R}$ and admits as Lipschitz constant $L_{k,g,f}$ only depending of k, g and f.

(H6) **Consistency** : $\forall u \in [0,1]$, $H_L(u,u) = k_L g(u) + f(u)$ and $H_R(u,u) = k_R g(u) + f(u)$.

(H7) **Monotonicity:** $(u, v, k_1, k_2) \mapsto H(u, v, k_1, k_2)$, from $[0,1]^4$ to $\mathbb{R}$, is nondecreasing with respect to u, k_1, k_2, and nonincreasing with respect to v.

3.2.2 Monotonicity of the scheme and L^∞ estimate

Lemma 3.1. *Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.2 and let $\Delta t \in \mathbb{R}_+^*$ be the time step. Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$.
Let $u_{\mathcal{T},\Delta t}$ be the finite volume approximated solution defined by (3.7). Under the CFL condition*

$$\Delta t \le \frac{\inf_{i\in\mathbb{Z}} h_i}{2L_{k,g,f}} \tag{3.8}$$

the function G is nondecreasing with respect to its three first arguments and the approximation $u_{\mathcal{T},\Delta t}$ satisfies

$$0 \leq u_{\mathcal{T},k} \leq 1 \quad \text{for a.e. } x \in \mathbb{R} \text{ and a.e. } t \in \mathbb{R}_+. \tag{3.9}$$

For the proof, we assume for simplicity that G is $\mathcal{C}^1$. Under the CFL condition, the partial differentials of G defined by (3.6) are non negative. Then, the monotonicity of function G and the following equalities : $G(0,0,0,.,.,.) = 0$ and $G(1,1,1,.,.,.) = 1$ are used.
G is nondecreasing with respect to its three first arguments :

- $$\begin{aligned}\frac{\partial G}{\partial u_i^n} &= 1 - \frac{\Delta t}{h_i} H_u(u_i^n, u_{i+1}^n, k_i, k_{i+1}) + \frac{\Delta t}{h_i}(H_v(u_{i-1}^n, u_i^n, k_{i-1}, k_i) \\ &\geq 1 - 2\frac{\Delta t}{h_i} L_{k,g,f} \geq 0,\end{aligned}$$

  under the CFL condition.

- $$\frac{\partial G}{\partial u_{i+1}^n} = -\frac{\Delta t}{h_i} H_v(u_i^n, u_{i+1}^n, k_i, k_{i+1}) \geq 0,$$

  because H is nonincreasing with respect to its second argument.

- $$\frac{\partial G}{\partial u_{i-1}^n} = \frac{\Delta t}{h_i} H_u(u_{i-1}^n, u_i^n, k_{i-1}, k_i) \geq 0,$$

  because H is nondecreasing with respect to its first argument.

By hypothesis, $0 \leq u_i^0 \leq 1$ a.e. on $\mathbb{R}$. For all i in $\mathbb{Z}$, using monotonicity argument, we obtain :

$$\begin{aligned}G(0,0,0,k_{i-1},k_i,k_{i+1}) \leq u_i^1 &= G(u_i^0, u_{i-1}^0, u_{i+1}^0, k_{i-1}, k_i, k_{i+1}) \\ &\leq G(1,1,1,k_{i-1},k_i,k_{i+1})\end{aligned}$$

with $G(0,0,0,k_{i-1},k_i,k_{i+1}) = 0$ and $G(1,1,1,k_{i-1},k_i,k_{i+1}) = 1$, because $g(0) = g(1) = 0$ (see (H1)).
Inequality (3.9) is deduced by induction on n.

3.2.3 Weak BV estimates

Theorem 3.1. *Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given values. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.2 such that $\alpha h \leq h_i$ for all $i \in \mathbb{Z}$. Let $\Delta t \in \mathbb{R}_+^*$ satisfying the CFL condition*

$$\Delta t \leq \frac{(1-\xi)\alpha \inf_{i\in\mathbb{Z}} h_i}{2L_{k,g,f}}. \tag{3.10}$$

Let $\{u_i^n,\ i \in \mathbb{Z},\ n \in \mathbb{N}\}$ be given by the finite volume scheme (3.5). Let $R \in \mathbb{R}_+^*$ and $T \in \mathbb{R}_+^*$ and assume $h < R$ and $\Delta t < T$. Let $i_0, i_2 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ such that: $-R \in \bar{K}_{i_0}$, $R \in \bar{K}_{i_2}$ and $T \in]N_T\Delta t, (N_T+1)\Delta t]$. Then there exists $C \in R_+^*$, only depending on g, f, R, T, u_0, ξ and α, such that

$$\begin{aligned}
&\sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)| \\
&\qquad\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(q) + f(q) - H_L(p,q)| \\
+ &\sum_{n=0}^{N_T} \Delta t \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_R g(q) + f(q) - H_R(p,q)| \\
&\qquad\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_R g(q) + f(q) - H_R(p,q)| \le \frac{C}{\sqrt{h}},
\end{aligned} \tag{3.11}$$

with for a, b real values, $\mathcal{C}(a,b) = \{(p,q) \in [a\bot b, a\top b]; (q-p)(b-a) \ge 0\}$.

To establish this estimate, some tools introduced in [EGH00] for conservation laws are used. But in this preceding work, they strongly use that k is Lipschitz continuous which is not the case in this work. Firstly, we focus the study on the left and on the right on the line $\{x = 0\}$. Secondly, the scheme is studied around $\{x = 0\}$.

3.2.4 Proof of Theorem 3.1

In order to prove (3.11), equality (3.5) is multiplied by $h_i u_i^n$ and the result is summed over $i = i_0, \dots, -1$ or over $i = 1, \dots, i_2$, and over $n = 0, \dots, N_T$.

Remark 3.2. *In this part, C_j denotes constant only depending on k, g, f, T, R, u_0, ξ, α.*

On the one hand, $\boxed{\text{for } i = i_0, \dots, -1,\ k_{i-1} = k_i = k_{i+1} = k_L}$, the sum gives:

$$B_1 + B_2 = 0$$

where

$$B_1 = \sum_{n=0}^{N_T} \sum_{i=i_0}^{-1} h_i (u_i^{n+1} - u_i^n) u_i^n, \tag{3.12}$$

$$B_2 = \sum_{n=0}^{N_T} \sum_{i=i_0}^{-1} \Delta t \left(H_L(u_i^n, u_{i+1}^n) - H_L(u_{i-1}^n, u_i^n) \right) u_i^n. \tag{3.13}$$

Each term is studied separately.

1. **Study of term B_2**
   A change of index permits to obtain:

$$
\begin{aligned}
B_2 &= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} \Delta t(H_L(u_{i-1}^n,u_i^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N_T}\sum_{i=i_0-1}^{-2} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n)))u_{i+1}^n \\
&= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} \Delta t(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n)))u_i^n \\
&\qquad -(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n)))u_{i+1}^n \\
&- \sum_{n=0}^{N_T} \Delta t(H_L(u_{i_0-1}^n,u_{i_0}^n) - (k_L g(u_{i_0}^n) + f(u_{i_0}^n)))u_{i_0}^n \\
&+ \sum_{n=0}^{N_T} \Delta t(H_L(u_{-1}^n,u_0^n) - (k_L g(u_0^n) + f(u_0^n)))u_0^n \\
&= B_2^1 + B_2^2,
\end{aligned}
$$

with

$$
\begin{aligned}
B_2^1 = \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} \Delta t\Big(&\big(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n))\big)u_i^n \\
&-\big(H_L(u_i^n,u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n))\big)u_{i+1}^n\Big),
\end{aligned}
$$

and

$$
|B_2^2| \;\le\; C_1.
$$

Denoting by Φ_L a primitive of the function $(.)k_L g'(.) + (.)f'(.)$, an integration by parts yields, for all a, b real values

$$
\begin{aligned}
\Phi_L(b) - \Phi_L(a) &= \int_a^b s\,(k_L g'(s) + f'(s))\,ds \\
&= a\big(H_L(a,b) - (k_L g(a) + f(a))\big) \\
&- b\big(H_L(a,b) - (k_L g(b) + f(b))\big) \\
&- \int_a^b \big(k_L g(s) + f(s) - H_L(a,b)\big)\,ds.
\end{aligned}
$$

Then, B_2^1 becomes :

$$\begin{aligned} B_2^1 &= \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \Phi_L(u_{i+1}^n) - \Phi_L(u_{i+1}^n) \\ &+ \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \int_{u_i^n}^{u_{i+1}^n} \big(k_L g(s) + f(s) - H_L(u_i^n, u_{i+1}^n)\big)\, ds \\ &= B_2^{1,1} + B_2^{1,2}, \end{aligned}$$

with, immediately $|B_2^{1,1}| \le C_2$. For study term $B_2^{1,2}$, one needs the following result:

Lemma 3.2. *Let $f \in \mathcal{C}(\mathbb{R})$ and $j \in \mathcal{C}(\mathbb{R}^2)$ Lipschitz continuous which satisfies for all $s \in \mathbb{R}$ $j(s,s) = f(s)$ and which is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument. Let j_1 and j_2 be the Lipschitz constants of j with respect to its two variables. Let $(a,b) \in \mathbb{R}^2$, then f and j satisfy the following inequality :*

$$\begin{aligned} \int_a^b \big(f(s) - j(a,b)\big)\, ds \ \ge\ & \frac{1}{2(j_1 + j_2)} \Big(\max_{(p,q)\in\mathcal{C}(a,b)} \big(f(p) - j(p,q)\big)^2 \\ & + \max_{(p,q)\in\mathcal{C}(a,b)} \big(f(q) - j(p,q)\big)^2 \Big). \end{aligned}$$

The reader can find the proof of this lemma in the Handbook of numerical analysis [EGH00] (page 915).

Using $H_L(s,s) = k_L g(s) + f(s)$ with H_L nondecreasing with respect to its first argument and nonincreasing with respect to its second argument, and applying Lemma 3.2 to $k_L g + f$ and H_L, $B_2^{1,2}$, we get

$$\begin{aligned} B_2^{1,2} \ \ge\ & \frac{1}{2\, L_{k,g,f}} \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \Big(\max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(k_L g(p) + f(p) - H_L(p,q)\big)^2 \\ & + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(k_L g(q) + f(q) - H_L(p,q)\big)^2 \Big). \end{aligned}$$

Then, this yields

$$\begin{aligned} B_2 \ \ge\ & \frac{1}{2\, L_{k,g,f}} \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \Big(\max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(k_L g(p) + f(p) - H_L(p,q)\big)^2 \\ & + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(k_L g(q) + f(q) - H_L(p,q)\big)^2 \Big) \\ & - (C_1 + C_2). \end{aligned} \tag{3.14}$$

2. **Study of B_1**
Using the definition of B_1 (3.12), one has

$$\begin{aligned} B_1 &= -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2 + \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^{N_T+1})^2 \\ &\geq -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2. \end{aligned} \tag{3.15}$$

Using scheme (3.5), for $i \in \{i_0, \ldots, -1\}$, with the CFL condition (3.10), this yields

$$\begin{aligned} h_i(u_i^{n+1}-u_i^n)^2 &= \frac{\Delta t^2}{h_i}\Big([H_L(u_i^n, u_{i+1}^n) - (g_L(u_i^n) + f(u_i^n))] \\ &\quad -[H_L(u_{i-1}^n, u_i^n) - (g_L(u_i^n) + f(u_i^n))]\Big)^2 \\ &\leq \frac{(1-\xi)\Delta t}{L_{k,g,f}} \\ &\quad \times\Big([H_L(u_i^n, u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n))]^2 \\ &\quad +[H_L(u_{i-1}^n, u_i^n) - (k_L g(u_i^n) + f(u_i^n))]^2\Big). \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} h_i(u_i^{n+1}-u_i^n)^2 \\ \leq\ & \frac{(1-\xi)}{2L_{k,g,f}}\Big(\sum_{n=0}^{N_T}\Delta t\sum_{i=i_0}^{-1}[H_L(u_i^n, u_{i+1}^n) - (k_L g(u_i^n) + f(u_i^n))]^2 \\ &\quad +[H_L(u_i^n, u_{i+1}^n) - (k_L g(u_{i+1}^n) + f(u_{i+1}^n))]^2\Big) \\ &\quad +C_5 \\ \leq\ & \frac{(1-\xi)}{2L_{k,g,f}} \\ &\times\Big(\sum_{n=0}^{N_T}\Delta t\sum_{i=i_0}^{-1}\max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[k_L g(p) + f(p) - H_L(p,q)]^2 \\ &\quad + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[k_L g(q) + f(q) - H_L(p,q)]^2\Big) \\ &\quad +C_5. \end{aligned} \tag{3.16}$$

Using the preceding inequality, equation (3.15) gives

$$\begin{aligned} B_1 \quad \geq \quad & -\frac{(1-\xi)}{2L_{k,g,f}} \\ & \times \Big(\sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(p) + f(p) - H_L(p,q)]^2 \\ & \qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(q) + f(q) - H_L(p,q)]^2 \Big) - C_6, \end{aligned} \tag{3.17}$$

with $C_6 = C_5 + \dfrac{1}{2}\displaystyle\sum_{i=i_0}^{-1}(u_i^0)^2$.

3. **Final estimate**
Adding (3.14) and (3.17) and using $B_1 + B_2 = 0$, this yields

$$\begin{aligned} 0 \quad = \quad & B_1 + B_2 \\ \geq \quad & \frac{\xi}{2L_{k,g,f}} \\ & \times \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(p) + f(p) - H_L(p,q)]^2 \\ & \qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(q) + f(q) - H_L(p,q)]^2 \\ - \quad & \bar{C}_7. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(p) + f(p) - H_L(p,q)]^2 \\ & + \qquad \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(q) + f(q) - H_L(p,q)]^2 \leq C_7. \end{aligned} \tag{3.18}$$

On the second hand, for $i = 2, \ldots, i_2$, $k_{i-1} = k_i = k_{i+1} = k_R$, in the same manner as above

$$\begin{aligned} & \sum_{n=0}^{N_T} \Delta t \sum_{i=2}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_R g(p) + f(p) - H_R(p,q)]^2 \\ & \qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_8. \end{aligned} \tag{3.19}$$

Moreover

$$\sum_{n=0}^{N_T} \Delta t \max_{(p,q)\in\mathcal{C}(u_1^n,u_2^n)} [k_R g(p) + f(p) - H_R(p,q)]^2 + \max_{(p,q)\in\mathcal{C}(u_1^n,u_2^n)} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_9, \tag{3.20}$$

because $\sum_{n=0}^{N_T} \Delta t \leq T$.

Finally, adding (3.18), (3.19) and (3.20), this yields:

$$\begin{aligned} &\sum_{n=0}^{N_T} \Delta t \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(p) + f(p) - H_L(p,q)]^2 \\ &\quad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_L g(q) + f(q) - H_L(p,q)]^2 \\ + &\sum_{n=0}^{N_T} \Delta t \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_R g(p) + f(p) - H_R(p,q)]^2 \\ &\quad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [k_R g(q) + f(q) - H_R(p,q)]^2 \leq C_{12}. \end{aligned}$$

To obtain estimate (3.11) and conclude the proof of Theorem 3.1, it is sufficient to apply the Cauchy-Schwartz inequality to the preceding inequality.

3.2.5 Discrete entropy inequalities

Theorem 3.2. *Under (H4) to (H7), let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.2 and $\Delta t \in \mathbb{R}_+^*$ the time step. Let $\{u_i^n, i \in \mathbb{Z}, n \in \mathbb{N}\}$ be given by (3.5); then for all $\kappa \in [0,1]$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following inequality holds :*

$$|u_i^{n+1} - \kappa| \leq |u_i^n - \kappa| - \frac{\Delta t}{h_i}(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n) + \frac{\Delta t}{h_i}|\Delta h^i| \tag{3.21}$$

with

$$\begin{aligned} G_{i+\frac{1}{2}}^n &= H(u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_i, k_{i+1}) - H(u_i^n \bot \kappa, u_{i+1}^n \bot \kappa, k_i, k_{i+1}), \\ \textit{and } |\Delta h^i| &= |H(\kappa,\kappa,k_i,k_{i+1}) - H(\kappa,\kappa,k_{i-1},k_i)|. \end{aligned}$$

The proof is based on the monotonicity of the scheme and on the following equality: $u\top\kappa - u\bot\kappa = |u-\kappa|$ with $u\top\kappa = max(u,\kappa)$ and $u\bot\kappa = min(u,\kappa)$.

Let $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $\kappa \in [0,1]$ and $\lambda_i := \dfrac{\Delta t}{h_i}$.

The proof is divided into two steps according to the sign of Δh^i.

1. Assume that $\Delta h^i \geq 0$.
   On the one hand, by monotonicity, this yields :
$$\begin{aligned} u_i^{n+1} - \lambda_i\,\Delta h^i &\leq u_i^{n+1} = G(u_{i-1}^n, u_i^n, u_i^n, k_{i-1}, k_i, k_{i+1}) \\ &\leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \end{aligned} \tag{3.22}$$
   and
$$\kappa - \lambda_i\,\Delta h^i \leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}), \tag{3.23}$$
   then with (3.22) and (3.23)
$$(u_i^{n+1} - \lambda_i\,\Delta h^i)\top(\kappa - \lambda_i\,\Delta h^i) \leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}),$$
   and
$$(u_i^{n+1} \top \kappa) \leq G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) + \lambda_i\,\Delta h^i. \tag{3.24}$$
   On the other hand,
$$\kappa \geq \kappa - \lambda_i\,\Delta h^i \geq G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}),$$
   and
$$u_i^{n+1} \geq G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}),$$
   then
$$u_i^{n+1} \perp \kappa \geq G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_i^n \perp \kappa, k_{i-1}, k_i, k_{i+1}). \tag{3.25}$$
   Finally, combining (3.24) and (3.25) yields :
$$\begin{aligned} |u_i^{n+1} - \kappa| &= (u_i^{n+1} \top \kappa) - (u_i^{n+1} \perp \kappa) \\ &\leq \quad G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad - G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad + \lambda_i \Delta h^i \\ &\leq \quad G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad - G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad + \lambda_i\,|\Delta h^i|. \end{aligned} \tag{3.26}$$

2. If $\Delta h^i \leq 0$, in the same manner
$$\begin{aligned} |u_i^{n+1} - \kappa| &= (u_i^{n+1} \top \kappa) - (u_i^{n+1} \perp \kappa) \\ &\leq \quad G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad - G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad - \lambda_i \Delta h^i \\ &\leq \quad G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad - G(u_{i-1}^n \perp \kappa, u_i^n \perp \kappa, u_{i+1}^n \perp \kappa, k_{i-1}, k_i, k_{i+1}) \\ &\quad + \lambda_i\,|\Delta h^i|. \end{aligned} \tag{3.27}$$

Eventually, this yields for all $\kappa \in [0,1]$

$$\begin{aligned} |u_i^{n+1} - \kappa| \quad \leq \quad & G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) \\ & - G(u_{i-1}^n \bot \kappa, u_i^n \bot \kappa, u_{i+1}^n \bot \kappa, k_{i-1}, k_i, k_{i+1}) \\ & + \lambda_i \, |\Delta h^i|. \end{aligned}$$

Eventually,

$$\begin{aligned} & G(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa, k_{i-1}, k_i, k_{i+1}) - \, G(u_{i-1}^n \bot \kappa, u_i^n \bot \kappa, u_{i+1}^n \bot \kappa, k_{i-1}, k_i, k_{i+1}) \\ = \quad & |u_i^n - \kappa| - \lambda_i \big(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n \big). \end{aligned}$$

Then, for all $\kappa \in [0,1]$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$|u_i^{n+1} - \kappa| \leq |u_i^n - \kappa| - \lambda_i \big(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n \big) + \lambda_i \, |\Delta h^i|.$$

3.3 Entropy process solution

Now the convergence of the scheme to an entropy process solution is presented. This convergence result is obtained in the sense of "nonlinear weak-$\star$ convergence", defined in [EGH00], which is a convenient way to understand the convergence towards a Young's measure (see [DiP85]):

Definition 3.3. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$, $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in the* nonlinear weak-$\star$ sense *if*

$$\int_\Omega h(u_n(x))\psi(x)\, dx \to \int_0^1 \int_\Omega h(u(x,\alpha))\psi(x) dx\, d\alpha, \;\; \textit{as } n \to +\infty \\ \forall \psi \in L^1(\Omega),\ \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{3.28}$$

Otherwise speaking, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L^\infty(\Omega \times (0,1))$ in the nonlinear weak-$\star$ sense if, for every $h \in \mathcal{C}(\mathbb{R},\mathbb{R})$, the nonlinear expression $g(u_n)$ converges in $L^\infty(\Omega)$ weak-$\star$ to a limit which has the structure $\int_0^1 h(u(\cdot,\alpha))d\alpha$. The fact is, that any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$*$ sense :

Theorem 3.3. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence converging in the nonlinear weak-$\star$ sense.*

The notion of entropy process solution is adapted to problem (3.1) as follows :

Definition 3.4. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. in $\mathbb{R}$. Let $u \in L^\infty(\mathbb{R}_+^* \times \mathbb{R} \times (0,1))$. The function u is an entropy process solution of problem (3.1) if for any*

$\kappa \in [0,1]$ *and any* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ *non negative,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \int_0^1 |u(t,x,\alpha) - \kappa|\, \partial_t \varphi(t,x)\, dt\, dx\, d\alpha \\ + &\int_0^\infty \int_{\mathbb{R}} \int_0^1 (k(x)\, \Phi(u(t,x,\alpha),\kappa) + \Psi(u(t,x,\alpha),\kappa))\, \partial_x \varphi(t,x)\, dx\, dt\, d\alpha \\ + &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \;+\; |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \geq 0. \end{aligned} \tag{3.29}$$

Theorem 3.4. *Let* $u_0 \in L^\infty(\mathbb{R})$ *with* $0 \leq u_0 \leq 1$ *a.e. in* $\mathbb{R}$*. Let* $\xi \in (0,1)$ *and* $\alpha \in (0,1)$ *be given. Let* $(\mathcal{T}_m)_{m\in\mathbb{N}}$ *be a sequence of an admissible mesh in the sense of Definition 3.2 such that for all* $m \in \mathbb{N}$*,* $i \in \mathbb{N}$*,* $\alpha\,\mathrm{size}(\mathcal{T}_\mathrm{m}) \leq \mathrm{h_i^m}$*. Let* $(\Delta t_m)_{m\in\mathbb{N}}$ *be a sequence of real positive values satisfying the CFL condition (3.10).*
For all $m \in \mathbb{N}$*, let* $u_{\mathcal{T}_m,\Delta t_m}$ *be the finite volume approximated solution defined by (3.7). Then a subsequence of* $(u_{\mathcal{T}_m,\Delta t_m})_{m\in\mathbb{N}}$ *converges towards* $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$ *in the weak-*$\star$ *nonlinear sense, as* $\bar{h}_m := \mathrm{size}(\mathcal{T}_\mathrm{m}) \to 0$ *and* v *is an entropy process solution to problem (3.1).*

3.3.1 Proof of Theorem 3.4

By monotonicity of the scheme and as $0 \leq u_0 \leq 1$ a.e., $|u_{\mathcal{T}_m,\Delta t_m}| \leq 1$ for all $m \in \mathbb{N}$. Then by convergence in the non linear weak-$\star$ sense, there exists a subsequence of $(u_{\mathcal{T}_m,\Delta t_m})_{m\in\mathbb{N}}$ and $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$ such that this subsequence converges to v in the weak-$\star$ nonlinear sense.

To establish that v is an entropy process solution, equation (3.21) is multiplied by $\dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \int_{K_i} \varphi(t,x)\, dt\, dx$ and one sums over i and n. The new issues (compared in [EGH00]) are the study around $x = 0$ and the study of the last term given by $\displaystyle\sum_{i\in\mathbb{Z}} \sum_{n\in\mathbb{N}} |\Delta h^i| \frac{1}{h_i} \int_{t^n}^{t^{n+1}} \int_{K_i} \varphi(t,x)\, dt\, dx.$

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $m \in \mathbb{N}$. Let $\mathcal{T}_m = \mathcal{T}$ and $\Delta t_m = \Delta t$. As $\mathrm{supp}(\varphi)$ is compact, there exists $T > 0$ and $R > 0$ such that $\mathrm{supp}\varphi \subset [0,\mathrm{T}] \times [-\mathrm{R}+\mathrm{h}, \mathrm{R}-\mathrm{h}]$. Let i_0, i_2 and N_T be as defined in Theorem 3.1.
Let $\kappa \in [0,1]$, multiplying equation (3.21) by $\dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \int_{K_i} \varphi(t,x)\, dt\, dx$, and summing over $i = i_0, \ldots, i_2$ and $n = 0, \ldots, N_T$, yields :

$$A_1 + A_2 \leq A_3.$$

Each term is studied separately.

Study of term A_1

$$\begin{aligned}
A_1 &= \sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\Big(|u_i^{n+1}-\kappa|-|u_i^n-\kappa|\Big)\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx\\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}|u_i^n-\kappa|\int_{t^n}^{t^{n+1}}\int_{K_i}\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t}\,dt\,dx\\
&\quad -\sum_{i=i_0}^{i_2}|u_i^0-\kappa|\frac{1}{\Delta t}\int_0^k\int_{K_i}\varphi(t,x)\,dt\,dx\\
&= B_1+B_2.
\end{aligned}\tag{3.30}$$

In fact, for this term, the convergence of $u_{\mathcal{T},\Delta t}$ to v for the weak-$\star$ non linear convergence as h tends to zero is used.
On the one hand

$$\begin{aligned}
B_2 &= -\sum_{i=i_0}^{i_2}|u_i^0-\kappa|\frac{1}{\Delta t}\int_0^k\int_{K_i}\varphi(t,x)\,dt\,dx\\
&= -\frac{1}{\Delta t}\int_0^k\int_{-R}^{R}|u_{\mathcal{T},0}-\kappa|\varphi(t,x)\,dt\,dx,
\end{aligned}\tag{3.31}$$

with $u_{\mathcal{T},0}=\sum_{i\in\mathbb{Z}}u_i^0 1_{K_i}$.
However $u_{\mathcal{T},0}$ converges towards u_0 in $L^1_{loc}(\mathbb{R})$ and $\dfrac{1}{\Delta t}\displaystyle\int_0^{\Delta t}\varphi(t,x)\,dt$ converges towards $\varphi(0,x)$ as size($\mathcal{T}$) tends to zero. This yields

$$B_2 \;\to\; \int_{\mathbb{R}}|u_0(x)-\kappa|\varphi(0,x)\,dt\,dx,\quad \text{as } h \text{ tends to zero.},$$

On the other hand,

$$\begin{aligned}
B_1 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}|u_i^n-\kappa|\int_{t^n}^{t^{n+1}}\int_{K_i}\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t}\,dt\,dx\\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}|u_{\mathcal{T},k}(t,x)-\kappa|\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t}\,dt\,dx\\
&= -\int_0^T\int_{-R}^{R}|u_{\mathcal{T},k}(t,x)-\kappa|\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t}\,dt\,dx.
\end{aligned}$$

$u_{\mathcal{T},k}$ converges towards v in the nonlinear weak-$\star$ sense as $h\to 0$, then

$$\int_0^T\int_{-R}^{R}|u_{\mathcal{T},k}(t,x)-\kappa|\,dt\,dx\underset{h\to 0}{\to}\int_0^T\int_{-R}^{R}\int_0^1|v(t,x,\alpha)-\kappa|\,dt\,dx.$$

and by use of the regularity of the function φ

$$\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t} \underset{h\to 0}{\to} \partial_t\varphi(t,x).$$

then

$$B_1 \underset{h\to 0}{\to} -\int_0^T\int_{-R}^R\int_0^1 |v(t,x,\alpha)-\kappa|\partial_t\varphi(t,x)\,dt\,dx.$$

One concludes

$$\begin{aligned}\lim_{h\to 0} A_1 &= -\int_0^T\int_{\mathbb{R}}\int_0^1 |v(t,x,\alpha)-\kappa|\partial_t\varphi(t,x)\,dt\,dx\,d\alpha \\ &\quad -\int_0^T |u_0(x)-\kappa|\varphi(0,x)\,dx. \end{aligned} \tag{3.32}$$

Study of term A_2

Term A_2 is defined by:

$$\begin{aligned} A_2 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\frac{1}{h_i}(G^n_{i+1/2}-G^n_{i-1/2})\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N_T}\frac{1}{h_i}(G^n_{i+1/2}-G^n_{i-1/2})\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx, \end{aligned} \tag{3.33}$$

because $supp(\varphi)\subset[-R+h,R-h]$.

This term A_2 is new compared with a conservation law with Lipschitz continuous flux function. The discontinuity of the function k introduces new difficulties. Then, several steps are needed to establish that

$$\lim_{h\to 0} A_2 = -\int_0^1\int_0^\infty\int_{\mathbb{R}}\big(k(x)\Phi(v(t,x,\alpha),\kappa)+\Psi(v(t,x,\alpha),\kappa)\big)\partial_x\varphi(t,x)\,dx\,dt\,d\alpha.$$

• At first

$$\lim_{h\to 0}|A_2-A_{20}|=0 \tag{3.34}$$

with A_{20} defined as follows:

$$\begin{aligned} A_{20} &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}G^n_{i+1/2}\int_{t^n}^{t^{n+1}}\int_{K_i}\partial_x\varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N_T}\big(G^n_{i+1/2}-G^n_{i-1/2}\big)\int_{t^n}^{t^{n+1}}\varphi(t,x_{i+1/2})\,dt. \end{aligned}$$

The difference between these terms is majored as follows:

$$
\begin{aligned}
& |A_2 - A_{20}| \\
\leq \; & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|G^n_{i+1/2} - G^n_{i-1/2}\big| \\
& \qquad \int_{t^n}^{t^{n+1}} \Big(\big|\varphi(t, x_{i+1/2}) - \frac{1}{h_i} \int_{K_i} \varphi(t,x)\, dx\big| \Big)\, dt \\
\leq \; & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|G^n_{i+1/2} - G^n_{i-1/2}\big| \\
& \qquad \Big(\int_{t^n}^{t^{n+1}} \frac{1}{h_i} \int_{K_i} \big|\varphi(t, x_{i+1/2}) - \varphi(t,x)\big|\, dx \Big)\, dt \\
\leq \; & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|G^n_{i+1/2} - G^n_{i-1/2}\big| \mathrm{Lip}(\varphi)\, \Delta t\, h \\
\leq \; & \mathrm{Lip}(\varphi)\, h \Big(\sum_{i=i_0+1}^{-2} \sum_{n=0}^{N_T} \Delta t\, \big|G^n_{i+1/2} - G^n_{i-1/2}\big| \\
& \qquad + \sum_{2}^{i_2-1} \sum_{n=0}^{N_T} \Delta t\, \big|G^n_{i+1/2} - G^n_{i-1/2}\big| \Big) \\
+ \; & \mathrm{Lip}(\varphi)\, h \sum_{i=-1}^{1} \sum_{n=0}^{N_T} \Delta t\, \big|G^n_{i+1/2} - G^n_{i-1/2}\big|. \qquad (3.35)
\end{aligned}
$$

∗ For $i = i_0, \cdots, -2$, $k_i = k_{i+1} = k_L$ and

$$
\begin{aligned}
|G^n_{i+1/2} - G^n_{i-1/2}| \;\leq\; & |H_L(u^n_i \top \kappa, u^n_{i+1} \top \kappa) - (k_L g(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
+ \; & |H_L(u^n_i \perp \kappa, u^n_{i+1} \perp \kappa) - (k_L g(u^n_i \perp \kappa) + f(u^n_i \perp \kappa))| \\
+ \; & |H_L(u^n_{i-1} \top \kappa, u^n_i \top \kappa) - (k_L g(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
+ \; & |H_L(u^n_{i-1} \perp \kappa, u^n_i \perp \kappa) - (k_L g(u^n_i \perp \kappa) + f(u^n_i \perp \kappa))|.
\end{aligned}
$$

then

$$\begin{aligned}
&\sum_{i=i_0+1}^{-2}\sum_{n=0}^{N_T} k|G^n_{i+1/2}-G^n_{i-1/2}| \\
\leq\ & 2\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} k\big(|H_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n))| \\
&\qquad +|H_L(u_i^n,u_{i+1}^n)-(k_L g(u_{i+1}^n)+f(u_{i+1}^n))|\big) \\
\leq\ & 2\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} k\big(\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)}|k_L g(p)+f(p)-H_L(p,q)| \\
&\qquad +\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)}|k_L g(q)+f(q)-H_L(p,q)|\big) \\
\leq\ & 2C\frac{1}{\sqrt{h}} \qquad (3.36)
\end{aligned}$$

using the weak-BV estimate (3.11).

∗ For $i=2,\cdots,i_2$, $k_{i-1}=k_i=k_{i+1}=k_R$. In the same manner as above

$$\begin{aligned}
&\sum_{i=2}^{i_2-1}\sum_{n=0}^{N_T} \Delta t|G^n_{i+1/2}-G^n_{i-1/2}| \\
\leq\ & 2\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} \Delta t\big(|H_R(u_i^n,u_{i+1}^n)-(k_R g(u_i^n)+f(u_i^n))| \\
&\qquad +|H_R(u_i^n,u_{i+1}^n)-(k_R g(u_{i+1}^n)+f(u_{i+1}^n))|\big) \\
\leq\ & 2\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} \Delta t\big(\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)}|k_R g(p)+f(p)-H_R(p,q)| \\
&\qquad +\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)}|k_R g(q)+f(q)-H_R(p,q)|\big) \\
\leq\ & 2C\frac{1}{\sqrt{h}}. \qquad (3.37)
\end{aligned}$$

∗ We can notice that

$$\sum_{i=-1}^{1}\sum_{n=0}^{N_T} \Delta t\left|G^n_{i+1/2}-G^n_{i-1/2}\right| \leq C\sum_{n=0}^{N_T}\Delta t \leq CT \qquad (3.38)$$

∗ Finally, with (3.36), (3.37) and (3.38), inequality (3.35) becomes:

$$|A_2-A_{20}|\leq C\sqrt{h}\longrightarrow 0, \text{ as } h\to 0.$$

• Now, we will prove that

$$\lim_{h\to 0} |A_{20} - \bar{A}_{20}| = 0 \tag{3.39}$$

with $\bar{A}_{20}$ defined as follows:

$$\begin{aligned}
\bar{A}_{20} &:= -\int_0^t \int_{\mathbb{R}} \big(k(x)\Phi(v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\big(k(x)\Phi(v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx
\end{aligned}$$

The proof of this equality has to take into account the different value of i.

$*$ For $i = i_0, \cdots, -1$, one has $k_{i-1} = k_i = k_L$,
$*$ For $i = 1, \cdots, i_2$, one has $k_{i-1} = k_i = k_R$,
So $A_{20} = A_{20}^1 + A_{20}^2 + A_{20}^3$ and $\bar{A}_{20} = \bar{A}_{20}^1 + \bar{A}_{20}^2 + \bar{A}_{20}^3$ with

$$\begin{aligned}
A_{20}^1 &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} G_{i+\frac{1}{2}}^n \int_{t^n}^{t^{n+1}}\int_{K_i}\partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\big(H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\qquad\qquad -H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^2 &= -\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\big(H(u_0^n\top\kappa, u_1^n\top\kappa, k_L, k_R) - H(u_0^n\bot\kappa, u_1^n\bot\kappa, k_L, k_R)\big) \\
&\qquad\qquad \partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^3 &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} G_{i+\frac{1}{2}}^n \int_{t^n}^{t^{n+1}}\int_{K_i}\partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\big(H_R(u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\qquad\qquad -H_R(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\bar{A}_{20}^1 &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big(g_L(v\top\kappa) - g_L(v\bot\kappa) \\
&\qquad + f(v\top\kappa) - f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha, \\
\bar{A}_{20}^2 &= -\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\int_0^1 \big(k(x)\Phi(v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dxd\alpha, \\
\bar{A}_{20}^3 &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big(g_R(v\top\kappa) - g_R(v\bot\kappa) \\
&\qquad + f(v\top\kappa) - f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha.
\end{aligned}$$

$*$ At first, the difference $A_{20}^1 - \bar{A}_{20}^1$ is studied :

$$\begin{aligned}
&|A_{20}^1 - \bar{A}_{20}^1| \leq \\
&\quad \sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \Big|\big(H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big) \\
&\quad -\big(g_L(v\top\kappa) - g_L(v\bot\kappa) + f(v\top\kappa) - f(v\bot\kappa)\big)\Big||\partial_x\varphi(t,x)|\,dt\,dx\,d\alpha
\end{aligned}$$

We can notice that

$$\begin{aligned}
&\Big|\big(H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big) \\
&\quad -\big(g_L(v\top\kappa) - k_L g(v\bot\kappa) + f(v\top\kappa) - f(v\bot\kappa)\big)\Big| \\
\leq\ & \big|H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - (k_L g + f)(u_i^n\top\kappa)\big| \\
&+ \big|(g_L + f)(u_i^n\top\kappa) - (k_L g + f)(v\top\kappa)\big| \\
&+ \big|H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) - (k_L g + f)(u_i^n\bot\kappa)\big| \\
&+ \big|(g_L + f)(u_i^n\bot\kappa) - (k_L g + f)(v\bot\kappa)\big|.
\end{aligned} \tag{3.40}$$

Moreover, an individually study shows

$$\begin{aligned}
&|H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - H_L(u_i^n\top\kappa, u_i^n\top\kappa)| \leq \\
&\qquad \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)|,
\end{aligned}$$

and

$$\begin{aligned}
&|H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) - H_L(u_i^n\bot\kappa, u_i^n\bot\kappa)| \leq \\
&\qquad \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)|.
\end{aligned}$$

Then, equation (3.40) becomes:

$$\begin{aligned}
& \Big|\big(H_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - H_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big) \\
& -\big(g_L(v\top\kappa) - k_L g(v\bot\kappa) + f(v\top\kappa) - f(v\bot\kappa)\big)\Big| \\
\leq\ & 2 \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)| \\
& +(\max\{k_L,k_R\}\mathrm{Lip}(g) + \mathrm{Lip}(f))\big|(u_i^n\top\kappa) - (v\top\kappa)\big| \\
& +(\max\{k_L,k_R\}\mathrm{Lip}(g) + \mathrm{Lip}(f))\big|(u_i^n\bot\kappa) - (v\bot\kappa)\big| \\
\leq\ & 2 \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)| \\
& +2\,(\max\{k_L,k_R\}\mathrm{Lip}(g) + \mathrm{Lip}(f))\big|u_i^n - v\big|.
\end{aligned}$$

Finally

$$\begin{aligned}
& \big|A_{20}^1 - \bar{A}_{20}^1\big| \\
\leq\ & 2||\partial_x\varphi||_\infty \sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} \Delta t\, h_i \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)| \\
+\ & 2||\partial_x\varphi||_\infty \sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} (\max\{k_L,k_R\}\mathrm{Lip}(g) + \mathrm{Lip}(f)) \\
& \qquad \int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big|u_i^n - v(t,x,\alpha)\big|\,dt\,dx\,d\alpha \\
\leq\ & 2h||\partial_x\varphi||_\infty \sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} \Delta t \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |k_L g(p) + f(p) - H_L(p,q)| \\
+\ & 2||\partial_x\varphi||_\infty (\max\{k_L,k_R\}\mathrm{Lip}(g) + \mathrm{Lip}(f)) \\
& \sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big|u_{\mathcal{T},k}(t,x) - v(t,x,\alpha)\big|\,dt\,dx\,d\alpha.
\end{aligned}$$

As the nonlinear weak-$\star$ convergence implies that $u_{\mathcal{T},k}$ converges to v in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}\times[0,1])$, and using estimate (3.11), we can conclude that $\lim_{h\to 0} |A_{20}^1 - \bar{A}_{20}^1| = 0$.

$*$ By the same way, by replacing H_L by H_R and k_L by k_R, we obtain $\lim_{h\to 0} |A_{20}^3 - \bar{A}_{20}^3| = 0$

$*$ It remains to study the limit of A_{20}^2 and $\bar{A}_{20}^2$.

$$\begin{aligned}|A_{20}^2| &\leq \sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\big|H(u_0^n\top\kappa,u_1^n\top\kappa,k_L,k_R)-H(u_0^n\bot\kappa,u_1^n\bot\kappa,k_L,k_R)\big| \\ &\qquad\qquad |\partial_x\varphi(t,x)|\,dt\,dx \\ &\leq C||\partial_x\varphi||_\infty\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}dt\,dx \\ &\leq C||\partial_x\varphi||_\infty Th_0 \\ &\leq C||\partial_x\varphi||_\infty Th\longrightarrow 0,\ \text{as } h\to 0,\end{aligned}$$

and

$$\begin{aligned}|\bar{A}_{20}^2| &\leq \sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\int_0^1|(k(x)\Phi(v,\kappa)+\Psi(v,\kappa))\,\partial_x\varphi(t,x)|\,dt\,dxd\alpha \\ &\leq C||\partial_x\varphi||_\infty\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\int_0^1 dt\,dxd\alpha \\ &\leq C||\partial_x\varphi||_\infty Th\longrightarrow 0,\ \text{as } h\to 0.\end{aligned}$$

To conclude, equality (3.39) had been shown.

• With (3.34) and (3.39), we obtain

$$\lim_{h\to 0}A_2=-\int_0^t\int_{\mathbb{R}}(k(x)\Phi(v,\kappa)+\Psi(v,\kappa))\,\partial_x\varphi(t,x)\,dt\,dx. \tag{3.41}$$

Study of term A_3

Term A_3 is defined by

$$A_3=\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}|\Delta h^i|\int_{t^n}^{t^{n+1}}\frac{1}{h_i}\int_{K_i}\varphi(t,x)\,dt\,dx. \tag{3.42}$$

To find its limit, A_3 is divided it into three terms according to values of i.

1. For $i\in\{i_0,\dots,-1\}$, $\Delta h^i=H(\kappa,\kappa,k_L,k_L)-H(\kappa,\kappa,k_L,k_L)=0$,
2. For $i\in\{2,\dots,i_2\}$, $\Delta h^i=H(\kappa,\kappa,k_R,k_R)-H(\kappa,\kappa,k_R,k_R)=0$,
3. $|\Delta h^0|=|H(\kappa,\kappa,k_L,k_R)-H(\kappa,\kappa,k_L,k_L)|=|H(\kappa,\kappa,k_L,k_R)-k_Lg(\kappa)+f(\kappa)|$, and $|\Delta h^1|=|H(\kappa,\kappa,k_R,k_R)-H(\kappa,\kappa,k_L,k_R)|=|k_Rg(\kappa)+f(\kappa)-H(\kappa,\kappa,k_L,k_R)|$.

Assuming $k_L>k_R$, (it is similar if $k_L<k_R$), with hypothesis (H4)

$$\begin{aligned}|\Delta h^0| &= (k_Lg(\kappa)+f(\kappa))-H(\kappa,\kappa,k_L,k_R) \\ \text{and } |\Delta h^1| &= H(\kappa,\kappa,k_L,k_R)-(k_Rg(\kappa)+f(\kappa)).\end{aligned}$$

Moreover

$$\begin{aligned}
& |\Delta h^0| \int_{K_0} \varphi(t,x)dx + |\Delta h^1| \int_{K_1} \varphi(t,x)dx \\
= \;& g(\kappa)\Big(k_L \int_{x_{-1/2}}^{0} \varphi(t,0)\,dx - k_R \int_0^{x_{3/2}} \varphi(t,0)dx\Big) \\
& +(H(\kappa,\kappa,k_L,k_R) - f(\kappa))\Big(\int_0^{x_{3/2}} \varphi(t,0)\,dx - \int_{x_{-1/2}}^{0} \varphi(t,0)\,dx\Big) \\
\to \;& g(\kappa)(k_L - k_R)\varphi(t,0), \quad \text{as } h \text{ tends to } 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\lim_{h\to 0} A_3 &= (k_L - k_R)g(\kappa) \int_0^{+\infty} \varphi(t,0)\,dt \\
&= |k_L - k_R| g(\kappa) \int_0^{+\infty} \varphi(t,0)\,dt. \qquad (3.43)
\end{aligned}$$

Final estimate

Using $A_1 + A_2 \le A_3$ and the limits established in previous sections (see equations (3.32), (3.41) and (3.43)), the function v satisfies the following inequality: for all $\kappa \in [0,1]$, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$

$$\begin{aligned}
& \int_0^1 \int_0^\infty \int_{\mathbb{R}} |v(t,x,\alpha) - \kappa|\, \partial_t \varphi(t,x)\, d\alpha\, dt\, dx \\
+ & \int_0^1 \int_0^\infty \int_{\mathbb{R}} (k(x)\Phi(v(t,x,\alpha),\kappa) + \Psi(v(t,x,\alpha),\kappa))\, \partial_x \varphi(t,x)\, d\alpha\, dx\, dt \\
+ & \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \;+ \int_0^\infty |k_L - k_R| g(\kappa) \varphi(t,0)\, dt \ge 0.
\end{aligned}$$

So the function $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$ is a weak entropy process solution to problem (3.1).

3.4 Convergence of the scheme

Theorem 3.5. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. in $\mathbb{R}$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ the unique entropy solution to problem (3.1). Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.2 such that $\alpha h \le h_i$ for all $i \in \mathbb{Z}$. Let $\Delta t > 0$ satisfying the CFL condition (3.10).*
Let $u_{\mathcal{T},\Delta t}$ be the finite volume approximated solution defined by (3.7). Then $u_{\mathcal{T},\Delta t} \to u$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $1 \le p < \infty$ (and in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ for the weak-$\star$ topology), as $h = \mathrm{size}(\mathcal{T}) \to 0$.

To establish this result, a theorem of comparison between two entropy process solutions is used. This comparison is obtained in a previous work [BV05] :

Theorem 3.6 (Comparison). *Under hypotheses (H1), (H2), (H3), let u and $v \in L^\infty(Q \times (0,1))$ be entropy process solutions of problem (3.1), associated to the initial conditions $u_0 \in L^\infty(\mathbb{R};[0,1])$ (resp. $v_0 \in L^\infty(\mathbb{R};[0,1])$). Then, with $R, T > 0$*

$$\int_0^1 \int_0^1 \int_0^T \int_{-R}^{R} (u(t,x,\lambda) - v(t,x,\zeta))^+ dx dt d\lambda d\zeta \leq T \int_{-R-CT}^{R+CT} (u_0(x) - v_0(x))^+ dx, \tag{3.44}$$

where $C := \max\{k_R, k_L\}\mathrm{Lip}(g) + \mathrm{Lip}(f)$.

Corollary 3.1. *If u and $v \in L^\infty(Q \times (0,1))$ are entropy process solutions of problem (3.1), then $u(t,x,\lambda) = v(t,x,\zeta)$ for a.e. $(t,x,\lambda,\zeta) \in Q \times (0,1) \times (0,1)$. So $u = v$ is a classical entropy solution.*

Proof of Theorem 3.5:
Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ a sequence of admissible mesh and $(\Delta t_m)_{m\in\mathbb{N}}$ a sequence of real positive values such that for all m, Δt_m satisfies the CFL condition (3.10). We assume that $\mathrm{size}(\mathcal{T}_\mathrm{m}) = \mathrm{h}^\mathrm{m} \to 0$.
Using Theorem 3.4 ans Corollary 3.1, a subsequence of $(u_{\mathcal{T}_m,\Delta t_m})_{m\in\mathbb{N}}$ converges towards an entropy process solution. Using Theorem 3.6, the entropy process solution is unique and is the entropy solution to problem (3.1). Then the subsequence converges towards the unique entropy solution to problem (3.1). Finally, as the sequence has a unique value of adherence, the whole sequence $(u_{\mathcal{T}_m,\Delta t_m})_{m\in\mathbb{N}}$ converges towards the entropy solution to problem (3.1) for the weak-$\star$ non linear topology.
Then

$$\int_0^\infty \int_\mathbb{R} h(u_{\mathcal{T}_m,\Delta t_m}(t,x))\psi(t,x)\,dx\,dt \to \int_0^\infty \int_\mathbb{R} h(u(t,x))\psi(t,x)\,dx\,dt$$
$$\forall \psi \in L^1(\mathbb{R}^+ \times \mathbb{R}), \quad \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{3.45}$$

Setting $h(s) = s^2$ in (3.45) and then $h(s) = s$ and ψu instead of ψ in (3.45) one obtains:

$$\int_0^\infty \int_\mathbb{R} \big(u_{\mathcal{T}_m,\Delta t_m}(t,x) - u(t,x)\big)^2 \psi(t,x)\,dx\,dt \to 0, \text{ as } m \to \infty,$$

for any function $\psi \in L^1(\mathbb{R}_+ \times \mathbb{R})$. From equation (3.45), and thanks to the L^∞ boundedness of $(u_{\mathcal{T}_m,\Delta t_m})_{m\in\mathbb{N}}$, the sequence $(u_{\mathcal{T}_m,\Delta t})_{m\in\mathbb{N}}$ converges to u in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $p \in [1,\infty[$.

3.5 Numerical methods

All the methods presented in this section are Finite Volume methods (see [EGH00]) for the hyperbolic equation (3.1) (with f is equal to zero, to simplify because the discontinuity of the flux doesn't concern the flux f), as scheme (3.5) presented in section 3.2.

For the sake of the simplicity, the presentation is restricted to uniform meshes (all methods may be naturally extended to non-uniform meshes). Let h be the space step, with $h = x_{i+1/2} - x_{i-1/2}$, $i \in \mathbb{Z}$, and let Δt be the time step, with $\Delta t = t^{n+1} - t^n$, $n \in \mathbb{N}$. Besides, let u_i^n denote the approximation of $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(t^n, x)\,dx$.
Integrating equation (3.1) over the cell $]x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[$ yields:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}\big(\varphi_{i+1/2}^n - \varphi_{i-1/2}^n\big)$$

where $\varphi_{i+1/2}^n$ is the numerical flux through the interface $\{x_{i+1/2}\} \times [t^n, t^{n+1}[$. We recall that the function k is approximated by a piecewise constant function. The numerical flux $\varphi_{i+1/2}^n$ depends on k_i, k_{i+1}, u_i^n, u_i^{n+1}.
Moreover, the CFL condition imposed in Theorem of convergence 3.4 is satisfied.

Notice that all the methods presented here rely on conservative schemes, since the problem is conservative. Finally, all the presented schemes are three-points schemes.

3.5.1 The Godunov scheme

The Godunov scheme [God59] is based on the resolution of the Riemann problem at each interface of the mesh. In fact we remark that problem (3.1), assuming $f = 0$, can be considered as the following resonant problem:

$$\partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \quad \partial_t = 0. \tag{3.46}$$

The Godunov Method applied to this resonant system had been studied by Lin, Temple and Wang ([LTW95a], [LTW95b]). A specific Godunov scheme associated to problem (3.1) had been studied by Towers using a discretization of k staggered with respect to u ([Tow00], [Tow01]). Here, we consider the Godunov Method applied to the following 2×2 system:

$$\begin{cases} \partial_t u + \partial_x\big(k(x)g(u)\big) = 0, \\ \partial_t k = 0, \qquad\qquad t > t^n,\ x \in \mathbb{R}, \\ \\ u(0,x) = \begin{cases} u_i^n & \text{if } x < x_{i+1/2} \\ u_{i+1}^n & \text{if } x > x_{i+1/2} \end{cases}, \quad k(x) = \begin{cases} k_i & \text{if } x < x_{i+1/2} \\ k_{i+1} & \text{if } x > x_{i+1/2} \end{cases}. \end{cases}$$

Let $u_{i+1/2}^n\big((x - x_{i+1/2})/(t - t^n); k_i, k_{i+1}, u_i^n, u_{i+1}^n\big)$ be the exact solution to the Riemann problem (see section 3.8 for an explicit presentation of the solution). Since the function k is discontinuous through the interface $\{x_{i+1/2}\} \times [t^n, t^{n+1}[$, the solution $u_{i+1/2}^n$ is discontinuous through this interface too. However, the problem is conservative, so the flux function is continuous through this interface, and writes:

$$\begin{aligned} \varphi_{i+1/2}^n &= k_i g(u_{i+1/2}^n(0^-; k_i, k_{i+1}, u_i^n, u_{i+1}^n)) \\ &= k_{i+1} g(u_{i+1/2}^n(0^+; k_i, k_{i+1}, u_i^n, u_{i+1}^n)). \end{aligned} \tag{3.47}$$

Remark 3.3. *To evaluate the numerical flux $\varphi^n_{i+1/2}$, we don't have to calculate the exact solution $u^n_{i+1/2}$ but only this value at $x = 0^-$ or at $x = 0^+$. As we remark in the section 3.8, it is simpler.*

Remark 3.4. *In the examples presented in section 3.6 and 3.7, we can show that the Godunov scheme is monotone.*

3.5.2 The VFRoe-ncv scheme

If we don't want to solve the Riemann problem at each step of the scheme, an alternative scheme is presented. This scheme is an approximate Godunov scheme, based on the exact solution to a linearized Riemann problem. A VFRoe-ncv scheme is defined by a change of variables (see [BGH00] and [GHS02]). The new variable is denoted by $\theta(k,u)$. For problem (3.1), we take $\theta(k,u) = kg(u)$ for the new variable. If v is defined by $v(t,x) = \theta(k(x), u(t,x))$, the VFRoe-ncv scheme is based on the exact resolution of the following linearized Riemann problem:

$$\begin{cases} \partial_t v + \big(\hat{k}\, g'(\hat{u})\big)\partial_x v = 0, & t > t^n,\ x \in \mathbb{R}, \\ v(0,x) = \begin{cases} \theta(k_i, u_i^n) & \text{if } x < x_{i+1/2} \\ \theta(k_{i+1}, u_{i+1}^n) & \text{if } x > x_{i+1/2} \end{cases}, \end{cases} \tag{3.48}$$

where $\hat{k} = (k_i + k_{i+1})/2$ and $\hat{u} = (u_i^n + u_{i+1}^n)/2$. As the Godunov scheme, the flux (which is represented by v) is continuous through the interface $\{x_{i+1/2}\} \times [t^n \times t^{n+1}[$ (this property is obtained by the good choice of θ). If $v^n_{i+1/2}\big((x - x_{i+1/2})/(t - t^n); k_i, k_{i+1}, u_i^n, u_{i+1}^n\big)$ is the exact solution to Riemann problem (3.48), as the function k is positive, the numerical flux of the VFRoe-ncv scheme is:

$$\begin{aligned} \varphi^n_{i+1/2} &= v^n_{i+1/2}(0; k_i, k_{i+1}, u_i^n, u_{i+1}^n) \\ &= \begin{cases} \theta(\kappa_i, u_i^n) & \text{if } g'(\hat{u}) > 0 \\ \theta(k_{i+1}, u_{i+1}^n) & \text{if } g'(\hat{u}) < 0, \end{cases} \end{aligned} \tag{3.49}$$

We can remark that the VFRoe-ncv scheme is reduced to the well-known upwind scheme for problem (3.48).
Finally, as function g is not genuinely nonlinear, the function g' can be equal to zero on an interval included in $[0,1]$. Then, if $g'(\hat{u}) = 0$, problem (3.48) is not ill-posed, we take for the numerical flux

$$\varphi^n_{i+1/2} = (k_i g(u_i^n) + k_{i+1} g(u_{i+1}^n))/2. \tag{3.50}$$

3.5.3 The God/VFRoe-ncv scheme

We will remark in section 3.8, that the resolution of the Riemann problem at the interface $\{x_{1/2}\} \times [t^n, t^{n+1}[$ where k is discontinuous, is long and difficult, then we introduce the God/VFRoe scheme.

For $i < 0$ and $i > 0$, the numerical flux is the Godunov flux (defined in subsection 3.5.1 with (3.47)):

$$\begin{aligned} \varphi^n_{i+1/2} &= k_i g(u^n_{i+1/2}(0^-; k_i, k_{i+1}, u^n_i, u^n_{i+1}) \\ &= k_{i+1} g(u^n_{i+1/2}(0^+; k_i, k_{i+1}, u^n_i, u^n_{i+1}), \end{aligned} \tag{3.51}$$

and for $i = 0$, the numerical flux is the VFRoe-ncv flux (defined in subsection 3.5.2 with (3.49) and (3.50)):

$$\begin{aligned} \varphi^n_{i+1/2} &= v^n_{i+1/2}(0; k_i, k_{i+1}, u^n_i, u^n_{i+1}) \quad \text{if } g'(\hat{u}) \neq 0, \\ \varphi^n_{i+1/2} &= (k_i g(u^n_i) + k_{i+1} g(u^n_{i+1}))/2 \quad \text{if } g'(\hat{u}) = 0. \end{aligned} \tag{3.52}$$

3.6 Numerical results for nor convex neither concave flux function

In this section, numerical results with g nor concave neither convex are presented. The graph of g is the following :

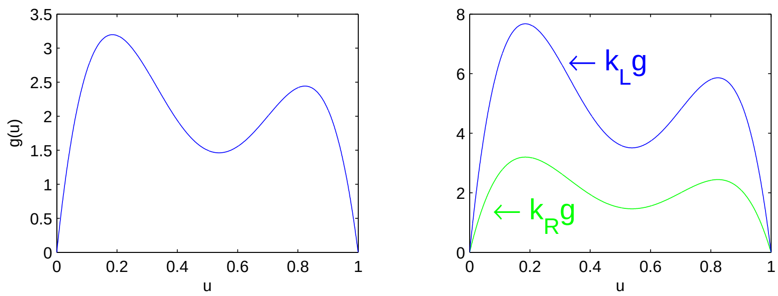

Figure 3.1: Graph of g and $k_L g$, $k_R g$

For numerical tests, g is given by $g(u) = -23.57u^4 + 48.33u^3 - 32.45u^2 + 7.69x$.

In the two following tests, the Riemann problem is numerically solved. The length of the domain is 10m. The mesh is composed of 100 cells and the CFL condition is set to 0.05. The variable u is plotted, in order to appreciate the behaviour of the Godunov scheme through the interface $\{x/t = 0\}$.
The initial conditions of the first Riemann problem are $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$ and $u_R = 0.4$. The results of Fig. 3.2 are plotted at $t = 4s$. The analytic solution to this Riemann problem is given in section 3.8. The numerical approximations provided by the three schemes are similar. We can observe that the three schemes present only one point in the shock between u_L and $u(t = 4s, 0^-)$, moreover this point is in the interval given by $[u_L, u(t = 4s, 0^-)]$.
The initial conditions of the second Riemann problem are $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$ and $u_R = 0.9$. The results of Fig. 3.3 are plotted at $t = 1s$. The analytic solution to

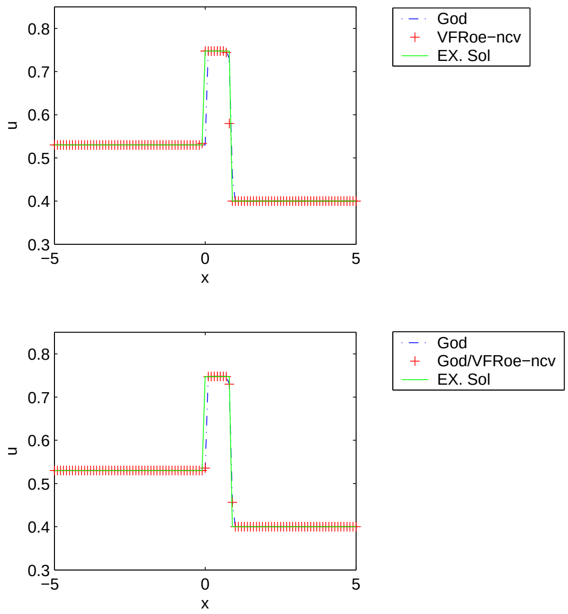

Figure 3.2: $k_L = 1.5$, $k_R = 1$, $u_L = 0.538$, $u_R = 0.4$, 100 cells.

this Riemann problem is given in section 3.8. The numerical approximation provided by the three schemes are similar and we observe the same behaviour than for the first Riemann problem presented.

3.7 Numerical results for a piecewise linear flux function

In this section, the function g is defined as follows:

$$g(u) = \begin{cases} 4u & \text{if} \quad 0 \leq u \leq 1/4, \\ 1 & \text{if} \quad 1/4 \leq u \leq 3/4, \\ -4u-4 & \text{if} \quad 3/4 \leq u \leq 1, \end{cases} \tag{3.53}$$

We have already remark that problem (3.1) can be considered as the following resonant system:

$$\partial_t u + \partial_x \big(k(x)g(u)\big) = 0, \quad \partial_t k = 0. \tag{3.54}$$

We notice that the system is resonant for $u \in [1/4, 3/4]$. We will show that the numerical methods are stable in spite of the resonance of the problem.
In the following test, the Riemann problem is numerically solved. The length of the domain is 10m. The mesh is composed of 100 cells and the CFL condition is set to 0.12. The variable u is presented in order to appreciate the behaviour of the Godunov and the VFRoe-ncv scheme through the interface $x/t = 0$.

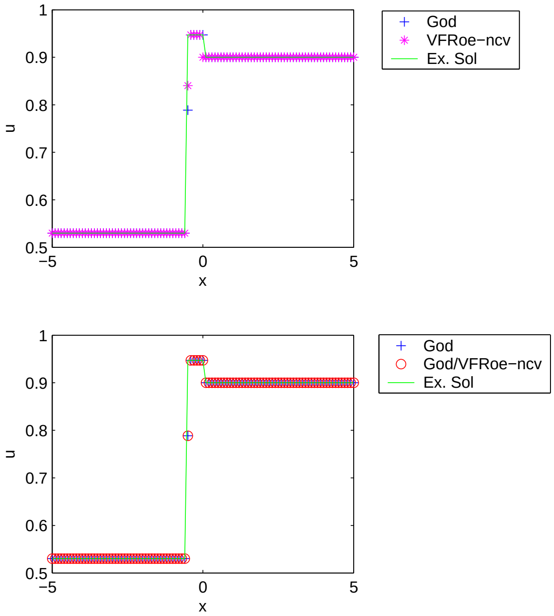

Figure 3.3: $k_L = 1.5$, $k_R = 1$, $u_L = 0.53$, $u_R = 0.9$, 100 cells.

The initial conditions of the two Riemann problems are $k_L = 1.5$, $k_R = 1$, $u_L = 3/8$ and $u_R = 5/8$. We remark that u_L, $u_R \in [1/4, 3/4]$. The results of Fig. 3.4 are plotted at $t = 2s$. The analytic solution to this Riemann problem is given in section 3.8. The numerical approximations provided by the Godunov scheme and the VFRoe scheme are similar. We may notice that the VFRoe-scheme introduce an error in the shock between u_L and $u(t = 2, x = 0^-)$. This error is due to the fact that $g'(u_L) = 0$ and $g'(u(t = 2, x = 0^-)) \neq 0$ and isn't due to the discontinuity of function k. Moreover, the behaviour of the scheme God/VFRoe, described in section 3.5.3, is similar that the behaviour of the Godunov scheme. This error is corrected. Then, even if g is constant on an interval included in $[0, 1]$, the behaviour of the schemes are similar as our attends.

We study now the ability of the schemes to converge towards the entropy solution. On the one hand, with Theorem 3.4 and as the Godunov scheme is monotone and satisfies hypothesis (H7), we know that the approximated solution given by this scheme converges to the entropy solution. But we don't know the order of this scheme. On the other hand, we don't know if the two others schemes are monotone, then Theorem 3.4 can't be use.
The computation of this test are based on the Riemann problem exposed just above. Some measurements of the numerical error provide that the methods tends to zero as Δx tends to zero. The L^1 discrete norm defined as follows: $\Delta x \sum_{i=1..N} |u_i^n - u^{ex}(x_i)|$

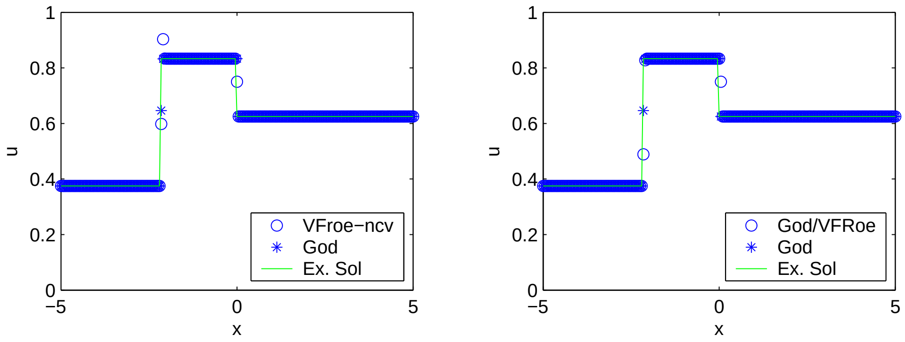

Figure 3.4: $k_L = 1.5$, $k_R = 1$, $u_L = 3/8$, $u_R = 5/8$, 50 cells

is used. But, numerical tests provided by all schemes presented are same behaviour. Several meshes are considering: involving 50, 100, 500, 1000. The axes of Fig. 3.5 have a logarithmic-scale. We observe a first order convergence for all schemes presented.

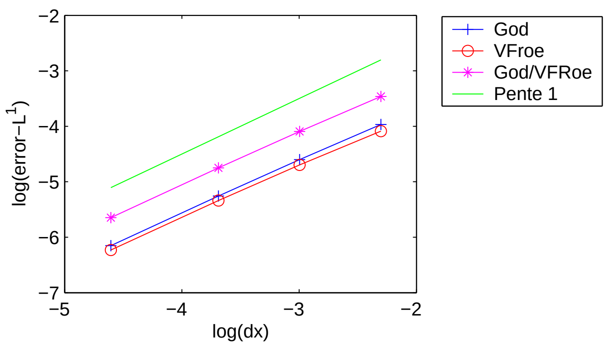

Figure 3.5: Error estimate in norm L^1

Remark 3.5. *We can observe the same results for g presented in section 3.6.*

3.8 The Riemann problem

In this section, the exact solution to the Riemann problem is presented :

$$\begin{cases} \partial_t u + \partial_x \big(k(x) g(u)\big) = 0, & x \in \mathbb{R},\, t \in \mathbb{R}_+ \\ u(t=0,x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x < 0 \end{cases}, \ k(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x < 0 \end{cases}, \end{cases}$$

where k_L, $k_R \in \mathbb{R}_+^*$ and u_L, $u_R \in [0, 1]$. We note that a general approach of this Riemann problem is given in [Die95].

3.8.1 Local entropy condition of the entropy solution

In order to know if a function u is the unique entropy solution of Riemann problem (3.55), we have to verify that the function u satisfies entropy inequalities (3.3). These conditions are difficult to satisfy. We can establish equivalent local conditions. In the following, we assume that if $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution, then u admits some traces along the line $\{x = 0\}$ (see [SV03, Bac04]). Let us define $u^- = u(t, x = 0^-)$ and $u^+ = u(t, x = 0^+)$. We can remark that u^- and u^+ are constant. Moreover, in the proof of uniqueness (see [SV03, Bac04]) some properties satisfied by the function u are established :

1. $\forall \kappa \in [0, 1]$, $I_u(\kappa) \geq 0$ with

$$I_u(\kappa) = k_L \Phi(u^-, \kappa) - k_R \Phi(u^+, \kappa) + |k_L - k_R| g(\kappa),$$

2. The Rankine-Hugoniot relation

$$k_L g(u^-) = k_R g(u^+). \tag{3.55}$$

If a function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0, 1])$ satisfies these two conditions and if u is a weak solution to problem (3.1) (with f equal to zero):

$$\int_{R_+} \int_R u(t, x) \partial_t \varphi(t, x) + k(x) g(u(t, x)) \partial_x \varphi(t, x)\, dt\, dx \;+ \int_R u_0(x) \varphi(0, x)\, dx = 0,$$

then the function u is the unique entropy solution to this problem. Now, we use these two conditions to solve the Riemann problem (3.55). To describe the solution, we assume for instance $k_L > k_R$.

Let $u_L, u_R \in [0, 1]$. Let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0, 1])$ be the entropy solution to the Riemann problem (3.55). Then u satisfies:

For $t \geq 0$, $x < 0$:

- u is the unique entropy solution to:

$$\begin{cases} \partial_t u + \partial_x \big(k_L g(u)\big) = 0 & t \in \mathbb{R}_+,\ x \in \mathbb{R}_-^*, \\ u(t = 0, x) = u_L & x \in \mathbb{R}_-^* \\ u(t, x = 0^-) = u^- & t \in \mathbb{R}_+ \end{cases} \tag{3.56}$$

- If u contains a rarefaction wave, $g'(u(t, x))$ must be negative for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_-^*$.
- If u contains a shock wave, the speed of the shock must be negative.

For $t \geq 0$, $x > 0$:

- u is the unique entropy solution to:

$$\begin{cases} \partial_t u + \partial_x \big(k_R g(u)\big) = 0 & t \in \mathbb{R}_+,\ x \in \mathbb{R}_+^*, \\ u(t = 0, x) = u_R & x \in \mathbb{R}_+^* \\ u(t, x = 0^+) = u^+ & t \in \mathbb{R}_+ \end{cases} \tag{3.57}$$

- If u contains a rarefaction wave, $g'(u(t,x))$ must be positive for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+^*$.
- If u contains a shock wave, the speed of the shock must be positive.

For $t \geq 0$, $x = 0$:
- $k_L g(u^-) = k_R g(u^+)$,
- if $u^- \leq u^+$, we only need to verify: $\forall \kappa \in [u^-, u^+]$

$$\begin{aligned} I_u(\kappa) &= -k_L g(u^-) + k_L g(\kappa) - k_R g(u^+) + k_R g(\kappa) + k_L g(\kappa) - k_R g(\kappa) \\ &= 2k_L(g(\kappa) - g(u^-)) \geq 0, \end{aligned}$$

using the Rankine-Hugoniot relation (3.55).
- if $u^- > u^+$, we only need to verify: $\forall \kappa \in [u^+, u^-]$

$$\begin{aligned} I_u(\kappa) &= k_L g(u^-) - k_L g(\kappa) + k_R g(u^+) - k_R g(\kappa) + k_L g(\kappa) - k_R g(\kappa) \\ &= 2k_R(g(u^+) - g(\kappa)) \geq 0, \end{aligned}$$

using the Rankine-Hugoniot relation (3.55).

3.8.2 Solution to the Riemann problem with g nor concave neither convex

In this section, the entropy solution to Riemann problem (3.55) is described, with g nor concave neither convex. The function g admits two local maximums in α and in γ and one local minimum in β with $\alpha \leq \beta \leq \gamma$ such that $g(\alpha) > g(\gamma) > g(\beta)$. A graph of g is represented in Fig. 3.1.

When the function k is equal to k_0, the construction of the solution to the Riemann problem is necessary. Let u_l and u_r two different states in $[0, 1]$. We link u_l and u_r by a shock wave and/or a rarefaction wave. We don't describe all possible situation, but we refer to [Ser96] for more details.

Then, the construction of the solution to Riemann problem (3.55) is reduced to the determination of u^- and u^+. We only focus on the case $k_L g(\beta) > k_R g(\alpha)$ (for others case, the solution may be constructed by the same way). In fact, with this assumption, the couple of root of $k_L g(u^-) = k_R g(u^+)$ are reduced to two possibilities in several cases:

- if $u_L \leq \alpha$:
  - if $u_R \leq \alpha$ and $k_L g(u_L) \leq k_R g(\alpha)$:
    - $u^- = u_L$,
    - u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave if $k_L g(u_L) > k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) < k_R g(u_R)$.
  - if $u_R \leq \alpha$ and $k_L g(u_L) > k_R g(\alpha)$:

  * u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$ and u_L and u^- are linked by a shock wave,
  * $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.
- <u>if $\alpha < u_R \leq \beta$ and $k_L g(u_L) < k_R g(u_R)$</u>:
  * $u^- = u_L$,
  * u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave.
- <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) < g(\gamma)$ and $k_L g(u_L) > k_R g(\gamma)$</u>:
  * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$ and u^- and u_L are linked by a shock wave,
  * $u^+ = \gamma$ and u^+ and u_R are linked by a shock wave.
- <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) < g(\gamma)$ and $k_L g(u_L) \leq k_R g(\gamma)$</u>:
  * $u^- = u_L$ is the greatest root of $k_L g(u^-) = k_R g(\gamma)$ and u^- and u_L are linked by a shock wave,
  * u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ included in $[\alpha, \beta]$ and u^+ and u_R are linked by a shock wave.
- <u>if $\alpha < u_R \leq \beta$, $k_L g(u_L) > k_R g(u_R)$ and $g(u_R) \geq g(\gamma)$</u>:
  * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave,
  * $u^+ = u_R$.
- <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) < k_R g(\beta)$</u>:
  * $u^- = u_L$,
  * u^+ is the smallest root of $k_R g(u^+) = k_L g(u_L)$ and u^+ and u_R are linked by a shock wave, then a rarefaction wave.
- <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) = k_R g(\beta)$</u>:
  * $u^- = u_L$,
  * $u^+ = \beta$ and u^+ and u_R are linked by a rarefaction wave.
- <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) < k_R g(u_R)$</u>:
  * $u^- = u_L$,
  * u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ in the interval $[\alpha, \beta]$, and u^+ and u_R are linked by a shock wave.
- <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) = k_R g(u_R)$</u>:
  * $u^- = u_L$,
  * $u^+ = u_R$.
- <u>if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$ and $k_L g(u_L) \leq k_R g(\gamma)$</u>:

    - $u^- = u_L$
    - u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ in $[\beta, \gamma]$, and u_+ and u_R are linked by a rarefaction wave.
  - if $\beta < u_R \leq \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$ and $k_L g(u_L) > k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L et u^- are linked by a shock wave.
    - $u^+ = \gamma$, and u_+ and u_R are linked by a rarefaction wave.
  - $u_R > \gamma$ and $k_L g(u_L) \leq k_R g(\beta)$ and $k_L g(u_L) \leq g(u_R)$:
    - $u^- = u_L$,
    - u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock.
  - $u_R > \gamma$ and $k_L g(u_L) \leq k_R g(\beta)$ and $k_L g(u_L) > g(u_R)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u^- and u_L are linked by a shock.
    - $u^+ = u_R$.
  - $u_R > \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) \leq k_R g(u_R)$:
    - $u^- = u_L$ is the greatest root of $k_R g(u_R) = k_L g(u^-)$ and u_L and u^- are linked by a shock,
    - u^+ is the root of $k_L g(u_L) = k_R g(u^+)$ included in $[\beta, \gamma]$, and u^+ and u_R are linked by a shock wave.
  - $u_R > \gamma$ and $k_L g(u_L) > k_R g(\beta)$ and $k_L g(u_L) > k_R g(u_R)$:
    - u^- is the greatest root of $k_R g(u_R) = k_L g(u^-)$ and u_L and u^- are linked by a shock wave,
    - $u^+ = u_R$.

- if $\alpha < u_L \leq \gamma$:
  - if $u_R \leq \alpha$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) \geq k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    - $u^+ = u_R$.
  - if $\alpha < u_R \leq \beta$ and $k_R g(u_R) < k_R g(\gamma)$:
    - u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,

    * $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

  - <u>if $\beta < u_R \leq \gamma$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a rarefaction wave and then a shock wave,
    * $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

  - <u>if $\gamma < u_R$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave,
    * $u^+ = u_R$.

- <u>if $\gamma < u_L$</u>:

  - <u>if $u_R \leq \alpha$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\alpha)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
    * $u^+ = \alpha$ and u^+ and u_R are linked by a rarefaction wave.

  - <u>if $\alpha < u_R \leq \beta$ and $k_R g(u_R) \geq k_R g(\gamma)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
    * $u^+ = u_R$.

  - <u>if $\alpha < u_R \leq \beta$ and $k_R g(u_R) < k_R g(\gamma)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(\gamma)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(\gamma)$,
    * $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

  - <u>if $\beta < u_R \leq \gamma$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(\gamma)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(\gamma)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(\gamma)$,
    * $u^+ = \gamma$, and u^+ and u_R are linked by a shock wave.

  - <u>if $u_R > \gamma$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$, and u_L and u^- are linked by a shock wave if $k_L g(u_L) < k_R g(u_R)$ or by a rarefaction wave if $k_L g(u_L) > k_R g(u_R)$,
    * $u^+ = u_R$.

3.8.3 Explicit form of the solution to the Riemann problem with g piecewise linear

In this section, we describe the entropy solution of Riemann problem (3.55), with g defined as in section 3.7 (see Eq. (3.53)).

We first present the construction when the function k is constant equal to k_0. Let u_l and u_r be two different states in $[0,1]$. We link u_l and u_r by a shock wave in all case because g is piecewise linear:

$$u(t,x) = \begin{cases} u_l & \text{if} \quad x/t < k_0 \dfrac{g(u_l) - g(u_r)}{u_l - u_r} \\ \\ u_r & \text{if} \quad x/t > k_0 \dfrac{g(u_l) - g(u_r)}{u_l - u_r} \end{cases} \tag{3.58}$$

The construction of the solution to the Riemann problem is reduced to the determination of u^- and u^+. We only focus on the case $k_L > k_R$ (if $k_L < k_R$, the solution may be constructed by the same way).

- <u>if $u_L < 1/4$</u>
  - <u>if $u_R \le 1/4$ and $k_L g(u_L) \le k_R g(1/4)$</u>:
    * $u^- = u_L$
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$, and u^+ and u_R are linked by a shock wave (defined by (3.58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>if $u_R < 1/4$ and $k_L g(u_L) > k_R g(1/4)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = 1/4$ and u^+ and u_R are linked by a shock wave (defined by (3.58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>if $u_R = 1/4$ and $k_L g(u_L) > k_R g(1/4)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.
  - <u>$1/4 < u_R \le 3/4$ and $k_L g(u_L) < k_R g(1/4)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock wave (defined by (3.58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>$1/4 < u_R \le 3/4$ and $k_L g(u_L) = k_R g(1/4)$</u>:

    * $u^- = u_L$,
    * $u^+ = u_R$.
  - <u>$1/4 < u_R \leq 3/4$ and $k_L g(u_L) > k_R g(1/4)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(3/4)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.
  - <u>$u_R > 3/4$ and $k_L g(u_L) < k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * u^+ is the smallest root of $k_L g(u_L) = k_R g(u^+)$ and u^+ and u_R are linked by a shock wave (defined by (3.58) with $u_l = u^+$, $u_r = u_R$ and $k_0 = k_R$).
  - <u>$u_R > 3/4$ and $k_L g(u_L) = k_R g(u_R)$</u>:
    * $u^- = u_L$,
    * $u^+ = u_R$.
  - <u>$u_R > 3/4$ and $k_L g(u_L) > k_R g(u_R)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.

- <u>$1/4 \leq u_L \leq 3/4$</u>:
  - <u>$u_R < 1/4$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(1/4)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = 1/4$ and u^+ and u_R are linked by shock wave.
  - <u>$1/4 \leq u_R$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.

- <u>$u_L > 3/4$</u>:
  - <u>$u_R < 1/4$ and $k_L g(u_L) > k_R g(u_R)$</u>:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(1/4)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),

    * $u^+ = 1/4$, and u^+ and u_R are linked by a shock wave.
- $u_R < 1/4$ and $k_L g(u_L) \leq k_R g(u_R)$:
    * u^- is the smallest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = 1/4$, u^+ and u_R are linked by a shock wave.
- $u_R < 1/4$ and $k_L g(u_L) < k_R g(u_R)$:
    * u^- is the smallest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.
- $1/4 \leq u_R$:
    * u^- is the greatest root of $k_L g(u^-) = k_R g(u_R)$ and u^- and u_L are linked by a shock wave (defined by (3.58) with $u_l = u_L$, $u^- = u_r$ and $k_0 = k_L$),
    * $u^+ = u_R$.

I would like to thank Thierry Gallouët, Nicolas Seguin and Julien Vovelle for theirs helps and theirs advices.

Bibliography

[AV03] Adimurthi and G.D. Veerappa Gowda, *Conservation law with discontinuous flux*, SIAM J; Numer Anal., **43**, (2003), no. 1, 27-70.

[AJV04] Adimurthi, J. Jaffre and G.D. Veerappa Gowda, *Godunov-type methods for conservation laws with a flux function discontinuous in space time*, SIAM J; Numer Anal., **42**, (2004), no. 1, 179-208.

[Bac04] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations (nov-déc 2004), no. 11-12, 1317–1338.

[BV05] F. Bachmann and J. Vovelle *Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients*, to appear in Communications in Partial Differential Equations.

[BGH00] T. Buffard, T. Gallouët and J.M. Hérard, *A sequel to a rough Godunov scheme. Application to real gas flows*, Computers and Fluids, **29-7**, (2000), 813-847.

[BKRT04] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers, *Well-possedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-tickener units*, Numer. Math., **97**, (2004), no. 1, 25-65.

[Die95] S. Diehl, *A conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Appl. Math., **56**, (1996), no. 2, 388-419.

[DiP85] R.J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223–270.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin, *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, 2000, pp. 713–1020.

[GHS02] T. Gallouët, J.M. Hérard and N. Seguin, *Some recent Finite Volume schemes to compute Euler equations using real gas EOS*, Int. J. Num. Meth. Fluids, **39-12**, (2002), 1073-1138.

[God59] S. K. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations hydrodynamics*, Mat. Sb. (N.S.) **47 (89)** (1959), 271-306.

[KRT02a] K.H. Karlsen, N.H. Risebro and J.D Towers. *Upwind difference approximations for degenerate parabolic convection-diffusion equations with discontinuous coefficient* IMA J. Numer. Anal., **22**, (2002), no. 4,623-664.

[KRT02b] K.H. Karlsen, N.H. Risebro and J.D Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron J. Differential Equations, (2002), pages No. 93, 23 pp (electronic).

[KRT03] K.H. Karlsen, N.H. Risebro and J.D Towers. *L1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid.Selsk, 1–49, 2003.

[KT04] K.H. Karlsen and J.D. Towers, *Convergence of the Lax-Friedrichs Scheme and Stability for Conservation Laws with a Discontinuous Space-Time Dependent Flux*, Chinese Ann. Math. Ser. B., **25**, (2004), no. 3, 287-318.

[KR01] K.H. Karlsen and N.H. Risebro. *Stability of a resonant system of conservation laws modelling polymer flow with gravitation*, J. Differential Equations, **120**, (2001), no. 2, 344-380.

[KR95] K.H. Karlsen and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness ans asymptotic behavior*, Comm. Partial Differential Equations, **20**, (1995), 11-12, 1959-1990.

[LTW95a] L.W. Lin, B.J. Temple, J.H. Wang, *A Comparison of convergence rates for Godunov's methods and Glimm's methods in resonant nonlinear systems of conservations laws*, SIAM J. Numer. Anal., **32**, (1995), 824-840.

[LTW95b] L.W. Lin, B.J. Temple, J.H. Wang, *Suppression of oscillations in Godunov's method for a resonant non-strictly hyperbolic system*, SIAM J. Numer. Anal., **32**, (1995), 841-864.

[SV03] N. Seguin and J. Vovelle, *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221–257.

[Ser96] D. Serre, *Systèms de lois de conservations I*, Diderot Editeur, Arts et Sciences, (1996).

[Tow00] J.D. Towers, *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681–698 (electronic).

[Tow01] J.D. Towers, *A difference scheme for conservation laws with a discontinuous flux: the nonconvex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197–1218 (electronic).

Chapter 4

Analyse d'une loi de conservation à flux discontinu de la forme $g(x, u)$: Unicité de la solution entropique

Abstract : In this paper, one studies a hyperbolic scalar equation in one space dimension with a flux function which is discontinuous with respect to the space variable. In the first part, one presents a convenient definition of weak entropy solution which ensures a uniqueness result. In the second part of this paper, one proves the convergence of some numerical results, whose a by product is the existence of a weak entropy solution.

4.1 Introduction

The Cauchy problem writes :

$$\begin{cases} \partial_t u + \partial_x\big(g(x,u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{4.1}$$

with initial value $u_0 \in L^\infty(\mathbb{R};[0,1])$. The functions f and g are supposed to satisfy the following hypotheses :

(H1) g is the discontinuous function defined by

$$g(x,u) = \begin{cases} g_L(u) & \text{if } x < 0 \\ g_R(u) & \text{if } x > 0 \end{cases} \quad \text{with } g_L \neq g_R,$$

$g_L, g_R \in \mathrm{Lip}([0,1])$ and $g_L(0) = g_R(0) = g_L(1) = g_R(1) = 0$,

(H2) $f \in \mathrm{Lip}([0,1])$.

One introduces the time - space domain $Q := (0,+\infty)\times\mathbb{R}$ and the line of discontinuity of the function g in the time - space domain, $\Sigma := (0,+\infty) \times \{0\}$.

Conservation laws with discontinuous flux function appear in the modeling of a two phase flow in porous media. The particular shape of function g comes from the model described in [GMT96] and [Bac04] in the particular case $g(x, u) = k(x)\bar{g}(u)$ with k a discontinuous function with respect to x. In this paper, the model takes into account relative permeabilities of two phases which depend with respect to x, what explains that g depends with respect to x and u a more global way. One just remarks that these hypotheses are natural with this model. One does not assume hypothesis of convexity or genuine non-linearity on g.

To study conservation law, a finite volume scheme is well adapted : one has been interested in this scheme for problem (4.1). One wants to know if there is convergence of the scheme and uniqueness of the "well" solution, under natural hypotheses. The notion of entropy solution is introduced for problem (4.1) with Definition 4.1.

The aim of this paper, in part I, is to show the uniqueness of the entropy solution with function g which isn't genuinely non linear with respect to u. This last point is a new point compared with preceding works on the subject (see in particular [AV02, Bac04, KR95, KRT03, SV03]). However, in [BV05], a new point of view for this type of problem is brought with $g(x, u) = k(x)\bar{g}(u)$ where k satisfies $k(x) = k_L$ for $x < 0$ and $k(x) = k_R$ for $x > 0$ with $k_L \neq k_R$. In this last work, one obtains existence and uniqueness of entropy solution without hypothesis of convexity nor genuine non linearity on $\bar{g}$. In this paper, the proof of the uniqueness is adapted and generalized to problem (4.1). Considering that $\bar{g}$ is not genuinely non linear brings as a main difficulty : the traces of entropy solution along the line of discontinuity of function k can not be considered. This implies that one can not consider problem (4.1) as two problems on both sides of the line $\{x = 0\}$. In particular, if g is assumed genuinely non linear, the technic of doubling variables of Kruzhkov for conservation law can be adapted to prove the uniqueness of entropy solution of problem (4.1). In fact, two entropy solutions are compared outside a neighborhood of $\{x = 0\}$ and by using the existence of traces of entropy solution, one obtains a comparison on $\mathbb{R}$ by passing to the limit on the support of the test function (see [Bac04, KRT03, SV03]). In this work (and in [BV05]), because g is not assumed genuinely non linear, this technic of doubling variables fails.

The main result of this paper is the following comparison between two entropy solutions :

Theorem 4.1. *Assume hypotheses (H1), (H2). Let u (resp. $v \in L^\infty(Q)$) be entropy solution of problem* (4.1)*, associated to the initial conditions $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ (resp. $v_0 \in L^\infty(\mathbb{R}; [0, 1])$). Then, given $R, T > 0$, one has*

$$\int_0^T \int_{-R}^{R} (u(t, x) - v(t, x))^{\pm} dxdt \leq T \int_{-R-CT}^{R+CT} (u_0(x) - v_0(x))^{\pm} dx, \qquad (4.2)$$

where $C := \max\{\text{Lip}(g_L,)\text{Lip}(g_R)\} + \text{Lip}(f)$.

However, Theorem 4.1 is not directly established. In fact, in part II of this paper, the existence of entropy solution is not obtained by the convergence of the finite volume scheme. The study of the scheme just permits to obtain the existence of entropy process solution. This notion is "weaker" than the entropy solution. This notion of entropy process solution is justified and introduced in section 4.2, and a theorem of comparison between two entropy process solutions is obtained (see Theorem 4.3). And finally, Theorem 4.1 is an easy consequence of Theorem 4.3 (see remark 4.2). To establish this comparison between two entropy process solution, one uses some tools which are introduced in [BV05]. As the technic of doubling variables to prove the uniqueness of entropy solution of problem (4.1) and other attempts for $g(x,u) = k(x)\bar{g}(u)$ in [BV05] have failed, one introduces the notion of kinetic process solution of problem (4.1). This notion is equivalent to the notion of entropy process solution (see section 4.3). Kinetic solution, for conservation laws with g Lipschitz continuous, has been introduced by Lions, Perthame, Tadmor [LPT94] and a proof of uniqueness of entropy solution has been established by Perthame in [Per98]. For conservation laws with discontinuous flux function, this notion has been introduced in [BV05] to establish the uniqueness of entropy solution to problem (4.1) with $g(x,u) = k(x)\bar{g}(u)$, and one adapts this notion to the new generalized problem (4.1).

4.2 Notion of solution

4.2.1 Entropy solution

Once problem (4.1) written and justified, the definition of entropy solution is introduced. In [BV05, Tow00] the non-negativity of the functions k and g implies that the term $(k_L - k_R)^{\pm} \int_0^{+\infty} g(\kappa)\varphi(t,0)\,dt$ is non negative. For problem (4.1), the notion of entropy solution is defined as follows :

Definition 4.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$. A function $u \in L^\infty(Q;[0,1])$ is said to be an entropy solution of problem (4.1) if it satisfies the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (u(t,x) - \kappa)^{\pm}\, \partial_t\varphi(t,x)\,dt\,dx \\ +\ &\int_0^\infty \int_{\mathbb{R}} (\Phi^{\pm}(x,u(t,x),\kappa) + \Psi^{\pm}(u(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt \\ +\ &\int_{\mathbb{R}} (u_0(x) - \kappa)^{\pm}\,\varphi(0,x)\,dx \\ +\ &\int_0^\infty (g_L(\kappa) - g_R(\kappa))^{\pm}\,\varphi(t,0)\,dt \geq 0, \end{aligned} \tag{4.3}$$

where respectively $\Phi^{\pm}$ and $\Psi^{\pm}$ denote the entropy flux associated with the Kruzhkov entropy,

$$\begin{aligned} \Phi^{\pm}(x,u,\kappa) &= \operatorname{sgn}_{\pm}(u-\kappa)(g(x,u) - g(x,\kappa)), \\ \Psi^{\pm}(u,\kappa) &= \operatorname{sgn}_{\pm}(u-\kappa)(f(u) - f(\kappa)). \end{aligned}$$

One just makes the reader notice that : An entropy solution of (4.1) is a weak solution of (4.1), i.e. : for all non-negative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

$$\int_0^{+\infty} \int_{\mathbb{R}} u(t,x)\partial_t\varphi(t,x) + (g(x,u(t,x)) + f(u(t,x)))\partial_x\varphi(t,x)\,dt\,dx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)\,dx = 0.$$

This equality is a consequence of the two inequalities obtained, first by developing the entropy inequality written with $\kappa = 0$, second by developing the entropy inequality written with $\kappa = 1$ on the basis of the bound $0 \leq u \leq 1$ a.e. Similarly, if a function $u \in L^\infty(Q)$ satisfies $0 \leq u \leq 1$ a.e. on the one hand and the entropy inequalities with classical Kruzhkov entropies : for all $\kappa \in [0,1]$, for all non-negative $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\,\partial_t\varphi(t,x)\,dt\,dx \\ +\ &\int_0^\infty \int_{\mathbb{R}} (\,\Phi(x,u(t,x),\kappa) + \Psi(u(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt \\ +\ &\int_{\mathbb{R}} |u_0(x) - \kappa|\,\varphi(0,x)\,dx\ +\ |g_R(\kappa) - g_L(\kappa)| \int_0^\infty \varphi(t,0)\,dt \geq 0, \end{aligned} \tag{4.4}$$

on the other hand, then u is a weak solution of problem (4.1) and therefore satisfies (4.3) (indeed $u^+ = (|u| + u)/2$ and $u^- = (|u| - u)/2$). Conversely, by adding the two inequalities of (4.3), we see that $u \in L^\infty(Q)$ is an entropy solution to problem (4.1) if, and only if, it satisfies $0 \leq u \leq 1$ a.e. and (4.4).

4.2.2 Entropy process solution

The following result is established by R. Eymard T. Gallouët and R. Herbin (see [EGH00]). It is a result based on Young measure and a result of Di Perna (see [DiP85]). It explains the introduction of entropy process solution.

Theorem 4.2. *Let Ω be an open subset of $\mathbb{R}^N$ ($N \geq 1$) and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence, already noted $(u_n)_{n\in\mathbb{N}}$, converging in the nonlinear weak-$\star$ sense, i.e. :*

$$\int_\Omega h(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega h(u(x,\alpha))\psi(x)dx\,d\alpha, \text{ as } n \to +\infty \quad \forall\psi \in L^1(\Omega),\ \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{4.5}$$

Theorem 4.2 is recalled to explain the introduction of the definition of entropy process solution for problem (4.1). This kind of convergence permits to pass to the limit in the numerical scheme and thus to show the existence of an entropy process solution (see part II). Then, to conclude to the convergence of the scheme, the uniqueness of entropy process solution is sufficient. The definition of entropy process solution follows :

Definition 4.2. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. Let $u \in L^\infty(Q \times (0,1);[0,1])$. The function u is an* entropy process solution *of problem (4.1) if for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\varphi \ge 0$,*

$$\begin{aligned} & \int_0^1 \int_Q (u(t,x,\lambda)-\kappa)^\pm \partial_t \varphi(t,x) dt dx d\lambda \\ + \quad & \int_0^1 \int_Q \left[\Phi^\pm(x,u(t,x,\lambda),\kappa) + \Psi^\pm(u(t,x,\lambda),\kappa)\right] \partial_x \varphi(t,x) dt dx d\lambda \\ + \quad & \int_{\mathbb{R}} (u_0(x)-\kappa)^\pm \varphi(0,x) dx \\ + \quad & (g_L(\kappa) - g_R(\kappa))^\pm \int_0^{+\infty} \varphi(t,0) dt \ge 0. \end{aligned} \tag{4.6}$$

Remark 4.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$ and $u \in L^\infty(Q \times (0,1);[0,1])$. The function u is an* entropy process solution *of problem (4.1) iff for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\varphi \ge 0$,*

$$\begin{aligned} & \int_0^1 \int_Q |u(t,x,\lambda)-\kappa| \partial_t \varphi(t,x) dt dx d\lambda \\ + \quad & \int_0^1 \int_Q \left[\Phi(x,u(t,x,\lambda),\kappa) + \Psi(u(t,x,\lambda),\kappa)\right] \partial_x \varphi(t,x) dt dx d\lambda \\ + \quad & \int_{\mathbb{R}} |u_0(x)-\kappa| \varphi(0,x) dx + |g_L(\kappa) - g_R(\kappa)| \int_0^{+\infty} \varphi(t,0) dt \ge 0. \end{aligned}$$

4.3 Uniqueness of entropy process solution

To establish Theorem 4.3, the main difficulty, compared in previous work [BV05], is that the functions g_L and g_R can cross on $[0,1]$. In the proof of uniqueness in [BV05], one assumes (for instance) that $k_L > k_R$, it rises that $\forall \kappa \in [0,1]$ $k_L \bar{g}(\kappa) > k_R \bar{g}(\kappa)$ which is an important point of the proof. This permits, when the entropy solution u satisfies inequality (4.3) with the half entropy $s \to (s-\kappa)^-$, having the term $(k_L - k_R)^- = 0$. Then, considering $g(x,u)$ instead of $k(x)\bar{g}(u)$ brings a new difficulty in the proof of the uniqueness of entropy solution. To circumvent this difficulty, functions $\theta^+ := \chi_{\{\xi :\, g_L(\xi) - g_R(\xi) < 0\}}$ and $1 - \theta^+$ are introduced. The result follows :

Theorem 4.3 (Comparison). *Assume hypotheses (H1), (H2). Let u (resp. $v \in L^\infty(Q \times (0,1))$) be entropy process solution of problem* (4.1)*, associated to the initial conditions $u_0 \in L^\infty(\mathbb{R};[0,1])$ (resp. $v_0 \in L^\infty(\mathbb{R};[0,1])$). Then, given $R, T > 0$, one has*

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^T \int_{-R}^R & (u(t,x,\lambda) - v(t,x,\zeta))^\pm dx dt d\lambda d\zeta \\ & \le T \int_{-R-CT}^{R+CT} (u_0(x) - v_0(x))^\pm dx, \end{aligned} \tag{4.7}$$

where $C := \max\{\text{Lip}(g_L,)\text{Lip}(g_R)\} + \text{Lip}(f)$.

Remark 4.2. *One remarks that if* $u_0 = v_0$ *in Theorem 4.3, one obtains* $u(t, x, \lambda) = v(t, x, \zeta)$ *for a.e.* $(t, x, \lambda, \zeta) \in Q \times (0, 1) \times (0, 1)$ *as desired. Then* u *does not depend on* λ, *i.e.* $\forall\lambda\ u(t, x, \lambda) = u(t, x)$ *with* $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. *And finally, Theorem 4.1 is immediately deduced by Theorem 4.3.*

We have not able to establish the inequality (4.7), with the habitual technic doubling variables. To establish this comparison, one introduces an equivalent notion : kinetic solution, motivated by the result of Perthame ([Per98]) as one explains in the introduction and in [BV05].

4.4 Kinetic solution

4.4.1 Equilibrium functions

If Ω is a subset of $\mathbb{R}^m$ $(m \geq 1)$ and $u : \Omega \to \mathbb{R}$ is measurable, the equilibrium function χ_u associated to u is the function $\Omega \times \mathbb{R} \ni (x, \xi) \mapsto \text{sgn}_+(u(x) - \xi) + \text{sgn}_-(\xi)$. Notice that χ_u is measurable and that $\chi_u \in L^\infty(\Omega \times \mathbb{R}; [-1, 1])$. In the following, one also considers for equilibrium functions, for a.e. $\xi \in \mathbb{R}$:

$$h_+(x, \xi) = \text{sgn}_+(u(x) - \xi),$$
$$h_-(x, \xi) = \text{sgn}_-(u(x) - \xi).$$

For $X = \bar{Q}$, $\mathcal{M}_+(X)$ denotes the set of positive Radon measures on X (which are finite on compact subsets of X) or, equivalently (by Riesz representation theorem) the cone of nonnegative linear form on $\mathcal{C}_c(X)$. Therefore $m \in \mathcal{C}(\mathbb{R}_\xi; w * -\mathcal{M}_+(X))$ means $m(\xi) \in \mathcal{M}_+(X)$ for every $\xi \in \mathbb{R}$ and, for every $\varphi \in \mathcal{C}_c(X)$,

$$\xi \mapsto \int_X \varphi dm(\xi)$$

is continuous.

4.4.2 Kinetic solution

Denote by a and b the derivatives of the flux functions (defined a.e.) :

$$a(x, \xi) := \partial_u g(x, \xi)\,, \qquad b(\xi) := f'(\xi)\,, \quad \xi \in \mathbb{R}\,.$$

Definition 4.3. *Let* $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ *and* $u \in L^\infty(Q \times (0, 1))$. *Let* h *and* h^0 *be the equilibrium functions associated with* u *and* u_0 *:*

$$h(t, x, \lambda, \xi) = \chi_{u(t,x,\lambda)}(\xi)\,, \quad h^0(x, \xi) = \chi_{u_0(x)}(\xi)\,.$$

The function u *is a* kinetic process solution *of (4.1) if there exists* $m_\pm \in \mathcal{C}(\mathbb{R}_\xi; w * -\mathcal{M}_+(\bar{Q}))$ *such that* $m + (\cdot, \xi)$ *vanishes for large*

ξ (resp. $m_-(\cdot,\xi)$ vanishes for large $-\xi$) and such that for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$,

$$\begin{aligned} &\int_0^1 \int_{Q\times\mathbb{R}_\xi} h_\pm(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi + \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h^0_\pm \varphi_{|t=0} \\ - &\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi (g_L(\xi)-g_R(\xi))^\pm \varphi_{|x=0} = \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \varphi dm_\pm. \end{aligned} \tag{4.8}$$

Proposition 4.1. *The notions of entropy process solution and kinetic process solution are be equivalent.*

This proof of this equivalence is obtain similarly than in [BV05].

Let $u \in L^\infty(Q\times(0,1))$ be a weak entropy process solution of (4.1). For $\kappa \in \mathbb{R}$, define the linear form m_+^κ on $\mathcal{C}_c^\infty(\bar{Q})$ by :

$$\begin{aligned} m_+^\kappa(\varphi) &= \int_0^1 \int_Q (u(t,x,\lambda)-\kappa)^+ \partial_t \varphi(t,x) dt dx d\lambda \\ &+ \int_0^1 \int_Q (\Phi^+(x,u(t,x,\lambda),\kappa) + \Psi(u(t,x,\lambda),\kappa))\partial_x \varphi(t,x) dt dx d\lambda \\ &+ \int_{\mathbb{R}} (u_0(x)-\kappa)^+ \varphi(0,x) dx + \int_\Sigma (g_L(\kappa)-g_R(\kappa))^+ \varphi(t,0) dt. \end{aligned} \tag{4.9}$$

Let $\kappa \in \mathbb{R}$ be fixed. Since u is a weak entropy process solution, m_+^κ is nonnegative. It therefore induces a nonnegative linear form on $\mathcal{C}_c(\bar{Q})$ which can be represented by a Borel measure, still denoted m_+^κ. We set $m^+(\xi) = m_+^\xi$ ($\xi \in \mathbb{R}$). For K a compact subset of $\bar{Q}$, there exists a nonnegative $\varphi \in \mathcal{C}_c^\infty(\bar{Q})$ such that $\varphi(t,x) = 1$ for all $(t,x) \in K$. If $|\kappa| \le R$ ($R > 0$) we thus have, by (4.9) :

$$m_+^\kappa(K) \le m_+^\kappa(\varphi) \le C_R \tag{4.10}$$

where the constant C_R depends on R (and φ) only. This yields $m_+ \in \mathcal{C}(\mathbb{R}_\xi; w* -\mathcal{M}_+(\bar{Q}))$. Indeed, if (ξ_n) is a sequence of real numbers converging to $\xi \in \mathbb{R}$, then there exists $R > 0$ such that $|\xi_n| \le R$ for every n and, by (4.10), $m^+(\varphi,\xi_n)$ is bounded and nonnegative for every nonnegative $\varphi \in \mathcal{C}_c(\bar{Q})$. By the Banach-Steinhaus theorem, there exists $m_\infty^+ \in \mathcal{M}_+(\bar{Q})$ such that $m^+(\varphi,\xi_n) \to m_\infty^+(\varphi)$ as $n \to +\infty$ for every $\varphi \in \mathcal{C}_c(\bar{Q})$. By (4.9), we have $m_\infty^+(\varphi) = m^+(\varphi,\xi)$ for every $\varphi \in \mathcal{C}_c^\infty(\bar{Q})$, this remains true for $\varphi \in \mathcal{C}_c(\bar{Q})$ by density : therefore $m_+ \in \mathcal{C}(\mathbb{R}_\xi; w* -\mathcal{M}_+(\bar{Q}))$. Besides, from (4.9), and the fact that $u \le A$ a.e. for an $A \in \mathbb{R}$ ($u \in L^\infty(Q\times(0,1))$ by hypotheses, it appears that $m_+(\xi)$ vanishes for $\xi > A$, in particular for large ξ.

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. We compute with an integration by part :

$$\begin{aligned}
& \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \phi(t,x,\xi) dm_+(t,x,\xi) \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} (u-\xi)^+ \partial_t \partial_\xi \phi + (\Phi^+(x,u,\xi) + \Psi(u,\xi))\partial_x \partial_\xi \phi \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} (u_0-\xi)^+ \partial_\xi \phi_{|t=0} + \int_\Sigma (g_L(\xi) - g_R(\xi))^+ \partial_\xi \phi_{|x=0} \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} \mathrm{sgn}_+(u-\xi)(\partial_t \phi + (a(x,\xi) + b(\xi))\partial_x \phi) \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} \mathrm{sgn}_+(u_0-\xi)\phi_{|t=0} - \int_\Sigma \partial_\xi (g_L(\xi) - g_R(\xi))^+ \phi_{|x=0} \\
= & \int_0^1 \int_{Q\times\mathbb{R}_\xi} h_+ \left(\partial_t \phi + (a(x,\xi) + b(\xi))\, \partial_x \phi\right) \\
+ & \int_{\mathbb{R}\times\mathbb{R}_\xi} h_+^0 \phi_{|t=0} - \int_\Sigma \partial_\xi (g_L(\xi) - g_R(\xi))^+ \phi_{|x=0}.
\end{aligned}$$

Therefore u is a kinetic process subsolution.

Conversely, suppose that $u \in L^\infty(Q\times(0,1))$ is a kinetic process solution. For $\kappa \in \mathbb{R}$, let $\xi \mapsto E_n(\xi)$ be a smooth and convex approximation of $\xi \mapsto (\xi-\kappa)^+$ such that $|E_n'(\xi)| \leq 1$ for any positive integer n. Let Ψ be a smooth function with support in $[-2,2]$, values in $[0,1]$ and that equals 1 on $[-1,1]$. Next, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, and define $\Psi_n(\xi) = \Psi(\xi/n)$. Now apply (4.8) to the test function $\phi(t,x,\xi) = \varphi(t,x)\Psi_n(\xi)E_n'(\xi)$:

$$\begin{aligned}
& \int_0^1 \int_Q \left[\int_{\mathbb{R}_\xi} \Psi_n E_n' h_+\right] \partial_t \varphi + \left[\int_{\mathbb{R}_\xi} (k(x)a(\xi) + b(\xi))\Psi_n E_n' h_+\right] \partial_x \varphi \\
+ & \int_{\mathbb{R}} \left[\int_{\mathbb{R}_\xi} \Psi_n E_n' h_+^0\right] \varphi_{|t=0} - \int_\Sigma \left[\int_{\mathbb{R}_\xi} \Psi_n \partial_\xi (g_L(\xi) - g_R(\xi))^+ E_n'\right] \varphi_{|x=0} \\
= & \int_{\bar{Q}\times\mathbb{R}_\xi} \varphi[\Psi_n' E_n' + \Psi_n E_n''] dm_+.
\end{aligned}$$

If moreover φ is assumed to be nonnegative, then $\displaystyle\int_{\bar{Q}\times\mathbb{R}_\xi} \varphi \Psi_n E_n'' dm_+ \geq 0$ and letting $n \to +\infty$, we get :

$$\begin{aligned}
& \int_0^1 \int_Q (u(t,x,\lambda) - \kappa)^+ \partial_t \varphi(t,x) \\
+ & \int_0^1 \int_Q (\Phi^+(x,u(t,x,\lambda),\kappa) + \Psi^+(u(t,x,\lambda),\kappa))\partial_x \varphi(t,x) dt dx d\lambda \\
+ & \int_{\mathbb{R}} (u_0(x) - \kappa)^+ \varphi(0,x) dx + (g_L(\kappa) - g_R(\kappa))^+ \int_0^{+\infty} \varphi(t,0) dt \geq 0,
\end{aligned}$$

which is (4.6).

Traces of equilibrium function

One introduces two functions : regularization and cut-off function. Let $\rho \in \mathcal{C}_c^\infty(0,1)$ be a nonnegative function with mass 1. For a small parameter ε, the regularizing kernel ρ_ε is defined by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$$

and the cut-off function ω_ε by

$$\omega_\varepsilon(x) = \int_0^{|x|} \rho_\varepsilon(\sigma)d\sigma\,.$$

Proposition 4.2. *Let $h \in L^\infty(Q \times (0,1) \times \mathbb{R}_\xi)$ satisfy (4.8). Then there exists two functions $h_\pm^{\tau_0} \in L^\infty(Q \times (0,1) \times \mathbb{R}_\xi)$ and $\Upsilon_\pm \in L^\infty(\Sigma \times (0,1) \times \mathbb{R}_\xi)$ such that, up to subsequences :*

$$\lim_{\eta\to 0^+} \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} \left[\int_0^{+\infty} h_\pm(t)\omega'_\eta(t)dt\right]\theta = \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_\pm^{\tau_0}\theta, \tag{4.11}$$

$$\lim_{\eta\to 0^+} \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \left[\int_{\mathbb{R}} (a(x,\xi)+b(\xi))h_\pm(x)\omega'_\eta(x)dx\right]\psi = \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_\pm\psi \tag{4.12}$$

for any $\theta \in L_c^1(\mathbb{R}_x\times(0,1)\times\mathbb{R}_\xi)$ and any $\psi \in L_c^1(\Sigma\times(0,1)\times\mathbb{R}_\xi)$ (the subsequences with respect to η are independent of θ and ψ respectively). Besides, denoting by $m_\pm^{\tau_0}$ (resp. $\overline{m}_\pm$) the restriction of $m_\pm$ to $\{0\}\times\mathbb{R}_x\times[0,1]\times\mathbb{R}_\xi$ (resp. $[0,+\infty)\times\{0\}\times[0,1]\times\mathbb{R}_\xi$), one has : $\forall\theta \in \mathcal{C}_c^\infty(\mathbb{R}_x\times\mathbb{R}_\xi)$, $\forall\psi \in \mathcal{C}_c^\infty([0,+\infty)\times\mathbb{R}_\xi)$

$$\int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_\pm^{\tau_0}\theta = \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_\pm^0\theta - \int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi\theta dm_\pm^{\tau_0} \tag{4.13}$$

and

$$\int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_\pm\psi = -\int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))^\pm\psi - \int_{[0,+\infty)\times\mathbb{R}_\xi} \partial_\xi\psi d\overline{m}_\pm\,. \tag{4.14}$$

The existence of $h_\pm^{\tau_0}$ and $\Upsilon_\pm$ follows from the local uniform boundedness of $\int_0^{+\infty} h_\pm(t)\omega'_\eta(t)dt$ and $\int_{\mathbb{R}}(a(\xi)k(x)+b(\xi)h_\pm()\omega'_\eta(x)dx$ in $L^\infty(\mathbb{R}_x\times(0,1)\times\mathbb{R}_\xi)$ and $L^\infty(\Sigma\times(0,1)\times\mathbb{R}_\xi)$ respectively. To prove (4.13), replace ϕ in (4.8) by the function $(t,x,\xi)\mapsto\theta(x,\xi)(1-\omega_\eta)(t)$, for $\theta\in\mathcal{C}_c^\infty(\mathbb{R}_x\times\mathbb{R}_\xi)$ and pass to the limit on η in the equation thus obtained. Similarly, use the test function $(t,x,\xi)\mapsto\psi(t,\xi)(1-\omega_\eta)(x)$ in (4.8) to get (4.14).

4.5 Proof of Theorem : Comparison

Let h_+ and j_- denote the equilibrium functions associated with u and v respectively and denote by m_+ and q_- the associated entropy defect measure. For $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ *with compact support in* $\mathbb{R}_t^* \times \mathbb{R}_x^* \times \mathbb{R}_\xi$, one has

$$\int_0^1 \int_{Q\times\mathbb{R}_\xi} h_+(\partial_t + (a(x,\xi) + b(\xi))\partial_x)\phi = \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \phi dm_+ \tag{4.15}$$

and

$$\int_0^1 \int_{Q\times\mathbb{R}_\xi} j_-(\partial_t + (a(x,\xi) + b(\xi))\partial_x)\phi = \int_{\bar{Q}\times\mathbb{R}_\xi} \partial_\xi \phi dq_-. \tag{4.16}$$

Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be a test function *with compact support in* $\mathbb{R}_t^* \times \mathbb{R}_x^* \times \mathbb{R}_\xi$ (θ vanishes in a neighborhood of $\mathbb{R}_t \times \{0\} \times \mathbb{R}_\xi$ and in a neighborhood of $\{0\} \times \mathbb{R}_x \times \mathbb{R}_\xi$). Denote by $\rho_{\beta,\nu,\sigma}$ the function $(t,x) \mapsto \rho_\beta(-t)\rho_\nu(x)\rho_\sigma(\xi)$ and by $\gamma_{\beta,\nu,\sigma}$ the function $(t,x) \mapsto \rho_{\beta,\nu,\sigma}(-t,-x,-\xi)$. For ν small enough, the function $(t,x) \mapsto \theta \star \gamma_{\beta,\nu,\sigma}$ still vanishes on $\mathbb{R}_t \times \{0\} \times \mathbb{R}_\xi$ and $\{0\} \times \mathbb{R}_x \times \mathbb{R}_\xi$ so that one can specify this test function in (4.16) to obtain

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^3} (\mathrm{sgn}_+(t) j_-) \star \rho_{\beta,\nu,\sigma} \partial_t \theta \\ + \quad &\int_0^1 \int_{\mathbb{R}^3} \big(a(x,\xi)(\mathrm{sgn}_+(t)j_-) + b(\xi)(\mathrm{sgn}_+(t)j_-)\big) \star \rho_{\beta,\nu,\sigma}\partial_x\theta \\ = \quad &\int_{\mathbb{R}^3} \partial_\xi \theta \; d(\mathrm{sgn}_+(t) q_-) \star \rho_{\beta,\nu,\sigma}. \end{aligned}$$

Still for ν small enough, one has

$$(\mathrm{sgn}_+(t)a(x,\xi)j_-) \star \rho_{\beta,\nu,\sigma}\partial_x\theta = a(x,\xi)(\mathrm{sgn}_+(t)j_-) \star \rho_{\beta,\nu,\sigma}\partial_x\theta + Q_-^{\beta,\nu,\sigma}(t,x,\xi)\partial_x\theta$$

with

$$Q_-^{\beta,\nu,\sigma}(t,x,\xi) = (\mathrm{sgn}_+(t)a(x,\xi)j_-) \star \rho_{\beta,\nu,\sigma} - a(x,\xi)(\mathrm{sgn}_+(t)j_-) \star \rho_{\beta,\nu,\sigma}$$

.

Remark 4.3. *By using Lebesgue dominated convergence theorem, one has*

$$\lim_{\beta,\nu,\sigma\to 0} Q_-^{\beta,\nu,\sigma}(t,x,\xi) \to 0 \text{ and } \lim_{\beta,\nu\to 0}(\lim_{\sigma\to 0} \partial_x Q_-^{\beta,\nu,\sigma}(t,x,\xi)) = 0.$$

The regularized equation follows :

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^{d+1}} j_-^{\beta,\nu,\sigma}(t,x,\zeta,\xi)(\partial_t + (a(x,\xi) + b(\xi))\partial_x)\theta(t,x,\xi) \\ + \quad &\int_0^1 \int_{\mathbb{R}^3} Q_-^{\beta,\nu,\sigma}(t,x,\xi)\partial_x\theta(t,x,\xi) = \int_{\mathbb{R}3} \partial_\xi\theta(t,x,\xi)\; dq_-^{\beta,\nu,\sigma}(t,x,\xi), \end{aligned} \tag{4.17}$$

where $j_-^{\beta,\nu,\sigma} = (\mathrm{sgn}_+(t)j_-) \star \rho_{\beta,\nu,\sigma}$, $q_-^{\beta,\nu,\sigma} = (\mathrm{sgn}_+(t)q_-) \star \rho_{\beta,\nu,\sigma}$. Similarly, with obvious notations, the following regularized kinetic equation is satisfied by h_+ (for ε small enough) :

$$\begin{aligned}&\int_0^1 \int_{\mathbb{R}^3} h_+^{\alpha,\varepsilon,\delta}(t,x,\lambda,\xi)(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\theta(t,x,\xi)\\ + &\int_0^1 \int_{\mathbb{R}^3} H_+^{\alpha,\varepsilon,\delta}(t,x,\xi)\partial_x\theta(t,x,\xi) = \int_{\mathbb{R}^3} \partial_\xi\theta(t,x,\xi)\, dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi)\,,\end{aligned} \tag{4.18}$$

with

$$\lim_{\alpha,\varepsilon,\sigma\to 0} H_+^{\alpha,\varepsilon,\delta} = 0.$$

To compare two solutions, $A^+ = \{\xi :\ (g_L(\xi)-g_R(\xi))^+ = 0\} = \{\xi :\ g_L(\xi)-g_R(\xi) < 0\}$ is introduced. A^+ is open and included in $[0,1]$ and let $\theta^+ = \chi_{A^+}$. The aim is to take this function θ^+ (resp. $(1-\theta^+)$) as a test function to eliminate, in a first time, the term $\partial_\xi(g_L(\xi)-g_R(\xi))^+$ (resp. $\partial_\xi(g_L(\xi)-g_R(\xi))^-$) in the definition of kinetic process solution. In a second time, one studies separately the new term which appears when one forgets θ^+.
As θ^+ is not regular, a sequence of function $(\theta_n^+)_{n\in\mathbb{N}}$ is introduced which converges to θ^+ such that for all $n \in \mathbb{N}$, $\theta_n^+ = 0$ on $(A^+)^c$. Then Lemma 4.1 follows :

Lemma 4.1. *There exists a sequence $(\theta_n^+)_{n\in\mathbb{N}}$ such that $0 \le \theta_n^+ \le \theta^+$ (i.e. $\theta_n^+ = 0$ on $(A^+)^c$), $\theta_n^+ \in \mathcal{C}^1(\mathbb{R})$ for all $n \in \mathbb{N}$ and $(\theta_n^+)_{n\in\mathbb{N}}$ converges to θ^+ a.e.*

Proof of Lemma 4.1 :
A is open and include in $[0,1]$, then $A^+ = \bigcup_{i=0}^{\infty} I_i$ with I_i an opened interval and $\bar{I}_i \cap \bar{I}_j = \varnothing$ for $i \neq j$. One notes $A_n = \bigcup_{i=0}^{n} I_i$, then $\chi_{A_n} \nearrow \chi_{A^+} = \theta^+$.
Let $n > 0$, A_n is a finished union of disjointed intervals so one can build a regular function θ_n^+ such that

$$0 \le \theta_n^+ \le \chi_{A_n} \quad \text{and} \quad ||\theta_n^+ - \chi_{A_n}||_{L^1} \le \frac{1}{n}.$$

This yields

$$||\theta_n^+ - \theta^+||_{L^1} \le ||\theta_n^+ - \chi_{A_n}||_{L^1} + ||\chi_{A^+} - \chi_{A_n}||_{L^1} \to 0 \text{ with n} \to +\infty$$

then θ_n^+ converges towards θ^+ and $0 \le \theta_n^+ \le \theta^+$.

Remark 4.4. *One notes that on ∂I_i, $g_L = g_R$ for all i by construction. Moreover one can define a sequence of disjointed intervals $(J_i)_{i\in\mathbb{N}}$ such that $[0,1] = \bigcup_{i=0}^{\infty} I_i \cup \bigcup_{i=0}^{\infty} \bar{J}_i$ and one also has on ∂J_i, $g_L = g_R$ for all i.*

Let $\psi \in \mathcal{C}_c^2(\mathbb{R})$ such that $\psi(x) = 1$ for all $x \in [-1,1]$ and $\psi(x) = 0$ for all $|x| > 2$. Let $\psi_R(x) = \psi(x/R)$ for all $R > 0$.

Let $\varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function with compact support in $\mathbb{R}_t^* \times \mathbb{R}_x^* \times \mathbb{R}_\xi$. One applies (4.18) to the test function $\theta = -j_-^{\beta,\nu,\sigma}\varphi_1\, \theta_n^+\, \psi_R$ and integrates the result with respect to $\zeta \in [0,1]$; one applies (4.17) to the test function $\theta = -h_+^{\alpha,\varepsilon,\delta}\varphi_1\, \theta_n^+\, \psi_R$ and integrates the result with respect to $\lambda \in [0,1]$. Finally, one sums the two resulting equations to get

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3} \varphi_1\theta_n^+\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)(-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma}) \\
+ \quad & 2\int_{\mathbb{R}^3} (-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma})\theta_n^+\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi_1 \\
+ \quad & \int_0^1\int_{\mathbb{R}^3} Q_-^{\beta,\nu,\sigma}\partial_x(-h_+^{\alpha,\varepsilon,\delta}\theta_n^+\psi_R\varphi_1) + \int_0^1\int_{\mathbb{R}^3} H_+^{\alpha,\varepsilon,\delta}\partial_x(-j_-^{\beta,\nu,\sigma}\theta_n^+\psi_R\varphi_1) \\
= \quad & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1\partial_\xi(-j_-^{\beta,\nu,\sigma})\theta_n^+\psi_R\, dm_+^{\alpha,\varepsilon,\delta} \\
+ \quad & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1\partial_\xi(-h_+^{\alpha,\varepsilon,\delta})\theta_n^+\,\psi_R\, dq_-^{\beta,\nu,\sigma} \\
+ \quad & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+^{\alpha,\varepsilon,\delta} \\
+ \quad & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1\partial_\xi(\theta_n^+\,\psi_R)(-h_+^{\alpha,\varepsilon,\delta})\, dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.19}$$

Since $m_+^{\alpha,\varepsilon,\delta}, q_-^{\beta,\nu,\sigma} \geq 0$ and $\partial_\xi(-j_-^{\beta,\nu,\sigma}), \partial_\xi(-h_+^{\alpha,\varepsilon,\delta}) \geq 0$, the first two terms of the right hand-side of (4.19) are nonnegative. One integrates by parts with respect to (t,x) in the left hand-side (an operation which is admissible since φ_1 vanishes in the neighborhood of the line of discontinuity of the function g) to get

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3} (-h_+^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma})\theta_n^+\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi_1 \\
+ \quad & \int_0^1\int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+^{\alpha,\varepsilon,\delta}\theta_n^+\psi_R\varphi_1 \\
+ \quad & \int_0^1\int_{\mathbb{R}^3} H_+^{\alpha,\varepsilon,\delta}\partial_x(-j_-^{\beta,\nu,\sigma}\theta_n^+\psi_R\varphi_1) \\
\geq \quad & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+^{\alpha,\varepsilon,\delta} \\
+ \quad & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1\partial_\xi(\theta_n^+\,\psi_R)(-h_+^{\alpha,\varepsilon,\delta})\, dq_-^{\beta,\nu,\sigma}.
\end{aligned}$$

Let $\alpha, \varepsilon, \delta$ tend to zero, one recalls that $\lim_{\alpha,\varepsilon,\delta\to 0} H_+^{\alpha,\varepsilon,\delta} = 0$, this yields :

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+ j_-^{\beta,\nu,\sigma})\theta_n^+ \psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi_1 \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+\theta_n^+\psi_R\varphi_1 \\
\geq & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1 \partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+ \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1 \partial_\xi(\theta_n^+\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.20}$$

Let us now remove the condition imposed on the test function : let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function, replace φ_1 by $(t,x) \mapsto \varphi(t,x)\omega_\eta(t)\omega_{\widetilde{\eta}}(x)$ in (4.20), use Proposition 4.2 and pass to the limit on accurate subsequences on η and $\widetilde{\eta}$ to get

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+ j_-^{\beta,\nu,\sigma})\theta_n^+ \psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+\theta_n^+\psi_R\varphi \\
+ & \int_0^1 \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\theta_n^+\psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\
+ & \int_0^1 \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_+(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\theta_n^+\psi_R\varphi(t,0)\,dt\,d\xi\,d\lambda\,d\zeta \\
\geq & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi \partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+ \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi \partial_\xi(\theta_n^+\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.21}$$

By (4.14), and since $\theta_n^+ = 0$ on $\{\xi : (g_L - g_R)^+ \neq 0\}$, one has

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_+(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)\theta_n^+\psi_R\,dt\,d\xi\,d\lambda\,d\zeta \\
= & -\int_0^1 \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\lambda\,d\zeta \\
& -\int_0^1 \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\lambda\,d\zeta \\
\leq & -\int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta,
\end{aligned}$$

because $\partial_\xi(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi)) \geq 0$. Similarly, by (4.13), one gets

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\theta_n^+\psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta\\
\leq\ &\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\theta_n^+\psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta\\
-\ &\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta\\
\leq\ &\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\theta_n^+\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
-\ &\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\ \,d\zeta,
\end{aligned}$$

and finally (4.21) becomes :

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})\theta_n^+\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi\\
+\ &\int_0^1\int_{\mathbb{R}^3}\partial_x(Q_-^{\beta,\nu,\sigma})h_+\theta_n^+\psi_R\varphi\,dx\,dt\,d\xi\,d\zeta\\
+\ &\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^{0}(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\theta_n^+\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
\geq\ &\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\ \,d\zeta\\
+\ &\int_0^1\int_{\Sigma\times\mathbb{R}_\xi}\varphi(t,0)\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta\\
+\ &\int_0^1 d\zeta\int_{\mathbb{R}^3}\varphi_1\partial_\xi(\theta_n^+\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+\\
+\ &\int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi_1\partial_\xi(\theta_n^+\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}.
\end{aligned}\tag{4.22}$$

In this step of the proof, the comparison between u and v on $\{\xi : g_L(\xi)-g_R(\xi)<0\}$ is obtained by using h_+ and j_-. In the same way, by using function $(1-\theta_n^+)$, the comparison on $\{\xi : g_L(\xi)-g_R(\xi)\geq 0\}$ is obtained. And since on the support of $(1-\theta^+)$, it is $(g_L(\xi)-g_R(\xi))^-=0$, one compares, at the beginning, h_- and j_-. By using, $h_-=h_+-1$, a comparison between h_+ and j_- on the support of $(1-\theta^+)$ is obtained, which permits to conclude.
Let $\theta\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ be a test function *with compact support in* $\mathbb{R}_t^*\times\mathbb{R}_x^*\times\mathbb{R}_\xi$. In the same manner as in (4.17), one obtains

$$\begin{aligned}
&\int_0^1\int_{\mathbb{R}^3} j_-^{\beta,\nu,\sigma}(t,x,\zeta,\xi)(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\theta(t,x,\xi)\\
+\ &\int_0^1\int_{\mathbb{R}^3} Q_-^{\beta,\nu,\sigma}\partial_x\theta=\int_{\mathbb{R}^3}\partial_\xi\theta(t,x,\xi)\ dq_-^{\beta,\nu,\sigma}(t,x,\xi),
\end{aligned}\tag{4.23}$$

and

$$\int_0^1 \int_{\mathbb{R}^3} h_-^{\alpha,\varepsilon,\delta}(t,x,\lambda,\xi)(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\theta(t,x,\xi)$$
$$+ \quad \int_0^1 \int_{\mathbb{R}^3} H_-^{\alpha,\varepsilon,\delta}\partial_x\theta = \int_{\mathbb{R}^3} \partial_\xi\theta(t,x,\xi)\, dm_-^{\alpha,\varepsilon,\delta}(t,x,\xi)\,, \tag{4.24}$$

with

$$\lim_{\beta,\nu\to 0}(\lim_{\sigma\to 0}\partial_x(Q_-^{\beta,\nu,\sigma})) = 0 \quad \text{and} \quad \lim_{\alpha,\varepsilon,\delta\to 0} H_-^{\alpha,\varepsilon,\delta} = 0. \tag{4.25}$$

Let $\varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function with compact support in $\mathbb{R}_t^* \times \mathbb{R}_x^* \times \mathbb{R}_\xi$. One applies (4.24) to the test function $\theta = -j_-^{\beta,\nu,\sigma}\varphi_1\,(1-\theta_n^+)\,\psi_R$ and integrates the result with respect to $\zeta \in [0,1]$; one applies (4.23) to the test function $\theta = -h_-^{\alpha,\varepsilon,\delta}\varphi_1\,(1-\theta_n^+)\,\psi_R$ and integrates the result with respect to $\lambda \in [0,1]$. Finally, one sums the two resulting equations to get

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^3}\varphi_1(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)(-h_-^{\alpha,\varepsilon,\delta}j_-^{\beta,\nu,\sigma})\\
+\;& 2\int_{\mathbb{R}^3}(-h_-^{\alpha,\varepsilon,\delta}j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi_1\\
+\;& \int_0^1\int_{\mathbb{R}^3}Q_-^{\beta,\nu,\sigma}\partial_x(-h_+^{\alpha,\varepsilon,\delta}(1-\theta_n^+)\psi_R\varphi_1)\\
+\;& \int_0^1\int_{\mathbb{R}^3}H_-^{\alpha,\varepsilon,\delta}\partial_x(-j_-^{\beta,\nu,\sigma}(1-\theta_n^+)\psi_R\varphi_1)\\
=\;& \int_0^1 d\zeta\int_{\mathbb{R}^3}\varphi_1\partial_\xi(-j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R\,dm_-^{\alpha,\varepsilon,\delta}\\
+\;& \int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi_1\partial_\xi(-h_-^{\alpha,\varepsilon,\delta})(1-\theta_n^+)\,\psi_R\,dq_-^{\beta,\nu,\sigma}\\
+\;& \int_0^1 d\zeta\int_{\mathbb{R}^3}\varphi_1\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma})dm_-^{\alpha,\varepsilon,\delta}\\
+\;& \int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi_1\partial_\xi((1-\theta_n^+)\psi_R)(-h_-^{\alpha,\varepsilon,\delta})\,dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.26}$$

Since $m_-^{\alpha,\varepsilon,\delta}, q_-^{\beta,\nu,\sigma} \geq 0$ and $\partial_\xi(-j_-^{\beta,\nu,\sigma}), \partial_\xi(-h_-^{\alpha,\varepsilon,\delta}) \geq 0$, the first two terms of the right hand-side of (4.26) are nonnegative. One integrates by parts with respect to (t,x) in the left hand-side (an operation which is admissible since φ_1 vanishes in the neighborhood of the line of discontinuity of the function g) to get

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-h_-^{\alpha,\varepsilon,\delta} j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi_1 \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma}) h_+^{\alpha,\varepsilon,\delta}(1-\theta_n^+)\psi_R\varphi_1 \\
+ & \int_0^1 \int_{\mathbb{R}^3} H_-^{\alpha,\varepsilon,\delta}\partial_x(-j_-^{\beta,\nu,\sigma}(1-\theta_n^+)\psi_R\varphi_1) \\
\geq & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}) dm_-^{\alpha,\varepsilon,\delta} \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_-^{\alpha,\varepsilon,\delta})\, dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.27}$$

Let $\alpha, \varepsilon, \delta$ tend to zero, by using (4.25), one has :

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-h_- j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi_1 \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma}) h_+(1-\theta_n^+)\psi_R\varphi_1 \\
\geq & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi_1\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}) dm_- \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi_1\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_-)\, dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.28}$$

Let us now remove the condition imposed on the test function : let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a nonnegative function, replace φ_1 by $(t,x) \mapsto \varphi(t,x)\omega_\eta(t)\omega_{\widetilde{\eta}}(x)$ in (4.28), use Proposition 4.2 and pass to the limit on accurate subsequences on η and $\widetilde{\eta}$ to get

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_- j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma}) h_+(1-\theta_n^+)\psi_R\varphi \\
+ & \int_0^1 \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+) \\
& \qquad \psi_R\varphi(0,x)\, dx\, d\xi\, d\lambda\, d\zeta \\
+ & \int_0^1 \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_-(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))(1-\theta_n^+) \\
& \qquad \psi_R\varphi(t,0)\, dt\, d\xi\, d\lambda\, d\zeta \\
\geq & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}) dm_- \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_-)\, dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.29}$$

One obtains a comparison between h_- and $j_-^{\beta,\mu,\sigma}$ with the traces of equilibrium function of h_- on the support of $(1-\theta_n^+)$. Then, in the formulation which defines $h_-^{\tau_0}$ and Γ_-, the term with $(g_L(\xi)-g_R(\xi))^- = 0$. However, one can compare u and v by using h_+ and j_-, (because one remarks that $\int_{\mathbb{R}} h_+(-j_-)\,d\xi = (u-v)^+$). Then in the previous inequality, one wants to replace h_- by h_+ both in the first and in the last terms, by using the equality $h_- = h_+ - 1$. On the one hand one has :

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_-j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi \\
= & \int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi. \qquad (4.30)
\end{aligned}$$

The first term is the term one wants to keep. One studies the second term

$$\begin{aligned}
& \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi \\
= & \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)\Big((\partial_t+(a(x,\xi)+b(\xi))\partial_x)(1-\theta_n^+)\psi_R\varphi\Big)*\gamma_{\beta,\nu,\sigma} \\
= & \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma} \\
+ & \mathfrak{R}^{n,R}_{\beta,\nu,\sigma}, \qquad (4.31)
\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{R}^{n,R}_{\beta,\nu,\sigma} = & \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma} \\
- & \int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)\Big((\partial_t+(a(x,\xi)+b(\xi))\partial_x)(1-\theta_n^+)\psi_R\varphi\Big)*\gamma_{\beta,\nu,\sigma}.
\end{aligned}$$

By using Lebesgue dominated convergence theorem, one remarks that

$$\lim_{\beta,\nu,\sigma\to 0}(\lim_{R\to+\infty}(\lim_{n\to+\infty}\mathfrak{R}^{n,R}_{\beta,\nu,\sigma})) = 0.$$

By using (4.8) for j_- (i.e. v is a kinetic solution), this yields :

$$\begin{aligned}
&\int_0^1\int_{\mathbb{R}^3}(\mathrm{sgn}_+(t)j_-)(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\\
= &-\int_{\mathbb{R}_x\times\mathbb{R}_\xi} j_-^0\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|t=0}\\
+ &\int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0}\\
+ &\int_{\bar{Q}\times\mathbb{R}_\xi}\partial_\xi\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)dq_-\\
\leq &\int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0}\\
+ &\int_{\bar{Q}\times\mathbb{R}_\xi}\partial_\xi((1-\theta_n^+)\psi_R)\varphi dq_-^{\beta,\nu,\sigma},
\end{aligned}$$

because $j_-^0\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)\geq 0$.

Finally, (4.29) becomes :

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_-j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi\\
\leq &\int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi\\
+ &\int_0^1\int_{\mathbb{R}^3}\partial_\xi((1-\theta_n^+)\psi_R)\varphi dq_-^{\beta,\nu,\sigma}\\
+ &\int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0}+\mathfrak{R}^{n,R}_{\beta,\nu,\sigma}. \qquad (4.32)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_-)\,dq_-^{\beta,\nu,\sigma}\\
= &\int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}\\
+ &\int_0^1\int_{\mathbb{R}^3}\partial_\xi((1-\theta_n^+)\psi_R)\varphi dq_-^{\beta,\nu,\sigma}. \qquad (4.33)
\end{aligned}$$

Then (4.27) becomes

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3}\partial_x(Q_-^{\beta,\nu,\sigma})h_+(1-\theta_n^+)\psi_R\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3}\partial_\xi((1-\theta_n^+)\psi_R)\varphi dq_-^{\beta,\nu,\sigma} \\
+ & \int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}^{n,R}_{\beta,\nu,\sigma} \\
+ & \int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\
+ & \int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi}\Upsilon_-(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(t,0)\,dt\,d\xi\,d\lambda\,d\zeta \\
\geq & \int_0^1 d\zeta\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma})dm_- \\
+ & \int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma} \\
+ & \int_0^1\int_{\mathbb{R}^3}\partial_\xi((1-\theta_n^+)\psi_R)\varphi dq_-^{\beta,\nu,\sigma}. \qquad (4.34)
\end{aligned}$$

After simplification, this yields :

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3}(-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3}\partial_x(Q_-^{\beta,\nu,\sigma})h_+(1-\theta_n^+)\psi_R\varphi \\
+ & \int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}^{n,R}_{\beta,\nu,\sigma} \\
+ & \int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta \\
+ & \int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi}\Upsilon_-(t,\lambda,\xi)(j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(t,0)\,dt\,d\xi\,d\lambda\,d\zeta \\
\geq & \int_0^1 d\zeta\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma})dm_- \\
+ & \int_0^1 d\lambda\int_{\mathbb{R}^3}\varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}. \qquad (4.35)
\end{aligned}$$

Now, it remains to study terms where the trace of equilibrium function appear. At first, one studies the term with the space-trace function. By (4.14), and by using the equality $\overline{m}_- = \overline{m}_+$ and since $1-\theta_n^+ = 0$ on $\{\xi : (g_L - g_R)^+ \neq 0\}$, we have

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \Upsilon_-(t,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi_{x=0}(1-\theta_n^+)\psi_R\,dt\,d\xi\,d\lambda\,d\zeta\\
=\;& -\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)\\
&\qquad (1-\theta_n^+)\psi_R\,dt\,d\xi\,d\lambda\,d\zeta\\
-\;& \int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))(1-\theta_n^+)\psi_R d\overline{m}_+\,dt\,d\xi\,d\lambda\,d\zeta\\
-\;& \int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\lambda\,d\zeta\\
\leq\;& -\int_0^1\int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)\\
&\qquad (1-\theta_n^+)\psi_R\,dt\,d\xi\,d\lambda\,d\zeta\\
-\;& \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta\,,
\end{aligned}$$

because $\partial_\xi(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi)) \geq 0$. Secondly, one studies the time-trace function of equilibrium function h_-. By (4.13), one gets :

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\\
&\qquad\qquad \psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta\\
\leq\;& \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
-\;& \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta.
\end{aligned}$$

Always in the aim to compare u and v by using h_+ and j_-, one uses the equality $h_-^0 = -1+h_+^0$, and as $(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x) \geq 0$, and the last inequality becomes

$$
\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_-^{\tau_0}(x,\lambda,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\\
&\qquad\qquad\qquad\qquad \psi_R\varphi(0,x)\,dx\,d\xi\,d\lambda\,d\zeta\\
\le\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
-\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} (-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
-\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta\\
\le\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
-\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta,
\end{aligned}
$$

Finally, using $m_+ = m_-$ (because an entropy process solution is a weak solution), (4.35) becomes

$$
\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+j_-^{\beta,\nu,\sigma})(1-\theta_n^+)\psi_R(\partial_t+(a(x,\xi)+b(\xi))\partial_x)\varphi\\
+\ & \int_0^1\int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+(1-\theta_n^+)\psi_R\varphi\\
+\ & \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}_{\beta,\nu,\sigma}^{n,R}\\
+\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))(1-\theta_n^+)\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta\\
\ge\ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\ \,d\zeta\\
+\ & \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta\\
+\ & \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)(1-\theta_n^+)\psi_R\,dt\,d\xi\,d\zeta\\
+\ & \int_0^1 d\zeta\int_{\mathbb{R}^3} \varphi\partial_\xi((1-\theta_n^+)\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+\\
+\ & \int_0^1 d\lambda\int_{\mathbb{R}^3} \varphi\partial_\xi((1-\theta_n^+)\,\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}.
\end{aligned}
\tag{4.36}
$$

In this step of the proof, a comparison between (4.22) and (4.36) leads to sum these inequalities. By using $\partial_\xi(\theta_n^+ + (1-\theta_n^+)) = 0$, this yields

$$\begin{aligned}
& \int_0^1\int_0^1\int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+ j_-^{\beta,\nu,\sigma})\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+\theta_n^+\psi_R\varphi \\
+ & \int_0^1\int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+(1-\theta_n^+)\psi_R\varphi \\
+ & \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta_n^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}^{n,R}_{\beta,\nu,\sigma} \\
+ & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta \\
\geq & \int_0^1\int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\;\,d\zeta \\
+ & \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta \\
+ & \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)(1-\theta_n^+)\psi_R\,dt\,d\xi\,d\zeta \\
+ & \int_0^1 d\zeta\int_{\mathbb{R}^3} \varphi\partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+ \\
+ & \int_0^1 d\lambda\int_{\mathbb{R}^3} \varphi\partial_\xi((\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}. \qquad (4.37)
\end{aligned}$$

Now one can pass to the limit on n in (4.37). Indeed, the regularity of function θ_n^+ is not necessary. The only term which brings difficulties is :

$$\begin{aligned}
& \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0)(1-\theta_n^+)\psi_R\,dt\,d\xi\,d\zeta \\
= & \int_0^1\int_0^{+\infty}\int_{\{\xi:g_L(\xi)-g_R(\xi)<0\}} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0) \\
& \qquad (1-\theta_n^+)\psi_R\,dt\,d\xi\,d\zeta,
\end{aligned}$$

but on $\{\xi : g_L(\xi)-g_R(\xi)<0\}$, $(1-\theta_n^+)$ tends to zero when n tends to infinity, then by using the Lebesgue dominated convergence Theorem, this yields :

$$\begin{aligned}
& \int_0^1\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))\varphi(t,0) \\
& \qquad (1-\theta_n^+)\psi_R\,dt\,d\xi\,d\zeta \longrightarrow_{n\to+\infty} 0.
\end{aligned}$$

So when n tends to infinity, (4.37) becomes :

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+ j_-^{\beta,\nu,\sigma})\psi_R(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+\psi_R\varphi \\
+ & \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta^+)\psi_R\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}^R_{\beta,\nu,\sigma} \\
+ & \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\psi_R\varphi(0,x)\,dx\,d\xi\,d\zeta \\
\geq & \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} \partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\ \,d\zeta \\
+ & \int_0^1 \int_{\Sigma\times\mathbb{R}_\xi} \varphi(t,0)\partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma}(t,0,\zeta,\xi))d\overline{m}_+\,dt\,d\xi\,d\zeta \\
+ & \int_0^1 d\zeta \int_{\mathbb{R}^3} \varphi\partial_\xi(\psi_R)(-j_-^{\beta,\nu,\sigma})dm_+ \\
+ & \int_0^1 d\lambda \int_{\mathbb{R}^3} \varphi\partial_\xi((\psi_R)(-h_+)\,dq_-^{\beta,\nu,\sigma}.
\end{aligned} \tag{4.38}$$

Finally, one remarks that $\partial_\xi\psi_R(\xi) = \dfrac{1}{R}\psi'(\xi/R)$ tends to zero when R tends to infinity and $\psi_R(\xi) \to 1$; by using Lebesgue dominated convergence Theorem, (4.38) becomes

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t)h_+ j_-^{\beta,\nu,\sigma})(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ & \int_0^1 \int_{\mathbb{R}^3} \partial_x(Q_-^{\beta,\nu,\sigma})h_+\varphi \\
+ & \int_0^1 \int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\beta,\nu,\sigma}(0,x,\zeta,\xi))\varphi(0,x)\,dx\,d\xi\,d\zeta \\
+ & \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi(g_L(\xi)-g_R(\xi))^-\big((1-\theta^+)\varphi\big)*\gamma_{\beta,\nu,\sigma}\big)_{|x=0} + \quad \mathfrak{R}_{\beta,\nu,\sigma} \geq 0,
\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{R}_{\beta,\nu,\sigma} \quad = & \int_0^1 \int_{\mathbb{R}^3} (\mathrm{sgn}_+(t)j_-)(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\big((1-\theta^+)\varphi\big)*\gamma_{\beta,\nu,\sigma} \\
- & \int_0^1 \int_{\mathbb{R}^3} (\mathrm{sgn}_+(t)j_-)\Big((\partial_t + (a(x,\xi)+b(\xi))\partial_x)(1-\theta^+)\varphi\Big)*\gamma_{\beta,\nu,\sigma}.
\end{aligned}$$

Remark 4.5. *The function ψ_R are introduced in order to assure the well definition of the considered integrals.*

Now, one passes to the limit on β, ν, σ.
In (4.38), $\sigma \to 0$, then $\nu \to 0$, using (4.25), this yields :

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-^\beta)(\partial_t + (a(\xi) + b(\xi))\partial_x)\varphi \\
+ \quad &\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(j_-^\beta(0,x,\zeta,\xi))\varphi(0,x)\, dx\, d\xi\, d\zeta \\
+ \quad &\int_{\Sigma \times \mathbb{R}_\xi} \partial_\xi (g_L(\xi) - g_R(\xi))^- \big((1-\theta^+)\varphi\big) * \rho_\beta(t)\big)_{|x=0} \\
+ \quad &\mathfrak{R}_\beta \geq 0
\end{aligned} \tag{4.39}$$

The limit as $\beta \to 0$ of the remaining terms is studied. First,

$$\begin{aligned}
&\lim_{\beta \to 0} \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-^\beta)(\partial_t + (a(x,\xi) + b(\xi))\partial_x)\varphi \\
&= \int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-)(\partial_t + (a(x,\xi) + b(\xi))\partial_x)\varphi\,.
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^\beta(0,x,\zeta,\xi))\varphi(0,x)\, dx\, d\xi\, d\zeta \\
= \quad &\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} \int_0^\infty h_+^0(x,\xi)(-j_-(s,x,\zeta,\xi))\rho_\beta(s)\varphi(0,x)\, dx\, d\xi\, d\zeta \\
= \quad &\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} \int_0^\infty h_+^0(x,\xi)(-j_-(s,x,\zeta,\xi))\omega'_\beta(s)\varphi(0,x)\, dx\, d\xi\, d\zeta,
\end{aligned}$$

therefore, for an appropriate subsequence, by Proposition 4.2, this yields

$$\begin{aligned}
&\lim_{\beta \to 0} \int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^\beta(0,x,\zeta,\xi))\varphi(0,x)\, dx\, d\xi\, d\zeta \\
= \quad &\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\tau_0}(x,\zeta,\xi))\varphi(0,x)\, dx\, d\xi\, d\zeta.
\end{aligned}$$

The trace $j_-^{\tau_0}$ satisfies the identity

$$j_-^{\tau_0} = j_-^0 + \partial_\xi q_-^0$$

from which one deduces

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^{\tau_0}(x,\zeta,\xi))\varphi(0,x)\, dx\, d\xi\, \zeta \\
\leq \quad &\int_{\mathbb{R}_x \times \mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^0(x,\xi))\varphi(0,x)\, dx\, d\xi\,.
\end{aligned}$$

Finally by using $\lim_{\beta\to 0} \mathfrak{R}_\beta = 0$, (4.39) becomes

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-)(\partial_t + (a(\xi)+b(\xi))\partial_x)\varphi \\
+ &\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^0(x,\xi))\varphi(0,x)\,dx\,d\xi \\
+ &\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi (g_L(\xi)-g_R(\xi))^- \big((1-\theta^+)\varphi\big)_{|x=0} \geq 0. \qquad (4.40)
\end{aligned}$$

To conclude, the last term becomes :

$$\begin{aligned}
&\int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi (g_L(\xi)-g_R(\xi))^- \big((1-\theta^+)\varphi\big)_{|x=0} \\
= &\int_0^\infty \int_{[0,1]} \partial_\xi (g_L(\xi)-g_R(\xi))^- (1-\theta^+)\varphi(t,0) \\
= &\int_0^\infty \varphi(t,0) \int_{[0,1]} \partial_\xi (g_L(\xi)-g_R(\xi))^- (1-\theta^+),
\end{aligned}$$

with

$$\int_{[0,1]} \partial_\xi (g_L(\xi)-g_R(\xi))^- (1-\theta^+) \;=\; \lim_{m\to+\infty} \sum_{i=0}^m \int_{J_i} \partial_\xi (g_L(\xi)-g_R(\xi))\,d\xi,$$

and for all i

$$\int_{J_i} \partial_\xi (g_L(\xi)-g_R(\xi))\,d\xi = 0,$$

because on ∂J_i, $g_R = g_L$ (see remark 4.4).

Then,

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_{\mathbb{R}^3} (-\mathrm{sgn}_+(t) h_+ j_-)(\partial_t + (a(x,\xi)+b(\xi))\partial_x)\varphi \\
+ &\int_{\mathbb{R}_x\times\mathbb{R}_\xi} h_+^0(x,\xi)(-j_-^0(x,\xi))\varphi(0,x)\,dx\,d\xi \geq 0. \qquad (4.41)
\end{aligned}$$

Collecting the previous results, and using the identities

$$\int_{\mathbb{R}} h_+(-j_-)d\xi = (u-v)^+\,, \qquad \int_{\mathbb{R}} h_+^0(-j_-^0)d\xi = (u_0-v_0)^+\,,$$
$$\int_{\mathbb{R}} a(x,\xi)h_+(-j_-)d\xi = \Phi^+(x,u,v)\,, \qquad \int_{\mathbb{R}} b(\xi)h_+(-j_-)d\xi = \Psi^+(u,v),$$

lead to the inequality

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^\infty\int_{\mathbb{R}}(u-v)^+\partial_t\varphi \\
+\quad&\int_0^1\int_0^1\int_0^\infty\int_{\mathbb{R}}(\Phi^+(x,u,v)+\Psi^+(u,v))\partial_x\varphi\,dx\,dt\,d\lambda\,d\zeta \\
+\quad&\int_{\mathbb{R}}(u_0-v_0)^+\varphi(0,x)dx\geq 0.
\end{aligned}$$

Finally, it's classical to obtain

$$\begin{aligned}
\int_0^1\int_0^1\int_0^T\int_{-R}^R(u(t,x,\lambda)-v(t,x,\zeta))^+dxdtd\lambda d\zeta& \\
\leq T\int_{-R-CT}^{R+CT}(u_0(x)-v_0(x))^+dx.&
\end{aligned}$$

In the same way, the same result with the half entropy $(u-\kappa)^-$ is obtained.

Bibliography

[AV02] Adimurthi, and G.D. Veerappa Gowda. *Conservation law with a discontinuous flux*, J. Math. Kyoto Univ., **42**, (2002), no. 4, 27-70.

[AP04] E. Audusse and B. Perthame. *Uniqueness for a scalar conservation law with discontinuous via adapted entropies*, Research report Inria RR-5261, (2004).

[Bac04] F. Bachmann. *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations, (nov-dec 2004), no. 11-12, 1317-1338.

[BV05] F. Bachmann and J. Vovelle. *Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients*, to appear in Communications in Partial Differential Equations.

[DiP85] R.J. DiPerna. *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223-270.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin. *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, (2000), 713-1020.

[GMT96] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Springer-Verlag, Berlin, (1996), with a preface by Charles-Michel Marle.

[KRT02b] K.H. Karlsen, N.H. Risebro and J.D Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron J. Differential Equations, (2002), **93**, 23 pp (electronic).

[KRT03] K.H. Karlsen, N.H. Risebro and J.D Towers. *L1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid.Selsk, (2003) 1-49.

[KR95] K.H. Karlsen and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness ans asymptotic behaviour*, Comm. Partial Differential Equations, **20**, (1995),1959-1990.

[Kru70] S.N. Kruzkov. *First order quasilinear equations with several independent variables* Mat. Sb. (N.S.),**81**, (1970), 228-255.

[LPT94] P.-L. Lions, B. Perthame, and E. Tadmor. *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc., **7**, (1994), no. 1, 169-191.

[Per98] B. Perthame. *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures Appl., **77**, (1998), no. 10, 1055-1064.

[Per02] B. Perthame. *Kinetic formulation of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, **21**, Oxford, (2002).

[SV03] N. Seguin and J. Vovelle. *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13**, (2003), no. 2, 221-257.

[Tem82] B. Temple. *Global solution of the Cauchy problem for a class of* 2×2 *nonstrictliy hyperbolic conservation laws.* Adv. in Appl. Math., **3**, (1982), no. 3.

[Tow00] J.D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681-698.

Chapter 5

Analyse d'une loi de conservation à flux discontinu de la forme $g(x,u)$: Existence d'une solution entropique et convergence d'un schéma Volume Fini

Abstract : In this paper, one studies a hyperbolic scalar equation in one space dimension with a flux function which is discontinuous with respect to the space variable. In the first part, one presents a convenient definition of weak entropy solution which ensures a uniqueness result. In the second part of this paper, one proves the convergence of some numerical results, whose a by product is the existence of a weak entropy solution.

5.1 Introduction

The Cauchy problem writes :

$$\begin{cases} \partial_t u + \partial_x\big(g(x,u) + f(u)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{5.1}$$

with initial value $u_0 \in L^\infty(\mathbb{R};[0,1])$. The functions f and g are supposed to satisfy the following hypotheses :

(H1) g is the discontinuous function defined by
$$g(x,u) = \begin{cases} g_L(u) & \text{if } x < 0 \\ g_R(u) & \text{if } x > 0 \end{cases} \quad \text{with } g_L \neq g_R,$$
$g_L, g_R \in \text{Lip}([0,1])$ and $g_L(0) = g_R(0) = g_L(1) = g_R(1) = 0$,

(H2) $f \in \text{Lip}([0,1])$.

One introduces the time - space domain $Q := (0,+\infty)\times\mathbb{R}$ and the line of discontinuity of the function g in the time - space domain, $\Sigma := (0,+\infty)\times\{0\}$.

One refers to the introduction of part I for the model of problem (5.1).

To study conservation law, finite volume scheme is well adapted. In this paper, one has been interested in this scheme for problem (5.1). One wants to know if there is convergence of the scheme and uniqueness of the "well" solution, i.e. entropy solution (see Definition 5.1), under natural hypotheses.

The uniqueness of entropy solution is established in part I, then the aim of this paper is to show the existence and the convergence of the scheme to the unique entropy solution. One specifies that the function g isn't genuinely non linear with respect to u. This last point is a new point compared with preceding works on the subject (see in particular [KR95, Tow01, KRT02a, KRT02b, KRT03, SV03, AJV04, Bac04]). In [BV05], the existence and the uniqueness have been established without assuming g genuinely non linear. However, the existence is not established with the convergence of a scheme. In particular, to establish the existence of entropy solution and to show the convergence of the scheme one does not use Temple function (as in [KR95, Tow00, Tow01, KRT02a, KRT02b, KRT03, SV03, Bac04]) because, principally, $\int_0^s |g'(s)|\,ds$ is not invertible if g is not genuinely non linear and moreover the existence of traces of entropy solution along the line $\{x=0\}$ is not assumed. In fact, with g not genuinely non linear, one don't know if these traces exist.

The definition of entropy solution is remained :

Definition 5.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. A function $u \in L^\infty(Q;[0,1])$ is said to be an entropy solution of problem (5.1) if it satisfies the following entropy inequalities : for all $\kappa \in [0,1]$, for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R})$,*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} (u(t,x)-\kappa)^\pm\, \partial_t\varphi(t,x)\,dt\,dx \\ + & \int_0^\infty \int_{\mathbb{R}} (\Phi^\pm(x,u(t,x),\kappa) + \Psi^\pm(u(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt \\ + & \int_{\mathbb{R}} (u_0(x)-\kappa)^\pm\,\varphi(0,x)\,dx + \int_0^\infty (g_L(\kappa)-g_R(\kappa))^\pm\,\varphi(t,0)\,dt \ge 0, \end{aligned} \quad (5.2)$$

where respectively $\Phi^{\pm}$ *and* $\Psi^{\pm}$ *denote the entropy flux associated with the Kruzhkov entropy,*

$$\begin{aligned}\Phi^{\pm}(x,u,\kappa) &= \operatorname{sgn}_{\pm}(u-\kappa)(g(x,u)-g(x,\kappa)),\\ \Psi^{\pm}(u,\kappa) &= \operatorname{sgn}_{\pm}(u-\kappa)(f(u)-f(\kappa)).\end{aligned}$$

The scheme for problem (5.1) considered is in first order Euler scheme explicit in time and finite volume scheme in space (see subsection 5.2.1). In section 5.3, the main property of this scheme is established : the "weak-BV estimate". This estimate permits to establish the existence of entropy process solution (see subsection 5.4.2) differently that in [BV05] and the convergence of the scheme.
One remarks that there exists some studies of this point, in the case of $g(x,u) = k(x)\bar{g}(u)$ with k discontinuous function and $\bar{g}$ is genuinely non linear or convex (see by example [Tow00, Tow01]). But it is just proved that a subsequence of the approximated function converges to a weak solution. In [AJV04], the authors establish the same result with g_L and g_R which only have one maximum on $[0,1]$ and not a local minimum (in particular, they impose that g_L and g_R are genuinely nonlinear).

In addition, in several works ([KRT02b, KRT03, KT04]), a conservation law with discontinuous flux function is introduced differently : the authors are interested in the problem $\partial_t u + \partial_x f(k,u) = 0$ with k a discontinuous function in space time. This problem is equivalent to a 2×2 resonant (non-strictly hyperbolic) system of conservation law

$$\partial_t k = 0 \quad \text{and} \quad \partial_t u + \partial_x f(k,u) = 0.$$

In [LTW95], the authors are interested in Glimm and Godunov scheme and in [KR95, KR01], in front tracking scheme. It is proved that these different schemes converge to a weak solution for such a problem and they assume that f is convex or genuinely non linear with respect to u. Recently, in [KRT03], the authors have established the existence and the uniqueness of entropy solution for the problem $\partial_t u + \partial_x f(k(x,t),u) = 0$ with k piecewise Lipschitz continuous and f genuinely non linear with respect to u. For such a problem, in [KT04], the authors have proved, for the first time, the convergence of the Lax-Friedrichs scheme to the entropy solution that they introduced in previous work ([KRT03]). Nevertheless, they always assume that the entropy solution must have some traces along the line of discontinuity of function k and that g is genuinely non linear.

In this paper, one proves that the approximated function, built with monotone finite volume scheme, converges to the unique entropy solution (see section 5.5). By using the convergence of the scheme, the existence of entropy process solution is obtained differently from [BV05] (see subsection 5.4.2). To establish this results, one uses some tools introduced by R. Eymard, T. Gallouët and R. Herbin (see [EGH00]) since Theorem 5.3, and by C. Chainais Hillaret and S. Champier in [CHC01] for conservation law with g Lipschitz continuous function.

5.2 Finite volume scheme

5.2.1 Presentation of the scheme

In this section, problem (5.1) is approximated by a finite volume scheme. Several works deal with this subject (see [Tow00, Tow01, SV03, AJV04, KT04]). In this paper, it is a general approach which includes previous works ([Tow00, Tow01, SV03, KT04]). However, in [KT04], the problem which is considered is not exactly the same: $u_t + f(k,u)_x = 0$ with k a discontinuous function on x. But the ideas, which are in our works, can be adapted for such a problem.

The definition of the mesh in space writes :

Definition 5.2. *An admissible mesh $\mathcal{T}$ of $\mathbb{R}$ is given by an increasing sequence of real values $(x_{i+1/2})_{i\in\mathbb{Z}}$, such that $\mathbb{R} = \bigcup_{i\in\mathbb{Z}}[x_{i-1/2}, x_{i+1/2}]$ and satisfies $x_{1/2} = 0$. The mesh $\mathcal{T}$ is the set of $\mathcal{T} = \{K_i, i \in \mathbb{Z}\}$ of subsets of $\mathbb{R}$ defined by $K_i = (x_{i-1/2}, x_{i+1/2})$ for all $i \in \mathbb{Z}$. The length of K_i is denoted by h_i, so that $h_i = x_{i+1/2} - x_{i-1/2}$ for all $i \in \mathbb{Z}$. One notes $h = \mathrm{size}(\mathcal{T}) = \mathrm{sup_{i\in\mathbb{Z}}\, h_i}$.*

Remark 5.1. *The choice of $x_{1/2} = 0$ does not lose generality.*

Consider an admissible mesh $\mathcal{T}$ in the sense of Definition 5.2 and let $k \in \mathbb{R}_+^*$ be the time step. In the general case, the finite volume 3-points scheme for the discretization of problem (5.1) can be written : $\forall i \in \mathbb{Z}$, $\forall n \in \mathbb{N}$

$$\begin{cases} \dfrac{h_i}{k}(u_i^{n+1} - u_i^n) + Q_{i+1/2}(u_i^n, u_{i+1}^n) - Q_{i-1/2}(u_{i-1}^n, u_i^n) = 0, \\ u_i^0 = \dfrac{1}{h_i}\displaystyle\int_{K_i} u_0(x)\,dx, \end{cases} \tag{5.3}$$

where u_i^n is expected to be an approximation of u at time $t_n = nk$ in cell K_i. The quantity $G_{i+1/2}(u_i^n, u_{i+1}^n)$ is the numerical flux at point $x_{i+1/2}$ and time t_n associated to the function $g(x,u) + f(u)$.

The approximate finite volume solution is defined by

$$u_{\mathcal{T},k}(x,t) = u_i^n \text{ for } x \in K_i \text{ and } t \in [nk, (n+1)k). \tag{5.4}$$

The writing (5.3) is equivalent to :

$$\begin{aligned} u_i^{n+1} &:= H_i(u_{i-1}^n, u_i^n, u_{i+1}^n) \\ &= u_i^n - \frac{k}{h_i}(Q_{i+1/2}(u_i^n, u_{i+1}^n) - Q_{i-1/2}(u_{i-1}^n, u_i^n)). \end{aligned} \tag{5.5}$$

The flux functions satisfy the following hypotheses :

(H3) **Flux adapted to g :**
$*\ \forall i \leq -1, \quad Q_{i+1/2} = Q_L, \quad \forall i \geq 1, \quad Q_{i+1/2} = Q_R,$
$*\ \forall \kappa \in [0,1], Q_{1/2}(\kappa,\kappa) \in [g_R(\kappa) + f(\kappa), g_L(\kappa) + f(\kappa)]$ or $Q_{1/2}(\kappa,\kappa) \in [g_L(\kappa) + f(\kappa), g_R(\kappa) + f(\kappa)]$.

(H4) **Regularity of flux functions :** Functions Q_L, Q_R and $Q_{1/2}$ are locally Lipschitz continuous from $\mathbb{R}^2$ to $\mathbb{R}$ and respectively admits for Lipschitz constant L_L only depending of g_L and f, L_R only depending of g_R and f, $L_{1/2}$ only depending of g and f.

(H5) **Consistency of flux Q_L and Q_R :** $\forall u \in [0,1]$, $Q_L(u,u) = g_L(u) + f(u)$ and $Q_R(u,u) = g_R(u) + f(u)$.

(H6) **Monotonicity** $(u,v) \mapsto Q_L(u,v)$, $(u,v) \mapsto Q_R(u,v)$, and $(u,v) \mapsto Q_{1/2}(u,v)$ from $[0,1]^2$ to $\mathbb{R}$, are nondecreasing respect to u and nonincreasing with respect to v.

Remark 5.2. *Hypothesis (H3) is satisfied by the schemes called scheme 1, scheme 2 and the Godunov scheme are be presented in [SV03] , for $g(u) = u(1-u)$ and $f = 0$. In fact, for the Godunov scheme, the authors impose the continuity of the numerical flux through the interface as follows :*

$$* \quad Q_{1/2}(u_0^n, u_1^n) = Q_L(u_0^n, u_1^{n-}) = Q_R(u_1^{n+}, u_2^n) \text{ with } g_L(u_1^{n-}) = g_R(u_1^{n+}).$$

and this hypothesis implies the second point of hypothesis (H3) for the Godunov scheme. In [AJV04], schemes which are presented also satisfies all these hypotheses.
In [Tow00, Tow01], the author considers a staggered scheme which satisfies (H3) (with theses notations, if one takes $x_0 = 0$). Then, the study presented here requires several adaptations to establish the convergence of this staggered scheme to the unique entropy solution of problem (5.1).

5.2.2 Monotonicity of the scheme and L^∞ estimate

Hypothesis (H7) ensures the monotonicity of the scheme under CFL condition (5.6) and this hypothesis is satisfied in all schemes presented in previous works ([Tow00, Tow01, AJV04]. The monotonicity permits to establish the L^∞ estimate under the CFL condition (5.6). One remarks that this property of the scheme is classical in the theory of the numerical schemes for conservation law with continuous flux function (see [EGH00]).

Lemma 5.1. *Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.2 and let $k \in \mathbb{R}_+^*$ be the time step. Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$.*
Let $u_{\mathcal{T},k}$ be the finite volume approximate solution defined by (5.4). Under the CFL condition

$$k \leq \frac{\inf_{i \in \mathbb{Z}} h_i}{2 \max\{L_L, L_R, L_{1/2}\}}, \tag{5.6}$$

the function H_i, for all $i \in \mathbb{Z}$ is nondecreasing with respect this all arguments and the approximation $u_{\mathcal{T},k}$ satisfies

$$0 \leq u_{\mathcal{T},k} \leq 1 \quad \textit{for a.e. } x \in \mathbb{R} \textit{ and a.e. } t \in \mathbb{R}_+, \tag{5.7}$$

as the entropy solution associated with initial condition u_0.

One wants to show that H_i, which is defined by (5.5), is nondecreasing with respect to all these arguments. To simplify, one assumes that for all $i \in \mathbb{Z}$, $Q_{i+1/2}$ is $\mathcal{C}^1$. This yields :

- $$\begin{aligned}\frac{\partial H_i}{\partial u_i^n} &= 1 - \frac{k}{h_i}(Q_{i+1/2})_u(u_i^n, u_{i+1}^n) + \frac{k}{h_i}(Q_{i-1/2})_v(u_{i-1}^n, u_i^n) \\ &\geq 1 - 2\frac{k}{h_i}\max\{L_{g_L}, L_{g_R}, L_{1/2}\} \geq 0,\end{aligned}$$

  under the CFL condition.

- $$\frac{\partial H_i}{\partial u_{i+1}^n} = -\frac{k}{h_i}(Q_{i+1/2})_v(u_i^n, u_{i+1}^n) \geq 0,$$

  because $Q_{i+1/2}$ is nonincreasing with respect to it second argument.

- $$\frac{\partial H_i}{\partial u_{i-1}^n} = \frac{k}{h_i}(Q_{i-1/2})_u(u_{i-1}^n, u_i^n) \geq 0,$$

  because $Q_{i-1/2}$ is nondecreasing with respect to it first argument.

By hypothesis, $0 \leq u_i^0 \leq 1$ a.e. on $\mathbb{R}$, then for all $i \in \mathbb{Z}$, by monotonicity one obtains :

$$H_i(0,0,0) \leq u_i^1 = H_i(u_i^0, u_{i-1}^0, u_{i+1}^0) \leq H_i(1,1,1)$$

with $H_i(0,0,0) = 0$ and $H_i(1,1,1) = 1$, because g_L and g_R are equal to zero in 0 and in 1 (see (H1)).
By induction on n, this gives (5.7).

5.2.3 Discrete entropy inequalities

In this part, some entropy inequalities satisfied by the approximate solution are established by using the monotonicity of the scheme (this point is classical, see [EGH00]) and by using the value of $H_i(\kappa,\kappa,\kappa) \neq \kappa$, generally.

This entropy inequalities are satisfied on each cell $[t^n, t^{n+1}[\times K_i$.

Theorem 5.1. *Under (H3) to (H7), let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.2 and $k \in \mathbb{R}_+^*$ be the time step. Let $\{u_i^n, i \in \mathbb{Z}, n \in \mathbb{N}\}$ be given by (5.3); then for all $\kappa \in [0,1]$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following inequality holds :*

$$|u_i^{n+1} - \kappa| \leq |u_i^n - \kappa| - \frac{k}{h_i}(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n) + \frac{k}{h_i}|\delta_i| \tag{5.8}$$

with

$$\begin{aligned} G_{i+\frac{1}{2}}^n &= Q_{i+1/2}(u_i^n \top \kappa, u_{i+1}^n \top \kappa) - Q_{i+1/2}(u_i^n \bot \kappa, u_{i+1}^n \bot \kappa), \\ \delta_i &= Q_{i+1/2}(\kappa, \kappa) - Q_{i-1/2}(\kappa, \kappa). \end{aligned}$$

Remark 5.3. *It is clear, with hypothesis (H3), that $\delta_i \neq 0$ just for $i = 0$ and $i = 1$.*

Proof :
Let $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $\kappa \in [0,1]$ and $\lambda_i := \dfrac{k}{h_i}$.
The proof is based on the monotonicity of the scheme, on the equality : $a \top b - a \bot b = |a - b|$ (for all a, b real values) and on the value of $H_i(\kappa, \kappa, \kappa) = \kappa - \lambda_i\, \delta_i$.
The proof is divided in two steps according to the sign of δ_i.

1. Assume that $\delta_i \geq 0$.
   On the one hand, by monotonicity, one gets :

$$\begin{aligned} u_i^{n+1} - \lambda_i\, \delta_i &\leq u_i^{n+1} = H_i(u_{i-1}^n, u_i^n, u_i^n) \\ &\leq H_i(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa) \end{aligned} \tag{5.9}$$

and

$$\kappa - \lambda_i\, \delta_i \leq H_i(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa), \tag{5.10}$$

then with (5.9) and (5.10)

$$(u_i^{n+1} - \lambda_i\, \delta_i) \top (\kappa - \lambda_i\, \delta_i) \leq H_i(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa),$$

and

$$(u_i^{n+1} \top \kappa) \leq H_i(u_{i-1}^n \top \kappa, u_i^n \top \kappa, u_{i+1}^n \top \kappa) + \lambda_i\, \delta_i. \tag{5.11}$$

On the other hand,

$$\kappa \geq \kappa - \lambda_i\, \delta_i \geq H_i(u_{i-1}^n \bot \kappa, u_i^n \bot \kappa, u_{i+1}^n \bot \kappa),$$

and

$$u_i^{n+1} \geq H_i(u_{i-1}^n \bot \kappa, u_i^n \bot \kappa, u_{i+1}^n \bot \kappa),$$

then

$$u_i^{n+1} \bot \kappa \geq H_i(u_{i-1}^n \bot \kappa, u_i^n \bot \kappa, u_i^n \bot \kappa). \tag{5.12}$$

Finally, withdrawing (5.11) and (5.12) this yields :

$$\begin{aligned}
|u_i^{n+1}-\kappa| &= (u_i^{n+1}\top\kappa)-(u_i^{n+1}\bot\kappa) \\
&\leq \quad H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\quad - H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
&\quad + \lambda_i\delta_i \\
&\leq \quad H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\quad - H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
&\quad + \lambda_i\,|\delta_i|. \qquad (5.13)
\end{aligned}$$

2. If $\delta_i \leq 0$, one proceeds in the same manner to obtain (5.13) : one gets

$$\begin{aligned}
|u_i^{n+1}-\kappa| &= (u_i^{n+1}\top\kappa)-(u_i^{n+1}\bot\kappa) \\
&\leq \quad H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\quad - H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
&\quad - \lambda_i\delta_i \\
&\leq \quad H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\quad - H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
&\quad + \lambda_i\,|\delta_i|. \qquad (5.14)
\end{aligned}$$

Eventually, this yields for all $\kappa \in [0,1]$

$$\begin{aligned}
|u_i^{n+1}-\kappa| \leq &\quad H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&- H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
&+ \lambda_i\,|\delta_i|.
\end{aligned}$$

Eventually,

$$\begin{aligned}
&H_i(u_{i-1}^n\top\kappa, u_i^n\top\kappa, u_{i+1}^n\top\kappa) - H_i(u_{i-1}^n\bot\kappa, u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) \\
= \; &|u_i^n-\kappa| - \lambda_i\big(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n\big).
\end{aligned}$$

Then, forall $\kappa \in [0,1]$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$|u_i^{n+1}-\kappa| \leq |u_i^n-\kappa| - \lambda_i\big(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n\big) + \lambda_i\,|\delta_i|.$$

5.3 Weak BV estimate

In this section, an estimate on the discrete derivates of the approximation solution are established. It is called "weak-BV estimate". This weak-BV estimate is a crucial point for the proof of convergence of the scheme. For conservation law with continuous flux function $g_L = g_R$, the proof of the weak BV estimate is established by R. Eymard, T. Gallouët and R. Herbin (see [EGH00]). Here, their methods are adapted to problem (5.1), the problem is considered as two problem : on the left and on the right of line $\{x = 0\}$ (because in this domain one has a conservation law with Lipschitz continuous flux function). Then, a study of the scheme around $\{x = 0\}$ is necessary.

In what follows, one introduces :

Definition 5.3. *Let a, b be real values, then one defines*
$\mathcal{C}(a,b) = \{(p,q) \in [a\bot b, a\top b]; (q-p)(b-a) \geq 0\}$.

Theorem 5.2. *Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given value. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 5.2 such that for all $i \in \mathbb{Z}$ $\alpha h \leq h_i$. Let $k \in \mathbb{R}_+^*$ satisfying the CFL condition*

$$k \leq \frac{(1-\xi)\alpha h}{2\max\{L_L, L_R, L_{1/2}\}}. \tag{5.15}$$

Let $\{u_i^n,\ i \in \mathbb{Z},\ n \in \mathbb{N}\}$ be given by the finite volume scheme (5.3). Let $R \in \mathbb{R}_+^*$ and $T \in \mathbb{R}_+^*$ and assume $h < R$ and $k < T$. Let $i_0, i_2 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ be such that $-R \in \bar{K}_{i_0}$, $R \in \bar{K}_{i_2}$ and $T \in (N_T k, (N_T+1)k]$. Then there exists $C \in R_+^*$, only depending on g, f, R, T, u_0, ξ and α, such that

$$\begin{aligned}
&\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)| \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_L(q) + f(q) - Q_L(p,q)| \\
+ &\sum_{n=0}^{N_T} k \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_R(p) + f(p) - Q_R(p,q) \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_R(q) + f(q) - Q_R(p,q)| \leq \frac{C}{\sqrt{h}}.
\end{aligned} \tag{5.16}$$

Remark 5.4 (Formal derivations of the weak BV estimate). *Approximating the solution of problem (5.1) by the finite volume scheme (5.3) (with $h_i = h$ for all i, for the sake simplicity), is equivalent (as far as approximation is concerned) to solving the equation (5.17)*

$$\partial_t u + \partial_x\big(g^\varepsilon(x,u) + f(u)\big) - \varepsilon\partial_{xx} u = 0 \tag{5.17}$$

where $\varepsilon = (h-k)/2$ under the CFL condition (5.15). One assumes that u is regular enough, with null limits for $u(t,x)$ and its derivates as $x \to \pm\infty$ and g^ε a regular function which approximates g when ε tends to zero such that $g^\varepsilon(x,u) = g(x,u)$ for

$|x| > \varepsilon$, $u \in [0,1]$ and $g^\varepsilon(x,u) \in [g_L(u), g_R(u)]$ or $\in [g_L(u), g_R(u)]$ for $|x| \leq \varepsilon$, $u \in [0,1]$. Multiplying (5.17) by u and summing over $(0,T) \times \mathbb{R}$ yields

$$\frac{1}{2}\int_{\mathbb{R}} u^2(T,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} u^2(0,x)\,dx + \int_0^T \int_{\mathbb{R}} \varepsilon(\partial_x u)^2(t,x)\,dx\,dt$$
$$+ \int_0^T \int_{\mathbb{R}} \partial_x\big(g^\varepsilon(x,u) + f(u)\big)\,u\,dx\,dt = 0$$

On the one hand, one remarks by the regularity of function u that

$$\int_0^T \int_{\mathbb{R}} \partial_x(f(u))\,u\,dx\,dt = 0.$$

On the other hand, $\partial_x(g^\varepsilon(x,u))\,u = \partial_x g^\varepsilon(x,u)u + \partial_u \bar{g}^\varepsilon$ with $\bar{g}^\varepsilon$ a regular function defined by $\partial_u \bar{g}^\varepsilon = u\partial_u g^\varepsilon(x,u)$. This yields :

$$\Big|\int_0^T \int_{\mathbb{R}} \partial_x g^\varepsilon(x,u)\,u\,dx\,dt\Big| \leq ||u||_\infty ||\partial_x g^\varepsilon||_1 \leq C_1.$$

Finally, one obtains with T sufficiently large :

$$\int_0^T \int_{\mathbb{R}} \varepsilon(\partial_x u)^2(t,x)\,dx\,dt \leq C_2$$

with C_2 depending only on g, f and u_0. This is the continuous analogous of (5.16) and one remarks that one just needs that $u_0 \in L^\infty(\mathbb{R})$ to obtain this estimate.

5.3.1 Proof of Theorem 5.2

In order to prove (5.16), one multiplies equality (5.3) by $h_i u_i^n$ and sums the result over $i = i_0, \dots, -1$ or over $i = 1, \dots, i_2$, and over $n = 0, \dots, N_T$.

Remark 5.5. *In this part, C_j denotes constants only depending on g, f, T, R, u_0, ξ, α.*

On the one hand, for $i = i_0, \dots, -1$, $Q_{i+1/2} = Q_L$ and $Q_{i-1/2} = Q_L$, the sum gives :

$$B_1 + B_2 = 0$$

where

$$B_1 = \sum_{n=0}^{N_T} \sum_{i=i_0}^{-1} h_i (u_i^{n+1} - u_i^n) u_i^n, \tag{5.18}$$

$$B_2 = \sum_{n=0}^{N_T} \sum_{i=i_0}^{-1} k\left(Q_L(u_i^n, u_{i+1}^n) - Q_L(u_{i-1}^n, u_i^n)\right) u_i^n. \tag{5.19}$$

We will study each term separately.

1. **Study of term B_2**
   A change of index permits to obtain :

$$\begin{aligned}
B_2 &= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} k(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} k(Q_L(u_{i-1}^n,u_i^n)-(g_L(u_i^n)+f(u_i^n)))u_i^n \\
&= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} k(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n)))u_i^n \\
&- \sum_{n=0}^{N_T}\sum_{i=i_0-1}^{-2} k(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_{i+1}^n)+f(u_{i+1}^n)))u_{i+1}^n \\
&= \sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} k(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n)))u_i^n \\
&\qquad -(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_{i+1}^n)+f(u_{i+1}^n)))u_{i+1}^n \\
&- \sum_{n=0}^{N_T} k(Q_L(u_{i_0-1}^n,u_{i_0}^n)-(g_L(u_{i_0}^n)+f(u_{i_0}^n)))u_{i_0}^n \\
&+ \sum_{n=0}^{N_T} k(Q_L(u_{-1}^n,u_0^n)-(g_L(u_0^n)+f(u_0^n)))u_0^n \\
&= B_2^1+B_2^2,
\end{aligned}$$

with

$$B_2^1=\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} k\Big(\big(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_i^n)+f(u_i^n))\big)u_i^n \\ -\big(Q_L(u_i^n,u_{i+1}^n)-(g_L(u_{i+1}^n)+f(u_{i+1}^n))\big)u_{i+1}^n\Big),$$

and

$$|B_2^2| \;\le\; C_1.$$

Denoting by Φ_L a primitive of the function $(.)g_L'(.)+(.)f'(.)$, an integration by parts yields, for all a,b real values

$$\begin{aligned}
\Phi_L(b)-\Phi_L(a) &= \int_a^b s\,(g_L'(s)+f'(s))\,ds \\
&= a\big(Q_L(a,b)-(g_L(a)+f(a))\big) \\
&- b\big(Q_L(a,b)-(g_L(b)+f(b))\big) \\
&- \int_a^b \big(g_L(s)+f(s)-Q_L(a,b)\big)\,ds.
\end{aligned}$$

Then, B_2^1 becomes :

$$\begin{aligned} B_2^1 &= \sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \Phi_L(u_{i+1}^n) - \Phi_L(u_{i+1}^n) \\ &+ \sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \int_{u_i^n}^{u_{i+1}^n} \big(g_L(s) + f(s) - Q_L(u_i^n, u_{i+1}^n)\big)\, ds \\ &= B_2^{1,1} + B_2^{1,2}, \end{aligned}$$

with, immediately $|B_2^{1,1}| \leq C_2$. For study term $B_2^{1,2}$, one need the following result :

Lemma 5.2. *Let $f \in \mathcal{C}(\mathbb{R})$ and $j \in \mathcal{C}(\mathbb{R}^2)$ Lipschitz continuous which satisfies for all $s \in \mathbb{R}$ $j(s,s) = f(s)$ and which is nondecreasing with respect it first argument and nonincreasing with respect it second argument. Let j_1 and j_2 be the Lipschitz constants of j with respect to its two. Let $(a,b) \in \mathbb{R}^2$, then f and j satisfy the following inequality :*

$$\begin{aligned} \int_a^b \big(f(s) - j(a,b)\big)\, ds \;\; &\geq \;\; \frac{1}{2(j_1 + j_2)} \Big(\max_{(p,q)\in\mathcal{C}(a,b)} \big(f(p) - j(p,q)\big)^2 \\ &\quad + \max_{(p,q)\in\mathcal{C}(a,b)} \big(f(q) - j(p,q)\big)^2 \Big). \end{aligned}$$

The reader can find the proof of this lemma in the Handbook of numerical analysis [EGH00] (page 915).

By using $Q_L(s,s) = g_L(s) + f(s)$ and Q_L nondecreasing with respect it first argument and nonincreasing with respect it second argument. Applying Lemma 5.2 to $g_L + f$ and Q_L, $B_2^{1,2}$ gives :

$$\begin{aligned} B_2^{1,2} \;\; \geq \;\; \frac{1}{2\,L_L} \sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \Big(& \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(g_L(p) + f(p) - Q_L(p,q)\big)^2 \\ & + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(g_L(q) + f(q) - Q_L(p,q)\big)^2 \Big). \end{aligned}$$

Then, this yields :

$$\begin{aligned} B_2 \;\; \geq \;\; & \frac{1}{2\,L_L} \sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \Big(\max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(g_L(p) + f(p) - Q_L(p,q)\big)^2 \\ & \qquad + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} \big(g_L(q) + f(q) - Q_L(p,q)\big)^2 \Big) \\ & -(C_1 + C_2). \end{aligned} \tag{5.20}$$

2. **Study of** B_1
   By using the definition of B_1 (5.18), one has

$$\begin{aligned} B_1 &= -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2 + \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^{N_T+1})^2 \\ &\geq -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1}(u_i^{n+1}-u_i^n)^2 - \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2. \end{aligned} \tag{5.21}$$

By using scheme (5.3), for $i \in \{i_0, \dots, -1\}$, with the CFL condition (5.15), this yields

$$\begin{aligned} h_i(u_i^{n+1}-u_i^n)^2 &= \frac{k^2}{h_i}\Big([Q_L(u_i^n, u_{i+1}^n) - (g_L(u_i^n) + f(u_i^n))] \\ &\quad -[Q_L(u_{i-1}^n, u_i^n) - (g_L(u_i^n) + f(u_i^n))]\Big)^2 \\ &\leq \frac{(1-\xi)k}{\max\{L_L, L_R, L_{1/2}\}} \\ &\quad \times\Big([Q_L(u_i^n, u_{i+1}^n) - (g_L(u_i^n) + f(u_i^n))]^2 \\ &\quad +[Q_L(u_{i-1}^n, u_i^n) - (g_L(u_i^n) + f(u_i^n))]^2\Big). \end{aligned}$$

Moreover, one has

$$\begin{aligned} &\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1}[Q_L(u_i^n, u_{i+1}^n) - (g_L(u_{i+1}^n) + f(u_{i+1}^n))]^2 \\ &\quad -[Q_L(u_{i-1}^n, u_i^n) - (g_L(u_i^n) + f(u_i^n))]^2 \leq C_4. \end{aligned}$$

Then, one obtains

$$\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{N_T}\sum_{i=i_0}^{-1} h_i(u_i^{n+1}-u_i^n)^2 \\
\leq\ & \frac{(1-\xi)}{2\max\{L_L, L_R, L_{1/2}\}}\Big(\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1}[Q_L(u_i^n, u_{i+1}^n) - (g_L(u_i^n) + f(u_i^n))]^2 \\
&\qquad +[Q_L(u_i^n, u_{i+1}^n) - (g_L(u_{i+1}^n) + f(u_{i+1}^n))]^2\Big) \\
&\qquad +C_5 \\
\leq\ & \frac{(1-\xi)}{2\max\{L_L, L_R, L_{1/2}\}} \\
&\times\Big(\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(p) + f(p) - Q_L(p,q)]^2 \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(q) + f(q) - Q_L(p,q)]^2\Big) \\
&\qquad +C_5.
\end{aligned} \tag{5.22}$$

By using the preceding inequality, (5.21) gives

$$\begin{aligned}
B_1 \ \geq\ & -\frac{(1-\xi)}{2\max\{L_L, L_R, L_{1/2}\}} \\
&\times\Big(\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(p) + f(p) - Q_L(p,q)]^2 \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(q) + f(q) - Q_L(p,q)]^2\Big) - C_6,
\end{aligned} \tag{5.23}$$

with $C_6 = C_5 + \frac{1}{2}\sum_{i=i_0}^{-1}(u_i^0)^2$.

3. **Final estimate**
By adding (5.20) and (5.23) and by using $B_1 + B_2 = 0$, this yields :

$$\begin{aligned}
0 \ =\ & B_1 + B_2 \\
\geq\ & \frac{\xi}{2\max\{L_L, L_R, L_{1/2}\}} \\
&\times\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(p) + f(p) - Q_L(p,q)]^2 \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)}[g_L(q) + f(q) - Q_L(p,q)]^2 \\
-\ & \bar{C}_7.
\end{aligned}$$

Then, one obtains :

$$\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_L(p)+f(p)-Q_L(p,q)]^2 \\ + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_L(q)+f(q)-Q_L(p,q)]^2 \leq C_7. \tag{5.24}$$

For $i = 2, \dots, i_2$, $Q_{i+1/2} = Q_R$ and $Q_{i-1/2} = Q_R$, in the same manner as bellows, one shows :

$$\sum_{n=0}^{N_T} k \sum_{i=2}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_R(p)+f(p)-Q_R(p,q)]^2 \\ + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_R(q)+f(q)-Q_R(p,q)]^2 \leq C_8. \tag{5.25}$$

Moreover

$$\sum_{n=0}^{N_T} k \max_{(p,q)\in\mathcal{C}(u_1^n,u_2^n)} [g_R(p)+f(p)-Q_R(p,q)]^2 \\ + \max_{(p,q)\in\mathcal{C}(u_1^n,u_2^n)} [g_R(q)+f(q)-Q_R(p,q)]^2 \leq C_9, \tag{5.26}$$

because $\sum_{n=0}^{N_T} k \leq T$.

Finally, adding (5.24), (5.25) and (5.26), this yields :

$$\sum_{n=0}^{N_T} k \sum_{i=i_0}^{-1} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_L(p)+f(p)-Q_L(p,q)]^2 \\ + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_L(q)+f(q)-Q_L(p,q)]^2 \\ + \sum_{n=0}^{N_T} k \sum_{i=1}^{i_2} \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_R(p)+f(p)-Q_R(p,q)]^2 \\ + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} [g_R(q)+f(q)-Q_R(p,q)]^2 \leq C_{12}.$$

To obtain estimate (5.16) and conclude the proof of Theorem 5.2, it is sufficient to apply the Cauchy-Schwartz inequality to the preceding inequality.

5.4 Entropy process solution

5.4.1 A property of bounded sequences in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$

Definition 5.4. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$, $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges towards u in the* nonlinear weak-$\star$ sense *if*

$$\int_\Omega h(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega h(u(x,\alpha))\psi(x)dx\,d\alpha, \quad \text{as } n \to +\infty$$
$$\forall \psi \in L^1(\Omega),\ \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{5.27}$$

Otherwise speaking, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L^\infty(\Omega \times (0,1))$ in the nonlinear weak-$\star$ sense if, for every $h \in \mathcal{C}(\mathbb{R},\mathbb{R})$, the nonlinear expression $g(u_n)$ converges in $L^\infty(\Omega)$ weak-$*$ to a limit which has the structure $\int_0^1 h(u(\cdot,\alpha))d\alpha$. The fact is, that any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$*$ sense :

Theorem 5.3. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence converging in the nonlinear weak-$\star$ sense.*

This result is established by R. Eymard T. Gallouët and R. Herbin (see [EGH00]). It is a result based on Young measure and a result of Di Perna (see [DiP85]).

This kind of convergence permits to pass to the limit in the numerical scheme and thus to show the existence of an entropy process solution, as follows.

5.4.2 Existence of entropy process solution

The notion of entropy process solution is introduced. This notion appears when one considers the convergence of the scheme by using the weak-$\star$ non linear convergence.

Definition 5.5. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ a.e. on $\mathbb{R}$. Let $u \in L^\infty(Q \times (0,1);[0,1])$. The function u is an* entropy process solution *of problem (5.1) if for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\varphi \geq 0$,*

$$\begin{aligned}
&\int_0^1 \int_Q (u(t,x,\lambda)-\kappa)^\pm \partial_t\varphi(t,x)dtdxd\lambda \\
+ &\int_0^1 \int_Q \left[\Phi^\pm(x,u(t,x,\lambda),\kappa) + \Psi^\pm(u(t,x,\lambda),\kappa)\right] \partial_x\varphi(t,x)dtdxd\lambda \\
+ &\int_\mathbb{R} (u_0(x)-\kappa)^\pm \varphi(0,x)dx \\
+ &(g_L(\kappa)-g_R(\kappa))^\pm \int_0^{+\infty} \varphi(t,0)dt \geq 0.
\end{aligned} \tag{5.28}$$

Remark 5.6. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$ and $u \in L^\infty(Q \times (0,1);[0,1])$. The function u is an* entropy process solution *of problem (5.1) iff for any $\kappa \in [0,1]$ and any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\varphi \ge 0$,*

$$\int_0^1 \int_Q |u(t,x,\lambda) - \kappa| \partial_t \varphi(t,x) dt dx d\lambda$$
$$+ \int_0^1 \int_Q [\Phi(x, u(t,x,\lambda), \kappa) + \Psi(u(t,x,\lambda),\kappa)]\, \partial_x \varphi(t,x) dt dx d\lambda$$
$$+ \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(0,x) dx + |g_L(\kappa) - g_R(\kappa)| \int_0^{+\infty} \varphi(t,0) dt \ge 0.$$

In following Theorem 5.4, the existence of entropy process solution by convergence of the scheme is established.

Theorem 5.4. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. Let $\xi \in (0,1)$ and $\alpha \in (0,1)$. Let $(\mathcal{T}_m, k_m)_m$ be a sequence of admissible meshes and time steps such that for all $m \in \mathbb{N}$, for all $i \in \mathbb{Z}$ $\alpha\, size(\mathcal{T}_m) \le h_i^m$. Assume that k_m satisfies (5.15), for $\mathcal{T} = \mathcal{T}_m$ and $k = k_m$, and size($\mathcal{T}_m$)$\to 0$ as $m \to +\infty$.*
Let $u_{\mathcal{T}_m,k_m}$ be the finite volume approximate solution defined by (5.4). Then there exists $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$ and a subsequence of $(u_{\mathcal{T}_m,k_m})_{m\in N}$ which converges to v for the weak-$\star$ nonlinear convergence as $h^m = \mathrm{size}(\mathcal{T}_\mathrm{m}) \to 0$, and v is an entropy process solution.

By Lemma 5.1, the sequence $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$ is bounded by 1 in $L^\infty(\mathbb{R}_+^* \times \mathbb{R})$. Therefore, by Theorem 5.3, there exists $v \in L^\infty(\mathbb{R}_+^* \times \mathbb{R} \times (0,1))$ such that a subsequence of $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$ converges, as m tends to ∞, towards v in the nonlinear weak-$\star$ sense. In fact, function v is an entropy process solution to problem (5.1).

Remark 5.7. *This proof of existence of entropy process solution is based on some tools used in [EGH00] to prove the existence of entropy solution of a conservation law with a Lipschitz continuous flux function. To establish that v is an entropy process solution, (5.8) is multiplied by $\frac{1}{k}\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx$ and one sums on i and n. One studies each term separately. The main new points (compared with [EGH00]) are the study around $\{x = 0\}$ (see subsection 5.4.2) and the study of the last term of discrete entropy inequality (5.8) given by $\sum_{i\in\mathbb{Z}}\sum_{n\in\mathbb{N}}|\delta_i|\frac{1}{h_i}\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx$ (see subsection 5.4.2).*

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $m \in \mathbb{N}$. Let $\mathcal{T}_m = \mathcal{T}$ et $k_m = k$. As supp(φ) is compact, there exists $T > 0$ and $R > 0$ such that supp$\varphi \subset [0,\mathrm{T}] \times [-\mathrm{R}+\mathrm{h}, \mathrm{R}-\mathrm{h}]$. Let i_0, i_2 and N_T as defined in Theorem 5.2.

Let $\kappa \in [0,1]$, multiplying (5.8) by $\frac{1}{k}\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx$, and summing for $i = i_0, \dots, i_2$ and $n = 0, \dots, N_T$, yields :

$$A_1 + A_2 \le A_3.$$

We will study each term separately.

Study of term A_1

$$\begin{aligned}
A_1 &= \sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\Big(|u_i^{n+1}-\kappa|-|u_i^n-\kappa|\Big)\frac{1}{k}\int_{t^n}^{t^{n+1}}\int_{K_i}\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}|u_i^n-\kappa|\int_{t^n}^{t^{n+1}}\int_{K_i}\frac{\varphi(t+k,x)-\varphi(t,x)}{k}\,dt\,dx \\
&\quad -\sum_{i=i_0}^{i_2}|u_i^0-\kappa|\frac{1}{k}\int_0^k\int_{K_i}\varphi(t,x)\,dt\,dx \\
&= B_1+B_2. \qquad (5.29)
\end{aligned}$$

In fact, for this term, one just uses the fact that $u_{\mathcal{T},k}$ converges to v for the weak-$\star$ non linear convergence as h tends to zero.
On the one hand, one gets

$$\begin{aligned}
B_2 &= -\sum_{i=i_0}^{i_2}|u_i^0-\kappa|\frac{1}{k}\int_0^k\int_{K_i}\varphi(t,x)\,dt\,dx \\
&= -\frac{1}{k}\int_0^k\int_{-R}^{R}|u_{\mathcal{T},0}-\kappa|\varphi(t,x)\,dt\,dx, \qquad (5.30)
\end{aligned}$$

with $u_{\mathcal{T},0}=\sum_{i\in\mathbb{Z}}u_i^0 1_{K_i}$.
However $u_{\mathcal{T},0}$ converges towards u_0 in $L^1_{loc}(\mathbb{R})$ and $\frac{1}{k}\int_0^k\varphi(t,x)\,dt$ converges towards $\varphi(0,x)$ when size($\mathcal{T}$) tends to zero. This yields :

$$B_2 \;\to\; \int_{\mathbb{R}}|u_0(x)-\kappa|\varphi(0,x)\,dt\,dx, \quad \text{with } h \text{ tends to zero.},$$

On the other hand,

$$\begin{aligned}
B_1 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}|u_i^n-\kappa|\int_{t^n}^{t^{n+1}}\int_{K_i}\frac{\varphi(t+k,x)-\varphi(t,x)}{k}\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}|u_k(t,x)-\kappa|\frac{\varphi(t+k,x)-\varphi(t,x)}{k}\,dt\,dx \\
&= -\int_0^T\int_{-R}^{R}|u_{\mathcal{T},k}(t,x)-\kappa|\frac{\varphi(t+k,x)-\varphi(t,x)}{k}\,dt\,dx.
\end{aligned}$$

$u_{\mathcal{T},k}$ converges towards v in the nonlinear weak-$\star$ sense wit $h\to 0$, then

$$\int_0^T\int_{-R}^{R}|u_{\mathcal{T},k}(t,x)-\kappa|\,dt\,dx \to_{h\to 0}\int_0^T\int_{-R}^{R}\int_0^1|v(t,x,\alpha)-\kappa|\,dt\,dx.$$

and by use the regularity if the function φ, one gets : Moreover

$$\frac{\varphi(t+k,x)-\varphi(t,x)}{k} \rightarrow_{h\to 0} \partial_t\varphi(t,x).$$

then

$$B_1 \rightarrow_{h\to 0} -\int_0^T \int_{-R}^R \int_0^1 |v(t,x,\alpha)-\kappa| \partial_t\varphi(t,x)\,dt\,dx.$$

One conclues

$$\begin{aligned}\lim_{h\to 0} A_1 &= -\int_0^T \int_{\mathbb{R}} \int_0^1 |v(t,x,\alpha)-\kappa| \partial_t\varphi(t,x)\,dt\,dx\,d\alpha \\ &\quad -\int_0^T |u_0(x)-\kappa|\varphi(0,x)\,dx. \qquad (5.31)\end{aligned}$$

Study of term A_2

Term A_2 is defined by :

$$\begin{aligned}A_2 &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T} \frac{1}{h_i}(Q^n_{i+1/2}-Q^n_{i-1/2}) \int_{t^n}^{t^{n+1}} \int_{K_i} \varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N_T} \frac{1}{h_i}(Q^n_{i+1/2}-Q^n_{i-1/2}) \int_{t^n}^{t^{n+1}} \int_{K_i} \varphi(t,x)\,dt\,dx, \qquad (5.32)\end{aligned}$$

because $supp(\varphi) \subset [-R+h, R-h]$.

The discontinuity of function g and the definition of $Q^n_{i+1/2}$ bring difficulties. Then, ones specifies various steps to establish that

$$\lim_{h\to 0} A_2 = -\int_0^1 \int_0^\infty \int_{\mathbb{R}} \big(\Phi(x,v(t,x,\alpha),\kappa)+\Psi(v(t,x,\alpha),\kappa)\big)\partial_x\varphi(t,x)\,dx\,dt\,d\alpha.$$

• At first, one proves :

$$\lim_{h\to 0} |A_2 - A_{20}| = 0 \qquad (5.33)$$

with A_{20} defined as follows :

$$\begin{aligned}A_{20} &= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T} Q^n_{i+1/2} \int_{t^n}^{t^{n+1}} \int_{K_i} \partial_x\varphi(t,x)\,dt\,dx \\ &= -\sum_{i=i_0+1}^{i_2-1}\sum_{n=0}^{N_T} \big(Q^n_{i+1/2}-Q^n_{i-1/2}\big) \int_{t^n}^{t^{n+1}} \varphi(t,x_{i+1/2})\,dt.\end{aligned}$$

The difference between these terms is majored as follows :

$$\begin{aligned}
& |A_2 - A_{20}| \\
\leq\ & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|Q^n_{i+1/2} - Q^n_{i-1/2}\big| \\
& \qquad \int_{t^n}^{t^{n+1}} \Big(\Big|\varphi(t, x_{i+1/2}) - \frac{1}{h_i}\int_{K_i} \varphi(t,x)\,dx\Big|\Big)\,dt \\
\leq\ & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|Q^n_{i+1/2} - Q^n_{i-1/2}\big| \\
& \qquad \Big(\int_{t^n}^{t^{n+1}} \frac{1}{h_i}\int_{K_i} \big|\varphi(t, x_{i+1/2}) - \varphi(t,x)\big|\,dx\Big)\,dt \\
\leq\ & \sum_{i=i_0+1}^{i_2-1} \sum_{n=0}^{N_T} \big|Q^n_{i+1/2} - Q^n_{i-1/2}\big| \mathrm{Lip}(\varphi)\,k\,h \\
\leq\ & \mathrm{Lip}(\varphi)\,h \Big(\sum_{i=i_0+1}^{-2} \sum_{n=0}^{N_T} k\,\big|Q^n_{i+1/2} - Q^n_{i-1/2}\big| \\
& \qquad + \sum_{2}^{i_2-1} \sum_{n=0}^{N_T} k\,\big|Q^n_{i+1/2} - Q^n_{i-1/2}\big| \Big) \\
+\ & \mathrm{Lip}(\varphi)\,h \sum_{i=-1}^{1} \sum_{n=0}^{N_T} k\,\big|Q^n_{i+1/2} - Q^n_{i-1/2}\big|.
\end{aligned} \tag{5.34}$$

$*$ For $i = i_0, \cdots, -2$, $Q_{i+1/2} = Q_{i-1/2} = Q_L$ and

$$\begin{aligned}
|Q^n_{i+1/2} - Q^n_{i-1/2}| \ \leq\ & |Q_L(u^n_i \top \kappa, u^n_{i+1} \top \kappa) - (g_L(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
+\ & |Q_L(u^n_i \bot \kappa, u^n_{i+1} \bot \kappa) - (g_L(u^n_i \bot \kappa) + f(u^n_i \bot \kappa))| \\
+\ & |Q_L(u^n_{i-1} \top \kappa, u^n_i \top \kappa) - (g_L(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
+\ & |Q_L(u^n_{i-1} \bot \kappa, u^n_i \bot \kappa) - (g_L(u^n_i \bot \kappa) + f(u^n_i \bot \kappa))|.
\end{aligned}$$

A study case by case, shows that

$$\begin{aligned}
& |Q_L(u^n_i \top \kappa, u^n_{i+1} \top \kappa) - (g_L(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
\leq\ & |Q_L(u^n_i, u^n_{i+1}) - (g_L(u^n_i) + f(u^n_i))|, \\
& |Q_L(u^n_i \bot \kappa, u^n_{i+1} \bot \kappa) - (g_L(u^n_i \bot \kappa) + f(u^n_i \bot \kappa))| \\
\leq\ & |Q_L(u^n_i, u^n_{i+1}) - (g_L(u^n_i) + f(u^n_i))|, \\
& |Q_L(u^n_{i-1} \top \kappa, u^n_i \bot \kappa) - (g_L(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
\leq\ & |Q_L(u^n_{i-1}, u^n_i) - (g_L(u^n_i) + f(u^n_i))|, \\
& |Q_L(u^n_{i-1} \bot \kappa, u^n_i \bot \kappa) - (g_L(u^n_i \top \kappa) + f(u^n_i \top \kappa))| \\
\leq\ & |Q_L(u^n_{i-1}, u^n_i) - (g_L(u^n_i) + f(u^n_i))|,
\end{aligned}$$

then

$$
\begin{aligned}
&\sum_{i=i_0+1}^{-2}\sum_{n=0}^{N_T} k|Q_{i+1/2}^n - Q_{i-1/2}^n| \\
\leq\ & 2\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} k\big(|Q_L(u_i^n,u_{i+1}^n) - (g_L(u_i^n)+f(u_i^n))| \\
&\qquad +|Q_L(u_i^n,u_{i+1}^n) - (g_L(u_{i+1}^n)+f(u_{i+1}^n))|\big) \\
\leq\ & 2\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} k\big(\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_L(p)+f(p)-Q_L(p,q)| \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_L(q)+f(q)-Q_L(p,q)|\big) \\
\leq\ & 2C\frac{1}{\sqrt{h}}
\end{aligned}
\tag{5.35}
$$

by using the weak-BV estimate (5.16).

$*$ For $i = 2,\cdots,i_2$, $Q_{i+1/2} = Q_{i-1/2} = Q_R$. In the same manner as what precedes, one obtains :

$$
\begin{aligned}
&\sum_{i=2}^{i_2-1}\sum_{n=0}^{N_T} k|Q_{i+1/2}^n - Q_{i-1/2}^n| \\
\leq\ & 2\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} k\big(|Q_R(u_i^n,u_{i+1}^n) - (g_R(u_i^n)+f(u_i^n))| \\
&\qquad +|Q_R(u_i^n,u_{i+1}^n) - (g_R(u_{i+1}^n)+f(u_{i+1}^n))|\big) \\
\leq\ & 2\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} k\big(\max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_R(p)+f(p)-Q_R(p,q)| \\
&\qquad + \max_{(p,q)\in\mathcal{C}(u_i^n,u_{i+1}^n)} |g_R(q)+f(q)-Q_R(p,q)|\big) \\
\leq\ & 2C\frac{1}{\sqrt{h}}.
\end{aligned}
\tag{5.36}
$$

$*$ One remarks :

$$
\sum_{i=-1}^{1}\sum_{n=0}^{N_T} k\left|Q_{i+1/2}^n - Q_{i-1/2}^n\right| \leq C\sum_{n=0}^{N_T} k \ \leq CT \tag{5.37}
$$

$*$ Finally, with (5.35), (5.36) and (5.37), (5.34) becomes :

$$
|A_2 - A_{20}| \leq C\sqrt{h} \longrightarrow 0, \text{ when } h \to 0.
$$

• Now, one shows that

$$\lim_{h\to 0} |A_{20} - \bar{A}_{20}| = 0 \tag{5.38}$$

with $\bar{A}_{20}$ defined as follows :

$$\begin{aligned}
\bar{A}_{20} &:= -\int_0^t \int_{\mathbb{R}} \big(\Phi(x,v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i} \big(\Phi(x,v,\kappa) + \Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx
\end{aligned}$$

To prove this equality, one proceeds according to the value of i.

$*$ For $i = i_0, \cdots, -1$, one has $Q_{i+1/2} = Q_L$ and for all $x \in K_i$, $\Phi(x,v,\kappa) = \Phi_L(v,\kappa) = g_L(v\top\kappa) - g_L(v\bot\kappa)$ and $\Psi(v,\kappa) = f(v\top\kappa) - f(v\bot\kappa)$.

$*$ For $i = 1, \cdots, i_2$, one has $Q_{i+1/2} = Q_R$ and for all $x \in K_i$, $\Phi(x,v,\kappa) = \Phi_R(v,\kappa) = g_R(v\top\kappa) - g_R(v\bot\kappa)$ and $\Psi(v,\kappa) = f(v\top\kappa) - f(v\bot\kappa)$.

One obtains $A_{20} = A_{20}^1 + A_{20}^2 + A_{20}^3$ and $\bar{A}_{20} = \bar{A}_{20}^1 + \bar{A}_{20}^2 + \bar{A}_{20}^3$ with

$$\begin{aligned}
A_{20}^1 &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T} G_{i+\frac{1}{2}}^n \int_{t^n}^{t^{n+1}}\int_{K_i} \partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i} \big(Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\qquad\qquad -Q_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^2 &= -\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0} \big(Q_{1/2}(u_0^n\top\kappa, u_1^n\top\kappa) - Q_{1/2}(u_0^n\bot\kappa, u_1^n\bot\kappa)\big) \\
&\qquad\qquad \partial_x\varphi(t,x)\,dt\,dx, \\
A_{20}^3 &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T} G_{i+\frac{1}{2}}^n \int_{t^n}^{t^{n+1}}\int_{K_i} \partial_x\varphi(t,x)\,dt\,dx \\
&= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i} \big(Q_R(u_i^n\top\kappa, u_{i+1}^n\top\kappa) \\
&\qquad\qquad -Q_R(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\partial_x\varphi(t,x)\,dt\,dx,
\end{aligned}$$

and

$$\begin{aligned}\bar{A}_{20}^1 &= -\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big(g_L(v\top\kappa) - g_L(v\bot\kappa)\\ &\qquad +f(v\top\kappa) - f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha,\\ \bar{A}_{20}^2 &= -\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_0}\int_0^1 \big(\Phi(x,v,\kappa)+\Psi(v,\kappa)\big)\partial_x\varphi(t,x)\,dt\,dxd\alpha,\\ \bar{A}_{20}^3 &= -\sum_{i=1}^{i_2}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \big(g_R(v\top\kappa) - g_R(v\bot\kappa)\\ &\qquad +f(v\top\kappa) - f(v\bot\kappa)\big)\,\partial_x\varphi(t,x)\,dt\,dx\,d\alpha.\end{aligned}$$

$*$ At first, one studies the difference $A_{20}^1 - \bar{A}_{20}^1$:

$$\begin{aligned}&|A_{20}^1 - \bar{A}_{20}^1| \leq\\ &\sum_{i=i_0}^{-1}\sum_{n=0}^{N_T}\int_{t^n}^{t^{n+1}}\int_{K_i}\int_0^1 \Big|\big(Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - Q_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\\ &\quad -\big(g_L(v\top\kappa) - g_L(v\bot\kappa) + f(v\top\kappa) - f(v\bot\kappa)\big)\Big||\partial_x\varphi(t,x)|\,dt\,dx\,d\alpha\end{aligned}$$

One has

$$\begin{aligned}&\Big|\big(Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - Q_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa)\big)\\ &\quad -\big(g_L(v\top\kappa) - g_L(v\bot\kappa) + f(v\top\kappa) - f(v\bot\kappa)\big)\Big|\\ \leq\ &\big|Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - (g_L+f)(u_i^n\top\kappa)\big|\\ &+\big|(g_L+f)(u_i^n\top\kappa) - (g_L+f)(v\top\kappa)\big|\\ &+\big|Q_L(u_i^n\bot\kappa, u_{i+1}^n\bot\kappa) - (g_L+f)(u_i^n\bot\kappa)\big|\\ &+\big|(g_L+f)(u_i^n\bot\kappa) - (g_L+f)(v\bot\kappa)\big|.\end{aligned}\tag{5.39}$$

Moreover,

$$\begin{aligned}&Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - g_L(u_i^n\top\kappa)\\ &= \begin{cases} 0 & \text{if} \quad \kappa \geq u_i^n \text{ and } \kappa \geq u_{i+1}^n\\ Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - g_L(u_i^n\top\kappa) & \text{if} \quad \kappa \in [u_i^n, u_{i+1}^n] \text{ and } u_i^n \leq u_{i+1}^n\\ Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - g_L(u_i^n\top\kappa) & \text{if} \quad \kappa \in [u_{i+1}^n, u_i^n] \text{ and } u_{i+1}^n \leq u_i^n\\ Q_L(u_i^n, u_{i+1}^n) - g_L(u_i^n) & \text{if} \quad \kappa < u_i^n \text{ and } \kappa < u_{i+1}^n\end{cases}\end{aligned}$$

In all cases, this yields :

$$\begin{aligned}&|Q_L(u_i^n\top\kappa, u_{i+1}^n\top\kappa) - Q_L(u_i^n\top\kappa, u_i^n\top\kappa)| \leq\\ &\qquad \max_{(p,q)\in\mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)|,\end{aligned}$$

and

$$|Q_L(u_i^n \bot \kappa, u_{i+1}^n \bot \kappa) - Q_L(u_i^n \bot \kappa, u_i^n \bot \kappa)| \leq \max_{(p,q)\in \mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)|.$$

Then, (5.39) becomes :

$$\begin{aligned} & \Big| \big(Q_L(u_i^n \top \kappa, u_{i+1}^n \top \kappa) - Q_L(u_i^n \bot \kappa, u_{i+1}^n \bot \kappa)\big) \\ & - \big(g_L(v \top \kappa) - g_L(v \bot \kappa) + f(v \top \kappa) - f(v \bot \kappa)\big)\Big| \\ \leq \ & 2 \max_{(p,q)\in \mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)| \\ & + (\mathrm{Lip}(g_L) + \mathrm{Lip}(f)) \big|(u_i^n \top \kappa) - (v \top \kappa)\big| \\ & + (\mathrm{Lip}(g_L) + \mathrm{Lip}(f)) \big|(u_i^n \bot \kappa) - (v \bot \kappa)\big| \\ \leq \ & 2 \max_{(p,q)\in \mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)| \\ & + 2\, (\mathrm{Lip}(g_L) + \mathrm{Lip}(f)) \big|u_i^n - v\big|. \end{aligned}$$

Finally, one obtains :

$$\begin{aligned} & \big|A_{20}^1 - \bar{A}_{20}^1\big| \\ \leq \ & 2||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N_T} k h_i \max_{(p,q)\in \mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)| \\ + \ & 2||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N_T} (\mathrm{Lip}(g_L) + \mathrm{Lip}(f)) \\ & \qquad \int_{t^n}^{t^{n+1}} \int_{K_i} \int_0^1 \big|u_i^n - v(t,x,\alpha)\big|\, dt\, dx\, d\alpha \\ \leq \ & 2h||\partial_x \varphi||_\infty \sum_{i=i_0}^{-1} \sum_{n=0}^{N_T} k \max_{(p,q)\in \mathcal{C}(u_i^n, u_{i+1}^n)} |g_L(p) + f(p) - Q_L(p,q)| \\ + \ & 2||\partial_x \varphi||_\infty (\mathrm{Lip}(g_L) + \mathrm{Lip}(f)) \\ & \sum_{i=i_0}^{-1} \sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}} \int_{K_i} \int_0^1 \big|u_{\mathcal{T},k}(t,x) - v(t,x,\alpha)\big|\, dt\, dx\, d\alpha. \end{aligned}$$

By using the estimate (5.16) and the nonlinear weak-$\star$ convergence which implies that $u_{\mathcal{T},k}$ converges to v in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$, this yields $\lim_{h\to 0} |A_{20}^1 - \bar{A}_{20}^1| = 0$.

$*$ In the same manner, by replacing Q_L by Q_R and g_L by g_R, one shows $\lim_{h\to 0} |A_{20}^3 - \bar{A}_{20}^3| = 0$

$*$ It remains to study the limit of A_{20}^2 and $\bar{A}_{20}^2$. One has

$$\begin{aligned}
|A_{20}^2| &\leq \sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}} \int_{K_0} \left| Q_{1/2}(u_0^n \top \kappa, u_1^n \top \kappa) - Q_{1/2}(u_0^n \perp \kappa, u_1^n \perp \kappa) \right| \\
&\qquad\qquad |\partial_x \varphi(t,x)| \, dt \, dx \\
&\leq C ||\partial_x \varphi||_\infty \sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}} \int_{K_0} dt \, dx \\
&\leq C ||\partial_x \varphi||_\infty T h_0 \\
&\leq C ||\partial_x \varphi||_\infty T h \longrightarrow 0, \text{ when } h \to 0,
\end{aligned}$$

and

$$\begin{aligned}
|\bar{A}_{20}^2| &\leq \sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}} \int_{K_0} \int_0^1 |(\Phi(x,v,\kappa) + \Psi(v,\kappa)) \, \partial_x \varphi(t,x)| \, dt \, dx d\alpha \\
&\leq C ||\partial_x \varphi||_\infty \sum_{n=0}^{N_T} \int_{t^n}^{t^{n+1}} \int_{K_0} \int_0^1 dt \, dx d\alpha \\
&\leq C ||\partial_x \varphi||_\infty T h \longrightarrow 0, \text{ when } h \to 0.
\end{aligned}$$

To sum up, one has shown (5.38).

• With (5.33) and (5.38), one concludes that

$$\lim_{h \to 0} A_2 = -\int_0^t \int_{\mathbb{R}} (\Phi(x,v,\kappa) + \Psi(v,\kappa)) \, \partial_x \varphi(t,x) \, dt \, dx. \tag{5.40}$$

Study of term A_3

Term A_3 is defined by

$$A_3 = \sum_{i=i_0}^{i_2} \sum_{n=0}^{N_T} |\delta_i| \int_{t^n}^{t^{n+1}} \frac{1}{h_i} \int_{K_i} \varphi(t,x) \, dt \, dx. \tag{5.41}$$

To find the limit of A_3, one will divide it into three parts according to values of i.

1. For $i \in \{i_0, \ldots, -1\}$, $\delta_i = Q_{i+1/2}(\kappa,\kappa) - Q_{i-1/2}(\kappa,\kappa) = Q_L(\kappa,\kappa) - Q_L(\kappa,\kappa) = 0$,
2. For $i \in \{2, \ldots, i_2\}$, $\delta_i = Q_{i+1/2}(\kappa,\kappa) - Q_{i-1/2}(\kappa,\kappa) = Q_R(\kappa,\kappa) - Q_R(\kappa,\kappa) = 0$,
3. $|\delta_0| = |Q_{1/2}(\kappa,\kappa) - Q_{-1/2}(\kappa,\kappa)| = |Q_{1/2}(\kappa,\kappa) - Q_L(\kappa,\kappa)| = |Q_{1/2}(\kappa,\kappa) - (g_L(\kappa) + f(\kappa))|$, and $|\delta_1| = |Q_{3/2}(\kappa,\kappa) - Q_{1/2}(\kappa,\kappa)| = |Q_R(\kappa,\kappa) - Q_{1/2}(\kappa,\kappa)| = |g_R(\kappa) + f(\kappa) - Q_{1/2}(\kappa,\kappa)|$.

Assuming $g_L(\kappa) > g_R(\kappa)$, (it is similar if $g_L(\kappa) \leq g_R(\kappa)$), with hypothesis (H3), this yields :

$$|\delta_0| = g_L(\kappa) + f(\kappa) - Q_{1/2}(\kappa,\kappa) \text{ and } |\delta_1| = Q_{1/2}(\kappa,\kappa) - (g_R(\kappa) + f(\kappa)).$$

Then,

$$\begin{aligned}
A_3 &= \int_0^T \Big(\frac{1}{h_0} \int_{K_0} (g_L(\kappa) + f(\kappa) - Q_{1/2}(\kappa,\kappa))\varphi(t,x)\,dx \\
&\quad + \frac{1}{h_1} \int_{K_1} (Q_{1/2}(\kappa,\kappa) - g_R(\kappa) - f(\kappa))\varphi(t,x)\,dx \Big) \\
&= g_L(\kappa) \int_0^T \frac{1}{h_0} \int_{x_{-1/2}}^0 \varphi(t,x)\,dx\,dt - g_R(\kappa) \int_0^T \frac{1}{h_1} \int_0^{x_{3/2}} \varphi(t,x)\,dx\,dt \\
&- (Q_{1/2}(\kappa,\kappa) - f(\kappa)) \int_0^T \Big(\frac{1}{h_0} \int_{x_{-1/2}}^0 \varphi(t,x)\,dx - \frac{1}{h_1} \int_0^{x_{3/2}} \varphi(t,x)\,dx \Big) dt.
\end{aligned} \tag{5.42}$$

By continuity of function φ, $\frac{1}{h_0} \int_{x_{-1/2}}^0 \varphi(t,x)\,dx$ and $\frac{1}{h_1} \int_0^{x_{3/2}} \varphi(t,x)\,dx$ tend to $\varphi(t,0)$ when h tends to zero. This yields :

$$\begin{aligned}
\lim_{h\to 0} A_3 &= \big(g_L(\kappa) - g_R(\kappa)\big) \int_0^{+\infty} \varphi(t,0)\,dt \\
&= |g_L(\kappa) - g_R(\kappa)| \int_0^{+\infty} \varphi(t,0)\,dt.
\end{aligned} \tag{5.43}$$

Final estimate

By using that $A_1 + A_2 \leq A_3$ and the limits established in previous sections ((5.31), (5.40) and (5.43)), one concludes that function v satisfies : for all $\kappa \in [0,1]$, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$

$$\begin{aligned}
&\int_0^1 \int_0^\infty \int_{\mathbb{R}} |v(t,x,\alpha) - \kappa|\, \partial_t \varphi(t,x)\,d\alpha\,dt\,dx \\
+ &\int_0^1 \int_0^\infty \int_{\mathbb{R}} (\Phi(x, v(t,x,\alpha),\kappa) + \Psi(v(t,x,\alpha),\kappa))\, \partial_x \varphi(t,x)\,d\alpha\,dx\,dt \\
+ &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\,dx + \int_0^\infty |g_L(\kappa) - g_R(\kappa)| \varphi(t,0)\,dt \geq 0.
\end{aligned}$$

Finally, one shows that function $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$ is a weak entropy process solution of problem (5.1).

Remark 5.8. *In this work, the existence of entropy solution is obtained differently from [BV05] : one uses the approximation built with numerical scheme and one proves the convergence of the scheme. However, if g_n is an approximation of function g, such that g_n is Lipschitz continuous for all n, one can prove, similarly as in [BV05] the existence of entropy solution.*

5.4.3 An other approximation of problem (5.1)

In this part, the existence of entropy process solution to problem (5.1) is established differently. This other proof does not use the numerical scheme and then can be presented an interest if one does not introduce the numerical scheme. This proof is deduced by the proof of existence in [BV05].

Let H the Heaveside function defined by :

$$H(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0. \end{cases}$$

Let $(H_n)_{n\in\mathbb{N}}$ a sequence of regular non decreasing functions such as this sequence converges towards H. Then, one defines the sequence of functions $(g_n)_{n\in\mathbb{N}}$: for all $n \in \mathbb{N}$

$$g_n(x,u) = H_n(x)g_R(u) + (1 - H_n(x))g_L(u) \qquad \forall (x,u) \in \mathbb{R} \times [0,1].$$

Then $(g_n)_{n\in\mathbb{N}}$ converges to g on $\mathbb{R} \times [0,1]$ and for all $n \in \mathbb{N}$, for all $\kappa \in [0,1]$ $g_n(.,\kappa)$ is monotone according to the sign of $g_L(\kappa) - g_R(\kappa)$.

Considers the following regularized problem :

$$\begin{cases} \partial_t u^n + \partial_x\big(g_n(x,u^n) + f(u^n)\big) = 0 \qquad (t,x) \in Q \\ u^n(0,x) = u_0(x), \qquad x \in \mathbb{R}\,, \end{cases} \tag{5.44}$$

Results of Kruzhkov in [Kru70] ensure that there exists an unique entropy solution $u^n \in L^\infty(Q)$ to problem (5.1), which, besides, satisfies the following entropy inequalities :

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (u^n(t,x) - \kappa)^{\pm}\, \partial_t\varphi(t,x)\, dt\, dx \\ + \quad &\int_0^\infty \int_{\mathbb{R}} (\Phi_n^{\pm}(x, u^n(t,x), \kappa) + \Psi^{\pm}(u^n(t,x),\kappa))\, \partial_x\varphi(t,x)\, dx\, dt \\ + \quad &\int_{\mathbb{R}} (u_0(x) - \kappa)^{\pm}\, \varphi(0,x)\, dx \\ - \quad &\int_0^\infty \int_{\mathbb{R}} \operatorname{sgn}_{\pm}(u^n - \kappa)\partial_x((g_n(x,\kappa))\varphi(t,x)\, dx\, dt \geq 0. \end{aligned} \tag{5.45}$$

with

$$\Phi_n^{\pm}(x,u,\kappa) = \operatorname{sgn}_{\pm}(u-\kappa)(g_n(x,u) - g_n(x,\kappa)).$$

Proposition 5.1. *Let $u_0 \in L^\infty(\mathbb{R};[0,1])$. Then, for all $n \in \mathbb{N}$, the entropy solution $u^n \in L^\infty(Q)$ of problem (5.44) satisfies :*

$$0 \leq u^n \leq 1 \quad a.e.$$

One introduces too $\mathfrak{L} = \max(\mathrm{Lip}(g_R), \mathrm{Lip}(g_L)) + \mathrm{Lip}(f)$.
One choices $\kappa = 0$ in (2.10), since $(u_0 - 0)^- = 0$ and $g_n(x,0) = 0$, this follows :

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (u^n(t,x))^- \, \partial_t \varphi(t,x)\, dt\, dx \\ + \quad &\int_0^\infty \int_{\mathbb{R}} (\Phi_n^-(x, u^n(t,x), 0) + \Psi^{\pm}(u^n(t,x),0))\, \partial_x \varphi(t,x)\, dx\, dt \geq 0 \\ . \end{aligned} \tag{5.46}$$

Let $R, T > 0$, let $r \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ be such that : r is non-decreasing, $r \equiv 1$ on $[0, R + \mathfrak{L}]$, $r \equiv 0$ on $[R + \mathfrak{L}T + 1, +\infty)$. The choice $\varphi(x,t) = \frac{T-t}{T}\chi_(0,T)(t) r(|x| + \mathfrak{L}t)$ in (5.46) gives

$$\begin{aligned} &-\frac{1}{T}\int_{\mathbb{R}}\int_0^T (u^n)^- r(|x| + \omega t)\, dx\, dt \\ &+ \int_{\mathbb{R}}\int_0^T \frac{T-t}{T} r'(|x| + \mathfrak{L}t) \\ &\qquad\qquad \big(\mathfrak{L}(u^n)^- + \mathrm{sgn}(x)\left(\Phi_n^-(x, u^n, 0) + \Psi_n^-(u^n, 0)\right)\big) \geq 0\,. \end{aligned}$$

Since $|\Phi_n^-(u^n, 0)| \leq \max(\mathrm{Lip}(g_L), \mathrm{Lip}(g_R))u^-$, $|\Psi^-(u,0)| \leq \mathrm{Lip}(f)u^-$ and since $r'(|x| + \mathfrak{L}t) \leq 0$ the second term of the left hand-side of the previous inequality is non-negative. Since $r(|x| + \mathfrak{L}t) = 1\,,\ \forall (x,t) \in (-R,R) \times (0,T)$ and since $r \geq 0$, the first term is upper bounded by $-\dfrac{1}{T}\displaystyle\int_{-R}^R \int_0^T u^-\, dx\, dt$ which is, by consequent, non-negative. Therefore, we have $u^- = 0$ on $(-R,R) \times (0,T)$. Letting $R, T \to +\infty$, we have $u \geq 0$ a.e.
Similarly, by choosing $\kappa = 1$ in (2.10) (with the semi-entropies $u \mapsto (u^n - 1)^+$), one proves $u^n \leq 1$ a.e.

With Proposition 5.1, this yields :

Proposition 5.2. *There exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, already noted $(u_n)_{n\in\mathbb{N}}$, and $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$ such as $(u_n)_{n\in\mathbb{N}}$ converges to u in the nonlinear weak-$\star$ sense. Moreover, u is an entropy process solution of problem (5.1).*

One recalls that for all $n \in \mathbb{N}$, u_n satisfies (2.10), then one studies the limit of each term of this inequality.

On the one hand, by nonlinear weak-$\star$ convergence, this yields :

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (u^n(t,x)-\kappa)^{\pm}\, \partial_t\varphi(t,x)\,dt\,dx \\
& + \int_0^\infty \int_{\mathbb{R}} (\Phi_n^{\pm}(u^n(t,x),\kappa)+\Psi^{\pm}(u^n(t,x),\kappa))\,\partial_x\varphi(t,x)\,dx\,dt \\
\Longrightarrow_{n\to\infty} & \int_0^1 \int_0^\infty \int_{\mathbb{R}} (u(t,x,\alpha)-\kappa)^{\pm}\,\partial_t\varphi(t,x)d\alpha\,dt\,dx \\
& + \int_0^1 \int_0^\infty \int_{\mathbb{R}} (\Phi^{\pm}(x,u(t,x,\alpha),\kappa)+\Psi^{\pm}(u(t,x,\alpha),\kappa)) \\
& \qquad \partial_x\varphi(t,x)\,d\alpha\,dx\,dt.
\end{aligned}$$

On the other hand, notice that

$$-\mathrm{sgn}_{\pm}(u^n-\kappa)\partial_x(g_n(x,\kappa))\varphi \le [\partial_x(g_n(x,\kappa)]^{\mp}\varphi.$$

As the function $g_n(.,\kappa)$ is monotone non-decreasing or non-increasing according to the sign of $g_R(\kappa)-g_L(\kappa)$, we have $[\partial_x(g_n(x,\kappa))]^{\mp} = \mathrm{sgn}_{\mp}(g_L(\kappa)-g_R(\kappa))H_n'(x)$. Therefore, the last term in equality (2.10) admits the bound

$$\begin{aligned}
& -\int_0^\infty \int_{\mathbb{R}} \mathrm{sgn}_{\pm}(u^{\varepsilon}-\kappa)\partial_x(g_n(x,\kappa))\varphi(t,x)\,dx\,dt \\
\le\ & \mathrm{sgn}_{\mp}(g_L(\kappa)-g_R(\kappa)) \int_0^\infty \int_{\mathbb{R}}^{\varepsilon} H_n'(x)\varphi(t,x)\,dx\,dt \\
=\ & \mathrm{sgn}_{\pm}(g_L(\kappa)-g_R(\kappa)) \int_0^\infty \int_{\mathbb{R}} H_n(x)\partial_x\varphi(t,x)\,dt\,dx \\
\longrightarrow_{n\to\infty}\ & (g_L(\kappa)-g_R(\kappa))^{\pm} \int_0^\infty \varphi(t,0)\,dt.
\end{aligned}$$

Then, with n tends to infinity, (2.10) gets that u satisfies $0 \le u \le 1$ a.e. and :

$$\begin{aligned}
& \int_0^1 \int_0^\infty \int_{\mathbb{R}} (u(t,x,\alpha)-\kappa)^{\pm}\,\partial_t\varphi(t,x)d\alpha\,dt\,dx \\
+ & \int_0^1 \int_0^\infty \int_{\mathbb{R}} (\Phi^{\pm}(x,u(t,x,\alpha),\kappa)+\Psi^{\pm}(u(t,x,\alpha),\kappa))\,\partial_x\varphi(t,x)\,d\alpha\,dx\,dt. \\
+ & \int_{\mathbb{R}} (u_0(x)-\kappa)^{\pm}\,\varphi(0,x)\,dt \\
+ & (g_L(\kappa)-g_R(\kappa))^{\pm} \int_0^\infty \varphi(t,0)\,dt \ge 0, \\
& \qquad \forall\kappa\in[0,1],\ \forall\varphi\in\mathcal{C}_c^\infty(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+).
\end{aligned}$$

To conclude, $u \in L^\infty(\mathbb{R}_+\times\mathbb{R}\times(0,1))$ is an entropy process solution.

5.5 Consequences of uniqueness

5.5.1 Existence of entropy solution

In part I, it is proved that the entropy process solution is the entropy solution (see section "Uniqueness of entropy process solution" of chapter 4). Then, this yields :

Theorem 5.5. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. Then there exists $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}; [0,1])$ entropy solution of problem (5.1).*

5.5.2 Convergence of the scheme

Theorem 5.6. *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. on $\mathbb{R}$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be the unique entropy solution of problem (5.1). Let $\xi \in (0,1)$ and $\alpha \in (0,1)$ be given values. Let $(\mathcal{T}_m, k_m)_m$ be a sequence of admissible meshes and time steps such that for all $m \in \mathbb{N}$, for all $i \in \mathbb{Z}$, $\alpha\, size(\mathcal{T}_m) \le h_i^m$. Assume that k_m satisfies the CFL condition (5.15), for $\mathcal{T} = \mathcal{T}_m$ and $k = k_m$, and $size(\mathcal{T}_m) \to 0$ as $m \to +\infty$.*
Let $u_{\mathcal{T}_m,k_m}$ be the finite volume approximate solution defined by (5.4). Then the sequence $(u_{\mathcal{T}_m,k_m})_{m\in N}$ converges to u in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $p \in [1,\infty)$ (and in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ for the weak-$\star$ topology) as $h^m = \mathrm{size}(\mathcal{T}_\mathrm{m}) \to 0$.

By Theorem 5.4, one knows that a subsequence of $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$ converges to the entropy process solution. By uniqueness of entropy process solution, established in chapter 4, the whole sequence that converges to the entropy process solution. Moreover, one remarks that the entropy process solution is in fact the entropy solution. Then, the sequence $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$ converges to the unique entropy solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ for the weak-$\star$ nonlinear convergence. Moreover, this yields :

$$\int_0^\infty \int_{\mathbb{R}} h(u_{\mathcal{T}_m,k_m}(t,x))\psi(t,x)\,dx\,dt \to \int_0^\infty \int_{\mathbb{R}} h(u(t,x))\psi(t,x)\,dx\,dt$$
$$\forall \psi \in L^1(\mathbb{R}^+ \times \mathbb{R}), \quad \forall h \in \mathcal{C}(\mathbb{R},\mathbb{R}). \tag{5.47}$$

Taking $h(s) = s^2$ in (5.47) and then $h(s) = s$ and ψu instead of ψ in (5.47) one obtains :

$$\int_0^\infty \int_{\mathbb{R}} \big(u_{\mathcal{T}_m,k_m}(t,x) - u(t,x)\big)^2 \psi(t,x)\,dx\,dt \to 0, \text{ as } m \to \infty,$$

for any function $\psi \in L^1(\mathbb{R}_+ \times \mathbb{R})$. From (5.47), and thanks to the L^∞ boundedness of $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$, the convergence of $(u_{\mathcal{T}_m,k_m})_{m\in\mathbb{N}}$ towards u in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $p \in [1,\infty)$ is deduced.

Bibliography

[AV02] Adimurthi, and G.D. Veerappa Gowda, *Conservation law with a discontinuous flux*, J. Math. Kyoto Univ., **42**, (2002), no. 4, 27-70.

[AJV04] Adimurthi, J. Jaffré and G.D. Veerappa Gowda, *Godunov-type methods for conservation laws with a flux function discontinuous in space time*, SIAM J; Numer Anal., **42**, (2004), no. 1, 179-208.

[Bac04] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential Equations (nov-dec 2004), no. 11-12, 1317-1338.

[BKRT] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers, *Well-possedness in* BV_t *and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units*, Numer. Math. **97** (2004), no. 1, 25-65.

[BV05] F. Bachmann and J. Vovelle *Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients*, to appear in Communications in Partial Differential Equations.

[CHC01] C. Chainais-Hillaret and S. Champier. *Finite volume schemes for nonhomogeneous scalar conservation laws : Error estimate*, Numer. Math., **88**, (2001), no. 4, 607-639.

[DiP85] R.J. DiPerna. *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88**, (1985), no. 3, 223-270.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin. *Finite volume methods*, Handbook of numerical analysis, Vol. VII, North-Holland, Amsterdam, (2000), 713-1020.

[KRT02a] K.H. Karlsen, N.H. Risebro and J.D Towers. *Upwind difference approximations for degenerate parabolic convection-diffusion equations with discontinuous coefficient*, IMA J. Numer. Anal., **22**, (2002), no. 4,623-664.

[KRT02b] K.H. Karlsen, N.H. Risebro and J.D Towers. *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron J. Differential Equations, (2002), **93**, 23 pp (electronic).

[KRT03] K.H. Karlsen, N.H. Risebro and J.D Towers. *L1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid.Selsk, (2003), 1-49, .

[KT04] K.H. Karlsen and J.D. Towers. *Convergence of the Lax-Friedrichs Scheme and Stability for Conservation Laws with a Discontinuous Space-Time Dependent Flux* preprint (http://www.math.ntnu.no/conservation/2004/005.html) (2004).

[KR01] K.H. Karlsen and N.H. Risebro. *Stability of a resonant system of conservation laws modeling polymer flow with gravitation*, J. Differential Equations, **120**, (2001), no. 2, 344-380.

[KR95] K.H. Karlsen and N.H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness ans asymptotic behaviour*. Comm. Partial Differential Equations, **20**, (1995), (11-12):1959-1990.

[LTW95] L.W. Lin, B.J. Temple, J.H. Wang. *A Comparison of convergence rates for Godunov's methods and Glimm's method in resonant nonlinear systems of conservations laws*, SIAM J. Numer. Anal., **32**, (1995), 824-840.

[SV03] N. Seguin and J. Vovelle. *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., **13** (2003), no. 2, 221-257.

[Tem82] B. Temple, *Global solution of the Cauchy problem for a class of* 2×2 *nonstrictliy hyperbolic conservation laws*. Adv. in Appl. Math., **3**, (1982), no. 3.

[Tow00] J.D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., **38**, (2000), no. 2, 681-698.

[Tow01] J.D. Towers. *A difference scheme for conservation laws with a discontinuous flux: the nonconvex case*, SIAM J. Numer. Anal., **39**, (2001), no. 4, 1197-1218.