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PROPERTIES OF THERMO-ELASTIC WAVES IN SALINE ICE

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ABSTRACT
The thermal expansion of saline ice is accompanied by the migration of liquid brine through porous space in the ice. Two previous models of this thermal expansion, proposed by Malmgren (1927) and Cox (1983), assume, respectively, zero and infinite permeability of saline ice by liquid brine. In the present paper theoretical investigations, based on Darcy's law, are used to describe thermo-elastic waves in saline ice, generated as the ice surface warms or cools. Characteristics of these thermo-elastic waves are analyzed for different values of the permeability of sea ice by brine, including zero and infinite values. The model matches known behaviour with these extreme permeabilities, and extends this understanding to sea ice with finite, non-zero permeability.

INTRODUCTION
When the temperature varies, sea ice expands and contracts. The volumetric thermal expansion coefficient of pure ice is constant across all temperatures up to melting, at approximately $1.58 \times 10^{-4} C^{-1}$. In this paper, we evaluate thermal expansion of sea ice by treating it as a permeable pure ice matrix enclosing pockets of saline water (brine). When the sea ice warms, the ice matrix melts. Since water (or brine) is denser than ice, this melting leads to a local reduction in volume. This in turn leads to a local pressure variation, which can drive brine through the ice. A full understanding of thermal expansion must therefore account for the permeability of the ice matrix to brine.

Two previous estimations of the thermal expansion of sea ice are known from the literature. For the purposes of this paper, they can be considered as two extremes on a continuum of ice permeability.

1) **Malmgren**, **1927**, proposes a model in which all brine is contained within the sea ice: this is equivalent to an impermeable ice matrix. Under this assumption, the thermal expansion coefficient of ice is constant and positive at low temperatures, but as the ice approaches 0^oC, ice melts and contributes to the brine volume, leading to an overall contraction with increasing temperature.

2) **Cox, 1983**, suggests that brine is free to move within the ice matrix, and thus has no effect on the thermal expansion. The thermal expansion coefficient of sea ice, says Cox, is constant and uninfluenced by the presence of brine.

Malmgren assumes *zero permeability*, while Cox assumes *infinite permeability.* Laboratory Experiments on thermal expansion performed with fiber optic FBG sensors (Marchenko et al, 2012-2014) have demonstrated values of the coefficient of thermal expansion lying between the values determined by the formulas of Malmgren and Cox.

In this paper, we consider a half-space of sea ice forced by a small temperature change at its upper surface. The permeability of sea ice to brine is explicitly modeled. We investigate the migration of temperature and pressure waves into the ice, and the variations in sea ice constituency.

We note that sea ice frequently contains air pockets alongside the brine. In this paper we do not account for air migration through the ice and brine. However, the analysis including air follows a similar line to the analysis presented here, and we plan to present this in future work.

MODEL EQUATIONS

The proposed model relies on conservation of mass, momentum, energy and salinity. We present equations describing these balances in turn.

Mass balance

The mass balance equation for sea ice is formulated as follows

$$\partial\rho_{si}/\partial t+\nabla\cdot\left(\rho_b\mathbf{v}_b\nu_b+\rho_i\mathbf{v}_i\nu_i\right)=0,\ \nabla=\left(\partial/\partial x,\partial/\partial y,\partial/\partial z\right), \tag{1}$$

where t is the time, x, y, and z are the spatial coordinates, ρ_{si}, ρ_b and ρ_i are the densities of sea ice, brine and pure ice, and $\mathbf{v}_b$ and $\mathbf{v}_i$ are the velocities of the brine and ice. Volumetric concentrations of pure ice (ν_i) and brine (ν_b) satisfy the condition

$$\nu_i+\nu_b=1, \tag{2}$$

and the sea ice density is determined by the formula

$$\rho_{si}=\rho_i\nu_i+\rho_b\nu_b, \tag{3}$$

where the densities of brine and pure ice are known functions of the temperature T and pressure p which will be introduced later.

The mass balance of the brine trapped in the ice is formulated as

$$\partial\left(\rho_b\nu_b\right)/\partial t+\nabla\cdot\left(\rho_b\mathbf{v}_b\nu_b\right)=Q_b, \tag{4}$$

where the source or sink of the brine Q_b due to the ice melting or brine refreezing is determined by the formula

$$Q_b=-\rho_i\partial\nu_i/\partial t. \tag{5}$$

From (1), (4) and (5) the mass balance of the pure ice follows:

$$\partial\left(\rho_i\nu_i\right)/\partial t+\nabla\cdot\left(\rho_i\mathbf{v}_i\nu_i\right)=-Q_b. \tag{6}$$

Momentum balance
The brine is interpreted as a viscous liquid with the coefficients of isothermal compressibility (K_b) and thermal expansion (α_b) determined by the formulas

$$K_b = \rho_b / (\partial \rho_b / \partial p), \alpha_b = -\rho_b^{-1} \partial \rho_b / \partial T . \tag{7}$$

The velocity of the brine is given as a sum of the ice velocity and the brine velocity relative to the ice:

$$\mathbf{v}_b = \mathbf{v}_i + \mathbf{v}_{b,v} . \tag{8}$$

The momentum balance of the brine is given by Darcy's law:

$$\nu_b \mathbf{v}_{b,v} = -k_{si,b} \mu_b^{-1} \nabla p \ , \tag{9}$$

where $k_{si,b}$ is the permeability of sea ice by brine, μ_b=1.8·10^{-3} Pa·s is the dynamic viscosity of brine, and p is the brine pressure.

The permeability of saline ice by brine is determined by the formula

$$k_{si,b} = k_{si,b0} e^{15\sqrt{\nu_b}} \ , \tag{10}$$

where the coefficient $k_{si,b0}$=10^{-13} m^2 characterizes ice permeability at low values of liquid brine content (Zhu et al, 2006). The dependence of the permeability $k_{si,b}$ on the liquid brine content ν_b is shown in Fig. 1a.

The ice is considered as a thermo-elastic material where internal stresses are formed due to the pressure in the brine and due to external stresses applied at the boundaries. It is assumed that ice deformations caused by thermal changes are so slow that inertial effects of elastic wave propagation can be ignored. At the same time ice deformations are fast enough to ignore creep effects in the ice. Therefore, equations of static equilibrium based on Hook's law are used:

$$2\mu \nu_i \nabla \cdot \boldsymbol{\varepsilon}_d - \nabla p_i = \nabla(\nu_b p), \ \boldsymbol{\sigma}_d = 2\mu \boldsymbol{\varepsilon}_d , \tag{11}$$
$$-p_i = \mathrm{K}\varepsilon_{kk} - \mathrm{K}\alpha_i (T - T_0), \ \mathrm{K} = \lambda + 2\mu/3 , \tag{12}$$

where p_i is the ice pressure, $\boldsymbol{\sigma}_d$ and $\boldsymbol{\varepsilon}_d$ are the deviators of ice stresses and ice strains, ε_{kk} is the trace of the ice strains, λ, μ and K are the elastic constants of ice, α_i is the volumetric coefficient of thermal expansion of ice, and T_0 is a reference temperature. The factor ν_i in formula (11) takes into account a reduction of the shear modulus due to the ice porosity. Numerical values of the shear modulus, the bulk modulus, and the coefficient of thermal expansion are given by the formulas

$$\mu = 3.8 \,\mathrm{GPa} , \ \mathrm{K} = 8.8 \,\mathrm{GPa} , \ \alpha_i = 1.58 \cdot 10^{-4} \ ^\circ\mathrm{C}^{-1} . \tag{13}$$

Energy balance
The energy balance is given by the heat transfer equation

$$\langle \rho c \rangle_{si} dT/dt - \rho_i L_i \partial \nu_i / \partial t = \nabla \cdot \left(\lambda_{si} \nabla T \right) - \mathrm{K} \nu_i \alpha_i (T - T_0) \nabla \cdot \mathbf{v}_i + 2\nu_b \mu_b \mathbf{e}_b \cdot \mathbf{e}_b, \tag{14}$$

where the material derivative is expressed by the formula $d/dt=\partial/\partial t+\nu_b \mathbf{v}_b \cdot \nabla$, and the coefficients of specific heat capacity and thermal conductivity are calculated by the formulas

$$\langle \rho c \rangle_{si} = \rho_i c_i \nu_i + \rho_b c_b \nu_b, \; \lambda_{si} = \lambda_i \nu_i + \lambda_b \nu_b. \tag{15}$$

Here c_i and c_b are the specific heat capacities, λ_i and λ_b are the thermal conductivities of ice and brine, L_i is the latent heat of ice, and $\mathbf{e}_b$ represents the strain rates of the brine. The second term on the left hand side of equation (14) describes the latent heat rate due to the brine freezing or ice melting. The last two terms on the right hand side of equation (14) are related to the heat produced due to the thermal expansion of ice and due to viscous energy dissipation in the brine.

Further it is assumed that the specific heat capacity and thermal conductivity of brine are equal to the specific heat capacity and thermal conductivity of water. The specific heat capacities and thermal conductivities of ice and water are given by the formulas

$$c_w = 4.23 \text{ kJ/(kg}^\circ\text{C)}, \; c_i = 2.12 \text{ kJ/(kg}^\circ\text{C)}, \; \lambda_w = 0.58 \text{ W/(m}^\circ\text{C)}, \; \lambda_i = 2.24 \text{ W/(m}^\circ\text{C)}. \tag{16}$$

The latent heat of ice is equal to $L_i = 333.4$ kJ/kg.

Salt balance
The equation of salt balance is written in the form

$$\partial \left(\rho_b \sigma_b \nu_b \right) / \partial t + \nabla \cdot \left(\rho_b \sigma_b \nu_b \mathbf{v}_b \right) = 0, \tag{17}$$

where σ_b is the brine salinity. The term $\rho_b \sigma_b \nu_b$ is equal to the mass of salt per unit volume of sea ice. The term $\rho_b \sigma_b \nu_b \mathbf{v}_b$ is equal to the salt flux.

State equations
The brine density ρ_b is calculated with the formula (Schwerdtfeger, 1963)

$$\rho_b = \rho_w (1 + S), \tag{18}$$

where ρ_w=1000 kg/m^3 is the water density and S is the fractional salt content of the brine. The fractional salt content S is calculated from the condition of thermodynamic equilibrium

$$S = \alpha T, \; \alpha = -1.82 \cdot 10^{-2} \text{ C}^{-1}, \; T > -8.2^\circ \text{ C}. \tag{19}$$

where the temperature T is calculated in Celsius degrees. Salinities of brine and sea ice are determined by the formulas

$$\sigma_b = S/(1 + S), \; \sigma_{si} = \nu_b \rho_b \sigma_b / \rho_{si}. \tag{20}$$

COEFFICIENT OF THERMAL EXPANSION IN A MODEL WITH ZERO PERMEABILITY OF ICE BY BRINE

From formulas (8) and (9) it follows that $\mathbf{v}_b=\mathbf{v}_i$ when the permeability of sea ice by brine is zero ($k_{si,b}=0$). The equation of mass balance (1) and equation of salt balance (17) are reduced to the forms

$$\partial\rho_{si}/\partial t+\nabla\cdot(\rho_{si}\mathbf{v}_i)=0,\quad \partial(\rho_b\sigma_b\nu_b)/\partial t+\nabla\cdot(\rho_b\sigma_b\nu_b\mathbf{v}_i)=0. \tag{21}$$

From (21) it immediately follows that sea ice salinity determined by the second formula in (20) satisfies the equation

$$d\sigma_{si}/dt=0,\ d/dt=\partial/\partial t+\mathbf{v}_i\cdot\nabla, \tag{22}$$

meaning that the sea ice salinity is conserved. In this case from formulas (2) and (3) and the second formula in (20), the sea ice density is expressed as follows

$$\rho_{si}=\frac{(1-\nu_a)\rho_b\rho_i\sigma_b}{\sigma_{si}\rho_i+(\sigma_b-\sigma_{si})\rho_b}. \tag{23}$$

From the first equation (23) it follows that the coefficient of thermal expansion of sea ice can be calculated with the formula

$$\alpha_{si}=-\rho_{si}^{-1}d\rho_{si}/dT. \tag{24}$$

If the dependence of ice density on temperature is specified by the formula $\rho_i=\rho_{i,0}(1-\alpha_i T)$, where $\rho_{i,0}$=917 kg/m^3, then formula (24) gives the coefficient of thermal expansion derived by Malmgren (1927).

THERMO-ELASTIC WAVES IN SALINE ICE

We now investigate the response of the sea ice to temperature changes without the assumption of zero permeability. If the temperature of the upper surface of an ice sheet changes, then these temperature changes propagate down into the ice sheet, over time and with damping. Stresses due to these temperature changes also propagate, and, in the case of sea permeable ice, may cause brine migration. This complicated interplay is modelled in this section of the paper. It is assumed that the steady state of a saline-ice-filled half space $x>0$ is described by the following characteristics

$$T=T_0,\ \nu_i=\nu_{i0},\ \nu_b=\nu_{b0},\ \mathbf{v}_i=\mathbf{v}_b=0,\ \boldsymbol{\varepsilon}_d=0,\ p=p_i=0,\ \boldsymbol{\sigma}_d=0. \tag{25}$$

Let us consider deformations of a saline ice half-space (x>0) initiated by small temperature variations around $T=T_0$ at x>0

$$T=T_0+\delta T,\ \delta T=A_T\cos\omega t,\ x=0, \tag{26}$$

where A_T is the amplitude and ω is the frequency of the temperature wave at the ice surface.

We investigate the evolution of saline ice characteristics caused by periodical variations of the temperature at the surface described by formula (26) and assume that the saline ice characteristics are represented as follows

$$\mathbf{X} = \mathbf{X}_0 + \delta\mathbf{X}. \tag{27}$$

Each component of vector $\mathbf{X}$ represents some characteristic of the saline ice, and each component of vector $\delta\mathbf{X}$ represents a perturbation of the ice characteristic near its value at the steady state specified by vector $\mathbf{X}_0$ according to the formulas

$$\mathbf{X}_0 = (T_0, p = 0, p_i = 0, \nu_{b0}, \nu_{i0}, v_{ix} = 0),\ \delta\mathbf{X} = (\delta T, \delta p, \delta p_i, \delta\nu_b, \delta\nu_i, \delta v_{ix}) \tag{28}$$

where the last term gives the velocity of the ice in the x-direction. A set of equations used for the investigation of the temporal and spatial evolution of $\delta\mathbf{X}$ is specified further in the case when all variables depend only on the time t and spatial coordinate x. Substitution of Darcy's laws (9) into equations (4) and (17) and linearization leads to the equations

$$\left(\rho_{b0,T}\frac{\partial \delta T}{\partial t} + \rho_{b0,p}\frac{\partial \delta p}{\partial t}\right)\nu_{b0} + \rho_{b0}\frac{\partial \delta\nu_b}{\partial t} + \rho_{i0}\frac{\partial \delta\nu_i}{\partial t} + \rho_{b0}\nu_{b0}\delta e_{kk} = \gamma_{b0}\frac{\partial^2 \delta p}{\partial x^2}, \tag{29}$$

,

$$\left((\rho_b\sigma_b)_{0,T}\frac{\partial \delta T}{\partial t} + (\rho_b\sigma_b)_{0,p}\frac{\partial \delta p}{\partial t}\right)\nu_{b0} + \rho_{b0}\sigma_{b0}\frac{\partial \delta\nu_b}{\partial t} + \rho_{b0}\sigma_{b0}\nu_{b0}\delta e_{kk} = \sigma_{b0}\gamma_{b0}\frac{\partial^2 \delta p}{\partial x^2}. \tag{30}$$

The difference of equation (30) and equation (29) multiplied by σ_{b0} gives

$$\rho_{b0}\nu_{b0}\sigma_{b0,T}\delta T = \rho_{i0}\sigma_{b0}\delta\nu_i. \tag{31}$$

Here and afterwards we use the designations

$$(\bullet)_0 = (\bullet)\big|_{\mathbf{X}=\mathbf{X}_0},\ (\bullet)_{0,T} = \left.\frac{\partial(\bullet)}{\partial T}\right|_{\mathbf{X}=\mathbf{X}_0},\ (\bullet)_{0,p} = \left.\frac{\partial(\bullet)}{\partial p}\right|_{\mathbf{X}=\mathbf{X}_0},\ \delta e_{kk} = \nabla\cdot\delta\mathbf{v}_i,\ \gamma_{b0} = \frac{\rho_{b0}k_{si,b0}}{\mu_b}. \tag{32}$$

The heat transfer equation (14) is written after linearization as follows

$$\langle\rho c\rangle_{si0}\,\partial\delta T/\partial t - \rho_{i0}L_i\partial\delta\nu_i/\partial t = \lambda_{si0}\partial^2\delta T/\partial x^2. \tag{33}$$

Components of the deviators of strain and stresses in the ice are expressed by the formulas

$$\delta\varepsilon_{d,xx} = \frac{2}{3}\frac{\partial\delta u_x}{\partial x},\ \delta\varepsilon_{d,yy} = \delta\varepsilon_{d,zz} = -\frac{1}{3}\frac{\partial\delta u_x}{\partial x},\ \delta\varepsilon_{d,xy} = \delta\varepsilon_{d,xz} = \delta\varepsilon_{d,yz} = 0,\ \delta\varepsilon_{kk} = \frac{\partial\delta u_x}{\partial x}, \tag{34}$$

$$\delta\sigma_{d,xx} = \frac{4\mu}{3}\frac{\partial\delta u_x}{\partial x},\ \delta\sigma_{d,yy} = \delta\sigma_{d,zz} = -\frac{2\mu}{3}\frac{\partial\delta u_x}{\partial x},\ \delta\sigma_{d,xy} = \delta\sigma_{d,xz} = \delta\sigma_{d,yz} = 0$$

where δu_x is the x-component of the displacement vector $\delta\mathbf{u}$. The ice velocity δv_{ix}=⁀(δu_x)/⁀t.

The equations of static equilibrium (11) and Hook's law (12) are differentiated with respect to time and written in the form

$$2\mu \nu_{i0}\delta e_{d,xx} - \partial \delta p_i / \partial t = \nu_{b0}\partial \delta p / \partial t, \tag{35}$$

$$-\partial \delta p_i / \partial t = \mathrm{K}\delta e_{kk} - \mathrm{K}\alpha_i \partial \delta T / \partial t, \tag{36}$$

where $\delta e_{d,xx} = \partial(\delta\varepsilon_{d,xx})/\partial t$ and $\delta e_{kk} = \partial(\delta\varepsilon_{kk})/\partial t$.

Equation (2) is written in the form

$$\delta \nu_i + \delta \nu_b = 0. \tag{37}$$

Equations (29), (31), and (35) - (37) form a closed system describing the evolution of the vector $\delta\mathbf{X}$ specified by the second formula (28). We consider the solution in the form

$$\delta\mathbf{X} = \mathrm{Re}\left\lfloor \mathbf{A}e^{i(\omega t + kx)} \right\rfloor, \ \mathbf{A} = \left(A_T, A_p, A_{pi}, A_{vb}, A_{vi}, A_{vx}\right). \tag{38}$$

Substitution of (38) in equations (31) and (33) leads to the dispersion equation

$$\omega = ik^2 / X_T, \ X_T \lambda_{si0} = \langle \rho c \rangle_{si0} - \rho_{b0}\nu_{b0}\sigma_{b0}^{-1}\sigma_{b0,T}L_i. \tag{39}$$

Substitution of (38) into (29), (31), (35)-(37) leads to

$$A_{vx} = i\omega A_{ux}, \ A_{vy} = A_{vz} = 0, \ A_{ux} = -\frac{3i\theta}{k}\left(\alpha_i + \frac{\nu_{b0}\alpha_{pT}}{\mathrm{K}}\right)A_T, \tag{40}$$

$$A_p = \alpha_{pT}A_T, \ A_{pi} = \theta(4\mu\nu_{i0}\alpha_i - 3\nu_{b0}\alpha_{pT})A_T, \tag{41}$$

$$\theta = \frac{\mathrm{K}}{4\mu\nu_{i0} + 3\mathrm{K}}, \ \alpha_{pT} = \frac{\Delta_{pT,1}}{\Delta_{pT,2}}, \ \Delta_{pT,1} = \frac{\rho_{i0}(\rho_b\sigma_b)_{0,T} - \rho_{b0}^2\sigma_{b0,T}}{\rho_{i0}\rho_{b0}\sigma_{b0}} + 3\alpha_i\theta,$$

$$\Delta_{pT,2} = \frac{ik^2\gamma_{b0}}{\omega\rho_{b0}\nu_{b0}} - \frac{\rho_{b0,p}}{\rho_{b0}} - \frac{3\nu_{b0}\theta}{\mathrm{K}}.$$

A_{ux} is the amplitude of the ice displacement u_x. The term proportional to $\rho_{b0,p}$ in the expression for $\Delta_{pT,2}$ is important only when the brine is compressible. State equation (18) assumes an incompressible model for the brine.

From Hook's law it follows that the diagonal components of the stresses, $\delta\sigma_{xx}$, $\delta\sigma_{yy}$ and $\delta\sigma_{zz}$ are expressed by the formulas

$$\delta\sigma_{xx} = \left(\frac{4\mu\nu_{i0}}{3} + \mathrm{K}\right)\frac{\partial \delta u_x}{\partial x} - \mathrm{K}\alpha_i\delta T, \ \delta\sigma_{yy} = \delta\sigma_{zz} = \left(-\frac{2\mu\nu_{i0}}{3} + \mathrm{K}\right)\frac{\partial \delta u_x}{\partial x} - \mathrm{K}\alpha_i\delta T. \tag{42}$$

Substituting formulas (38) and (40) into (42) we find the amplitudes of ice stresses $A_{\sigma xx}$, $A_{\sigma yy}$ and $A_{\sigma zz}$

$$A_{\sigma xx}=\nu_{b0}A_p,\ A_{\sigma yy}=A_{\sigma zz}=\frac{\nu_{b0}(3\mathrm{K}-2\mu\nu_{i0})A_p-6\mu\nu_{i0}\alpha_i\mathrm{K}A_T}{4\mu\nu_{i0}+3\mathrm{K}}. \tag{43}$$

The first formula (43) shows that the amplitude $A_{\sigma xx}$ of the normal stress $\delta\sigma_{xx}$ at the ice surface x=0 is proportional to the amplitude A_p of the pore pressure. The normal stress at the ice surface appears due to the hydrostatic pressure at the bottom of thin brine layer expelled from the ice. When the pore pressure increases the thickness of the liquid layer also increases and the pressure at the ice surface grows. A decrease of the pore pressure causes suction of brine into the ice from the liquid layer, decreasing of the liquid layer thickness and decreasing of the pressure at the ice surface.

From formulas (40) and (41) follows that A_{ux} and A_p tends to infinity when $\Delta_{pT,2}$=0. In this case the dispersion equation describes pressure waves propagating in sea ice without temperature variations, i.e. when A_T=0 and kA_{ux}=-$3i\theta\nu_{b0}A_p$,

$$\omega=ik^2/X_p,\ X_p=\frac{\rho_{b0}\nu_{b0}}{\gamma_{b0}}\left(\frac{\rho_{b0,p}}{\rho_{b0}}+\frac{3\nu_{b0}\theta}{\mathrm{K}}\right). \tag{44}$$

There is a resonance between the temperature waves and the pressure waves when both of the dispersion equations (39) and (44) are satisfied. The critical value of the sea ice permeability by brine when resonance occurs is found from equations (39) and (44)

$$k_{si,b,cr}=\frac{\mu_b\nu_{b0}\lambda_{si0}(\rho_{b0,p}\rho_{b0}^{-1}+3\nu_{b0}\theta\mathrm{K}^{-1})}{\langle\rho c\rangle_{si0}-\rho_{b0}\nu_{b0}\sigma_{b0}^{-1}\sigma_{b0,T}L_i}. \tag{45}$$

Let us compare the wave numbers of the temperature and the pressure waves of the same frequency ω for large and small values of the sea ice permeability $k_{si,b}$, which is related to the coefficient γ_{b0} by the last formula (32). From dispersion equation (39) it follows that the wave number of the temperature waves doesn't depend on the sea ice permeability. From dispersion equation (44) it follows that the wave number of the pressure waves tends to infinity when the ice permeability tends to zero and vice versa. Thus the pressure waves are very short and can't penetrate into the ice when the sea ice permeability is low. When the ice has high permeability the pressure waves are long and can penetrate into the ice over large distances. Limit values of the sea ice permeability $k_{si,b}$=0 and $k_{si,b}$=$\mathbb{N}$ are associated with the models of Malmgren (1927) and Cox (1983).

NUMERICAL ESTIMATES

It is helpful at this stage to try to quantify some of the effects determined by the previous equations. In numerical estimates the formula of Frankenstein and Garner (1967) is used to specify liquid brine content as a function of sea ice temperature and salinity

$$\nu_{b0}=\sigma_{si0}\left(48.185|T_0|^{-1}+0.532\right). \tag{46}$$

Thick lines in Fig. 1b show the critical values of the sea ice permeability by brine $k_{si,b,cr}$ calculated from formulas (45) and (46) with different salinities of sea ice σ_{si} calculated with formula (20). The resonance between the pressure waves and the temperature waves is not possible.

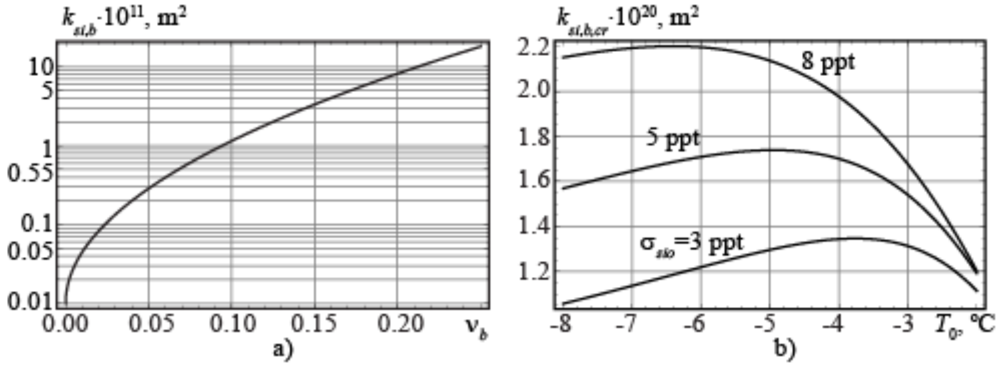

Figure 1. Sea ice permeability $k_{si,b}$ versus liquid brine content (a). Critical values of the sea ice permeability $k_{si,b,cr}$ versus the temperature constructed with different sea ice salinities (b).

The ratio of the displacement amplitude to temperature amplitude iA_{ux}/A_T characterizes vertical displacement of the surface due to periodical changes of the surface temperature of 1°C with period $T_\omega=2\pi/\omega$. The imaginary unit i shows the phase shift of $\pi/2$ with respect to the temperature phase. Figure 2 shows the ratio iA_{ux}/A_T versus the mean temperature of the ice T_0 calculated for the wave period T_ω=1 h and different values of the ice salinity. The graphs in Fig. 2a are calculated using the permeability of sea ice determined by formula (10), and the graphs in Fig. 2b are calculated with zero permeability (Malmgren's model). Numerical estimates show that graphs in Fig. 2a are practically not changing when the permeability increases, i.e. the graphs in Fig. 2a also represent Cox's model. The main difference between Fig. 2a and Fig. 2b is that the graphs in Fig. 2b extend into the range of negative values of iA_{ux}/A_T , i.e. in the range of relatively high temperatures the ice surface displaces in different directions in the models of Malmgrem and Cox. This property has been observed in experiments.

The ratio of the pore pressure amplitude to temperature amplitude $\alpha_{pT}=A_p/A_T$ characterizes the variation of the pore pressure in the brine due to periodical changes of the surface temperature of 1°C with period $T_\omega=2\pi/\omega$. Figure 3 shows the ratio α_{pT} versus the mean temperature of the ice T_0 calculated for the wave period T_ω=1 h and different values of the ice salinity. Graphs in Fig. 3a and Fig. 3b are calculated using the same values of the permeability of sea ice as Fig. 2a and Fig. 2b. Fig. 3b shows much higher absolute values of the pore pressure in comparison to Fig. 3a, and very little dependence on sea ice salinity. Negative values of α_{pT} in Fig. 3b show that the pore pressure increases when the temperature decreases and vice versa. The explanation of this is that ice melts around brine pockets when the temperature increases, and brine refreezes in the brine pockets when the temperature decreases.

The ratio of the ice pressure amplitude to temperature amplitude A_{pi}/A_T characterizes the variation of the ice pressure due to the periodical changes of the surface temperature of 1°C with period $T_\omega=2\pi/\omega$. Figure 4 shows the ratio α_{pT} versus the mean temperature of the ice T_0 calculated for the wave period T_ω=1 h and different values of the ice salinity. The graphs in Fig. 4a and Fig. 4b are calculated using the same values of the permeability of sea ice as Fig. 2a and Fig. 2b. Fig. 4a and Fig. 4b show similar absolute values of the ice pressure, and very different dependencies of the ice pressures on the sea ice salinity. In Cox's model the ice pressure increases with the temperature decrease and vice versa (Fig. 4a), while in Malmgren's model the direction of the ice pressure changes depends on the ice temperature

T_0. If the ice temperature is low then Malmgren's model shows the same changes in the ice pressure as Cox's model. If the ice temperature is high enough the pressure and temperature variations have the same direction. There are critical values of the ice temperature depending on the ice salinity when the pressure amplitudes are equal to zero.

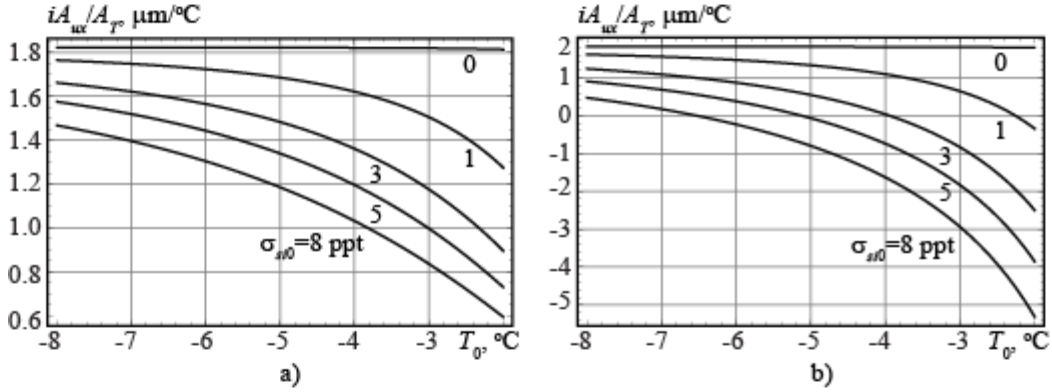

Figure 2. The ratio of the displacement amplitude to temperature amplitude iA_{ux}/A_T versus the temperature calculated with the sea ice permeability determined by formula (10) (a) and with zero permeability (b).

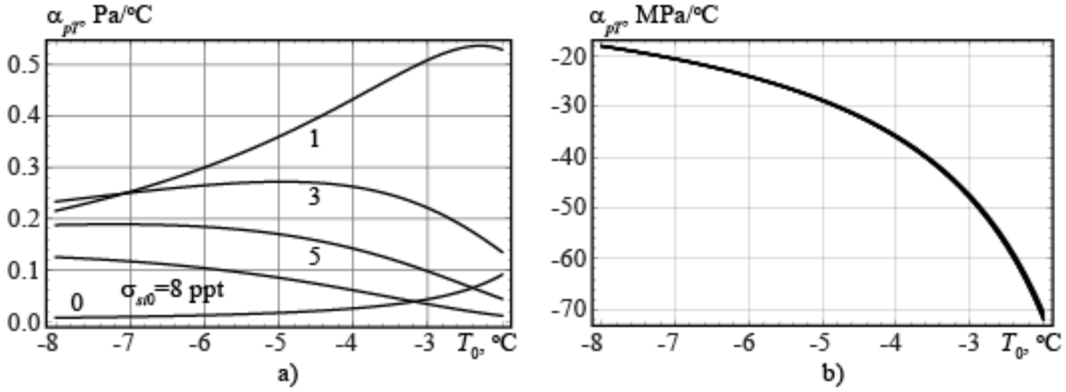

Figure 3. The ratio of the pore pressure amplitude to temperature amplitude $\alpha_{pT}=A_p/A_T$ versus the temperature calculated with the sea ice permeability determined by formula (10) (a) and with zero permeability (b).

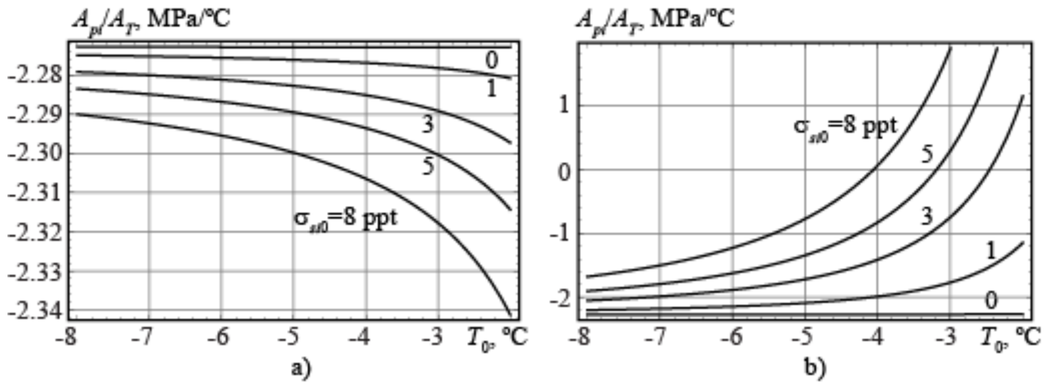

Figure 4. The ratio of the ice pressure amplitude to temperature amplitude $\alpha_{pT}=A_p/A_T$ versus the temperature calculated with the sea ice permeability determined by formula (10) (a) and with zero permeability (b).

CONCLUSIONS

A thermodynamic model of saline ice taking into account thermal changes, phase changes and liquid brine migration in the porous space is formulated and basic equations are derived. The brine migration is described by Darcy's law in the model. It is shown that the coefficient of thermal expansion of sea ice given by the new model is similar to the coefficient of thermal expansion derived by Malmgrem (1927) when the sea ice permeability by brine tends to zero. In the case when the sea ice permeability is not zero the ice salinity is not constant and thermal behavior of sea ice depends on the permeability.

We have investigated the characteristics of thermal deformations of a sea ice half-space caused by periodical changes of the surface temperature, and analyzed the dependence of these characteristics on the sea ice permeability. It is shown that the displacement and pressure characteristics of the ice caused by temperature waves are practically similar for infinite permeability of sea ice by brine (Cox, 1983) and for the values of sea ice permeability observed in experiments (see, e.g., Zhu et al, 2006). The displacement and pressure characteristics of temperature waves calculated with Cox's and Malmgren's models are very different. The transition between the models occurs at very low permeabilities ($\approx 10^{-20}$m^2).

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